Definition 1. Given \( w \in \ell^\infty \), define a bounded linear operator \( B_w : \ell^2 \to \ell^2 \) by
\[
B_w(x)(i) = w(i) \cdot x(i + 1).
\]
Such a \( B_w \) is called a unilateral weighted shift. A vector \( x \in \ell^2 \) is hypercyclic for \( B_w \) iff the set
\[
\{ B^k_w(x) : k \in \omega \}
\]
of forward iterates is dense in \( \ell^2 \). Let \( \text{HC}(w) \) denote the set of all hypercyclic vectors for \( B_w \).

It is routine to check that \( \text{HC}(w) \) is a \( G_\delta \) set for any \( w \in \ell^\infty \). The question addressed in the present paper is how much the complexity of \( \text{HC}(w) \) can be increased by looking at those sequences which are hypercyclic for many \( w \) simultaneously. Concretely, for \( W \subseteq \ell^\infty \), let
\[
X_W = \bigcap_{w \in W} \text{HC}(w).
\]
It turns out that \( X_W \) can be made arbitrarily complicated by making \( W \) sufficiently complex (Theorem 4). Even for a \( G_\delta \) set \( W \), however, the set \( X_W \) can still be non-Borel (Theorem 5).

It is necessary to introduce a few preliminaries and some terminology before proceeding. Let \( \| \cdot \|_2 \) denote the usual \( \ell^2 \) norm. In what follows, this notation will be used for finite sequences as well, i.e., for \( s \in \mathbb{R}^{<\omega} \),
\[
\| s \|_2 = \sqrt{s(0)^2 + \ldots + s(n)^2}
\]
assuming \( s \) is of length \( n + 1 \).

The notation \( |s| \) will be used to denote both the length of a string (if \( s \in 2^{<\omega} \)) and the length of an interval (if \( s \subseteq \omega \) is an interval). The notation \( \| x \|_\infty \) will denote the \( \ell^\infty \) or sup-norm of \( x \). Again, this definition makes sense for any string \( x \) – either finite or infinite. There is a relationship between the 2-norm and the sup-norm of a finite string which will be useful in what follows. Indeed, if \( s \) is a finite string of real numbers, having length \( n \), a computation shows that
\[
\| s \|_2 \leq n^{1/2} \| s \|_\infty.
\]
One of the key descriptive set theoretic concepts in this paper is that of a “pointclass”. There are many variations on the definition of “pointclass”. For the purposes of the present work, use the following definition of a pointclass $\Gamma$:

**Definition 2.** A *pointclass* $\Gamma$ is a collection of subsets of Polish (separable completely metrizable) spaces such that

- $\Gamma$ is closed under continuous preimages,
- $\Gamma$ is closed under finite unions and
- $\Gamma$ is closed under finite intersections.

Given a pointclass $\Gamma$, the *dual pointclass* $\bar{\Gamma}$ consists of those $Y$ contained in some Polish space $X$ such that $X \setminus Y \in \Gamma$. A pointclass is *non-self-dual* iff there is a Polish space $X$ and a set $Y \subseteq X$ such that $Y \in \Gamma$ but $Y \notin \bar{\Gamma}$ (equivalently, $X \setminus Y \notin \Gamma$).

To take a few examples, “closed” and “open” are dual pointclasses as are “$F_\sigma$” and “$G_\delta$”. All four of these classes are non-self-dual.

**Proposition 3.** For a Borel set $W \subseteq \ell^\infty$, the intersection $\bigcap_{w \in W} HC(w)$ is co-analytic.

**Proof.** To see this, observe that, for $y \in \ell^2$,

$$y \in \bigcap_{w \in W} HC(w) \iff (\forall w \in \ell^\infty)(w \in W \implies y \in HC(w)).$$

The key observation is that, although $\ell^\infty$ is not Polish, its Borel structure is the same as that inherited from $\mathbb{R}^\omega$ (which is Polish). Therefore, the claim that $\bigcap_{w \in W} HC(w)$ is co-analytic follows by regarding $W$ and $\ell^\infty$ as subsets of $\mathbb{R}^\omega$ and using the fact that the relation

$$P(y, w) \iff y \in HC(w)$$

is itself $G_\delta$.

The next two theorems show that the upper bound from the last proposition cannot be improved.

**Theorem 4.** Given a non-self-dual pointclass $\Gamma$ which contains the closed sets, there is a set $W \subseteq \ell^\infty$ such that $\bigcap_{w \in W} HC(w)$ is not in $\Gamma$.

**Theorem 5.** There is a Borel set $W$ such that $\bigcap_{w \in W} HC(w)$ is properly co-analytic, i.e., not analytic.

The key to proving Theorems 4 and 5 lies with the next three lemmas.
Lemma 6. If \( s \in \mathbb{R}^n \) and \( \|s\|_\infty < n^{-1/2}\varepsilon \), then \( \|s\|_2 < \varepsilon \).

Proof. Suppose that \( s \in \mathbb{R}^n \) and \( \|s\|_\infty < n^{-1/2}\varepsilon \), i.e., \( |s(i)| < n^{-1/2}\varepsilon \) for all \( i < n \). It follows that
\[
\|s\|_2 = \sqrt{s(0)^2 + \ldots + s(n-1)^2} < \sqrt{n \cdot (n^{-1/2}\varepsilon)^2} = \varepsilon
\]
This proves the lemma. \( \square \)

Lemma 7. If \( A \) is a countable set and \( f : 2^A \rightarrow \ell^2 \) is such that

1. \( f \) is continuous with respect to the product topologies on \( 2^A \) and \( \ell^2 \) (inherited from \( \mathbb{R}^\omega \)) and

2. there exists \( y \in \ell^2 \) such that \( |f(x)(i)| \leq y(i) \) for all \( x \in 2^A \) and \( i \in \omega \),

then \( f \) is continuous with respect to the norm-topology on \( \ell^2 \).

Proof. Let \( y \in \ell^2 \) be as in the statement of the lemma. Towards the goal of showing that \( f \) is \( \ell^2 \)-continuous, fix \( \varepsilon > 0 \) and let \( n \) be such that
\[
\|y \upharpoonright [n, \infty)\|_2 < \varepsilon/4.
\]
Since \( f \) is continuous into the product topology on \( \ell^2 \), let \( F \subseteq A \) be finite and such that, for \( x_1, x_2 \in 2^A \), if \( x_1 \upharpoonright F = x_2 \upharpoonright F \), then
\[
|f(x_1)(i) - f(x_2)(i)| < n^{-1/2}\varepsilon/2
\]
for all \( i < n \). In particular, \( x_1 \upharpoonright F = x_2 \upharpoonright F \) guarantees
\[
\|f(x_1) - f(x_2) \upharpoonright n\|_2 < \varepsilon/2
\]
by Lemma 6. It now follows that, whenever \( x_1, x_2 \in 2^A \) and \( x_1 \upharpoonright F = x_2 \upharpoonright F \),
\[
\|f(x_1) - f(x_2)\|_2 \leq \|f(x_1) - f(x_2) \upharpoonright n\|_2 + \|f(x_1) \upharpoonright [n, \infty)\|_2
+ \|f(x_2) \upharpoonright [n, \infty)\|_2
< \varepsilon/2 + 2\|y \upharpoonright [n, \infty)\|_2
< \varepsilon/2 + 2\varepsilon/4 = \varepsilon.
\]
Since \( \varepsilon \) was arbitrary this completes the proof. Note that a stronger result was in fact proved: \( f \) is uniformly continuous with respect to the standard ultrametric on \( 2^A \). \( \square \)
Lemma 8. Given a countable set $A$. It is possible to assign to each $a \subseteq A$, sequences $y_a \in \ell^2$ and $w_a \in \{1, 2\}^\omega$ such that

$$y_a \in \text{HC}(w_b) \iff a \not\supseteq b$$

Moreover the maps $a \mapsto y_a$ and $a \mapsto w_a$ are homeomorphism between $2^A$ and their ranges.

Before proving this lemma, it will be helpful to introduce an alternative topological basis for $\ell^2$. Given a finite string $q \in \mathbb{Q}^{<\omega}$ of rationals and a (rational) number $\varepsilon > 0$, let

$$U_{q,\varepsilon} = \{ x \in \ell^2 : \| (x \upharpoonright |q|) - q \|_\infty < \varepsilon |q|^{-1/2} \text{ and } \| x \upharpoonright [|q|, \infty) \|_2 < \varepsilon \}$$

First note that each $U_{q,\varepsilon}$ is open. In order to check that the $U_{q,\varepsilon}$ form a basis for $\ell^2$, fix a basic open ball

$$V = \{ x \in \ell^2 : \| x - x_0 \|_2 < \varepsilon \}$$

where $x_0 \in \ell^2$ and $\varepsilon > 0$ are fixed. Let $n \in \omega$ be such that

$$\| x_0 \upharpoonright [n, \infty) \|_2 < \varepsilon/4$$

and choose $q \in \mathbb{Q}^n$ such that

$$\| x_0 \upharpoonright n - q \|_2 < \varepsilon/4.$$ 

First of all, it follows from the definition of $U_{q,\varepsilon}$ that $x_0 \in U_{q,\varepsilon/4}$. To see that $U_{q,\varepsilon/4} \subseteq V$, observe that if $x \in U_{q,\varepsilon/4},$

$$\| x - x_0 \| \leq \| (x - x_0) \upharpoonright n \|_2 + \| (x - x_0) \upharpoonright [n, \infty) \|_2$$

$$\leq n^{1/2} \| (x - x_0) \upharpoonright n \|_\infty + \| x \upharpoonright [n, \infty) \|_2 + \| x_0 \upharpoonright [n, \infty) \|_2$$

$$< n^{1/2} (\| (x \upharpoonright n) - q \|_\infty + \| (x_0 \upharpoonright n) - q \|_\infty) + \varepsilon/4 + \varepsilon/4$$

$$< n^{1/2} ((\varepsilon/4)n^{-1/2} + (\varepsilon/4)n^{-1/2}) + \varepsilon/2$$

$$= \varepsilon$$

As $x \in U_{q,\varepsilon/4}$ was arbitrary, it follows that $U_{q,\varepsilon/4} \subseteq V$. Since $V$ was an arbitrary open ball, this shows that the $U_{q,\varepsilon}$ form a topological basis for $\ell^2$.

Proof of Lemma 8. Let $\pi : \omega \to \mathbb{Q}^{<\omega}$ be a surjection. Let $A$ be the fixed countable set from the statement of the lemma. for “coding” purposes, fix a bijection

$$\langle :, :, : \rangle : \omega \times (\mathbb{Q} \cap (0, 1)) \times A \to \omega.$$ 

Given $n \in \omega$, let $p_n \in \omega$, $\varepsilon_n > 0$ and $i_n \in A$ be such that

$$n = \langle p_n, \varepsilon_n, i_n \rangle.$$
Finally, let
\[ \rho_n = \min\{\varepsilon_r : r < n\}. \]

The first step of the proof is to choose a suitable partition
\[ I_0, J_0, I_1, J_1, \ldots \]
of \(\omega\) into consecutive intervals, i.e., such that \(\min(J_n) = \max(I_n) + 1\) and \(\min(I_{n+1}) = \max(J_n) + 1\). Each \(J_n\) will be chosen with \(|J_n| = |\pi(p_n)|\). The lengths of the \(I_n\) will be chosen recursively and, for concreteness, of minimal length satisfying

1. \(|I_n| \geq |I_{n-1}|\),
2. \(|I_n| > \max(J_{n-1})\) and
3. \(2^{-|I_n|} \cdot \|\pi(p_n)\|_2 \leq 2^{-n-1} \cdot \rho_n \cdot 2^{-\max(J_{n-1})} \cdot 2^{-|I_{n-1}|}\).

for \(n > 1\). The length of \(I_0\) is arbitrary – \(I_0\) can even be the empty interval.

The next step is to define the desired \(y_a\) and \(w_a\) for each \(a \subseteq A\). For \(n = \langle p, \varepsilon, i \rangle\), define \(y_a\) on each \(I_n\) and \(J_n\) by

1. \((\forall n)(y_a \upharpoonright I_n = \bar{0})\),
2. \((\forall n)(i \in a \implies y_a \upharpoonright J_n = \bar{0})\) and
3. \((\forall n)(i \notin a \implies y_a \upharpoonright J_n = 2^{-|I_n|} \cdot \pi(p)).\)

The first important observation about the map \(a \mapsto y_a\) is that it is continuous. To see this, first observe that every initial segment of \(y_a\) is determined by an initial segment of \(a\). This implies that \(a \mapsto y_a\) is continuous into the product topology on \(\ell^2\) (which it inherits from \(\mathbb{R}^\omega\)). Now invoke Lemma 7 and use the fact that \(y_a\) is always termwise bounded by \(y_0 \in \ell^2\). It now follows that \(a \mapsto y_a\) is in fact continuous with respect to the norm-topology on \(\ell^2\).

It also follows from the definition of \(y_a\) that the function \(a \mapsto y_a\) is injective. As the domain of this map \((2^A)\) is compact, \(a \mapsto y_a\) must therefore be a homeomorphism with its range.

Now define \(w_a \in \{1, 2\}^\omega\) (for \(a \subseteq A\)) by making sure that the restrictions \(w_a \upharpoonright I_n \cup J_n\) satisfy

1. \((\forall n)(i_n \notin a \implies w_a \upharpoonright I_n \cup J_n = \bar{1})\),
2. \((\forall n)(i_n \in a \implies (\forall j \in J_n) (|\{i < j : w_a(i) = 2\}| = |I_n|)\) and
3. if \(i, j \in I_n\) with \(i < j\) and \(w_a(j) = 2\), then \(w_a(i) = 2\).
The continuity of \( a \mapsto w_a \) follows from the fact that initial segments of \( w_a \) are completely determined by initial segments of \( a \).

The next three claims will complete the proof. The proofs of these three claims all follow similar arguments using the definitions of the \( y_a \) and \( w_a \).

**Claim.** Each \( y_a \) is in \( \ell^2 \).

It suffices to show that the \( \ell^2 \) norm of \( y_a \) is finite. Indeed, by the triangle inequality and the third part of the definition of \( y_a \),

\[
\|y_a\|_2 \leq \sum_{n \in \omega} \|y_a \upharpoonright J_n\|_2 \\
\leq \sum_{n \in \omega} 2^{-|I_n|} \cdot \|\pi(p_n)\|_2 \\
\leq \sum_{n \in \omega} 2^{-n-1} \cdot \rho_n \cdot 2^{-\max(J_n-1)} \cdot 2^{-|I_{n-1}|} \\
\leq \sum_{n \in \omega} 2^{-n-1} \\
\leq 1
\]

This proves the claim.

**Claim.** If \( a, b \subseteq A \) with \( a \supseteq b \), then \( y_a \notin HC(w_b) \).

For this claim, it suffices to show that \( \|B_{w_b}^k(y_a)\|_2 \leq 1 \) or \( B_{w_b}^k(y_a)(0) = 0 \) for each \( k \in \omega \). This will establish that there is no \( k \in \omega \) such that \( B_{w_b}^k(y_a) \) is in the open set

\[ U = \{ y \in \ell^2 : \|y\|_2 > 1 \text{ and } y(0) \neq 0 \}. \]

To this end, fix \( k \in \omega \) and let \( n \in \omega \) be such that \( k \in I_n \cup J_n \). First of all, if \( i_n \in a \), then \( y_a \upharpoonright I_n \cup J_n = 0 \) and hence

\[ B_{w_b}^k(y_a)(0) = w_b(0) \cdot \ldots \cdot w_b(k-1) \cdot y_a(k) = 0. \]

On the other hand, if \( i_n \notin a \supseteq b \), then \( w_b \upharpoonright I_n \cup J_n = \bar{1} \) and hence

\[ |\{ j < k : w_b(j) = 2 \}| \leq \max(J_{n-1}). \]

To obtain an estimate on \( \|B_{w_b}^k(y_a)\|_2 \), a couple preliminary observations will be useful. Suppose \( t \in \omega \) is such that \( k + t \in I_r \) for some \( r \in \omega \). In this case,

\[ B_{w_b}^k(y_a)(t) = 0 \]

since \( y_a(k + t) = 0 \). If \( k + t \in J_n \) (where \( k \in I_n \cup J_n \)), then

\[ |B_{w_b}^k(y_a)(t)| \leq 2^{\max(J_{n-1})} \cdot |y_a(k + t)| \]
since $w_b \upharpoonright I_n \cup J_n = 1$. Finally, if $k + t \in J_r$ for some $r > n$, then

$$|B^k_{w_b}(y_a)(t)| \leq 2^k \cdot |y_a(k + t)|$$

$$\leq 2^{\max(J_{r-1})}$$

since $k \leq \max(J_n) \leq \max(J_{r-1})$. It now follows by the triangle inequality that

$$\|B^k_{w_b}(y_a)\|_2 \leq \sum_{r \geq n} 2^{\max(J_{r-1})} \cdot \|y_a \upharpoonright J_r\|_2$$

$$\leq \sum_{r \geq n} 2^{\max(J_{r-1})} \cdot 2^{-r-1} \cdot \rho_r \cdot 2^{-\max(J_{r-1})} \cdot 2^{I_{r-1}}$$

$$\leq \sum_{r \geq n} 2^{-r-1}$$

$$\leq 1$$

This completes the proof of the claim.

**Claim.** If $a, b \subseteq A$ with $a \not\supseteq b$, then $y_a \in \mathbf{HC}(w_b)$.

For this final claim, it suffices to show that, for each $q \in \mathbb{Q}^{<\omega}$ and $\varepsilon > 0$, there is a $k \in \omega$ such that $B^k_{w_b}(y_a)$ is in the open set

$$U_{q,\varepsilon} = \{x \in \ell^2 : \|(x \upharpoonright |q|) - q\|_\infty < \varepsilon |q|^{-1/2} \text{ and } \|x \upharpoonright [|q|, \infty)\|_2 < \varepsilon\}$$

as these open sets form a topological basis for $\ell^2$ by remarks preceding the proof. Indeed, fix $q \in \mathbb{Q}^{<\omega}$ and let $p \in \omega$ be such that $\pi(p) = q$. Fix $i \in b \setminus a$ and let $n = \langle p, \varepsilon, i \rangle$. Since $i \in b$ and $i \not\in a$, the second case in the definition of $w_b \upharpoonright I_n \cup J_n$ and the second case in the definition of $y_a \upharpoonright J_n$ are active. In particular, for each $j \in J_n$,

$$|\{t < j : w_b(t) = 2\}| = |I_n|.$$  

It follows that

$$B^\min(J_n)_{w_b}(y_a) = \pi(p) \cdot y$$

for some $y \in \ell^2$. To show that $B^\min(J_n)_{w_b}(y_a) \in U_{q,\varepsilon}$ (for any given $\varepsilon > 0$), it now suffices to show that $\|y\|_2 < \varepsilon$, since $q \prec B^\min(J_n)_{w_b}(y_a)$ by choice of $n$. Indeed,
observe that, again by the triangle inequality,
\[
\|y\|_2 \leq 2^{\lceil I_n \rceil} \cdot \sum_{r > n} \|y_{a \upharpoonright J_r}\|_2 \\
\leq 2^{\lceil I_n \rceil} \cdot \sum_{r > n} 2^{-|I_r|} \cdot \|\pi(p_r)\|_2 \\
\leq 2^{\lceil I_n \rceil} \cdot \sum_{r > n} 2^{-r-1} \cdot \rho_r \cdot 2^{-\max(J_{r-1})} \cdot 2^{-|I_{r-1}|} \\
\leq 2^{\lceil I_n \rceil} \cdot \sum_{r > n} 2^{-r-1} \cdot \rho_n \cdot 2^{-|I_n|} \\
\leq \varepsilon \cdot \sum_{r > n} 2^{-r-1} \\
< \varepsilon
\]
since \(\rho_n \leq \varepsilon = \varepsilon_n\). This complete the proof of the claim and proves Lemma 8. \(\square\)

**Proof of Theorem 4.** Let \(P \subseteq 2^\omega\) be a perfect set such that \(a \nsubseteq b\) for any two distinct \(a, b \in P\). The construction of such a set is a standard inductive argument (similar to the construction of a perfect independent set). Let \(y_a\) and \(w_a\) be as in the lemma for all \(a \subseteq \omega\). It follows from the independence of \(P\) that \(y_a \in HC(w_b)\) iff \(a \neq b\) for all \(a, b \in P\).

Given a non-self-dual pointclass \(\Gamma\) which contains the closed sets, fix \(Y \subseteq P\) with \(Y \in \Gamma \setminus \bar{\Gamma}\). Since \(P\) is closed, it follows that \(P \setminus Y \in \bar{\Gamma} \setminus \Gamma\). Let
\[
W = \{w_a : a \in Y\}.
\]
Now consider the set
\[
X_W = \bigcap_{w \in W} HC(w).
\]
For \(a \in P\), notice that \(y_a \in X_W\) iff \(a \notin Y\). Hence,
\[
X_W \cap \{y_a : a \in P\} = \{y_a : a \in P \text{ and } a \notin Y\} = \{y_a : a \in P \setminus Y\}
\]
It follows that \(X_W \notin \Gamma\) since \(\{y_a : a \in P\}\) is closed and \(\{y_a : a \in P \setminus Y\} \in \bar{\Gamma} \setminus \Gamma\) (because \(a \mapsto y_a\) is a homeomorphism). This completes the proof of the theorem. \(\square\)

**Proof of Theorem 5.** The key to this proof is an application of Lemma 8 with the countable set \(A\) taken to be \(\omega^{<\omega}\). With this in mind, let
\[
Wf = \{T \subseteq \omega^{<\omega} : T \text{ is a well-founded subtree}\}
\]
and
\[
C = \{p \subseteq \omega^{<\omega} : p \text{ is a maximal } \prec\text{-chain}\}.
\]
In other words, $C$ may be identified with the set of infinite branches through $\omega^{<\omega}$. The set $Wf$ proper co-analytic while $C$ is $G_\delta$. Let $W = \{w_p : p \in C\}$ and notice that $W$ is also $G_\delta$ since $p \mapsto w_p$ is a homeomorphism by Lemma 8. To see that

$$X_W = \bigcap_{w \in W} \text{HC}(w)$$

is not analytic, observe that, for any subtree $T \subseteq \omega^{<\omega}$,

$$[T] = \emptyset \iff (\forall p \in C)(T \not\supset p)$$
$$\iff (\forall p \in C)(y_T \in \text{HC}(w_p)) \quad \text{(by Lemma 8)}$$
$$\iff (y_T \in X_W).$$

It follows that $Wf$ is a continuous preimage of $X_W$ under the map $T \mapsto y_T$. In turn, this implies that $X_W$ cannot be analytic.

\[ \square \]

References