ROUGH AST NOTES

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Contents

1. Definitions and Examples

I'll assume most people are familiar with the basics of Hilbert space geometry. For completeness, let's record the definition of a Hilbert space:

Definition 1.1 (Hilbert Spaces)**.** Let H be a (complex) vector space equipped with a **positive definite sesquilinear form** $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$, that is for $\eta_1, \eta_2, \xi_1, \xi_2 \in H$ and $\lambda \in \mathbb{C}$ we have

(1) $\langle \lambda\eta_1 + \eta_2, \xi_1\rangle = \lambda\langle \eta_1, \xi_1\rangle + \langle \eta_2, \xi_1\rangle$.
(2) $\langle \eta_1, \lambda\xi_1 + \xi_2\rangle = \overline{\lambda}\langle \eta_1, \xi_1\rangle + \langle \eta_1, \xi_2\rangle$.
(3) $\overline{\langle \eta_1, \xi_1\rangle} = \langle \xi_1, \eta_1\rangle$
(4) $\langle \eta_1, \eta_1\rangle \geq 0$

If moreover, we have $\langle \eta_1, \eta_1\rangle = 0$ if and only if $\eta_1 = 0$, then we say $\langle \cdot, \cdot\rangle$ is an **inner product**. If H is complete with respect to the norm $\langle \eta_1, \eta_1\rangle^{1/2} = \|\eta_1\|$, then we call H a **Hilbert space**.

Proposition 1.2 (Cauchy-Schwarz inequality)**.** *Recall that for a positive definite sesquilinear form on a vector space V we have*

$$|\langle \eta, \xi\rangle| \leq \langle \eta, \eta\rangle^{1/2}\langle \xi, \xi\rangle^{1/2}$$

for all $\eta, \xi \in V$.

Example 1.3. All separable infinite dimensional Hilbert spaces are isometrically isomorphic to

$$\ell^2(\mathbb{N}) = \{\xi : \mathbb{N} \to \mathbb{C} : \|\xi\|^2 := \sum_n |\xi(n)|^2 < \infty\}.$$

For each $n \in \mathbb{N}$ we let $e_n \in \ell^2(\mathbb{N})$ be defined by $e_n(n) = 1$ and $e_n(m) = 0$ if $n \neq m$. Then $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis for $\ell^2(\mathbb{N})$.

Definition 1.4. Let A be a complex Banach space equipped with a multiplication that makes A a ring. A is a **Banach algebra** if

$$\|ab\| \leq \|a\|\|b\| \quad \text{for all } a, b \in A. \tag{1.0.1}$$

A is called **unital**, if there is an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

Exercise 1.5. Let A be a Banach algebra. Show that multiplication is continuous.

Example 1.6. Let E be any Banach space and $T : E \to E$ a linear operator. We say T is **bounded** if

$$\|T\| := \sup_{\|x\|=1} \|T(x)\| < \infty. \tag{1.0.2}$$

We denote by $B(E)$ the Banach space of all bounded linear operators on E with the norm from (1.0.2). With multiplication defined by composition of operators, $B(E)$ is a Banach algebra. We will be particularly interested in the case when E is a Hilbert space. Note that if $\dim(E) > 1$, then $B(E)$ is non commutative

Example 1.7. Let X be a compact Hausdorff space. Let

$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous }\}.$$

Then $C(X)$ is a Banach algebra under pointwise addition and multiplication with norm defined by

$$\|f\| = \sup_{x\in X} |f(x)|.$$

Note that $C(X)$ is commutative.

Example 1.8. Let $\mathbb{D} \subseteq \mathbb{C}$ be the open unit disk. The **Disk algebra** is defined as

$$A(\overline{\mathbb{D}}) = \{f \in C(\overline{\mathbb{D}}) : f \text{ is analytic on } \mathbb{D}\}$$

By elementary complex function theory, $A(\overline{\mathbb{D}})$ is a Banach algebra. We note that $A(\overline{\mathbb{D}})$ is not isomorphic (as a Banach space) to any $C(X)$ space (see [37]).

Definition 1.9. A Banach algebra A is a **Banach *-algebra** if there is map $* : A \to A$ such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$ we have:

(1) $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$
(2) $(ab)^* = b^* a^*$
(3) $(a^*)^* = a$
(4) $\|a^*\| = \|a\|$

The map * is called an **involution**.

Exercise 1.10. Let A be a unital Banach *-algebra. Prove that $1^* = 1$.

Example 1.11. Let Γ be a group. Then the Banach space $\ell^1(\Gamma)$ is a Banach *-algebra with convolution as the product:

$$f * g(t) = \sum_{s \in \Gamma} f(s) g(s^{-1} t)$$

and involution defined as

$$f^*(t) = \overline{f(t^{-1})}$$

Definition 1.12. An involutive Banach algebra A is a C^*-algebra if:

$$\|a^* a\| = \|a\|^2 \quad \text{for all } a \in A. \tag{1.0.3}$$

Example 1.13 (Example 1.7 cont'd)**.** $C(X)$ is a C*-algebra with involution defined by

$$f^*(x) = \overline{f(x)} \quad \text{for } x \in X.$$

Example 1.14. If A is a C*-algebra and $B \subseteq A$ is a norm-closed, *-closed subalgebra, then B is a C*-algebra. If $a \in A$, then

$$a^* A a = \overline{\{}a^* x a : x \in A\} \tag{1.0.4}$$

is a C*-subalgebra of A. Such an algebra is a **hereditary** subalgebra of A (see Definition 3.37)

Definition 1.15. Let A be a C*-algebra and $X \subseteq A$ be a subset. We denote by $C^*(X) \subseteq A$ as the C*-algebra generated by X.

Exercise 1.16. Let Γ be a nontrivial group. Prove that the Banach *-algebra $\ell^1(\Gamma)$ is not a C*-algebra.

Example 1.17. Let H be a Hilbert space and $T \in B(H)$. Let $T^* \in B(H)$ be the adjoint of T (i.e. the unique operator satisfying the equations $\langle T(\eta), \xi \rangle = \langle \eta, T^*(\xi) \rangle$ for all $\eta, \xi \in H$.) Then $B(H)$ is a C*-algebra under the involution $T \mapsto T^*$. Let's prove that.

$$\begin{aligned} \|T\|^2 &= \sup_{\|\xi\|=1} \|T\xi\|^2 \\ &= \sup_{\|\xi\|=1} \langle T^* T \xi, \xi \rangle \\ &\leq \|T^* T\| \quad \text{CS-inequality} \\ &\leq \|T^*\| \|T\| \end{aligned}$$

One then uses the above inequality to deduce both $\|T\| = \|T^*\|$ and $\|T\|^2 = \|T^* T\|$.

Example 1.18 (Compact Operators)**.** Let H be an infinite dimensional Hilbert space. A linear operator $T : H \to H$ is called **compact** if $\overline{T(B_1)} \subseteq H$ is compact, where B_1 denotes the unit ball of H. Clearly all finite rank operators are compact. It is a non-trivial fact that a linear operator T is compact, if and only if it is a norm limit of finite rank operators (see [24, Theorem 3.3.3]). We denote by $K(H)$ the space of all compact operators on H.

Exercise 1.19. Prove that $K(H)$ is a C*-algebra and a two-sided ideal in $B(H)$.

Example 1.20. Let $A_1, ..., A_n$ be C*-algebras. We define the **direct sum** of these algebras as

$$\bigoplus_{i=1}^{n} A_i = \{(a_1, ..., a_n) : a_i \in A_i\}$$

with algebraic operations defined pointwise and $\|(a_1, ..., a_n)\| = \max\{\|a_1\|, ..., \|a_n\|\}$.

Example 1.21 (Finite Dimensional C*-algebras)**.** In Example 1.17, when $\dim(H) = n$, we write M_n for $B(H)$. We recall from elementary linear algebra that for an $n \times n$ matrix $a \in M_n$, one has $(a^*)_{ij} = \overline{a_{ji}}$ ("the adjoint of A is its conjugate transpose"). If A is any finite-dimensional C*-algebra, then there are integers $n_1, ..., n_k$ such that

$$A \cong \bigoplus_{i=1}^{k} M_{n_i}.$$

See [31, Theorem I.11.2] for a proof of this characterization of finite dimensional C*-algebras.

Example 1.22 (Homogeneous C*-algebras)**.** Let X be a compact Hausdorff space and A a C*-algebra. We define

$$C(X, A) = \{f : X \to A : f \text{ is continuous }\}$$

Then $C(X, A)$ is a C*-algebra under pointwise operations and sup norm. In the case where $A = M_n$, we call $C(X, M_n)$ a **homogeneous** C*-algebra (this terminology comes from the fact that all irreducible representations of a homogeneous C*-algebra have the same dimension).

Example 1.23 (Subhomogeneous C*-algebras)**.** A C*-algebra A is called **subhomogeneous** if it is a subalgebra of a homogeneous C*-algebra.

Example 1.24 (Dimension Drop Algebras)**.** Fix $n \in \mathbb{N}$ and consider

$$D = \{f \in C([0, 1]\, M_n) : f(0), f(1) \in \mathbb{C}1\}.$$

Then $D \subseteq C([0, 1], M_n)$ is a subhomogeneous C*-algebra.

Example 1.25. The above example can be generalized in a variety of ways. For example, fix a matrix algebra M_k and sub C*-algebras $B_1, ..., B_n$. Let X be a compact Hausdorff space and $X_1, ..., X_n$ be (closed) subsets of X. Then define the subalgebra of $C(X, M_k)$ as the set of continuous functions $f : X \to M_k$ such that $f(x) \in B_i$ when $x \in X_i$ for $i = 1, ..., n$. Of course, one can replace M_k with any C*-algebra (you just may not have a subhomogeneous C*-algebra). What we have just described is a special case of the construction known as **continuous fields** of C*-algebras. See [11, Chapter 10] for lots of information about continuous fields.

1.1. **Unitization.** C*-algebras need not be unital (e.g. Example 1.18). Sometimes it's convenient to work with unital C*-algebras. The following simple construction shows that we can always adjoin a unit to a C*-algebra in a minimal way. Let A be a C*-algebra (unital or not). We define the unitization of A as the vector space $A \oplus \mathbb{C}$ with multiplication and involution defined by

$$(x_1, \lambda_1)(x_2, \lambda_2) = (x_1 x_2 + \lambda_2 x_1 + \lambda_1 x_2, \lambda_1 \lambda_2), \quad (x, \lambda)^* = (x^*, \overline{\lambda}).$$

We write $\widetilde{A}$ for the unitization of A. There isn't an immediately obvious choice for a norm on $\widetilde{A}$ that makes it a C*-algebra. Nonetheless, it can be done as follows (see [28, Exercise 1.3]): Let $x = (a, \lambda) \in \widetilde{A}$ and define

$$\|x\| := \max\{|\lambda|, \sup_{b \in A, \|b\| \leq 1} \|ab\|\}.$$

Exercise 1.26. Show that $\widetilde{A}$ is a C*-algebra.

2. Spectrum

I highly recommend Gert Pedersen's *Analysis Now* [24, Chapter 4 + Exercises] as a reference for everything in this section. Murphy's book [21] is also usually included as a standard reference for the material in this section (I haven't personally read it though).

Definition 2.1. Let A be a unital Banach algebra. An element $a \in A$ is called **left invertible** if there is a $b \in A$ such that $ba = 1$ (and similarly we define **right invertible**). We say a is **invertible** if it is both left and right invertible.

Exercise 2.2. Let A be a unital Banach algebra and $a \in A$ with $\|a\| < 1$. Then $1 - a$ is invertible with $(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n$. In particular, if $\|b - 1\| < 1$, then b is invertible.

Definition 2.3. Let A be a C*-algebra or a unital Banach algebra. We define the **spectrum** of a as

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible }\}$$

If A is not unital, it is understood that $\lambda - a \in \widetilde{A}$. The **spectral radius** of a, is defined by

$$r(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(a)\}.$$

Remark 2.4. If $a \in M_n$, then $\mathrm{sp}(a)$ is just the set of eigenvalues of a.

Exercise 2.5. Let $f \in C(X)$ for some compact Hausdorff space. Show that $\mathrm{sp}(f) = \mathrm{Range}(f)$.

Lemma 2.6. *Let a, b be in a Banach algebra A. Then $sp(ab) \cup \{0\} = sp(ba) \cup \{0\}$.*

Proof. If $\lambda \notin \mathrm{sp}(ab) \cup \{0\}$, then one checks that

$$(\lambda - ba)^{-1} = \frac{1}{\lambda}(1 + b(\lambda - ab)^{-1} a).$$

□

Definition 2.7. For a subset $X \subseteq \mathbb{C}$ and $\lambda \in \mathbb{C}$ we define the sets $\lambda + X = \{\lambda + x : x \in X\}$, $\mathrm{conj}(X) = \{\overline{x} : x \in X\}$, and $\lambda X = \{\lambda x : x \in X\}$

Exercise 2.8. Let A be a unital Banach algebra and $\lambda \in \mathbb{C}$. Then $\mathrm{sp}(\lambda - a) = \lambda - \mathrm{sp}(a)$ and $\mathrm{sp}(\lambda a) = \lambda \mathrm{sp}(a)$. If, in addition, A is a Banach *-algebra, then $\mathrm{conj}(\mathrm{sp}(a)) = \mathrm{sp}(a^*)$.

Theorem 2.9 ([24, 4.1.13])**.** *Let a be an element of a Banach algebra. Then $sp(a)$ is nonempty and compact. In particular, the spectral radius is well-defined.*

Exercise 2.10. Show that $r(a) \leq \|a\|$. [Hint: if $|\lambda| > \|A\|$, use (1.0.1) to show that $(\lambda - a)^{-1} = \sum_{k=0}^{\infty} \lambda^{-(n+1)} a^n$]

Theorem 2.11 (Spectral Radius Formula [24, 4.1.13])**.** *Let a be an element of a Banach algebra. Then*

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Definition 2.12. Let A be a C*-algebra and $a \in A$. We call a **normal** if $a^*a = aa^*$.

Exercise 2.13. If a is normal then $\|a\| = r(a)$.

Hint:

(1) Suppose first that $a = a^*$ and use (1.0.3) to prove that $\|a^{2^n}\| = \|a\|^{2^n}$
(2) Let a be arbitrary and apply the above to a^*a to show $\|a^{2^n}\| = \|a\|^{2^n}$.
(3) Use Theorem 2.11.

Remark 2.14. Note that normality is necessary in the above exercise as for $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, we have $r(x) = 0 < 1 = \|x\|$.

Exercise 2.15. Construct a function $f : \mathbb{C}^4 \to [0, \infty)$ involving only field operations of $\mathbb{C}$, conjugation and square roots of positive numbers that calculates the norm of a 2×2 complex matrix. [Hint:Use Exercise 2.13 and (1.0.3)]

2.1. **Commutative C*-algebras.** Let A be a commutative, unital Banach algebra. Let $\widehat{A}$ denote the space of characters of A, i.e.

$$\widehat{A} = \{\gamma : A \to \mathbb{C} : \gamma \text{ is a non-zero algebra homomorphism } \},$$

Then $\widehat{A}$ is a compact Hausdorff space (equipped with the topology of pointwise convergence).

Exercise 2.16. Let X be a compact Hausdorff space and $x \in X$. Define $\gamma_x \in \widehat{C(X)}$ by $\gamma_x(f) = f(x)$. Then the map $x \mapsto \gamma_x$ is a homeomorphism. [Hint: Use the Riesz representation theorem]

Definition 2.17. Let A and B be C*-algebras. A ***-homorphism** $\pi : A \to B$ is a linear map that satisfies

$$\pi(ab) = \pi(a)\pi(b) \quad \text{and} \quad \pi(a^*) = \pi(a)^* \quad \text{for all} \quad a, b \in A.$$

If π is injective and surjective, then we call π an **isomorphism**. In this case we say that A and B are **isomorphic**, and write $A \cong B$. In the case that $B = B(H)$ we call π a **representation** of A. When we write "(π, H) is a representation of A" we understand that $\pi : A \to B(H)$.

Definition 2.18. Let $\phi : X \to Y$ be a continuous function between topological spaces. Define the *-homomorphism $\phi^* : C(Y) \to C(X)$ by

$$\phi^*(f) = f \circ \phi.$$

Exercise 2.19. If ϕ is surjective, then ϕ^* is an isometry.

Proposition 2.20. *Let $\pi : C(X) \to C(Y)$ be a *-homomorphism. Then there is a continuous function $\pi_* : Y \to X$ such that $(\pi_*)^* = \pi$.*

Proof. Define $\pi_* : \widehat{C(Y)} \to \widehat{C(X)}$ by $\pi_*(\gamma)(f) = \gamma(\pi(f))$ and apply Exercise 2.16. □

Proposition 2.21. *Let $\pi : C(X) \to C(Y)$ be an injective *-homomorphism. Then π_* is surjective, whence π is isometric.*

Proof. If π_* is not surjective, there is an $x \in X$ and $f \in C(X)$ such that $f(x) = 1$ and $f|_{\pi_*(Y)} \equiv 0$. Then by Proposition 2.20, we have

$$\pi(f)(y) = (\pi_*)^*(f)(y) = f(\pi_*(y)) = 0, \quad \text{for all } y \in Y,$$

violating the injectivity of π. The last claim follows from Exercise 2.19. □

For a commutative Banach algebra A, the **Gelfand transform** is the map

$$\Gamma : A \to C(\widehat{A}), \quad \text{defined by} \quad \Gamma(a)(\gamma) = \gamma(a).$$

The following is a cornerstone of operator algebra theory:

Theorem 2.22 (Gelfand, see [24, 4.3.13])**.** *The Gelfand transform is an isomorphism if and only if A is a C*-algebra. In particular if A is a unital commutative C*-algebra, then A is isomorphic to $C(X)$ for some compact Hausdorff space.*

Example 2.23. Let A be a C*-algebra and $a \in A$ a normal element. Then $C^*(a, 1)$ (see Definition 1.15) is a commutative C*-algebra.

Theorem 2.24 (Spectral Theorem; Hilbert, J. von Neumann)**.** *Let $a \in A$ be normal. Then there is a *-isomorphism $\pi : C(sp(a)) \to C^*(a, 1)$, such that $\pi(1) = 1$ and $\pi(id) = a$. For a function $f \in C(sp(a))$, it is common to write $f(a)$ for the operator $\pi(f)$. An immediate consequence of this isomorphism is the fact that $sp(f(a)) = sp(f) = f(sp(a))$.*

Corollary 2.25 (Spectral Theorem for Matrices)**.** *Let $a \in M_n$ be normal and $sp(a) = \{\lambda_1, ..., \lambda_k\}$ and let $p_1, ..., p_k \in M_n$ be the projections onto the respective eigenspaces of a. Then*

$$a = \sum_{i=1}^{k} \lambda_i p_i.$$

2.2. **Special operators and Order.** The spectral theorem from elementary linear algebra already shows the usefulness of the adjoint operation: A matrix is unitarily diagonalizable if and only if it commutes with its adjoint. We've already seen the spectral theorem for infinite-dimensional C*-algebras, but the adjoint provides another source of structure for a C*-algebra: a powerful order structure on its self-adjoint operators. It would be tough to overstate the utility of this order structure. Therefore, we will take care in proving some of the most used facts about it.

Definition 2.26. Let A be a C*-algebra and $a, u \in A$. We say

(1) a is **self-adjoint** if $a^* = a$
(2) a is **positive** if a is self-adjoint and $\text{sp}(a) \subseteq [0, \infty)$
(3) u is **unitary** if $u^*u = uu^* = 1$ (this only makes sense in a unital C*-algebra)

Let A_{sa} denote the space of self-adjoint elements of A and $A^+ \subseteq A_{sa}$ denote the positive elements of A. For $a \in A^+$, we typically write $a \geq 0$. If b and c are self-adjoint, then we write $b \leq c$ if $c - b \geq 0$. As we will prove below, A^+ is a positive cone in A_{sa}, making A_{sa} an ordered $\mathbb{R}$-vector space (although very rarely a Banach lattice).

Exercise 2.27. Let $a \in A$ be normal and $f \in C(\mathrm{sp}(a))$ be positive (i.e. $f(\lambda) \geq 0$ for all $\lambda \in \mathrm{sp}(a)$). Show that $f(a) \geq 0$. [Hint:Use Exercise 2.5 and the spectral theorem]

Exercise 2.28 (Cheating)**.** Use the spectral theorem to prove the following facts

(1) If $a = a^*$, then $\mathrm{sp}(a) \subseteq \mathbb{R}$.
(2) If u is unitary, then $\mathrm{sp}(u) \subseteq \mathbb{T}$.
(3) If a is normal and $\mathrm{sp}(a) = \{\lambda\}$, then $a = \lambda$.

(RE: Cheating; A typical proof of the spectral theorem may involve first proving Items 1 and 2.)

Exercise 2.29. If you feel bad about me asking you to cheat in the last exercise, you can get over it by reproving Exercise 2.28 with the extra assumption that a belongs to a matrix algebra (in which case every element of its spectrum is an eigenvalue) and without the aid of the spectral theorem.

Exercise 2.30. Let $a \in A$. Define the **real part** of a as $\mathrm{Re}(a) = \frac{1}{2}(a+a^*)$ and the **imaginary part** of a as $\mathrm{Im}(a) = \frac{i}{2}(a^* - a)$. Show that $\mathrm{Re}(a), \mathrm{Im}(a) \in A_{sa}$ and $a = \mathrm{Re}(a) + i\mathrm{Im}(a)$.

Exercise 2.31. Let $a \in A_{sa}$. Let ${}_+, {}_- \in C(\mathrm{sp}(a))$ be defined by ${}_+(t) = \max\{0, t\}$ and ${}_-(t) = -\min\{0, t\}$. Prove that $a_+, a_- \in A^+$ with $a_+ a_- = 0$ and $a = a_+ - a_-$.

Definition 2.32. Let $a \in A^+$. Then the square root function $f(x) = \sqrt{x}$ is continuous on $\mathrm{sp}(a)$. We write $a^{1/2} \in A^+$ to denote $f(a)$.

Remark 2.33. If $a \geq 0$, then $a^{1/2}$ is the *unique* positive operator satisfying $(a^{1/2})^2 = a$.

Exercise 2.34. Let A be unital and $a \in A_{sa}$ with $\|a\| \leq 1$. Prove that $u = a + i(1 - a^2)^{1/2}$ is a unitary with $a = \frac{1}{2}(u + u^*)$. Deduce from Exercise 2.30 that the unitaries span A.

Exercise 2.35. Let $b \in A$ be invertible. Then $b(b^*b)^{-1/2}$ is unitary.

2.2.1. *Special Operators in $B(H)$.* It's helpful to understand what a self-adjoint, positive, etc. operator looks like as a linear operator on a Hilbert space. Suppose that $T = T^* \in B(H)$. Then for any $\eta \in H$ we have

$$\langle T(\eta), \eta \rangle = \langle \eta, T(\eta) \rangle = \overline{\langle T(\eta), \eta \rangle},$$

that is $\langle T(\eta), \eta \rangle \in \mathbb{R}$. This characterizes self-adjoint operators on a Hilbert space.

Suppose now that T is positive. Then (see Definition 2.32)

$$\langle T(\eta), \eta \rangle = \langle (T^{1/2})^2(\eta), \eta \rangle = \langle T^{1/2}(\eta), T^{1/2}(\eta) \rangle = \|T^{1/2}(\eta)\|^2 \geq 0.$$

This characterizes positive operators on a Hilbert space.

Suppose now that U is unitary. Then

$$\langle U(\xi), U(\eta) \rangle = \langle U^*U(\xi), \eta \rangle = \langle \eta, \xi \rangle.$$

That is, U leaves the inner product invariant. As a consequence, we have $\|\eta\| = \|U(\eta)\|$ for all $\eta \in H$. This doesn't tell the whole story about unitaries though. Notice that we haven't used the fact that $UU^* = 1$. If H is finite dimensional, then $U^*U = 1$ immediately implies $UU^* = 1$ since injectivity and surjectivity mean the same thing. Let's take a quick look at a classic example highliting the difference between finite and infinite dimensions (which is just highlighting the difference between finite and infinite sets)

Example 2.36 (Bilateral Shift)**.** Let $H = \ell^2(\mathbb{N})$ (see Definition 1.3) and define $S \in B(H)$ as

$$S(e_n) = e_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

One checks that S^* is defined by

$$S^*(e_1) = 0 \quad \text{and} \quad S^*(e_n) = e_{n-1} \quad \text{for } n > 1.$$

It then follows that $S^*S = 1$, but as $e_1 \in \ker(S^*)$ we have $SS^* \neq 1$. Hence S leaves the inner product invariant but is not a unitary.

Exercise 2.37. $\text{sp}(S) = \overline{\mathbb{D}}$ and S has no eigenvalues.

Hence to properly define unitaries via the inner product we must require

$$\langle U(\eta), U(\xi)\rangle = \langle U^*(\eta), U^*(\xi)\rangle = \langle \eta, \xi\rangle,$$

or equivalently that U is a surjective isometry.

2.2.2. *More on Positive Operators.* Here's a warmup for our first lemma:

Exercise 2.38. Let $a \in A_{sa}$. Show that

$$-\|a\| \leq a \leq \|a\|.$$

[Hint: Use Exercises 2.8 and 2.13]

Lemma 2.39. *Let $a \in A_{sa}$. The following are equivalent*

(1) $a \geq 0$
(2) $\|t - a\| \leq t$ *for all* $t \geq \|a\|$
(3) $\|t - a\| \leq t$ *for some* $t \geq \|a\|$

Proof. We'll repeatedly use Exercises 2.8 and 2.13.
(1)→(2): By assumption, we have $\text{sp}(a) \subseteq [0, \|a\|]$, hence for $t \in \mathbb{R}$ we have $\text{sp}(t - a) \subseteq [t - \|a\|, t]$. If $t \geq \|a\|$, then $\|t - a\| = r(t - a) \leq t$.
(3)→(1): By assumption $\text{sp}(t - a) \subseteq [-t, t]$, hence $\text{sp}(a) \subseteq [0, 2t]$. □

Lemma 2.40. *Let A be a C^*-algebra, then A^+ is a closed cone. That is,*

(1) *If $r \in \mathbb{R}^+$ and $a \in A^+$, then $ra \in A^+$.*
(2) *If $a, b \in A^+$, then $a + b \in A^+$*
(3) *A^+ is closed.*

Proof. (1) is immediate from Exercise 2.8. For (2) we notice that by Lemma 2.39 applied to both a and b, we have

$$\|\|a\| + \|b\| - (a + b)\| \leq \|a\| + \|b\|.$$

Then again by Lemma 2.39 appied to $t = \|a\| + \|b\|$, it follows that $a + b$ is positive.
To prove (3), notice that if a_i is a net converging to a, then we may assume that $\|a_i\| \leq \|a\|$ for all $i \in I$. One can now apply Lemma 2.39(2,3) to show that $a \geq 0$ if all of the $a_i \geq 0$. □

Now we can finally sort out the role of the involution in the order structure:

Theorem 2.41. *Let A be a C^*-algebra and $a \in A$. Then $a \geq 0$ if and only if $a = b^*b$ for some $b \in A$. Equivalently,*

$$A^+ = \{b^*b : b \in A\}$$

Proof. First let $a = b^*b$. Decompose $a = a_+ - a_-$ with $a_\pm \in A^+$ as in Exercise 2.31. We'll show that $a_- = 0$, thus forcing $a = a_+ \geq 0$. By Exercise 2.27, we have

$$(a_-)^3 \geq 0$$

Set $t = ba_-$. Then by Exercise 2.31 and the above line, we have

$$t^*t = a_-(b^*b)a_- = a_-(a_+ - a_-)a_- = -(a_-)^3 \leq 0. \tag{2.2.1}$$

Therefore, $\mathrm{sp}(t^*t) \subseteq (-\infty, 0]$.

We now show $tt^* \geq 0$. Decompose $t = h + ik$ with $h, k \in A_{sa}$ (see Exercise 2.30). We have $t^*t + tt^* = 2h^2 + 2k^2$. By Exercise 2.27, Lemma 2.40 and (2.2.1), we have $tt^* = h^2 + k^2 - t^*t \geq 0$. Therefore $\mathrm{sp}(tt^*) \subseteq [0, \infty)$. By Lemma 2.6, we have $\mathrm{sp}(t^*t) = \{0\}$, which forces $t^*t = 0$ by Exercise 2.28(3). Therefore, by the spectral theorem, we have $a_- = (a_-^3)^{1/3} = 0$.

If $a \geq 0$, then we can simply take $b = a^{1/2}$. □

Exercise 2.42. Let $\pi : A \to B$ be a *-homorphism. If $a \leq b$, then $\pi(a) \leq \pi(b)$.

Exercise 2.43. Show that if $a \leq b$, then $c^*ac \leq c^*bc$ for all $c \in A$.

2.2.3. *Projections and Murray-von Neumann Equivalence.*

Definition 2.44. An element $p \in A$ is called a **projection** if $p = p^* = p^2$.

Example 2.45. There is a one-to-one correspondence between projections in $B(H)$ and closed subspaces of H given by $P \mapsto \mathrm{Range}(P)$. To see surjectivity, let $K \subseteq H$ and decompose $H = K \oplus K^\perp$ and decompose $\eta \in H$ uniquely as $\eta = \eta_1 + \eta_2$ with $\eta_1 \in K$ and $\eta_2 \in K^\perp$. One then defines $P(\eta) = \eta_1$.

Exercise 2.46. Let $p, q \in A$ be projections. Prove that

1. $\mathrm{sp}(p) \subseteq \{0, 1\}$.
2. $p \leq q$ if and only if $pq = p$.
3. $pq = 0$ if and only if $p + q$ is a projection.

Exercise 2.47. Show 0 and 1 are the only projections in $C(X)$ if and only if X is connected.

Definition 2.48. An element $v \in A$ is called a **partial isometry** if v^*v is a projection. If $v^*v = 1$, then v is called an **isometry**.

Example 2.49. Clearly all unitaries are isometries. The bilateral shift (Example 2.36) is an example of a non-unitary isometry.

Lemma 2.50. *Let v be a partial isometry. Then v^* is a partial isometry.*

Proof. Set $z = (1 - vv^*)v$. Using the fact that v^*v is a projection, we have $z^*z = 0$, hence $z = 0$ or $v = vv^*v$. Hence $vv^*vv^* = vv^*$. □

Example 2.51 (Partial Isometries in $B(H)$)**.** Let $V \in B(H)$ be a partial isometry. Then we can decompose $H = K \oplus K^{\perp}$ with V restricted to K an isometry and $K^{\perp}$ equal to the kernel of V. Then V^* is an isometry when restricted to $V(K)$ and 0 on $V(K)^{\perp}$. We have V^*V the projection onto the range of V^* and VV^* the projection onto the range of V.

Definition 2.52. Let $p, q \in A$ be projections. We say that p and q are **Murray-von Neumann equivalent** if there is a partial isometry $v \in A$ such that $vv^* = p$ and $v^*v = q$. Murray-von Neumann equivalence is an equivalence relation.

Remark 2.53. Two projections $P, Q \in B(H)$ are Murray-von Neuamann equivalent if and only if they have the same rank.

Let's isolate a calculation involving isometries that will be needed later.

Proposition 2.54. *Let $u, s \in A$ with u a unitary and s an isometry. If s^*us is unitary, then $uss^* = ss^*u$.*

Proof. Let $p = ss^*$. One easily checks thats pup is a unitary in pAp (see (1.0.4)). We have $pu = pup + pu(1-p)$, hence

$$p = (pu)(u^*p) = (pup)(pu^*p) + pu(1-p)u^*p = p + pu(1-p)u^*p.$$

Hence $pu(1-p) = 0$ or equivalently $pu = pup$. By a similar argument with u^* replacing u we get $pu^*(1-p) = 0$, and by taking adjoints we get $(1-p)up = 0$ or $up = pup$. ☐

3. Homomorphisms and Ideals

The following theorem is not only extremely useful but an excellent illustration of how one typically exploits the C*-equality: Prove something for self-adjoint operators (using the spectral theorem), then use the fact that $\|a\|^2 = \|a^*a\|$ to relate the norm of a to the norm of the self-adjoint operator a^*a.

Theorem 3.1. *Let $\pi : A \to B$ be a unital *-homomorphism. Then $\|\pi\| \leq 1$. Moreover, if π is injective, then π is isometric.*

Proof. It is obvious that a *-homomorphism preserves invertibility, hence for any $a \in A$ we have $\mathrm{sp}(\pi(a)) \subseteq \mathrm{sp}(a)$. From this and Exercise 2.13 we have

$$\|\pi(a)\|^2 = \|\pi(a^*a)\| = r(\pi(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a^2\|$$

Assume now that π is injective. Then $C^*(a^*a, 1)$ and $C^*(\pi(a^*a), 1)$ are commutative C*-algebras and $\pi : C^*(a^*a, 1) \to C^*(\pi(a^*a), 1)$ is an injective *-homomorphism between commutative C*-algebras, hence it is isometric on $C^*(a^*a, 1)$ by Proposition 2.21 and Theorem 2.22. In particular,

$$\|a\|^2 = \|a^*a\| = \|\pi(a^*a)\| = \|\pi(a)\|^2.$$

☐

Corollary 3.2. *Let A be a unital C*-algebra. Then A has a unique norm that makes it a C*-algebra.*

Proof. Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two norms on A making it a C*-algebra. By Theorem 3.1, the identity map from $(A, \|\cdot\|_\alpha)$ to $(A, \|\cdot\|_\beta)$ is an isometry. ☐

Definition 3.3. Let A be a C*-algebra. Unless otherwise stated, by an **ideal** of a C*-algebra we always mean a closed 2-sided ideal of A (this is standard practice in the literature as well). It is a non-trivial (but not terribly difficult) fact that ideals are automatically *-closed and hence C*-algebras [1, Page 11]. If A is a C*-algebra and J is an ideal, then A/J is also a C*-algebra under the obvious algebraic operations and norm defined as the Banach space quotient norm (i.e. $\|a + J\| = \inf_{j\in J} \|a + j\|$). We call a C*-algebra **simple** if it only has trivial ideals.

Exercise 3.4. Show that there is a one-to-one correspondence between ideals of $C(X)$ and closed subsets of X.

Definition 3.5. An isomorphism from A to itself is called an **automorphism**.

Exercise 3.6. Let A be a C*-algebra and u a unitary. Prove that $Ad(u)(x) = u^*xu$ is an automorphism of A.

Definition 3.7. Let $\pi_i : A \to B$ be *-homomorphisms for $i = 1, 2$. We say π_1 is **unitarily equivalent** to π_2 is there is a unitary $u \in B$ such that $\pi_1 = Ad(u) \circ \pi_2$.

Example 3.8 (Homomorphisms of Matrix algebras)**.** Let $n = km$ be positive integers. Then there is a unique (up to unitary equivalence, Definition 3.7) *-homomorphism from M_k into M_n defined by

$$x \mapsto \begin{bmatrix} x & 0 & \dots & 0 \\ 0 & x & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x \end{bmatrix}.$$

3.1. **Coronas, Ultraproducts and Inductive Limits.**

Example 3.9. Let I be a set and $(A_i)_{i\in I}$ a family of C*-algebras. Define

$$\ell^\infty(A_i) := \{(a_i)_{i\in I} : a_i \in A_i \text{ and } \sup_{i\in I} \|a_i\| < \infty\}.$$

Then $\ell^\infty(A_i)$ is a C*-algebra under pointwise operations and sup norm. Define

$$c_0(A_i) = \{(a_i)_{i\in I} : (\forall\epsilon > 0)(|\{i \in I : \|a_i\| \geq \epsilon\}| < \infty)\}.$$

Let $\mathcal{U}$ be an ultrafilter, and define

$$\mathcal{U}(a_i) = \{(a_i)_{i\in I} : \lim_{i\to\mathcal{U}} \|a_i\| = 0\}.$$

Then $c_0(A_i)$ and $\mathcal{U}(A_i)$ are ideals of $\ell^\infty(A_i)$. The C*-algebra $\ell^\infty(A_i)/\mathcal{U}(A_i)$ is called the **ultraproduct** of the C*-algebras A_i with respect to the ultrafilter $\mathcal{U}$. The C*-algebra $\ell^\infty(A_i)/c_0(A_i)$ is caled the **corona algebra** of the family $(A_i)_{i\in I}$. In either case, we write $[(x_i)]$ for the equivalence class of the element $(x_i) \in \ell^\infty(A_i)$.

In the special case when all of the $A_i = A$ are the same algebra, we write $A_\mathcal{U}$ as the ultrapower. We think of $A \subseteq A_\mathcal{U}$ under the diagonal embedding. Let

$$A' \cap A_\mathcal{U} = \{x \in A_\mathcal{U} : xa = ax \text{ for all } a \in A\}. \tag{3.1.1}$$

In other words an equivalence class of a net $(x_i) \in A' \cap A_\mathcal{U}$ when $\lim_{i\to\mathcal{U}} \|x_ia - ax_i\| = 0$ for all $a \in A$.

Exercise 3.10. Let $u \in A_{\mathcal{U}}$ be a unitary (resp. positive, self-adjoint or a projection). Show that there is a representing set $(u_i)_{i\in I}$, with each u_i unitary (resp. positive, self-adjoint or a projection).
[Hint: Use Exercise 2.35 for the unitary case]

Exercise 3.11. Let $(H_i)_{i\in I}$ be a family of Hilbert spaces. Define the Hilbert space

$$\bigoplus_{i\in I} H_i = \{(\xi_i)_{i\in I} : \xi \in H_i \text{ and } \|(\xi_i)_{i\in I}\|^2 := \sum \|\xi_i\|^2 < \infty\}.$$

Prove that we have a natural inclusion of $\ell^\infty(B(H_i))$ into $B(\oplus_{i\in I} H_i)$.

Exercise 3.12 (More or less just a statement)**.** Let (π_i, H_i) be a family of representations of A. Define (i.e. show this all makes sense) the representation $(\oplus_{i\in I}\pi_i, \oplus_{i\in I}H_i)$ by

$$\bigoplus_{i\in I} \pi_i(a) = (\pi_i(a))_{i\in I}.$$

(here we are using the identifications of the last exercise). Show that $\ker(\oplus_{i\in I}\pi_i) = \cap_{i\in I}\ker(\pi_i)$.

Example 3.13. Let A_n be a sequence of C*-algebras with injective *-homorphisms $\pi_n : A_n \to A_{n+1}$. For $m > n$ we define $\pi_{n,m} : A_n \to A_m$ by $\pi_{n,m} = \pi_{m-1} \circ \cdots \circ \pi_{n+1} \circ \pi_n$. There is a unique (up to isomorphism) C*-algebra A and injective *-homomorphisms $\sigma_n : A_n \to A$ that satisfying the following two conditions:

1. For $m > n$, we have $\sigma_n = \sigma_m \circ \pi_{n,m}$.
2. The union $\bigcup_{n=1}^{\infty} \sigma_n(A_n)$ is dense in A.

We call A the **inductive limit** of the sequence (A_n, π_n) and it is common to write

$$A = \lim_{n\to\infty} (A_n, \pi_n).$$

It is typical to think of A_n as subalgebras of A (i.e. identifying A_n with it's image in A) and $A = \overline{\cup_{n=1}^{\infty} A_n}$. In particular we write $\pi_{n,\infty} : A_n \to A$ as the embedding of A_n into A as the limiting embedding of the $\pi_{n,m}$.

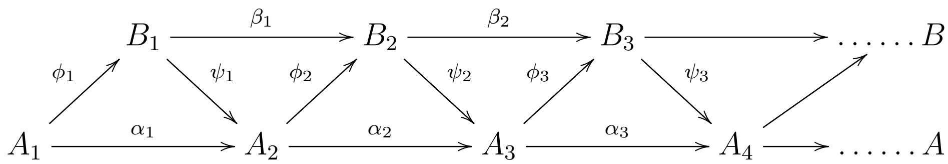
Remark 3.14. It's not necessary for the *-homomorphisms π_n to be injective. In these notes (and in most places) we are only concerned with the injective case.

Definition 3.15 (Nomenclature)**.** If A is an inductive limit of finite-dimensional C*-algebras, we call A an AF-algebra. Similarly if A is an inductive limit of homogeneous (resp. subhomogeneous) C*-algebras then we call it an AH (resp. ASH algebra). The "A" stands for the word "approximately."

Exercise 3.16. Use Example 3.9 to show the existence of inductive limits.

3.1.1. *Approximate Intertwining.* We now describe a very useful way, introduced by Elliott in [13], of showing that two inductive limit C*-algebras are isomorphic. One typically refers to the following method of proof as an "approximate intertwining argument." We simply provide enough details so the reader can fill in the rest on their own or at least believe that the argument works. For the full details, see Rørdam's monograph [27, Section 2.3].

Let $A = \lim(A_n, \alpha_n)$ and $B = \lim(B_n, \beta_n)$ be inductive limits of separable C*-algebras and suppose there exist injective *-homomorphisms $\phi_n : A_n \to B_n$ and $\psi_n : B_n \to A_{n+1}$ such that the following diagram



is approximately commutative in the following sense: For any $x \in A_n$,

$$\sum_{m>n} \|\psi_m \circ \phi_m \circ \alpha_{n,m}(x) - \alpha_{n,m+1}(x)\| < \infty,$$

and for all $y \in B_n$

$$\sum_{m>n} \|\phi_{m+1} \circ \psi_m \circ \beta_{n,m}(y) - \beta_{n,m+1}(y)\| < \infty.$$

Then A and B are isomorphic. One defines a map $\phi : A \to B$ and a map $\psi : B \to A$ by

$$\phi(a) = \lim_{m\to\infty} \beta_{m,\infty} \circ \phi_m \circ \alpha_{n,m}(a) \quad \text{for } a \in A_n$$

$$\psi(b) = \lim_{m\to\infty} \alpha_{m+1,\infty} \circ \psi_m \circ \beta_{n,m} \quad \text{for } b \in B_n.$$

Then ϕ and ψ are well-defined *-homomorphisms with $\psi^{-1} = \phi$, in particular A and B are isomorphic.

3.1.2. *Approximate Intertwining II.* Every C*-algebra is an inductive limit (just take the identity map as connecting *-homomorphisms). So there's not really a limit for the applications of approximate intertwining arguments. Let's take a look at how we'll exploit this argument. Let A and B be C*-algebras. Then A and B are, by definition, isomorphic if there are *-homomorphisms $\phi : A \to B$ and $\psi : B \to A$ such that $\phi\psi = id_B$ and $\psi\phi = id_A$. What approximate intertwining does is allow us to relax this rigid algebraic definition of isomorphism. We first have

Definition 3.17. Let $\phi, \psi : A \to B$ be two *-homomorphisms. We say that ϕ and ψ are **approximately unitarily equivalent** if there exist a sequence of unitaries (u_n) in B such that $u_n^* \phi(a) u_n \to \psi(a)$ for all $a \in A$.

Here is the special case that we'll be using:

Lemma 3.18. *Let A and B be separable C*-algebras and $\phi : A \to B$ and $\psi : B \to A$ *-homorphisms. If $\psi \circ \phi$ is approximately unitarily equivalent to id_A and $\phi \circ \psi$ is approximately unitarily equivalent to id_B, then A and B are isomorphic.*

Proof. Say (u_n) makes $\psi \circ \phi$ approximately unitarily equivalent to id_A and (v_n) makes $\phi \circ \psi$ approximately unitarily equivalent to id_B. Then (after passing to appropriate subsequences of the (u_n) and (v_n)) one can build a diagram as in (3.1.1) with $\alpha_n = Ad(u_n)$ and $\beta_n = Ad(v_n)$ and $\phi_n = \phi$ and $\psi_n = \psi$ for all $n \in \mathbb{N}$. $\square$

Remark 3.19. Lemma 3.18, of course, holds in more generality. There is no reason why we need approximate unitary equivalence (one could define "approximate automorphic equivalence" in the obvious way).

3.2. **Three important classes of simple C*-algebras.**

3.2.1. *Matrix algebras.* Fix $n \in \mathbb{N}$. For $1 \le i, j \le n$, let $e_{ij} \in M_n$ be the matrix with $(i,j)-$entry equal to 1 and all other entries 0. We call the e_{ij} **matrix units** for M_n. Note that $\{e_{ij}\}_{1 \le i,j \le n}$ is a basis for M_n and the multiplication is determined by the rules:

$$e_{ij}e_{k\ell} = \begin{cases} 0 & \text{if } j \neq k \\ e_{i\ell} & \text{if } j = k \end{cases}$$

One sees that the identity of M_n is written as $1 = \sum_{k=1}^{n} e_{kk}$. For a general $x \in M_n$ we write x_{ij} as the e_{ij}-coefficient of x.

Theorem 3.20. *For each $n \in \mathbb{N}$ the matrix algebra M_n is simple.*

Proof. Let $a \in M_n$ be nonzero. Then for some $1 \le i', j' \le n$ we have $a_{i'j'} \neq 0$. Then

$$1 = \frac{1}{a_{i'j'}} \sum_{k=1}^{n} e_{ki'} a e_{j'k}.$$

Hence 1 is in the ideal generated by a. □

3.2.2. *UHF algebras.*

Lemma 3.21. *Let $A = \lim_{n\to\infty}(A_n, \pi_n)$. If each A_n is simple, then A is simple.*

Proof. Let $J \neq A$ be an ideal of A and $\pi : A \to A/J$ the quotient homomorphism. Since each A_n is simple, π restricted to A_n be injective, hence an isometry by Theorem 3.1. Hence π is an isometry on a dense subspace of A, hence π is an isometry on A, hence injective. Therefore $J = 0$. □

Definition 3.22. Let $p_1, p_2, ...$ be a sequence of prime numbers For each $i \in \mathbb{N}$ define $n_i = p_1 \cdots p_i$. Let $\pi_i : M_{n_i} \to M_{n_{i+1}}$ be any *-homomorphism (see Example 3.8). Then the C*-algebra

$$A = \lim_{n\to\infty}(M_{n_i}, \pi_i)$$

is called the **UHF algebra** (UHF=Uniformly HyperFinite) of type $\{p_i\}$. James Glimm showed in [15, Theorem 1.12] that If A_1 is a UHF algebra of type $\{p_i\}$ and A_2 is a UHF algebra of type $\{q_i\}$, then $A_1 \cong A_2$ if and only if there is a bijection σ of $\mathbb{N}$ such that $p_{\sigma(i)} = q_i$ for all $i \in \mathbb{N}$.

The UHF algebra with $p_i = 2$ for all $i \in \mathbb{N}$, holds special significance and is called the **CAR algebra**. CAR stands for Canonical Anticommutation Relations and the terminology comes from statistical mechanics. See [3] for the full story on the CAR algebra and statistical mechanics.

Proposition 3.23. *UHF algebras are simple.*

Proof. Theorem 3.20 and Lemma 3.21. □

3.2.3. *Goodearl Algebras.* Over the years several ingenious inductive limit constructions involving (sub)homogeneous (and more generally residually finite dimensional) C*-algebras have led to the solution of numerous open problems in C*-algebra theory. To name a few: the Elliott-Evans decomposition of the irrational rotation algebras [14] as an A$\mathbb{T}$-algebra, Villadsen's algebras [33, 34] not only notable for the problems they solved but for the techniques introduced, the Jiang-Su algebra [17] and all of its implications for the classification problem, Dadarlat's Brattelli systems [9] that clarified the (lack of) relationship between nuclearity, exactness and certain finiteness properties of C*-algebras, Rordam's example [30] of a simple nuclear C*-algebra with a finite and infinite projection and Toms's example [32] which has brought the Cuntz semigroup to the forefront of C*-research recently.

There is no way we have time to recount the constructions above in these notes (in fact the poor state of the author's algebraic topology knowledge prohibits it in some cases). Instead we'll focus on a specific case of the so-called Goodearl algebras, named for Ken Goodearl after his paper [16]. We will focus here only on proving simplicity and giving the reader a very general way to construct simple C*-algebras.

Let X be a separable compact Hausdorff space and (x_n) a sequence such that $\{x_n, x_{n+1}, ...\}$ is dense for each $n \in \mathbb{N}$. Recall the C*-algebras $C(X, M_n)$ from Example 1.22.

Let $(k_{1,i}, k_{2,i})_{i=1}^{\infty}$ be a sequence from $\mathbb{N} \times \mathbb{N}$. Inductively define $N_1 = 1$ and $N_{i+1} = N_i(k_{1,i+1} + k_{2,i+1})$. We then define *-homomorphisms $\pi_i : C(X, M_{N_i}) \to C(X, M_{N_{i+1}})$ by

$$\pi_i(f)(x) = \begin{bmatrix} f(x_i) & & & & & \\ & \ddots & & & & \\ & & f(x_i) & & & \\ & & & f(x) & & \\ & & & & \ddots & \\ & & & & & f(x) \end{bmatrix}$$

where there are $k_{1,i+1}$ copies of $f(x_i)$ and $k_{2,i+1}$ copies of $f(x)$. We then define $A = \lim_{n\to\infty}(A_i, \pi_i)$. We'll prove that A is simple. Notice that if X is not trivial, then each A_i is not simple (see Exercise 3.4) so we can't simply invoke Lemma 3.21 on this one. The key is that it's the point evaluation homomorphisms that deliver simplicity.

Proposition 3.24. *A is simple.*

Proof. Let $f \in A_i$ for some i. Then choose m big enough so $0 \neq f(x_m) \in M_{N_i}$. Without loss of generality suppose that the $(1,1)$-entry of $f(x_m)$ is non-zero. Considering $M_{N_i} \subseteq C(X, M_{N_i})$ as constant functions with matrix units e_{ij}, we have

$$1 = \frac{1}{f(x_m)_{11}} \sum_{i=1}^{N_i} e_{i1}\pi_{i,m}(f)e_{1i}.$$

Hence the ideal generated by f is A. Let now J be a closed ideal of A not equal to A and let $\pi : A \to A/J$ be the quotient map. The above shows that $A_i \cap J = \{0\}$ for each i, hence π is injective and hence isometric by Theorem 3.1, on each A_i, making π isometric on A, so $J = \{0\}$. □

Remark 3.25. As Goodearl showed [16], by playing with the numbers $(k_{1,i}, k_{2,i})$ one can influence the properties of the limit algebra A. Notice that when X is a single point we have a UHF algebra.

3.2.4. *Cuntz Algebras.* These algebras are named for Joachim Cuntz who first studied them in [7, 8] and proved the properties that we will discuss below (as well as many other properties including calculation of K-theory and Ext groups). Everything below is taken from Cuntz's papers, with a couple of simplifications thrown in from [29] and [10, V.4].

Fix $n \geq 2$. The **Cuntz algebra** is the universal C*-algebra generated by elements $s_1, ..., s_n$ satisfying the relations

$$s_i^* s_i = 1 \quad \text{for all } 1 \leq i \leq n \quad \text{and} \quad \sum_{i=1}^{n} s_i s_i^* = 1. \tag{3.2.1}$$

So what do I mean by "the universal C*-algebra"? It basically means exactly what you think it should (but to properly define it we really should have the tools of Section 3.3 available to us). Let me show you that there exists *some* C*-algebra satisfying the relations in (3.2.1). Let $f_1, ..., f_n : \mathbb{N} \to \mathbb{N}$ be injective functions such that $f_1(\mathbb{N}), ..., f_n(\mathbb{N})$ form a partition of $\mathbb{N}$. Then define $s_i : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $s_i(e_k) = e_{f_i(k)}$. One easily checks that the s_i satisfy the relations (3.2.1).

The Cuntz algebra $\mathcal{O}_2$ holds special significance in C*-theory and extra-special significance in these notes. Let's spend a little time with it. Most of the things we'll say below all hold true for $\mathcal{O}_n$ with the obvious modifications. We don't want to get bogged down with too much notation, so we'll drop the variable n for the constant 2.

Definition 3.26. Fix an integer $n \geq 1$. Let $\mu \in \{1,2\}^n$. We say μ is a **word in** $\{1,2\}$ of **length** n and write $|\mu|$ to denote the length of a word. We simply refer to a word of any length as a **word**. For each word μ of length n, we write $s_\mu = s_{\mu(1)} \cdots s_{\mu(n)} \in \mathcal{O}_2$.

Lemma 3.27.

$$\mathcal{O}_2 = \overline{span}\{s_\mu s_\nu^* : \mu, \nu \text{ are words }\}. \tag{3.2.2}$$

Proof. By Lemma 2.50, $s_i s_i^*$ is a projection for $i = 1, 2$. Then by Exercise 2.46, we have $s_i s_i^* s_j s_j^* = 0$ when $i \neq j$. Therefore

$$s_i^* s_j = s_i^*(s_i s_i^* s_j s_j^*) s_j = 0.$$

By basic algebra one sees that the *-algebra generated by s_1 and s_2 is the span of elements of the form:

$$x = s_{i_1}^{\epsilon_1} s_{i_2}^{\epsilon_2} \cdots s_{i_k}^{\epsilon_k}$$

where $i_\ell \in \{1,2\}$ and $\epsilon_\ell \in \{1, *\}$. Then by repeated application of the identity $s_i^* s_j = \delta_{i,j} 1$, one sees that x will reduce to an element of the form $s_\mu s_\nu^*$ for words μ and ν. ☐

Exercise 3.28. Show that $\|s_\mu s_\nu^*\| = 1$ for all words μ and ν. (Use the C*-identity a few times)

Lemma 3.29. *Let $n \in \mathbb{N}$. Then*

$$\mathcal{A}_n = \{s_\mu s_\nu^* : |\mu| = |\nu| = n\}$$

is a C-algebra isomorphic to M_{2^n}. Furthermore*

$$\mathcal{A} = \{s_\mu s_\nu^* : |\mu| = |\nu|\}$$

is isomorphic to the CAR algebra (Section 3.2.2).

Proof. Fix your favorite bijection $\phi : \{1,2\}^n \to \{1, ..., 2^n\}$. One then sees (using Exercise 3.28) that the map $s_\mu s_\nu^* \mapsto e_{\phi(\mu),\phi(\nu)}$ defines an isomorphism from $\mathcal{A}_n$ onto M_{2^n}.

Define the map $\lambda : \mathcal{O}_2 \to \mathcal{O}_2$ by

$$\lambda(x) = s_1 x s_1^* + s_2 x s_2^* \tag{3.2.3}$$

and check that λ is a unital *-homomorphism. Check that $\lambda(\mathcal{A}_n) \subseteq \mathcal{A}_{n+1}$. Since there is a unique (up to unitary equivalence) unital *-homomorphism from M_{2^n} into $M_{2^{n+1}}$, it follows that $\mathcal{A} = \lim_{n\to\infty}(\mathcal{A}_n, \lambda)$ is isomorphic to M_{2^∞}. ☐

It is easy to check that for each $\theta \in \mathbb{R}$ we have $e^{i\theta}s_1$ and $e^{i\theta}s_2$ satisfy (3.2.1). Hence by the universal property of $\mathcal{O}_2$ there is an automorphism $\lambda_\theta : \mathcal{O}_2 \to \mathcal{O}_2$ defined by $\lambda_\theta(s_i) = e^{i\theta}s_i$ for $i = 1, 2$. Notice that

$$\lambda_\theta(s_\mu s_\nu^*) = e^{i\theta(|\mu|-|\nu|)}$$

Now define the linear map $\mathbb{E} : \mathcal{O}_2 \to \mathcal{O}_2$ by

$$\mathbb{E}(x) = \int_0^{2\pi} \lambda_\theta(x)d\theta. \tag{3.2.4}$$

We then have

$$\mathbb{E}(s_\mu s_\nu^*) = \begin{cases} 0 & \text{if } |\mu| \neq |\nu| \\ s_\mu s_\nu^* & \text{if } |\mu| = |\nu| \end{cases} \tag{3.2.5}$$

Theorem 3.30. $\mathbb{E}$ *is a faithful conditional expectation onto* $\mathcal{A}$*, that is*

(1) $\mathbb{E}^2 = \mathbb{E}$
(2) $\mathbb{E}(\mathcal{O}_2) = \mathcal{A}$.
(3) $\mathbb{E}$ *is* ***positive*** *(i.e. if* $a \geq 0$*, then* $\mathbb{E}(a) \geq 0$*).*
(4) $\|\mathbb{E}\| = 1$.
(5) *If* $x \geq 0$ *and* $E(x) = 0$*, then* $x = 0$.

Sketchy Proof. (1) and (2) are immediate from the discussion above. We prove (3). From Exercise 2.42 we know *-homomorphisms are positive. It's easy to see that if $r, s \geq 0$ and ϕ_1, ϕ_2 are positive maps, then $r\phi_1 + s\phi_2$ is a positive map. It then follows that integrating positive maps produces a positive map. By the same reasoning, it follows that integrating norm 1 maps produces a map of norm bounded by 1(which proves (4)). It is definitely not clear that condition (5) holds (but it does). We refer the reader to [7, Proposition 1.10] for a proof (and note that the reader will soon have all the tools available to understand it). ☐

Exercise 3.31. Suppose that $\phi : A \to B$ is a positive map, i.e. $\phi(a) \geq 0$ if $a \geq 0$. Show that $\phi(x^*) = \phi(x)^*$ for all $x \in A$.

Lemma 3.32 ([29, Lemma 4.4])**.** *Recall* λ *from* (3.2.3)*. For every* $n \geq 1$ *and* $x \in \mathcal{O}_2$ *we have*

$$\lambda^n(x) = \sum_{|\delta|=n} s_\delta x s_\delta^*$$

Moreover,

$$\lambda^n(\mathcal{O}_2) = \mathcal{A}_n'.$$

Here $\mathcal{A}_n' := \{y \in \mathcal{O}_2 : yz = zy \text{ for all } z \in \mathcal{A}_n\}$.

Proof. The first statement is easy to see by induction on n. For the second, let μ, ν be words of length n. Then using the first statement we have

$$s_\mu s_\nu^* \lambda^n(x) = s_\mu x s_\nu^* = \lambda^n(x) s_\mu s_\nu^*.$$

Conversely, suppose that $x \in \mathcal{A}_n'$. Let $y = s_1^{*n} x s_1^n$. Then,

$$\lambda^n(y) = \sum_{|\delta|=n} (s_\delta s_1^{*n}) x s_1^n s_\delta^* = x \sum_{|\delta|=m} (s_\delta s_1^{*n}) s_1^n s_\delta^* = x.$$

□

To prove that $\mathcal{O}_2$ is simple we will require another form of the conditional expectation $\mathbb{E}$. We first define

Definition 3.33. For each $m \geq 1$, define the finite dimensional sub*space* of $\mathcal{O}_2$

$$Y_m = \operatorname{span}\{s_\mu s_\nu^* : |\mu|, |\nu| \leq m\}.$$

Lemma 3.34. *There is a sequence of isometries (w_n) with $w_n \in \mathcal{A}_n'$ such that for every $y \in Y_n$ we have $\mathbb{E}(y) = w_n^* y w_n$.*

Proof. Set $w_n = \lambda^n(s_1^n s_2)$. By Lemma 3.32, we have $w_n \in \mathcal{A}_n'$. Hence it's trivial that $\mathbb{E}$ leaves $\mathcal{A}_n$ invariant. One checks that if $|\mu| < |\nu| \leq n$, then $w_n^* s_\mu s_\nu^* w_n = 0 = \mathbb{E}(s_\mu s_\nu^*)$. □

We now see that $\mathcal{O}_2$ enjoys a very strong form of simplicity:

Theorem 3.35. *For every non-zero $x \in \mathcal{O}_2$, there exist $a, b \in \mathcal{O}_2$ such that $axb = 1$. If $x \geq 0$, then we may take $a = b^*$. In particular, $\mathcal{O}_2$ is simple.*

Remark 3.36. Note that for $x = s_\mu s_\nu^*$, we have $s_\mu^*(s_\mu s_\nu^*) s_\nu = 1$. Not a proof of the above theorem, just a reason why it should be believable.

Proof. Note we may assume that $x \geq 0$. Indeed by Theorem 2.41, all positive elements are of the form y^*y and $a(y^*y)b = (ay^*)yb$. Furthermore, it is enough to find $a, b \in \mathcal{O}_2$ such that axb is invertible.

By Theorem 3.30, we have $\mathbb{E}(x) \neq 0$. Without loss of generality, assume that $\|\mathbb{E}(x)\| = 1$. Obtain an m large enough and $y \in Y_m$ such that $\|x - y\| \leq 1/4$. Note that $\|1/2(y^* + y) - x\| \leq \|y - x\|$ because $x = x^*$, hence we may assume $y = y^*$. Then $\mathbb{E}(y) \in (\mathcal{A}_m)_{s.a.}$ by Exercise 3.31, and $\|\mathbb{E}(y)\| \geq 3/4$ by Theorem 3.30(4).

By the spectral theorem for matrices (Corollary 2.25) there is a minimal projection $p \in \mathcal{A}_m$ commuting with $\mathbb{E}(y)$ such that $p\mathbb{E}(y) = \|\mathbb{E}(y)\|p$. (Lazy proof: At least one of $\pm\|\mathbb{E}(y)\|$ is an eigenvalue for $\mathbb{E}(y)$. Since $\mathbb{E}(y)$ is "close" to $\mathbb{E}(x)$, a positive operator, it follows that $\|\mathbb{E}(y)\|$ is an eigenvalue for $\mathbb{E}(y)$ and not $-\|\mathbb{E}(y)\|$.) Since $s_1^m s_1^{*m} \in \mathcal{A}_m$ is a minimal projection, there is a unitary $u \in \mathcal{A}_m$ such that $upu^* = s_1^m s_1^{*m}$. Obtain the sequence (w_n) as in Lemma 3.34. Let $z = \|\mathbb{E}(y)\|^{-1/2} s_1^{*m} u p w_m^*$. We leave it to the reader to check that $zyz^* = 1$. Therefore $\|1 - zxz^*\| \leq 1/3$, forcing zxz^* to be invertible by Exercise 2.2. □

Definition 3.37. Let $B \subseteq A$ be C*-algebras. We say B is a **hereditary** subalgebra of A, if whenever $0 \leq a \leq b$ with $b \in B$, then $a \in B$.

Most important facts about hereditary C*-algebras can be deduced from the following:

Theorem 3.38 ([23, Theorem 1.5.2])**.** *Let $B \subseteq A$ be hereditary. Then*

$$L(B) = \{x \in A : x^*x \in B^+\}$$

is a left ideal of A and $B = L(B)^* \cap L(B)$, where $L(B)^* = \{x^* : x \in L(B)\}$.

Definition 3.39. A projection $p \in A$ is called **infinite** if there is a $v \in A$ such that $v^*v = p$ and $vv^* < p$ (i.e. $vv^* \leq p$ and $vv^* \neq p$.) If A is unital, then we say A is **purely infinite** if every hereditary subalgebra contains an infinite projection.

Remark 3.40. If A is not unital, then Definition 3.39 is *not* the accepted definition of purely infinite. We're only interested in unital purely infinite C*-algebras, so we won't bother with a discussion of the non unital case.

Exercise 3.41. Let H be infinite dimensional. Show that $B(H)$ has an infinite projection, but is not purely infinite.

Theorem 3.42. *Let $A \neq \mathbb{C}$ be a unital C*-algebra, then for every nonzero $x \in A$, there are $a, b \in A$ such that $axb = 1$ if and only if A is purely infinite and simple. In particular the Cuntz algebras are purely infinite.*

Proof. We prove left to right. Let $B \subseteq A$ be a hereditary subalgebra and $b \in B^+$ nonzero and not invertible. We produce an infinite projection $p \in B$. Do the following:

- Use Theorem 3.35 to get $x \in A$ so $x^*bx = 1$.
- Set $s = b^{1/2}x$. Then $s^*s = 1$ and $p := ss^* \leq \|x\|^2 b$, so $p \in B$.
- One checks that $(sp)^*(sp), (sp)(sp)^* \in B$, so $sp \in B$ by Theorem 3.38.
- $p \neq 1$ since b is not invertible, hence $s(1-p)s^* \neq 0$ (since s is an isometry)
- Then $p = sps^* + s(1-p)s^*$, thus $sps^* < p$.

In the interest of time, we'll skip the proof of the other direction and refer the reader to [8] or [10, Theorem V.5.5] □

Exercise 3.43. Let H be an infinite dimensional Hilbert space. Show that the Calkin algebra $B(H)/K(H)$ is simple and purely infinite.

Theorem 3.44 ([8])**.** *Let A be a simple, unital, purely infinite C*-algebra with $\mathcal{O}_2 \subseteq A$. Then for any nonzero projection $p \in A$, there is an isometry $v \in A$ such that $vv^* = p$*

Remarks and a little proof. I'm not sure how to prove this without going through the machinery of K-theory. It's not very difficult (once you know what to do), but it would take some time. Roughly the idea is that every projection is (Murray von Neumann) equivalent to "twice" itself. In fact every projection p will be equivalent to $s_1ps_1^* + s_2ps_2^*$, which is "twice" itself in the sense that $s_1ps_1^*$ and $s_2ps_2^*$ are orthogonal and both are equivalent to p. One then uses the fact that all nonzero projections have this property, to show that they are all equivalent. In terms of K-theory, using the fact that in a unital, simple purely infinite C*-algebra K_0 is just the semigroup of Murray von Neumann equivalence classes of projections, this is saying that $K_0(A) = 0$. From which the conclusion follows. See [8] for all of these details. □

Remark 3.45. The above theorem does **not** carry over to $\mathcal{O}_n$ for $n \geq 3$. In particular if we let s_1, s_2, s_3 be the generating isometries for $\mathcal{O}_3$ it will in general not be the case that p is equivalent to $s_1ps_1^* + s_2ps_2^*$. Cuntz has made this precise in [8, Corollary 3.12].

3.3. **Building representations.** The goal of this section is to show that for every C*-algebra A there is an isometric embedding $\pi : A \to B(H)$ for some Hilbert space H. This provides then another characterization of C*-algebras: subalgebras of $B(H)$.

Let A be a C*-algebra. Since A is a ring, we have a natural way to view A as a subalgebra of $B(A)$. For each $a \in A$ define $M_a \in B(A)$ as $M_a(b) = ab$. But, except in the trivial case, the C*-norm on A will not be a Hilbert space norm. The idea then is to keep the action of A on itself as multiplication, but change the norm on A to make it into (after taking an appropriate quotient) an inner product space. For this, we need a positive definite sesquilinear form on A. This is our motivation for the next definition

Definition 3.46. Let A be a unital C*-algebra and $\phi : A \to \mathbb{C}$ be a linear function. We say that ϕ is a **positive functional** if $\phi(a) \geq 0$ for every $a \geq 0$. If $\phi(1) = 1$, then we say ϕ is a **state** (we are primarily interested in this special case). A state defines a positive definite sesquilinear form on A defined by

$$\langle a, b\rangle_\phi := \phi(b^*a).$$

Applying the Cauchy-Schwarz inequality to $\langle \cdot, \cdot\rangle_\phi$ we obtain

$$|\phi(b^*a)| \leq \phi(a^*a)^{1/2}\phi(b^*b)^{1/2}. \tag{3.3.1}$$

Theorem 3.47 (GNS-construction)**.** *Let A be a unital C*-algebra and ϕ a state on A. Then*

$$N_\phi := \{x \in A : \phi(x^*x) = 0\}$$

is a left ideal in A. Then A/N_ϕ is an inner product space equipped with the sequilinear form $\langle \cdot, \cdot\rangle_\phi$. Let H_ϕ denote the completion of A/N_ϕ with respect to the norm induced by $\langle \cdot, \cdot\rangle_\phi$ and for each $a \in A$, let $a_\phi \in H_\phi$ denote the image of a. The map $\pi_\phi : A \to B(A/N_\phi)$ defined by

$$\pi_\phi(a)(b_\phi) = (ab)_\phi.$$

*is a *-preserving, algebra homomorphism with $\|\pi_\phi(a)\| \leq \|a\|$ for all $a \in A$. Hence π_ϕ extends to a *-homomorphism from A into $B(H_\phi)$. Moreover we have*

$$\phi(a) = \langle \pi_\phi(a)1_\phi, 1_\phi\rangle_\phi, \quad \textit{for all } a \in A.$$

Remark 3.48. It is trivial to verify that the vector 1_ϕ is **cyclic**, that is $\pi_\phi(A)1_\phi = \{\pi_\phi(a)(1_\phi) : a \in A\}$ is dense in H_ϕ. It turns out that if we require cyclic vectors, then the GNS-construction is unique up to unitary equivalence. We make this precise.
Let $\pi : A \to B(H)$ be a representation of A and $\xi \in H$ a norm 1 vector. Suppose that

(1) ξ is a cyclic vector.
(2) $\phi(a) = \langle \pi(a)\xi, \xi\rangle$ for all $a \in A$.

Then there is a unitary operator $U : H_\phi \to H$ such that $U(1_\phi) = \xi$ and $U^*\pi(a)U = \pi_\phi(a)$ for all $a \in A$. The uniqueness is actually not that difficult to prove if you want to give it a try (Hint: Define U by $U\pi_\phi(a)(1_\phi) = \pi(a)\xi$). For this reason, one usually refers to *the* GNS-construction. All the details are in [31, Theorem I.9.14].

Remark 3.49. The Hilbert space H_ϕ constructed above is also sometimes referred to as $L^2(A, \phi)$.

Proof of GNS-construciton. Let $a \in N_\phi$ and let $b \in A$. By Exercises 2.38 and 2.43 we have

$$\phi(a^*b^*ba) \leq \|b^*b\|\phi(a^*a) = 0.$$

This shows that N_ϕ is a left ideal. It is trivial to see that π_ϕ is a ring homomorphism. To see that it preserves the *-operation note that

$$\langle \pi_\phi(a)(b_\phi), c_\phi\rangle_\phi = \phi(c^*ab) = \langle b_\phi, \pi_\phi(a^*)(c_\phi)\rangle_\phi,$$

that is $\pi_\phi(a^*) = \pi_\phi(a)^*$. Finally by Exercises 2.38 and 2.43 we have

$$\|\pi_\phi(a)(b_\phi)\|^2 = \|(ab)_\phi\|^2 = \phi(b^*a^*ab) \leq \|a\|^2\phi(b^*b) = \|a\|^2\|b_\phi\|.$$

□

So we now have a good way of constructing representations, we just have to check that we have sufficiently many states to build a faithful representation.

Proposition 3.50. *Let ϕ be a state on a unital C*-algebra A, then $\|\phi\| = 1$.*

Proof. Let $x \in A$ with $\|x\| = 1$. Then $\|x^*x\| = 1$, so $x^*x \leq 1$ by Exercise 2.38, so $\phi(x^*x) \leq 1$. Then by the Cauchy Schwarz inequality we have

$$|\phi(x)|^2 \leq \phi(1)\phi(x^*x) \leq 1.$$

□

The converse is also true, but in the interest of time we'll skip the proof:

Theorem 3.51 (See [24, Exercise 4.3.13]). *Let ϕ be a continuous funcitonal on A with $\|\phi\| = \phi(1) = 1$. Then ϕ is a state.*

Proposition 3.52. *Let $B \subseteq A$ be unital C*-algebras (sharing the same unit). If ϕ is a state on B, then there is a state extension $\widetilde{\phi}$ of ϕ to A, i.e. $\widetilde{\phi}$ is a state and $\widetilde{\phi}$ restricted to B is ϕ.*

Proof. Use the Hahn-Banach theorem to extend ϕ to a contractive functional which is automatically a state by Theorem 3.51. □

Example 3.53. Let X be a compact Hausdorff space and μ a finitely additive positive Borel measure with $\mu(X) = 1$. Then $\phi(f) = \int f d\mu$ is a state on $C(X)$. By the Riesz representation theorem, all states of $C(X)$ arise in this way.

Exercise 3.54. Let $f \in C(X)$. Show that there is a state ϕ on $C(X)$ so $|\phi(f)| = \|f\|$.

Proposition 3.55. *Let A be a C*-algebra and $a \in A$. Then there is a state ϕ on A with $\|a\|^2 = \phi(a^*a)$.*

Proof. By Exercise 3.54 there is a state ψ on $C^*(a^*a, 1)$ such that $\psi(a^*a) = \|a^*a\|$. By Proposition 3.52 we can extend ψ to a state on A. □

Theorem 3.56 (Gelfand-Naimark Theorem). *Let A be a C*-algebra. Then there is an isometric representation (π, H) of A.*

Proof. Let $a \in A$ be nonzero. Use Proposition 3.55 to choose a state ϕ_a so $\phi_a(a^*a) \neq 0$. Then

$$\langle \pi_{\phi_a}(a)(1_{\phi_a}), a_{\phi_a} \rangle = \phi_a(a^*a) \neq 0.$$

In particular, $\pi_{\phi_a}(a) \neq 0$. Hence the representation $\oplus_{a \in A \setminus \{0\}} \pi_{\phi_a}$ is injective by Exercise 3.12, and thus isometric by Theorem 3.1. □

Exercise 3.57. Let A be a separable C*-algebra. Show that there is a separable Hilbert space H and an injective *-homomorphism $\pi : A \to B(H)$.

Exercise 3.58. Let $\ell^\infty(\mathbb{N})$ be the C*-algebra of all bounded complex sequences and $c_0(\mathbb{N})$ those sequences that tend to 0. By identifying $\ell^\infty(\mathbb{N})$ with diagonal operators in $B(\ell^2(\mathbb{N}))$, it's clear we can represent $\ell^\infty(\mathbb{N})$ faithfully on a separable Hilbert space. Show that we cannot faithfully represent the corona algebra $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$ on a separable Hilbert space. [Hint: Show that $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$ has an uncountable family of pairwise orthogonal projections.]

Definition 3.59. Fix $n \in \mathbb{N}$ and A a C*-algebra. Define the *-algebra $M_n(A)$ of all $n \times n$ matrices with entries in A. Then multiplication and addition are defined as they are for any matrix ring. The adjoint operation is the "*-transpose", i.e. take the adjoint of every entry and then transpose the matrix. The only ingredient missing to make this a C*-algebra is a norm. Fix a faithful representation (π, H) of A (possible by Theorem 3.56). By basic linear algebra we have an identification of the *-algebra $M_n(B(H))$ with $B(H^n)$. In this way we define a norm on $M_n(B(H))$ and then put the inherited norm on $M_n(A)$. Notice that this norm is well-defined by uniqueness of C*-norms.

4. Tensor Products of C*-algebras

I recommend Takesaki's classic text [31, Section IV.4] as a reference for tensor products of C*-algebras. I haven't read Brown and Ozawa's treatment of tensor products but based on the quality of the chapters I have read, I suspect that [4, Chapter 3] is also an excellent reference. We won't prove anything here, just list some often-used-and-easy-to-believe facts about tensor products.

Let A and B be C*-algebras and let $A \odot B$ denote their algebraic tensor product (making $A \odot B$ an involutive algebra with $(a \otimes b)^* = a^* \otimes b^*$). We would like to equip $A \odot B$ with a norm making it a pre-C*-algebra. This is always possible, but in general there is more than one way. We don't just want any old norm on $A \odot B$, but a *reasonable* norm:

Definition 4.1. A norm $\|\cdot\|_\alpha$ on $A \odot B$ is a C*-norm if

(1) $\|a \otimes b\|_\alpha = \|a\|\|b\| \quad$ for all $a \in A, b \in B$.
(2) $\|x\|_\alpha^2 = \|x^*x\|_\alpha \quad$ for all $x \in A \odot B$.

4.1. $\otimes_{min}$.

Let H and K be Hilbert spaces. Then one defines the Hilbert space $H \otimes K$ as the completion of the vector space tensor product $H \odot K$ with inner product defined by

$$\langle \eta_1 \otimes \xi_1, \eta_2 \otimes \xi_2 \rangle = \langle \eta_1, \eta_2 \rangle_H \langle \xi_1, \xi_2 \rangle_K.$$

Exercise 4.2. If $\{e_i : i \in I\}$ is an orthonormal basis for H and $\{f_j : j \in J\}$ is an orthonormal basis for K, then $\{e_i \otimes f_j : (i,j) \in I \times J\}$ is an orthonormal basis for $H \otimes K$.

Definition 4.3. Let $T \in B(H)$ and $S \in B(K)$. We then form the operator $T \otimes S \in B(H \otimes K)$ defined by the equations

$$T \otimes S(\eta \otimes \xi) = T(\eta) \otimes S(\xi), \qquad \text{for } \eta \in H,\ \xi \in K.$$

Exercise 4.4. $\|T \otimes S\| = \|T\|\|S\|$.

Let now A and B be unital C*-algebras and let (π, H) be a representation of A and (σ, K) a representation of B. Then form the *-algebra homomorphism $\pi \otimes \sigma : A \odot B \to B(H \otimes K)$ by $\pi \otimes \sigma(a \otimes b) = \pi(a) \otimes \sigma(b)$. We define a seminorm on $A \odot B$ as

$$\|x\|_{min} := \sup\{\|\pi \otimes \sigma(x)\| : \pi, \sigma \text{ are representations of } A, B\} \tag{4.1.1}$$

Theorem 4.5 (Takesaki). *$\|\cdot\|_{min}$ is a C*-norm on $A \odot B$. For any other C*-cross norm $\|\cdot\|_\alpha$ on $A \odot B$ we have $\|\cdot\|_{min} \leq \|\cdot\|_\alpha$.*

Definition 4.6. We write $A \otimes B$ as the completion of $A \odot B$ with respect to the min norm.

Theorem 4.7. *Let (π, H) and (σ, K) be* any *faithful representations of A and B. Then $\pi \otimes \sigma : A \odot B \to B(H \otimes K)$ extends to an isometry on $A \otimes B$.*

In general, if A_1, A_2, B_1, B_2 are C*-algebras and $\pi_i : A_i \to B_i$ are *-homomorphisms, then the algebraic map $\pi_1 \otimes \pi_2 : A_1 \odot A_2 \to B_1 \odot B_2$ need not extend to a continuous homomorphism of $A_1 \otimes_{min} A_2$. We do record the following special case, which is an easy consequence of Theorem 4.7:

Corollary 4.8. *Let $A \subseteq B$ and C be C*-algebras, and let $\iota : A \to B$ be the inclusion map. Then the map $\iota \otimes id_C : A \otimes C \to B \otimes C$ is an injective *-homomorphism.*

Exercise 4.9. Show that $M_n(A)$ is isomorphic to $M_n \otimes A$. See Definition 3.59.

Theorem 4.10 (Takesaki). *If A and B are simple, then $A \otimes B$ is simple.*

4.2. $\otimes_{max}$.

Definition 4.11. Let A and B be unital C*-algebras. Define

$$\|x\|_{max} := \sup\{\|\pi(x)\| : \pi : A \odot B \to B(H) \text{ is a *-homomorphism } \}.$$

Denote by $A \otimes_{max} B$ the completion of $A \odot B$ with respect to the max norm.

Corollary 4.12. *Let $\pi : A \to B(H)$ and $\sigma : B \to B(H)$ be representations with commuting ranges. Then the map defined by $a \otimes b \mapsto \pi(a)\sigma(b)$ extends to a *-homomorphism of $A \otimes_{max} B$.*

Corollary 4.13. *For any C*-norm $\|\cdot\|_\alpha$, we have $\|\cdot\|_{min} \leq \|\cdot\|_\alpha \leq \|\cdot\|_{max}$.*

Corollary 4.14. *Let $\|\cdot\|_\alpha$ be a C*-norm on $A \odot B$. Then the identity map extends to a *-homomorphism from $A \otimes_{max} B$ onto $A \otimes_\alpha B$.*

Theorem 4.15 ([26, Corollary 11.9 + Exercise 11.1]). *Let $0 \to J \to A \to A/J \to 0$ be an exact sequence and B any C*-algebra. Then we have an inclusion $J \otimes_{max} B \subseteq A \otimes_{max} B$ and the following sequence is exact:*

$$0 \to J \otimes_{max} B \to A \otimes_{max} B \to (A/J) \otimes_{max} B.$$

5. Nuclear and Exact C*-algebras

For information on nuclear and exact C*-algebras I highly recommend any of the following books of Wassermann, Pisier or Brown-Ozawa [36, 26, 4] (although the scope of Pisier and Brown-Ozawa are much broader than Wassermann's short text).

Definition 5.1. Let A be a C*-algebra. We say A is **nuclear** if the extension of the identity map from $A \otimes_{max} B$ onto $A \otimes B$ is injective for every C*-algebra B.

Proposition 5.2. *Every finite dimensional C*-algebra is nuclear.*

Proof. Let A be finite dimensional and B any C*-algebra. Then $A \odot B$ is already complete under any C*-norm. As every C*-algebra has a unique C*-norm (see Corollary 3.2), the conclusion follows. □

Proposition 5.3. *Let A be an inductive limit of nuclear C*-algebras A_n. Then A is nuclear.*

Proof. Let B be a C*-algebra and $\pi : A \otimes_{max} B \to A \otimes B$ be the quotient map. By Corollary 4.8, we have $\pi|_{A_n \odot B}$ is an isometry, hence π is an isometry. $\square$

Corollary 5.4. *AF algebras are nuclear.*

Nuclear C*-algebras are also closed under quotients and extensions. As far as I know there is no easy proof of this fact, they all rely on Alain Connes's theorem of the equivalence of injectivity and hyper finiteness for von Neumann algebras [6].

Theorem 5.5 (Combination of results of Connes, Choi-Effros, Kirchberg)**.** *Let A be a C*-algebra and J an ideal of A. If A is nuclear, then so is J and A/J. On the other hand, if both J and A/J are nuclear, then so is A.*

Exercise 5.6. Once you convince yourself that the max and min tensor products are associative, it's easy to see that if A and B are nuclear, then so is $A \otimes B$.

Definition 5.7. A C*-algebra A is **exact** if for any exact sequence $0 \to J \to B \to B/J \to 0$ the following sequence is also exact:

$$0 \to J \otimes A \to B \otimes A \to (B/J) \otimes A \to 0.$$

Corollary 5.8 (to Theorem 4.15)**.** *Every nuclear C*-algebra is exact, and every subalgebra of an exact C*-algebra is exact.*

Remark 5.9. Nuclearity is **not** preserved by subalgebras. As a special case of a theorem of Blackadar [2] (that used Voiculescu's embedding result [35]), every unital, simple, infinite dimensional nuclear C*-algebra has a non-nuclear subalgebra. Not every exact C*-algebra is nuclear. An example is the reduced C*-algebra $C_r^*(\mathbb{F}_n)$ of a non-abelian free group (see [4] for details and definitions). Not every C*-algebra is exact (for example $B(H)$ is not exact).

5.1. **Completely positive maps.** Let A be an infinite dimensional, simple C*-algebra. Then there are no *-homomorphisms from A to M_n. In other words, *-homomorphisms can be too rigid as our morphisms in the category of C*-algebras. On the other hand, completely positive maps preserve a fair amount of the structure of a C*-algebra (not just the order structure, but the so-called *complete* order structure of a C*-algebra), while being plentiful enough to separate points. For much more information on completely positive maps, I recommend both Paulsen's monograph [22] and Effros and Ruan's [12].

Definition 5.10. Let $\phi : A \to B$ be a linear function. Then for each $n \in \mathbb{N}$ one obtains a linear function $id \otimes \phi : M_n \otimes A \to M_n \otimes B$ defined by $id \otimes \phi(x \otimes a) = x \otimes \phi(a)$.

Remark 5.11. Notice that under the identifications from Exercise 4.9, then $id \otimes M_n$ is just ϕ applied "entrywise", for example if $n = 2$, we have $id \otimes \phi : M_2(A) \to M_2(B)$ by

$$id \otimes \phi\Big(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \Big) = \begin{bmatrix} \phi(x_{11}) & \phi(x_{12}) \\ \phi(x_{21}) & \phi(x_{22}) \end{bmatrix}.$$

Definition 5.12. Let $\phi : A \to B$ be a linear operator. We say ϕ is **positive** if $\phi(a) \geq 0$ whenever $a \in A^+$. We say ϕ is **completely positive** if $id \otimes \phi : M_n \otimes A \to M_n \otimes B$ is positive for every $n \geq 1$.

Exercise 5.13. Let $\phi : M_2 \to M_2$ be the transpose map. Then ϕ is positive, but not completely positive.

Proposition 5.14. *Let $\pi : A \to B$ be a *-homomorphism. Then π is completely positive.*

Proof. By Exercise 2.42, π is positive. Notice that $id \otimes \pi$ is also a *-homomorphism, hence positive. $\square$

Proposition 5.15. *Let H and K be Hilbert spaces and let $V : K \to H$ be a bounded linear operator. Then the map $\phi : B(H) \to B(K)$ defined by $\phi(T) = V^*TV$ is completely positive.*

Proof. By Exercise 2.43, it follows that ϕ is positive. Let $n \geq 1$, then by Remark 5.11 we have

$$id \otimes \phi(X) = \begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix}^* X \begin{bmatrix} V & 0 & \dots & 0 \\ 0 & V & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & V \end{bmatrix}$$

which will also be positive by Exercise 2.43. $\square$

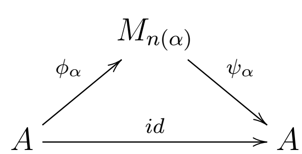
The most fundamental result about completely positive maps is the following

Theorem 5.16 (Stinespring's Theorem)**.** *Let $\phi : A \to B(H)$ be a linear map. Then ϕ is completely positive if and only if there is a Hilbert space K, a bounded linear operator $V : H \to K$ and a *-homomorphism $\pi : A \to B(K)$ such that $\phi(a) = V^*\pi(a)V$ for all $a \in A$. Moreover if ϕ is unital, then we can take V to be an isometry.*

Proof. Right to left is immediate from the previous two results. We won't prove the other direction, but mention that in the case $H = \mathbb{C}$ then ϕ is a state and the representation $(\pi, K) = (\pi_\phi, H_\phi)$ is simply the GNS representation, with V the map that sends $1 \in \mathbb{C}$ to the vector $1_\phi \in H_\phi$. The proof of the general case is very similar to the proof of the GNS construction (see [31, Theorem IV.3.6]). $\square$

Theorem 5.17 ([20, 5, 18])**.** *A C*-algebra A is nuclear if and only if it has the* ***completely positive approximation property*** *(CPAP); There is a net of completely positive maps $\phi_\alpha : A \to M_{n(\alpha)}$ and $\psi_\alpha : M_{n(\alpha)} \to A$ such that the following diagram approximately commutes:*

(5.1.1)



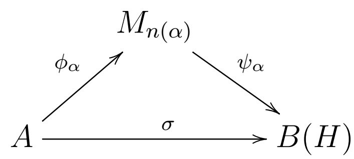
i.e., $\psi_\alpha \circ \phi_\alpha(a) \to a$ for every $a \in A$.

Theorem 5.18. *ASH algebras and Cuntz algebras are nuclear.*

Proof. It's more-or-less a real analysis exercise to show that commutative C*-algebras satisfy the CPAP. By Exercise 5.6, it follows that all homogeneous C*-algebras are nuclear and hence all AH algebras by Proposition 5.3. I can't think of any way to show that sub homogeneous C*-algebras are nuclear without using von Neumann algebra theory that we haven't touched on, so you'll just have to trust me on that one. Cuntz algebras, you'll really have to trust me. Typically one shows this by using the fact that $\mathcal{O}_2 \otimes K(H)$ arises as a "nice" crossed product (an important and ubiquitous construction of operator algebras that we've skipped entirely) and using this to show that $\mathcal{O}_2$ is nuclear (via the CPAP). □

Theorem 5.19 (Kirchberg, see [4, 36]). *A C*-algebra A is exact if and only if it is* ***nuclearly embeddable****, i.e. there is an injective *-homomorphism $\sigma : A \to B(H)$ and a net of completely positive maps $\phi_\alpha : A \to M_{n(\alpha)}$ and $\psi_\alpha : M_{n(\alpha)} \to B(H)$ such that the following diagram approximately commutes:*

(5.1.2)



i.e., $\psi_\alpha \circ \phi_\alpha(a) \to \sigma(a)$ for every $a \in A$.

In 2000, Kirchberg and Phillips significantly strengthened the above theorem in the separable case:

Theorem 5.20 (Kirchberg's Embedding Theorem, [19]). *Let A be a separable exact C*-algebra. Then there is an injective *-homomorphism $\pi : A \to \mathcal{O}_2$.*

Proof. I recommend reading the original, well-written Kirchberg-Phillips paper [19] for the proof of this. I also suggest having copies of the books of Wassermann [36], Rørdam[27], and Brown-Ozawa [4] handy as references for the background material. □

Remark 5.21. It's not known if the above theorem holds in the non-separable case: Let A be an exact C*-algebra, is there a nuclear C*-algebra B and an injective *-homomorphism $\sigma : A \to B$?

Lemma 5.22 ([19, Lemma 1.3]). *Let $\phi : M_n \to A$ be unital and completely positive. Then there is a partial isometry $v \in M_n \otimes M_n \otimes A$ such that $t^*(b \otimes 1 \otimes 1)t = e_{11} \otimes e_{11} \otimes \phi(b)$.*

Proof. Notice that $x = \sum_{i,j=1}^n e_{ij} \otimes e_{ij} \in (M_n \otimes M_n)^+$, since $\frac{1}{n}x$ is a projection. Hence $id \otimes T(x) \geq 0$ since T is completely positive (Side note: It is an old observation of Choi that the fact that $T(x) \geq 0$ actually characterizes completely positive maps from M_n into A). Set

$$0 \leq \sum_{i,j=1}^{n} e_{ij} \otimes a_{ij} = [id \otimes T(x)]^{1/2}.$$

Set $t = \sum_{i,j=1}^n e_{i1} \otimes e_{j1} \otimes a_{ij}$. One then checks (by squaring the above matrix equation and using the resulting n^2 operator equations) for a matrix unit $e_{k\ell}$, that $t^*(e_{k\ell} \otimes 1 \otimes 1)t = 1 \otimes 1 \otimes T(e_{k\ell})$, from which the conclusion follows. □

Remark 5.23. The point of the above lemma is a common one in arguments about completely positive maps from M_n into a C*-algebra: All of the information about T is contained in the positive element $id \otimes T(x)$. Use operator theoretic techniques on the operator $id \otimes T(x)$ to then deduce something useful about the map T. In the above case, we wanted to factor a map between C*-algebras, which was subsequently reduced to factoring an *element* (i.e. a map on a Hilbert space) of a C*-algebra; a much easier prospect.

6. The Main Point

6.1. **More on $\mathcal{O}_2$.** In this section we "prove" one of the so-called Kirchberg absorption theorems, namely that for any separable, simple, unital nuclear C*-algebra we have $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. Due to time constraints, and the depth of the results, we'll have a number of black boxes. Not only is Kirchberg's theorem deep, but it also relies on some of the deeper parts of operator algebra theory (To name a few: Kirchberg's embedding theorem, Effros-Haagerup lifting theorem, equivalence of injectivity and hyperfiniteness for von Neumann algebras. To do any one of these results justice is much more time than we have.)

In addition to the Kirchberg-Phillips paper [19] that we will use as a guide, I also highly recommend Rørdam's monograph [27] for everything we will discuss here and much much more. We must mention that even though the absorption theorem is the stopping point for us, it is actually a springboard for a fantastic classification result [25], all of which is explained in Rørdam's book.

Theorem 6.1 (Black Box # 1)**.** $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

Proof. This theorem is due to George Elliott, but the easiest available proof is due to Rørdam. The interested reader should consult Rørdam's monograph [27, Chapter 5] for a guide to his proof of this theorem. I should mention that this isomorphism is not explicit. In fact (as far as I know) we don't even have an explicit *-homomorphism from $\mathcal{O}_2 \otimes \mathcal{O}_2$ into $\mathcal{O}_2$. □

Definition 6.2. Let A be a unital C*-algebra. We inductively define $A^{\otimes n} = (A^{\otimes(n-1)}) \otimes A$. We think of $A^{\otimes n} \subseteq A^{\otimes(n+1)}$ as $a \mapsto a \otimes 1$ and write $A^{\otimes\infty}$ as the inductive limit of this sequence.

Corollary 6.3. $\mathcal{O}_2^{\otimes\infty} \cong \mathcal{O}_2$.

Definition 6.4. We say that B has an **asymptotically central inclusion** of A if there exists a sequence $\phi_n : A \to B$ of *-homomorphisms such that $\|\phi_n(a)b - b\phi_n(a)\| \to 0$ for all $a \in A$ and $b \in B$.

Corollary 6.5. *Let A be a C*-algebra. Then $A \otimes \mathcal{O}_2$ has an asymptotically central inclusion of $\mathcal{O}_2$.*

Proof. Let ϕ_n map $\mathcal{O}_2$ to the "n"-th tensor position of $\mathcal{O}_2^{\otimes\infty} \subseteq \mathcal{O}_2^{\otimes\infty} \otimes A \cong \mathcal{O}_2 \otimes A$. □

6.2. $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$**.**

Lemma 6.6. *Let A be a simple, purely infinite unital C^*-algebra. Then, there are isometries $t_1, ..., t_n \in A$ such that $1 > \sum_{i=1}^n t_i t_i^*$.*

Proof. Let $s \in A$ be an isometry such that $p := ss^* < 1$. By Theorem 3.42, there is an $a \in A$ so $a^*(1-p)a = 1$. Set $t_i = s^{i-1}(1-p)a$ for $i = 1, ..., n$. □

Theorem 6.7 (Gray Box)**.** *Let A be a unital, nuclear, purely infinite C*-algebra and let $T : A \to A$ be a unital completely positive map. Let $F \subseteq A$ be a finite set and $\epsilon > 0$. Then there is an isometry $s \in A$ such that*

$$\max_{a \in F} \|s^*as - T(a)\| < \epsilon.$$

Not a proof at all. Very, very, very roughly one uses the fact that T can be approximately factored through matrices. Then (a) Stinespring's (type) theorem for the map from A into M_n and Lemma 5.22 for the map from M_n into A and then the fact that in a simple purely infinite C*-algebra there is a ton of "room" to chop elements up and move them around to obtain s. ☐

Theorem 6.8. *Let A be a nuclear, simple, purely infinite unital C*-algebra. Then $B = A \otimes \mathcal{O}_2$ is simple and purely infinite.*

Proof. First note that B is simple by Theorem 4.10. Let $a \in B$. Since B is simple, there are elements $x_1, ..., x_n, y_1, ..., y_n \in B$ such that

$$\sum_{i=1}^{n} x_i a y_i = 1.$$

Get isometries $t_1, ..., t_n \in \mathcal{O}_2$ that satisfy Lemma 6.6. By Corollary 6.5 obtain a *-homomorphism $\phi : \mathcal{O}_2 \to B$ such that $\phi(t_1), ..., \phi(t_n)$ sufficiently commute with a. Let $x = \sum_{i=1}^{n} x_i \phi(t_i)^*$ and $y = \sum_{i=1}^{n} \phi(s_i) y_i$. As long as the $\phi(t_i)'s$ commute well with a, then we can make xay as close to 1 as we wish. In particular, xay will be invertible by Exercise 2.2, making B purely infinite by Theorem 3.42. ☐

Before we can finish we need one more black box:

Theorem 6.9 (Black Box # 2, [19])**.** *Let $\mathcal{U}$ be an non-principal ultrafilter on $\mathbb{N}$ and A be a purely infinite,simple, unital, separable, nuclear C*-algebra. Then $A' \cap A_{\mathcal{U}}$ (see Example* 3.9*) is unital, simple and purely infinite.*

Theorem 6.10. *Let $B = A \otimes \mathcal{O}_2$ and $\gamma : B \to B$ be a *-homomorphism. Then γ is approximately unitarily equivalent to id_B.*

Proof. Let $\mathcal{U}$ be a non principal ultrafilter on $\mathbb{N}$ and regard $A \subseteq A_{\mathcal{U}}$ via the diagonal embedding (see Example 3.9). By Exercise 3.10 and the separability of B, it suffices to find a unitary $u \in A_{\mathcal{U}}$ such that $u^* a u = \gamma(a)$ for all $a \in B$

By Theorem 6.7, there is a sequence of isometries (v_n) from B such that $v_n^* a v_n \to \gamma(a)$, for all $a \in B$

Then $v = [(v_n)] \in A_{\mathcal{U}}$ is an isometry and $v^* a v = \gamma(a)$ for all $a \in A$. Since γ is a *-homomophism, we have $\gamma(a)$ a unitary for each unitary $a \in A$. By Proposition 2.54, we have vv^* commutes with every unitary in A. Since the unitaries span A, we have $vv^* \in A'$. Since B has an asymptotically central inclusion of $\mathcal{O}_2$, it follows that there is an embedding of $\mathcal{O}_2$ into $A' \cap A_{\mathcal{U}}$. Since $A' \cap A_{\mathcal{U}}$ is simple, unital and purely infinite, by Theorem 3.44, there is an isometry $w \in A' \cap A_{\mathcal{U}}$, such that $ww^* = vv^*$. Then $u = w^* v$ is the desired unitary. ☐

Theorem 6.11. *Let A be simple, unital and nuclear. Then $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.*

Proof. By Theorem 5.20, we have a *-homomorphism $\sigma : A \otimes \mathcal{O}_2 \to \mathcal{O}_2$ and let $\iota : \mathcal{O}_2 \to A \otimes \mathcal{O}_2$ be the inclusion. By Theorem 6.1 and Theorem 6.10, we have both $\sigma \circ \iota$ and $\iota \circ \sigma$ approximately unitarily equivalent to the identity operators on their respective algebras. By Lemma 3.18, we have $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. ☐

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