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Perturbations of completely positive maps and strong NF algebras

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Abstract

Let $\phi : M_n \to B(H)$ be an injective completely positive contraction with $\|\phi^{-1} : \phi(M_n) \to M_n\|_{\mathrm{cb}} \leqslant 1 + \delta(\epsilon)$. We show that if either $\mathrm{C}^*(\phi(M_n))$ contains no non-trivial compact operators or $\phi(M_n)$ approximately contains a rank 1 projection, then there is a complete order embedding $\psi : M_n \to B(H)$ with $\|\phi - \psi\|_{\mathrm{cb}} < \epsilon$. We also give examples showing that such a perturbation does not exist in general. As an application, we show that every C*-algebra A with $\mathcal{OL}_\infty(A) = 1$ and a finite separating family of primitive ideals is a strong NF algebra, providing a partial answer to a question of Junge, Ozawa and Ruan.

1. *Introduction*

The main goal of this paper is to provide a clearer picture of the relationship between Blackadar and Kirchberg's strong NF algebras and Junge, Ozawa and Ruan's notion of $\mathcal{OL}_\infty$ structure. Our main technical tools in this endeavour are perturbations of completely positive maps. We feel that these perturbation results may have applications in other areas of operator algebras, and so we first describe them in some detail.

Let $B(H)$ denote the space of bounded linear operators on a Hilbert space H and let M_n denote $B(H)$ when the dimension of H is n. Following [**1**, 4.1] we say that a linear map ϕ between C*-algebras is a *complete order embedding* if ϕ is a completely positive complete isometry.

We start with the following question.

Question 1.1. Let $\epsilon > 0$ be given. Does there exist a $\delta > 0$ such that if $\phi : M_n \to B(H)$ is an injective completely positive contraction with $\|\phi^{-1} : \phi(M_n) \to M_n\|_{\mathrm{cb}} < 1 + \delta$, then there is a complete order embedding $\psi : M_n \to B(H)$ such that $\|\phi - \psi\| < \epsilon$? In particular, can δ be chosen independent of n?

In [**14**, Lemma 7.1] Kerr and Li showed that if ϵ depends on both δ and n, then one can answer Question 1.1 affirmatively. We show (see § 2.3.1) that the dependence on n in Kerr and Li's lemma cannot be omitted. Hence the answer to Question 1.1 is no, in general.

On the other hand we show that in two extreme cases, the answer to Question 1.1 is yes. First we show that if $\mathrm{C}^*(\phi(M_n)) \cap K(H) = \{0\}$ ($K(H)$ denotes the compact operators on H), then we can perturb ϕ independent of n (Theorem 2.22). At the other end of the spectrum, we show that if ϕ approximately contains a rank 1 projection, then we can perturb ϕ independent of n (Theorem 2.27).

These two special cases prove to be the most applicable, due to the well-known fact that for any irreducible representation (π, H) of a C*-algebra A we either have $\pi(A) \cap K(H) = \{0\}$ or $K(H) \subseteq \pi(A)$.

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In the second part of the paper, we apply our perturbation results to the study of strong NF algebras. Blackadar and Kirchberg introduced and studied strong NF algebras in [**1**–**3**]. A C*-algebra A is called strong NF if, for every finite subset $F \subset A$ and $\epsilon > 0$, there is a finite-dimensional C*-algebra B and a complete order embedding $\psi : B \to A$ such that $\max_{x\in F} \operatorname{dist}(x, \psi(B)) < \epsilon$. Strong NF algebras form a large and natural class of C*-algebras. Indeed, it was shown in [**2**] that strong NF algebras are precisely the nuclear inner quasidiagonal C*-algebras. In particular, every simple nuclear quasidiagonal C*-algebra is strong NF.

In [**12**], the authors generalized the notion of strong NF algebras by introducing the invariant $\mathcal{OL}_\infty(\cdot)$ for C*-algebras. Let A be a C*-algebra and let $\lambda > 1$. Then A is called an $\mathcal{OL}_{\infty,\lambda}$ space if, for every finite subset $F \subset A$, there is a finite-dimensional C*-algebra B and an injective linear map $\phi : B \to A$ such that $F \subset \phi(B)$ and $\|\phi\|_{\mathrm{cb}}\|\phi^{-1} : \phi(B) \to B\|_{\mathrm{cb}} < \lambda$. One then defines $\mathcal{OL}_\infty(A)$ to be the infimum of all λ such that A is an $\mathcal{OL}_{\infty,\lambda}$ space. It was shown in [**12**] that $\mathcal{OL}_\infty(A)$ is finite precisely when A is nuclear.

In this paper we are interested in C*-algebras A with $\mathcal{OL}_\infty(A) = 1$. It was shown in [**12**] that all C*-algebras with $\mathcal{OL}_\infty(A) = 1$ are (nuclear) and quasidiagonal. On the other hand in [**9**], examples were given of nuclear quasidiagonal C*-algebras A with $\mathcal{OL}_\infty(A) > 1$. At present we do not have a good description of those C*-algebras with $\mathcal{OL}_\infty(A) = 1$. A goal of this paper is to bring us closer to a 'good' description of these algebras.

A simple perturbation argument shows that all strong NF algebras A have $\mathcal{OL}_\infty(A) = 1$, and it was asked in [**12**] whether or not all C*-algebras with $\mathcal{OL}_\infty(A) = 1$ are strong NF algebras.

We use the perturbation arguments of the first part of the paper to show that if A has a finite separating family of primitive ideals and $\mathcal{OL}_\infty(A) = 1$, then A is a strong NF algebra (Theorem 3.10). The general case is still open, and we discuss the known necessary conditions at the end of the paper. We then apply Theorem 3.10 to show that $\mathcal{OL}_\infty$ is not continuous with respect to inductive limits and not multiplicative with respect to tensor products, even when one of the algebras is AF.

We use the following shorthand notation throughout the paper. Let $\phi : A \to B$ be a linear map. If ϕ is a unital completely positive map, we say that ϕ is a UCP map. If ϕ is a completely positive contraction, we say that ϕ is a CPC. We define

$$\phi^{(n)} := \mathrm{id}_{M_n} \otimes \phi : M_n \otimes A \longrightarrow M_n \otimes B.$$

If ϕ is injective with closed range, we define

$$\|\phi^{-1}\|_{\mathrm{cb}} := \|\phi^{-1} : \phi(A) \longrightarrow A\|_{\mathrm{cb}}.$$

We write $\ell^2(n)$ for an n-dimensional Hilbert space with standard basis vectors $e_1, \dots, e_n$ and $(e_{ij})_{i,j=1}^n$ as standard matrix units for M_n. We let A^+ denote the positive cone of the C*-algebra A and for $x, y \in A$ we write $[x, y] = xy - yx$.

2. *Perturbations*

The objects of study in this section are injective CPCs, $\phi : M_n \to B(H)$ with $\|\phi^{-1}\|_{\mathrm{cb}}$ close to 1. We first answer Question 1.1 affirmatively when $\dim(H) = n$. Then we gather some non-unital analogues of well-known theorems about UCP maps. We finish this section with the general answer to Question 1.1 discussed in § 1.

2.1. *The case where* $\dim(H) = n$

Lemma 2.1. *Let $0 < \epsilon < 1/2$, let H and K be Hilbert spaces and let $v : H \to K$ be an isometry. Suppose that there is a projection $p \in B(K)$ such that $\|vv^* - p\| < \epsilon$. Then there is an isometry $w : H \to K$ such that $\|v - w\| < 2\epsilon$ and $ww^* = p$.*

Proof. Consider the polar decomposition of $pv = w|pv|$. It follows from basic spectral theory that w has the desired properties. □

The following proposition (in the unital case) can be deduced from Erik Christensen's paper [**7**]. It appears that proving it directly versus applying [**7**, Lemma 3.3] requires the same amount of work, and so we prove it directly.

PROPOSITION 2.2. *Let $57^{-1/2} > \delta > 0$ and let $\phi : M_n \to M_n$ be a CPC with $\|\phi^{-1}\| < 1 + \delta$. Then there is a *-automorphism π of M_n such that $\|\pi - \phi\|_{\mathrm{cb}} < 57\sqrt{\delta}$.*

REMARK 2.3. We emphasize that the condition $\|\phi^{-1}\| < 1 + \delta$ in Proposition 2.2 is not a typo, that is, it is not necessary that the cb-norm of the inverse be close to 1, only that the inverse of the 1-norm is close to 1. This will not be particularly important for us, but it does have the happy consequence of less notation in Lemma 2.20.

Proof of Proposition 2.2. Suppose first that $\phi(1) = 1$. Let $(\sigma, H, v : \ell^2(n) \to H)$ be the Stinespring dilation of ϕ. Since $\phi(1) = 1$, it follows that v is an isometry.

We first show that the C*-algebra $\sigma(M_n)$ approximately commutes with the projection vv^*. To this end, let $u \in M_n$ be unitary. Then $\|\phi^{-1}(u)\| \leqslant 1 + \delta$. Hence, we have

$$\begin{aligned}
(1+\delta)^2 &\geqslant \|vv^*\sigma(\phi^{-1}(u))\|^2 \\
&= \|vv^*\sigma(\phi^{-1}(u))vv^* + vv^*\sigma(\phi^{-1}(u))(1 - vv^*)\|^2 \\
&= \|vuv^* + vv^*\sigma(\phi^{-1}(u))(1 - vv^*)\|^2 \\
&= \|vv^* + vv^*(\sigma(\phi^{-1}(u))(1 - vv^*)\sigma(\phi^{-1}(u))^* vv^*\| \\
&= 1 + \|vv^*(\sigma(\phi^{-1}(u))(1 - vv^*)\|^2, \qquad (2.1)
\end{aligned}$$

where the last equality follows by spectral theory performed in the C*-algebra $vv^*B(H)vv^*$. By (2.1) applied to both u and u^* we obtain

$$\|[\sigma(\phi^{-1}(u)), vv^*]\| \leqslant (2\delta + \delta^2)^{1/2} \leqslant 2\sqrt{\delta}. \qquad (2.2)$$

Applying the Russo–Dye theorem to $\phi(u)$, we obtain unitaries $v_1, \ldots, v_r \in M_n$ and positive scalars $\lambda_1, \ldots, \lambda_r$ that sum to 1 such that

$$u = \sum_{i=1}^{r} \lambda_i \phi^{-1}(v_i).$$

By (2.2) applied to $v_1, \ldots, v_r$, it follows that

$$\|[\sigma(u), vv^*]\| \leqslant 2\sqrt{\delta} \quad \text{for every unitary } u \in M_n.$$

Let μ denote normalized Haar measure on $\mathcal{U}_n$, the unitary group of M_n. Define

$$x = \int_{\mathcal{U}_n} \sigma(u)(vv^*)\sigma(u)^* \, d\mu(u).$$

Then, $x \in \sigma(M_n)'$ and

$$\|x - vv^*\| \leqslant \int_{\mathcal{U}_n} \|\sigma(u)(vv^*)\sigma(u)^* - vv^*\| \, d\mu(u) \leqslant 2\sqrt{\delta}.$$

Now, x may not be a projection, but it is close to one. It is clearly positive, and

$$\begin{aligned}\|x^2 - x\| &\leqslant \|x^2 - (vv^*)\| + \|x - vv^*\| \\ &\leqslant \|x^2 - xvv^*\| + \|xvv^* - vv^*\| + \|x - vv^*\| \\ &\leqslant 3(2\sqrt{\delta}) = 6\sqrt{\delta}. \end{aligned} \tag{2.3}$$

Since $0 \leqslant x \leqslant 1$, and by (2.3), it follows that

$$\mathrm{sp}(x) \subset [0, 12\sqrt{\delta}] \cup [1 - 12\sqrt{\delta}, 1].$$

Let p be the spectral projection of x associated with $[1 - 12\sqrt{\delta}, 1]$. Then, $p \in \sigma(M_n)'$ and $\|p - x\| \leqslant 12\sqrt{\delta}$, and hence

$$\|p - vv^*\| \leqslant 14\sqrt{\delta}.$$

We apply Lemma 2.1 to obtain an isometry $w : \ell^2(n) \to H$ such that $\|w - v\| \leqslant 28\sqrt{\delta}$ and $ww^* = p$. Now consider the map

$$\pi : M_n \longrightarrow M_n \text{ defined by } \pi(x) = w^*\sigma(x)w.$$

Then, $\|\pi - \phi\|_{\mathrm{cb}} \leqslant 56\sqrt{\delta}$, and since ww^* commutes with $\sigma(M_n)$, it follows that π is a *-homomorphism.

For the non-unital case, note that $\phi^{-1}(1)\phi^{-1}(1)^* \leqslant (1 + \delta)^2$. Then

$$1 = \phi(\phi^{-1}(1))\phi(\phi^{-1}(1))^* \leqslant \phi(\phi^{-1}(1)\phi^{-1}(1))^* \leqslant (1 + \delta)^2\phi(1).$$

By basic spectral theory, it follows that $\|\phi(1)^{-1/2} - 1\| \leqslant \delta$. Hence defining

$$\psi(x) = \phi(1)^{-1/2}\phi(x)\phi(1)^{-1/2},$$

it follows that ψ is unital and $\|\psi - \phi\|_{\mathrm{cb}} \leqslant 2\delta < \sqrt{\delta}$. We apply the first part of the proof to ψ to obtain the conclusion. □

Eventually, we will want to perturb maps where the dimension of H is arbitrary. Our method will be to first cut down by a rank n projection and then perturb the cut-down map via Theorem 2.2. The next lemma provides the justification for this method.

DEFINITION 2.4. Let $\phi : A \to B$ be a linear map between C*-algebras and let $p \in B$ be a projection. We define the map $\phi_p : A \to B$ as

$$\phi_p(x) = p\phi(x)p.$$

LEMMA 2.5. *Let $\phi : M_n \to B(H)$ be a CPC and let $\delta > 0$. Suppose that there is a rank n projection $p \in B(H)$ such that ϕ_p is injective with $\|\phi_p^{-1}\| \leqslant 1 + \delta$. Then there is a complete order embedding $\psi : M_n \to B(H)$ such that $\|\psi - \phi\|_{\mathrm{cb}} \leqslant 68\delta^{1/4}$ and $\psi = \psi_p + \psi_{(1-p)}$, where ψ_p is a non-zero *-homomorphism.*

Proof. By Proposition 2.2 there is a *-homomorphism $\sigma : M_n \to pB(H)p$ such that

$$\|\sigma - \phi_p\|_{\mathrm{cb}} \leqslant 57\sqrt{\delta}. \tag{2.4}$$

Let $k \in \mathbb{N}$ be arbitrary. Let $u \in M_k \otimes M_n$ be a unitary. Then

$$\begin{aligned} 1 &\geqslant \|\phi_p^{(k)}(u) + (1_k \otimes p)\phi^{(k)}(u)(1_k \otimes 1_n - 1_k \otimes p)\| \\ &\geqslant \|\sigma^{(k)}(u) + (1_k \otimes p)\phi^{(k)}(u)(1_k \otimes 1_n - 1_k \otimes p)\| - 57\sqrt{\delta} \\ &= \|1_k \otimes p + (1_k \otimes p)\phi^{(k)}(u)(1_k \otimes 1_n - 1_k \otimes p)\phi^{(k)}(u)^*(1_k \otimes p)\|^{1/2} - 57\sqrt{\delta} \\ &= \Big(1 + \|(1_k \otimes p)\phi^{(k)}(u)(1_k \otimes 1_n - 1_k \otimes p)\phi^{(k)}(u)^*(1_k \otimes p)\|\Big)^{1/2} - 57\sqrt{\delta}, \end{aligned}$$

where the last line follows by spectral theory. After rearranging and then applying the Russo–Dye theorem, it follows that

$$\|(1_k \otimes p)\phi^{(k)}(x)(1_k \otimes 1_n - 1_k \otimes p)\| \leqslant \sqrt{115}\delta^{1/4} \tag{2.5}$$

for every $x \in M_k \otimes M_n$ of norm 1. Define

$$\psi(x) = \sigma(x) + \phi_{(1-p)}(x).$$

Then ψ is a complete order embedding of the required form. By (2.4) and (2.5) it follows that

$$\|\psi - \phi\|_{\mathrm{cb}} \leqslant 57\sqrt{\delta} + \sqrt{115}\delta^{1/4} < 68\delta^{1/4}. \qquad \square$$

2.2. *Non-unital analogues*

Many of our maps of interest will be non-unital CPCs. For this reason we collect some non-unital analogues (Corollary 2.7 and Theorem 2.12) of well-known results about UCP maps that we could not find in the literature.

Lemma 2.6. *Let B be a unital C^*-algebra and let $\phi : M_n \to B$ be a complete order embedding. Let τ_n denote the normalized trace on M_n. Consider the map $\psi : M_n \to B$ defined by*

$$\psi(x) = \phi(x) + \tau_n(x)(1 - \phi(1)). \tag{2.6}$$

Then ψ is a (unital) complete order embedding.

Proof. We find that ψ is CP as it is the sum of CP maps. To see that it is completely isometric, let $x \in (M_k \otimes M_n)^+$, then

$$\|x\| \geqslant \|\psi^{(k)}(x)\| \geqslant \|\phi^{(k)}(x)\| = \|x\|.$$

By [**9**, Lemma 2.3], it follows that ψ is a complete isometry. $\square$

We now obtain a non-unital analogue of [**6**, 7.1].

Corollary 2.7. *Let B be a finite-dimensional C^*-algebra and let $\phi : M_n \to B$ be a complete order embedding. Then there is a rank n projection $p \in B$ such that ϕ_p is a non-zero *-homomorphism and $\phi = \phi_p + \phi_{(1-p)}$.*

Proof. If $n = 1$, this is trivial, and so assume that $n \geqslant 2$. Let ψ be as in (2.6). By [**6**, 7.1] there is a rank n projection $p \in B$ such that $\psi = \psi_p + \psi_{(1-p)}$ with ψ_p a non-zero *-homomorphism. Let $u \in M_n$ be a unitary with $\tau_n(u) = 0$. Then $\phi(u) = \psi(u)$ and

$$p\phi(1)p + p(1 - \phi(1))p = p\psi(1)p = p\psi(u)p\psi(u)^*p = p\phi(u)p\phi(u)^*p \leqslant p\phi(1)p. \tag{2.7}$$

Hence, $p(1 - \phi(1))p = 0$, and so $\psi_p = \phi_p$. $\square$

In general we cannot replace B with $B(H)$ for an infinite-dimensional Hilbert space in Corollary 2.7, because $\phi(e_{11})$ need not have 1 as an eigenvalue; but we can get as close as possible (Theorem 2.11). We start with the following perturbation lemma of Kerr and Li.

LEMMA 2.8 [14, Lemma 7.1]. *For every $\epsilon > 0$ and $n \in \mathbb{N}$, there is a $\delta(\epsilon, n) > 0$ such that if $\phi : M_n \to M_N$ is an injective UCP map with $\|\phi^{-1}\|_{\text{cb}} \leqslant 1 + \delta$, then there is a unital complete order embedding $\psi : M_n \to M_N$ with $\|\phi - \psi\|_{\text{cb}} \leqslant \epsilon$.*

It is also worth recalling the following result of Roger Smith that we will use repeatedly.

THEOREM 2.9 (Smith's lemma [20, Theorem 2.10]). *Let X be an operator space and let $\phi : X \to M_n$ be a linear map. Then*

$$\|\phi^{(n)}\| = \|\phi\|_{\text{cb}}.$$

DEFINITION 2.10. Fix $n \in \mathbb{N}$ and a Hilbert space H. Let $\mathcal{H}(n, H)$ denote the set of all complete order embeddings $\phi : M_n \to B(H)$ such that there is a rank n projection $p \in B(H)$, and so $\phi = \phi_p + \phi_{(1-p)}$ and ϕ_p is a non-zero *-homomorphism.

THEOREM 2.11. *We find that $\mathcal{H}(n, H)$ is cb-dense in the set of all complete order embeddings from M_n to $B(H)$.*

Proof. Suppose first that $\phi(1) = 1$. Let $0 < \epsilon < (150)^{-1/4}$ and let $\delta = \delta(\epsilon, n) > 0$ satisfy Lemma 2.8. From Theorem 2.9 it follows that

$$\|\phi^{-1} : \phi(M_n) \longrightarrow M_n\|_{\text{cb}} = \|(\phi^{(n)})^{-1} : M_n \otimes \phi(M_n) \longrightarrow M_n \otimes M_n\|.$$

Since the unit ball of $M_n \otimes \phi(M_n)$ is compact, it follows that there is a finite rank projection $q \in B(H)$ such that $\|\phi_q^{-1}\| \leqslant 1 + \delta$. Use Lemma 2.8 to obtain a unital complete order embedding $\psi : M_n \to qB(H)q$ with $\|\phi_q - \psi\|_{\text{cb}} \leqslant \epsilon$.

By [6, 7.1], there is a rank n projection $p \leqslant q$ such that $\psi = \psi_p + \psi_{(q-p)}$ and ψ_p is a non-zero *-homomorphism. Then $\|\phi_p - \psi_p\|_{\text{cb}} \leqslant \epsilon$. In particular, $\|\phi_p^{-1}\|_{\text{cb}} \leqslant 1 + 2\epsilon$. By Lemma 2.5, there is a $\widetilde{\psi} \in \mathcal{H}(n, H)$ such that $\|\phi - \widetilde{\psi}\|_{\text{cb}} \leqslant 150\epsilon^{1/4}$.

To prove the non-unital case, we first define ψ as in (2.6). Then apply the first part of the proof and use the 'approximate' version of (2.7) to obtain the conclusion. □

In the unital case, the following theorem is an immediate consequence of Arveson's extension theorem. The reason for the following theorem is that we do not know if 1 is in the range of ϕ, that is, $\phi(M_n)$ need not be a suboperator system of $B(H)$.

THEOREM 2.12. *Let $n \in \mathbb{N}$ and let $\phi : M_n \to B(H)$ be a complete order embedding. Then there is a UCP map $T : B(H) \to M_n$ such that $T\phi = \text{id}_{M_n}$.*

Proof. Let ϕ be a complete order embedding and let $(\psi^k)_{k=1}^{\infty}$ be a sequence from $\mathcal{H}(n, H)$ that converges to ϕ. Let $(p_k)_{k=1}^{\infty}$ be a sequence of rank n projections such that

$$\psi^k = \psi^k_{p_k} + \psi^k_{(1-p_k)} \quad \text{with } \psi^k_{p_k} \text{ a non-zero *-homomorphism.}$$

For each $k \in \mathbb{N}$ define $T_k : B(H) \to M_n$ by

$$T_k(x) = (\psi^k_{p_k})^{-1}(p_k x p_k). \tag{2.8}$$

Then each T_k is UCP and $T_k \psi^k = \mathrm{id}_{M_n}$, and hence

$$\|T_k \phi - \mathrm{id}_{M_n}\|_{\mathrm{cb}} \leqslant \|\phi - \psi^k\|_{\mathrm{cb}}.$$

Finally, let ω be a non-principal ultrafilter on $\mathbb{N}$. Then the map

$$T(x) = \lim_{k \to \omega} T_k(x)$$

has the desired properties. □

2.3. *The case where dim$(H) > n$*

We now turn our attention to the general case, where the range of ϕ is $B(H)$ for an arbitrary Hilbert space. We start with an example showing that the dependence on n in Lemma 2.8 cannot be omitted.

2.3.1. *Dependence on dimension.* Fix $n > 4$. Let

$$X_n = \{p \in M_n : p \text{ is a rank } n-1 \text{ projection}\}. \tag{2.9}$$

Let

$$p_1, \ldots, p_r \in X_n \quad \text{be a } \frac{1}{n}\text{-net for } X_n. \tag{2.10}$$

Define $\phi : M_r \to M_r \otimes M_n$ by

$$\phi(e_{ij}) = e_{ij} \otimes p_i p_j \quad \text{for } 1 \leqslant i, j \leqslant r.$$

Lemma 2.13. *We have that ϕ is a CPC.*

Proof. We have

$$\begin{aligned}\phi^{(r)}\left(\sum_{i,j=1}^r e_{ij} \otimes e_{ij}\right) &= \sum_{i,j=1}^r e_{ij} \otimes e_{ij} \otimes p_i p_j \\ &= \left(\sum_{i=1}^r e_{i1} \otimes e_{i1} \otimes p_i\right)\left(\sum_{i=1}^r e_{i1} \otimes e_{i1} \otimes p_i\right)^* \geqslant 0.\end{aligned}$$

Then ϕ is completely positive by [16, Theorem 3.14]. Furthermore, $\phi(1)$ is a projection, and hence ϕ is completely contractive. □

Lemma 2.14. *We have that ϕ is injective with $\|\phi^{-1}\|_{\mathrm{cb}} \leqslant n/(n-4)$.*

Proof. Let $k \in \mathbb{N}$ be arbitrary. Let $p \in M_k \otimes M_r$ be a rank 1 projection. Then there exist scalars $\alpha_{ij} \in \mathbb{C}$, with $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant r$ and $\sum_{i,j} |\alpha_{ij}|^2 = 1$, and an operator $v \in M_k \otimes M_r$, with $vv^* = p$ and

$$v = \sum_{i=1}^k e_{i1} \otimes \left(\sum_{j=1}^r \alpha_{ij} e_{j1}\right).$$

Let τ_n denote the normalized trace on M_n. Since each $p_i \in M_n$ is a rank $n-1$ projection, it follows that

$$\tau_n(p_1 p_i p_1) = \tau_n(p_i) - \tau_n((1-p_1)p_i) \geqslant \frac{n-1}{n} - \|p_i\|\tau_n(1-p_i) = \frac{n-2}{n} \quad \text{for } i = 1, \dots, r.$$

Therefore

$$\begin{aligned}
\|\phi^{(k)}(p)\| &\geqslant \|\phi^{(k)}(v)\phi^{(k)}(v)^*\| \\
&= \|\phi^{(k)}(v)^*\phi^{(k)}(v)\| \\
&= \left\| \left(\sum_{i=1}^k e_{1i} \otimes \left(\sum_{j=1}^r \overline{\alpha}_{ij} e_{1j} \otimes p_1 p_j \right) \right) \left(\sum_{i=1}^k e_{i1} \otimes \left(\sum_{j=1}^r \alpha_{ij} e_{j1} \otimes p_j p_1 \right) \right) \right\| \\
&= \left\| e_{11} \otimes e_{11} \otimes \left(\sum_{i=1}^k \sum_{j=1}^r |\alpha_{ij}|^2 p_1 p_j p_1 \right) \right\| \\
&\geqslant \tau_n \left(\sum_{i=1}^k \sum_{j=1}^r |\alpha_{ij}|^2 p_1 p_j p_1 \right) \\
&\geqslant \frac{n-2}{n}.
\end{aligned}$$

Now, let $a \in M_k \otimes M_r$ be positive and norm 1. Then there is a rank 1 projection $p \leqslant a$. Since ϕ is CP, it follows that $\|\phi^{(k)}(a)\| \geqslant \|\phi^{(k)}(p)\| \geqslant (n-2)/n$. Finally, by [**9**, Lemma 2.3] it follows that ϕ is injective with

$$\|\phi^{-1}\|_{\text{cb}} \leqslant \left(2 \left(\frac{n-2}{n} \right) - 1 \right)^{-1} = \frac{n}{n-4}. \qquad \square$$

Theorem 2.15. *Let $\psi : M_r \to M_r \otimes M_n$ be a complete order embedding. Then $\|\psi - \phi\| \geqslant 1 - 1/n$.*

Proof. Since ψ is a complete order embedding, Corollary 2.7 provides a rank r projection $p \in M_r \otimes M_n$ such that ψ_p is a *-monomorphism. In particular, there is a norm 1 vector $\xi \in \ell^2(r) \otimes \ell^2(n)$ such that

$$\|\psi(e_{i1})\xi\| = 1 \quad \text{for every } i = 1, \dots, r. \tag{2.11}$$

Decompose $\xi = e_1 \otimes \xi_1 + \eta$, where $\eta \perp e_1 \otimes \ell^2(n)$. Let $q \in X_n$ (see Definition (2.9)) be the rank $n-1$ projection onto $(\mathbb{C}p_1(\xi_1))^\perp$ (or if $p_1(\xi_1) = 0$, let q be any element of X_n). Then $qp_1(\xi_1) = 0$. By (2.10), there is an $1 \leqslant i' \leqslant r$ such that $\|q - p_{i'}\| < 1/n$.

Hence,

$$\|\phi(e_{i'1})\xi\| = \|(e_{i'1} \otimes p_{i'}p_1)\xi\| = \|e_{i'} \otimes (p_{i'}p_1\xi_1)\| < \|e_{i'} \otimes (qp_1\xi_1)\| + \frac{1}{n} = \frac{1}{n}.$$

Combining this with (2.11), it follows that

$$\|\psi - \phi\| \geqslant \|(\psi - \phi)(e_{i'1})\| > 1 - \frac{1}{n}. \qquad \square$$

REMARK 2.16. Note that $\phi(1)$ is a projection. Hence we may additionally assume that ϕ is unital above.

COROLLARY 2.17. *For every $\epsilon > 0$ there are $n, N \in \mathbb{N}$ and an injective UCP map $\phi : M_n \to M_N$ with $\|\phi^{-1}\|_{\text{cb}} \leqslant 1 + \epsilon$ and $\|\phi - \psi\| \geqslant 3/4$ for every complete order embedding ψ.*

2.3.2. *Independence of dimension in special cases.* Let $\phi : M_n \to B(H)$ be a CPC with $\|\phi^{-1}\|_{\text{cb}} \sim 1$. Corollary 2.17 shows that we cannot perturb ϕ to a complete order embedding unless we take the dimension into account. In this section, we show that in two special cases we can perturb ϕ independent of the dimension. In particular, we show if the range of ϕ is faithful modulo the compact operators (Theorem 2.22), then we can perturb independent of the dimension. Next we show (Theorem 2.25) that we can always perturb an amplification of ϕ regardless of the dimension. Finally, we use Theorem 2.25 to show that if the range of ϕ approximately contains a minimal projection, then we can perturb ϕ regardless of the dimension. The key to all the three results is Lemma 2.20.

DEFINITION 2.18. Let $n \in \mathbb{N}$. Let Tr denote the (non-normalized) trace on M_n. Let

$$\Lambda_n = \{x \in M_n^+ : \text{Tr}(x) = 1\}. \tag{2.12}$$

For a linear map $\phi : M_n \to B(H)$, let

$$\Lambda_n^\phi = \left\{ \sum_{i,j=1}^n \lambda_{ij} \phi(e_{1i})\phi(e_{j1}) : (\lambda_{ij}) \in \Lambda_n \right\}. \tag{2.13}$$

LEMMA 2.19. *Let $\phi : M_n \to B(H)$ be an injective CPC with $\|\phi^{-1}\|_{\text{cb}} \leqslant 1 + \delta$. Then Λ_n^ϕ is a compact convex subset of $B(H)^+$ with*

$$x \leqslant \phi(e_{11}) \text{ and } \|x\| > \frac{1}{1 + 3\delta} \quad \text{for every } x \in \Lambda_n^\phi. \tag{2.14}$$

Proof. It is clear that Λ_n is compact and convex, which implies that Λ_n^ϕ shares the same properties.

Let $\lambda = (\lambda_{ij})_{i,j=1}^n \in \Lambda_n$. By spectral theory, $\lambda = \sum_{\ell=1}^k r_\ell p_\ell$ with $r_\ell \in \mathbb{R}^+$ and p_ℓ orthogonal rank 1 projections. In particular, for each $\ell = 1, \ldots, k$ there are scalars $\alpha_1(\ell), \ldots, \alpha_n(\ell) \in \mathbb{C}$ such that

$$\lambda_{ij} = \sum_{\ell=1}^k \alpha_i(\ell)\overline{\alpha_j(\ell)}. \tag{2.15}$$

Define

$$v = \begin{bmatrix} \sum_{i=1}^n \alpha_i(1)e_{1i} & \sum_{i=1}^n \alpha_i(2)e_{1i} & \ldots & \sum_{i=1}^n \alpha_i(k)e_{1i} \end{bmatrix} \in M_{1,k}(M_n).$$

Then by (2.15), we have

$$\begin{aligned} e_{11} \otimes \left(\sum_{i,j=1}^n \lambda_{ij}\phi(e_{1i})\phi(e_{j1}) \right) &= \phi^{(k)}(v)\phi^{(k)}(v)^* \\ &\leqslant \phi^{(k)}(vv^*) = e_{11} \otimes (\text{Tr}(\lambda))\phi(e_{11}) = e_{11} \otimes \phi(e_{11}). \end{aligned} \tag{2.16}$$

This shows that $\Lambda_n^\phi \subset B(H)^+$ and also proves the first inequality from (2.14). For the second inequality, first note that $\|v\| = 1$. Combining this fact with (2.16), we have

$$\left\| \sum_{i,j=1}^{n} \lambda_{ij}\phi(e_{1i})\phi(e_{j1}) \right\| = \|\phi^{(k)}(v)\phi^{(k)}(v)^*\| \\ = \|\phi^{(k)}(v)\|^2 \geqslant (1+\delta)^{-2} > (1+3\delta)^{-1}.$$

This proves the second inequality from (2.14). □

LEMMA 2.20. *Let $\phi : M_n \to B(H)$ be a CPC and $\delta > 0$. Suppose that there is a norm 1 vector $\xi \in H$ such that*

$$\min_{x\in\Lambda_n^\phi} \langle x\xi, \xi\rangle \geqslant \frac{1}{1+\delta}. \tag{2.17}$$

Then there is a complete order embedding $\psi : M_n \to B(H)$ such that $\|\psi - \phi\|_{\mathrm{cb}} \leqslant 136\delta^{1/4}$.

Proof. We will construct a rank n projection $p \in B(H)$ that satisfies Lemma 2.5.

Since ϕ is contractive and by (2.17), for each $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ with $\sum_i |\alpha_i|^2 = 1$ we have

$$1 \geqslant \left\| \sum_{i=1}^{n} \alpha_i\phi(e_{i1})\xi \right\| = \left\langle \sum_{i,j=1}^{n} \overline{\alpha}_i\alpha_j\phi(e_{1i})\phi(e_{j1})\xi, \xi \right\rangle^{1/2} \geqslant \frac{1}{(1+\delta)^{1/2}}. \tag{2.18}$$

In particular, the vectors $\phi(e_{11})\xi, \dots, \phi(e_{n1})\xi$ are linearly independent, and so the subspace

$$K = \operatorname{span}\{\phi(e_{11})\xi, \dots, \phi(e_{n1})\xi\} \subseteq H$$

is n-dimensional. Let p be the orthogonal projection from H onto K.

Let $q \in M_2 \otimes M_n$ be a rank 1 projection. Then there are scalars $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{C}$ with $\sum_i |\alpha_i|^2 + |\beta_i|^2 = 1$ such that $q = vv^*$, where

$$v = e_{11} \otimes \left(\sum_{i=1}^{n} \alpha_i e_{i1} \right) + e_{21} \otimes \left(\sum_{i=1}^{n} \beta_i e_{i1} \right).$$

Set

$$\eta = e_1 \otimes \left(\sum_{i=1}^{n} \alpha_i\phi(e_{i1})\xi \right) + e_2 \otimes \left(\sum_{i=1}^{n} \beta_i\phi(e_{i1})\xi \right) \in \ell^2(2) \otimes K.$$

Then $\|\eta\| \leqslant 1$ by (2.18). Furthermore,

$$\langle \phi_p^{(2)}(q)\eta, \eta\rangle = \langle \phi^{(2)}(q)\eta, \eta\rangle \geqslant \langle \phi^{(2)}(v)\phi^{(2)}(v)^*\eta, \eta\rangle = \|\phi^{(2)}(v)^*\eta\|^2. \tag{2.19}$$

Note that $(\overline{\alpha}_i\alpha_j + \overline{\beta}_i\beta_j)_{i,j} \in \Lambda_n$. Hence it follows from (2.17) and the Cauchy–Schwarz inequality that

$$\|\phi^{(2)}(v)^*\eta\| = \left\| \sum_{i,j=1}^{n} (\overline{\alpha}_i\alpha_j + \overline{\beta}_i\beta_j)\phi(e_{1i})\phi(e_{j1})\xi \right\| \geqslant \frac{1}{1+\delta}.$$

Combining this with (2.19), it follows that

$$\|\phi_p^{(2)}(q)\| \geqslant (1+\delta)^{-2} \quad \text{for every rank 1 projection } q \in M_2 \otimes M_n.$$

Let $a \in M_2 \otimes M_n$ be positive and norm 1. Then there is a rank 1 projection $q \leqslant a$. Since ϕ_p is completely positive, it follows that

$$\|\phi_p^{(2)}(a)\| \geqslant \|\phi_p^{(2)}(q)\| \geqslant \frac{1}{(1+\delta)^2}.$$

Applying [9, Lemma 2.3], it follows that ϕ_p is injective with

$$\|\phi_p^{-1}\| \leqslant \left(\frac{2}{(1+\delta)^2} - 1\right)^{-1} \leqslant 1 + 7\delta.$$

Thus, by Lemma 2.5 there is a complete order embedding $\psi : M_n \to B(H)$ such that $\|\phi - \psi\|_{\mathrm{cb}} \leqslant 136\delta^{1/4}$. □

Since von Neumann's minimax theorem appeared in [15], numerous generalizations have followed. We use one such generalization due to Ky Fan in the following lemma.

Lemma 2.21. *Let A be a* C*-*algebra and let $X \subset A^+$ be compact and convex such that*

$$\|x\| \geqslant r \quad \text{for all } x \in X.$$

Then there is a state $\omega \in A^*$ such that

$$\omega(x) \geqslant r \quad \text{for all } x \in X.$$

If A is a von Neumann algebra, then ω can be chosen to be normal.

Proof. Let $S(A)$ denote the state space of A equipped with the $\sigma(A^*, A)$-topology. Then $S(A)$ is compact and convex. Consider the mapping

$$f : S(A) \times X \Longrightarrow \mathbb{R}^+ \quad \text{defined by } f(\omega, x) = \omega(x).$$

Then f is continuous and affine in each variable. By [10, Theorem 3] we have

$$\max_{\omega \in S(A)} \min_{x \in X} f(\omega, x) = \min_{x \in X} \max_{\omega \in S(A)} f(\omega, x) \geqslant r.$$

For the von Neumann case we can replace $S(A)$ with the space of normal states on A. □

Let us pause for a moment and map out the rest of the section. Let $\phi : M_n \to B(H)$ be a CPC with $\|\phi^{-1}\|_{\mathrm{cb}} \sim 1$. Then Lemmas 2.19 and 2.21 provide a state ω such that $\omega(x) \sim 1$ for $x \in \Lambda_n^\phi$. Lemma 2.20 then says we can perturb ϕ if ω can be chosen to be a vector state. For example, for the 'non-perturbable' map ϕ defined in § 2.3.1 we had $\omega = \langle (\cdot)\, e_1, e_1 \rangle \otimes \tau_n$, which is far away from a vector state. We complete this section with two extreme cases where ω can be chosen to be a vector state.

Theorem 2.22. *Let $\phi : M_n \to B(H)$ be a CPC with $\|\phi^{-1}\|_{\mathrm{cb}} < 1 + \delta$. Moreover, assume that $\mathrm{C}^*(\phi(M_n)) \cap K(H) = \{0\}$. Then there is a complete order embedding $\psi : M_n \to B(H)$ such that $\|\phi - \psi\|_{\mathrm{cb}} \leqslant 272\delta^{1/4}$.*

Proof. Consider Λ_n^ϕ as in (2.13). By Lemmas 2.19 and 2.21 there is a state $\omega \in \mathrm{C}^*(\phi(M_n))$ such that

$$\min_{x \in \Lambda_n^\phi} \omega(x) > \frac{1}{1+3\delta}.$$

Since $\mathrm{C}^*(\phi(M_n)) \cap K(H) = \{0\}$, Glimm's lemma (see [8, Lemma 11.2.1]) states that ω is a weak* limit of vector states. Since Λ_n^ϕ is compact, we obtain a unit vector $\xi \in H$ such that

$$\min_{x \in \Lambda_n^\phi} \langle x\xi, \xi \rangle \geqslant \frac{1}{1+3\delta}.$$

The conclusion now follows from Lemma 2.20. □

Before we consider the second extreme case, we first need to show that we can always perturb amplifications of injective CPC maps. We start with a technical lemma.

LEMMA 2.23. *Let B be a* C*-*algebra and let* $n \in \mathbb{N}$. *Suppose that* $\phi : M_n \oplus B \to B(H)$ *is a CPC with* $\|\phi^{-1}\|_{\mathrm{cb}} < 1+\delta$ *and there is a complete order embedding* $\psi : M_n \to B(H)$ *such that* $\|\phi|_{M_n} - \psi\|_{\mathrm{cb}} \leqslant \delta$. *Moreover, assume that there is a rank* n *projection* $q \in B(H)$ *such that* $\psi = \psi_q + \psi_{(1-q)}$ *with* ψ_q *a non-zero *-homomorphism. Then, we have*

$$\|(\phi_q)|_{0\oplus B}\|_{\mathrm{cb}} \leqslant 8\delta, \tag{2.20}$$

and

$$\|[\phi(x), q]\| \leqslant 8\sqrt{\delta} \quad \text{for all } x \in M_n \oplus B \quad \text{of norm 1.} \tag{2.21}$$

Proof. Let $k \in \mathbb{N}$ and $x \in 0 \oplus (M_k \otimes B)$ be positive and norm 1. Let $p = 1_n \oplus 0 \in M_n \oplus B$. Set $\widetilde{q} := 1_k \otimes q$. Since $\widetilde{q}\psi^{(k)}(1_k \otimes p) = \widetilde{q}$ and $\|\psi - \phi|_{M_n}\|_{\mathrm{cb}} \leqslant \delta$, we have

$$\begin{aligned}
1 &\geqslant \|\widetilde{q}\phi^{(k)}(1_k \otimes p + x)\| \\
&\geqslant \|\widetilde{q} + \widetilde{q}\phi^{(k)}(x)\| - \delta \\
&= \|\widetilde{q} + \widetilde{q}\phi^{(k)}(x)\widetilde{q} + \widetilde{q}\phi^{(k)}(x)(1-\widetilde{q})\| - \delta \\
&= \|\widetilde{q} + 2\widetilde{q}\phi^{(k)}(x)\widetilde{q} + \widetilde{q}\phi^{(k)}(x)\widetilde{q}\phi^{(k)}(x)\widetilde{q} + \widetilde{q}\phi^{(k)}(x)(1-\widetilde{q})\phi^{(k)}(x)\widetilde{q}\|^{1/2} - \delta \\
&= \Big(1 + \|2\widetilde{q}\phi^{(k)}(x)\widetilde{q} + \widetilde{q}\phi^{(k)}(x)\widetilde{q}\phi^{(k)}(x)\widetilde{q} + \widetilde{q}\phi^{(k)}(x)(1-\widetilde{q})\phi^{(k)}(x)\widetilde{q}\|\Big)^{1/2} - \delta.
\end{aligned}$$

After rearranging it follows that

$$\max\{\|2\widetilde{q}\phi^{(k)}(x)\widetilde{q}\|, \|\widetilde{q}\phi^{(k)}(x)(1-\widetilde{q})\phi^{(k)}(x)\widetilde{q}\|\} \leqslant 2\delta + \delta^2. \tag{2.22}$$

Since every $y \in 0 \oplus (M_k \otimes B)$ of norm 1 can be written as the sum of four positive elements, each with norm bound by 1, we have

$$\|(\phi_q)|_{0\oplus B}\|_{\mathrm{cb}} \leqslant 4(\delta + \delta^2/2) \leqslant 8\delta.$$

Let $(z, x) \in M_n \oplus B$ be positive and norm 1. By hypothesis, we have $\|[\phi(z, 0), q]\| \leqslant \delta$. By (2.22), we have $\|[\phi(0, x), q]\| \leqslant (2\delta + \delta^2)^{1/2}$. Again by decomposing an arbitrary $y \in M_n \oplus B$ into positive pieces, we obtain (2.21). □

DEFINITION 2.24. Let $\phi : X \to Y$ be a linear map and let $k \in \mathbb{N}$. Define $1_k \otimes \phi : X \to M_k(Y)$ by

$$(1_k \otimes \phi)(x) = 1_k \otimes \phi(x).$$

THEOREM 2.25. *Let* $n \in \mathbb{N}$ *and let* $\phi : M_n \to B(H)$ *be a CPC with* $\|\phi^{-1}\|_{\mathrm{cb}} < 1+\delta$. *Then there is a natural number* $k \in \mathbb{N}$ *and a complete order embedding* $\psi : M_n \to M_k \otimes B(H)$ *such that*

$$\|1_k \otimes \phi - \psi\|_{\mathrm{cb}} \leqslant 272\delta^{1/4}.$$

Proof. By Lemmas 2.19 and 2.21 we obtain a state $\omega \in B(H)_*$ such that

$$\min_{x\in\Lambda_n^\phi} \omega(x) > \frac{1}{1+3\delta}.$$

Since $\omega \in B(H)_*$, there are vectors $\xi_1, \xi_2, \ldots \in H$ such that

$$\omega(x) = \sum_{i=1}^{\infty} \langle x\xi_i, \xi_i \rangle \quad \text{for all } x \in B(H).$$

Since Λ_n^ϕ is compact, there is a $k \in \mathbb{N}$ such that

$$\min_{x \in \Lambda_n^\phi} \sum_{i=1}^{k} \langle x\xi_i, \xi_i \rangle \geqslant \frac{1}{1+3\delta}. \tag{2.23}$$

Let $\xi = \sum_{i=1}^k e_i \otimes \xi_i \in H^k$. Then, for any $x \in B(H)$, we have

$$\langle (1_k \otimes x)\xi, \xi \rangle = \sum_{i=1}^{k} \langle x\xi_i, \xi_i \rangle.$$

Combining this with (2.23) and Lemma 2.20, the conclusion follows. □

If we assume that ϕ is unital in Proposition 2.26, the conclusion is an easy consequence of [**21**, Proposition 1.19] and Wittstock's extension theorem, with a much better estimate.

PROPOSITION 2.26. *Let $n \in \mathbb{N}$ and let $\phi : M_n \to B(H)$ be an injective CPC with $\|\phi^{-1}\|_{\mathrm{cb}} < 1 + \delta$. Then there is a UCP map $T : B(H) \to M_n$ such that*

$$\|\mathrm{id}_{M_n} - T\phi\|_{\mathrm{cb}} \leqslant 272\delta^{1/4}.$$

Proof. By Theorem 2.25 there is a $k \in \mathbb{N}$ and a complete order embedding $\psi : M_n \to M_k \otimes B(H)$ such that

$$\|1_k \otimes \phi - \psi\|_{\mathrm{cb}} \leqslant 272\delta^{1/4}.$$

By Theorem 2.12 there is a UCP map $R : M_k \otimes B(H) \to M_n$ such that $R\psi = \mathrm{id}_{M_n}$. Define $T : B(H) \to M_n$ by $T(x) = R \circ (1_k \otimes x)$. Then we have

$$\|id_B - T\phi\|_{\mathrm{cb}} = \|R\psi - R \circ (1_k \otimes \phi)\|_{\mathrm{cb}} \leqslant \|\psi - 1_k \otimes \phi\|_{\mathrm{cb}} \leqslant 272\delta^{1/4}.$$ □

We are now ready to prove our perturbation theorem for the second extreme case.

THEOREM 2.27. *Let $\phi : M_n \to B(H)$ be a CPC with $\|\phi^{-1}\|_{\mathrm{cb}} < 1 + \delta$. Suppose that there is a rank 1 projection $p \in B(H)$ and $x \in M_n$ such that $\|\phi(x) - p\| < \delta$. Then there are the following:*

(i) *a rank 1 projection $r \in M_n$ such that $\|\phi(r) - p\| \leqslant 315\delta^{1/8}$;*

(ii) *a complete order embedding $\psi : M_n \to B(H)$ such that $\|\psi - \phi\|_{\mathrm{cb}} \leqslant 1360\delta^{1/32}$.*

Proof. (i) Let $T : B(H) \to M_n$ be as in Proposition 2.26. Then $T(p) \geqslant 0$ and

$$\begin{aligned}
\|\phi(T(p)) - p\| &\leqslant \|\phi(T(p)) - \phi(x)\| + \delta \\
&\leqslant \|T(p) - x\| + \delta \\
&\leqslant \|T(p) - T(\phi(x))\| + \|T(\phi(x)) - x\| + \delta \\
&\leqslant \|p - \phi(x)\| + 272\delta^{1/4} + \delta \leqslant 273\delta^{1/4}.
\end{aligned}$$

Then $\|T(p)\| \geqslant \|\phi(T(p))\| > 1 - 273\delta^{1/4}$.

$$\text{Let } q = \|T(p)\|^{-1}T(p); \quad \text{then } \|\phi(q) - p\| \leqslant 819\delta^{1/4}. \tag{2.24}$$

Let $\delta' = 819\delta^{1/4}$. Let $r \leqslant q$ be a rank 1 projection. Let $\xi \in H$ be norm 1 such that $p\xi = \xi$. Let $\eta \in H$ of norm 1 with $\langle \phi(r)\eta, \eta\rangle \geqslant (1+\delta)^{-1}$. We will now show that

$$(\exists \theta \in \mathbb{R})(\|\eta - e^{i\theta}\xi\| \leqslant 2\sqrt{\delta'}). \tag{2.25}$$

By (2.24), it follows that

$$\delta' \geqslant \langle (\phi(r) + \phi(q-r) - p)\eta, \eta\rangle \geqslant \langle \phi(r)\eta, \eta\rangle - \langle p\eta, \eta\rangle \geqslant (1+\delta)^{-1} - \langle p\eta, \eta\rangle.$$

Hence,

$$|\langle \xi, \eta\rangle| = \langle p\eta, \eta\rangle^{1/2} \geqslant \left(\frac{1}{1+\delta} - \delta'\right)^{1/2} \geqslant 1 - 2\delta'.$$

Choose θ such that $\langle e^{i\theta}\xi, \eta\rangle = |\langle \xi, \eta\rangle|$. Then

$$\|e^{i\theta}\xi - \eta\|^2 = 2 - 2|\langle \xi, \eta\rangle| \leqslant 4\delta'.$$

This proves (2.25).

Now suppose that $\xi^\perp$ is norm 1 and perpendicular to ξ. From (2.24) we deduce that

$$\langle \phi(r)\xi^\perp, \xi^\perp\rangle \leqslant \delta'. \tag{2.26}$$

Without loss of generality, suppose that $\phi(r)$ has a $\|\phi(r)\|$ eigenvalue with eigenvector ζ. By (2.25) it follows that

$$\|\phi(r)\xi - \xi\| \leqslant \|\phi(r)\zeta - \zeta\| + 4\sqrt{\delta'} \leqslant 5\sqrt{\delta'}. \tag{2.27}$$

Let t be the projection onto ζ. Then $\|t - p\| \leqslant 2\sqrt{\delta'}$ by (2.25). Therefore

$$\begin{aligned}\|(1-p)\phi(r)p\| &\leqslant \|(1-t)\phi(r)t\| + 2\|t-p\| \\ &\leqslant \sqrt{1 - \|\phi(r)\|} + 4\sqrt{\delta'} \leqslant 5\delta'.\end{aligned} \tag{2.28}$$

Combining (2.26), (2.27) and (2.28), it follows that

$$\|\phi(r) - p\| \leqslant 11\sqrt{\delta'} \leqslant 315\delta^{1/8}.$$

(ii) Note that, for any unitary $u \in M_n$, the existence of a perturbation for ϕ is equivalent to the existence of a perturbation for the map $x \mapsto \phi(uxu^*)$. Hence without loss of generality, assume that $r = e_{11}$. By Lemma 2.19, we have

$$\|x\| \geqslant (1+\delta)^{-1} \text{ and } x \leqslant \phi(e_{11}) \quad \text{for every } x \in \Lambda_n^\phi.$$

Let $x \in \Lambda_n^\phi$ and let $\eta \in H$ be norm 1 such that $\langle x\eta, \eta\rangle \geqslant (1+\delta)^{-1}$. Then

$$\langle \phi(e_{11})\eta, \eta\rangle \geqslant \langle x\eta, \eta\rangle \geqslant (1+\delta)^{-1}.$$

Hence, by (2.25) we have

$$\langle \phi(x)\xi, \xi\rangle \geqslant (1+\delta)^{-1} - 3(2\sqrt{\delta'}) \geqslant \frac{1}{1 + 12\sqrt{\delta'}} \quad \text{for every } x \in \Lambda_n^\phi.$$

Finally, by Lemma 2.20 there is a complete order embedding $\psi : M_n \to B(H)$ such that

$$\|\psi - \phi\|_{\text{cb}} \leqslant 136(12\sqrt{\delta'})^{1/2} \leqslant 1360\delta^{1/32}.$$ □

Corollary 2.28. *Let B be a finite-dimensional* C*-*algebra and let $\phi : B \to B(H)$ be a CPC with $\|\phi^{-1}\|_{\text{cb}} < 1 + \delta$. Suppose that there is a rank 1 projection $p \in B(H)$ and a $x \in B$ such that $\|\phi(x) - p\| < \delta$. Then there is a rank 1 projection $r \in B$ such that $\|\phi(r) - p\| \leqslant 315\delta^{1/8}$.*

3. *Strong NF algebras*

Before we can apply our perturbation results, we first need some technical theorems.

3.1. *Technical theorems*

DEFINITION 3.1. Let I be an index set and let $(n_i)_{i\in I}$ be a family of positive integers. Define the C*-algebra

$$\mathcal{B}(I,(n_i)) := \prod_{i\in I} M_{n_i}.$$

For a vector space X and a linear map $\phi : X \to \mathcal{B}(I,(n_i))$, for each $i \in I$, define the linear map $\phi_i : X \to M_{n_i}$ as the ith coordinate map.

Our first goal of this section is to prove the following theorem.

THEOREM 3.2. *Let $\epsilon > 0$ and $n \in \mathbb{N}$. There is a $\delta = \delta(n,\epsilon) > 0$ such that, for every C*-algebra $\mathcal{B} = \mathcal{B}(I,(n_i))$ and every injective linear map $\phi : M_n \to \mathcal{B}$ with $\|\phi^{-1}\|_{\mathrm{cb}} < 1+\delta$, there is an index $i \in I$ such that ϕ_i is injective with $\|\phi_i^{-1}\|_{\mathrm{cb}} < 1+\epsilon$.*

To prove Theorem 3.2, it is necessary to introduce ternary rings of operators (TROs). If ϕ is assumed unital above, then we can compose an (essentially identical) proof using the theory of completely positive maps. Unfortunately it seems that deducing the non-unital case from the unital case is a non-trivial matter, and we will definitely require the non-unital case for our applications. For this reason, we now recall some of the basic facts about TROs required for our proof.

Let A be a C*-algebra and let $V \subseteq A$ be a closed subspace. If V is closed under the triple product

$$(x,y,z) \mapsto xy^*z, \quad \text{where } x,y,z \in V,$$

then V is called a *ternary ring of operators* (henceforth TRO). We refer the reader to [**13**, Section 2] and the references therein for a detailed look at the development of the theory of TROs.

Consider the following subspaces of A:

$$V^* := \{v^* : v \in V\}, \quad C(V) = \overline{\mathrm{span}}\{vw^* : v,w \in V\}, \quad D(V) = \overline{\mathrm{span}}\{v^*w : v,w \in V\}. \quad (3.1)$$

It is straightforward to verify that $C(V)$ and $D(V)$ are C*-subalgebras of A. Furthermore, one checks that

$$A(V) = \begin{bmatrix} C(V) & V \\ V^* & D(V) \end{bmatrix} \subset M_2 \otimes A$$

is a C*-algebra.

A linear map $\phi : V \to W$ between TROs is called a *TRO-homomorphism* if

$$\phi(xy^*z) = \phi(x)\phi(y)^*\phi(z) \quad \text{for all } x,y,z \in V.$$

THEOREM 3.3 [**11**]. *Let $\phi : V \to W$ be a TRO homomorphism. Then there are *-homomorphisms*

$$\phi_C : C(V) \longrightarrow C(W) \quad \text{and} \quad \phi_D : D(V) \longrightarrow D(W)$$

defined by

$$\phi_C\left(\sum v_i w_i^*\right) = \sum \phi(v_i)\phi(w_i)^* \quad \text{and} \quad \phi_D\left(\sum v_i^* w_i\right) = \sum \phi(v_i)^*\phi(w_i).$$

Moreover, the following map:

$$\pi_\phi := \begin{bmatrix} \phi_C & \phi \\ \phi^* & \phi_D \end{bmatrix} : A(V) \longrightarrow A(W)$$

*is a well-defined *-homomorphism.*

The following theorem of Masamichi Hamana is the key element to our proof.

Theorem 3.4 [**11**, Theorem 3.2(ii)]. *Let V and W be TROs and let $\phi : V \to W$ be a complete isometry. Then there is a TRO-homomorphism $T : \mathrm{TRO}(\phi(V)) \to V$ extending ϕ^{-1}, where $\mathrm{TRO}(\phi(V))$ is the TRO generated by $\phi(V)$.*

We now recall some facts about n-homogeneous C*-algebras. Let X be a compact Hausdorff space and let $n \in \mathbb{N}$. Under the usual identification

$$C(X) \otimes M_n \cong \{f : X \longrightarrow M_n : f \text{ is continuous}\},$$

it is easy to see that there is a one-to-one correspondence between the elements $x \in X$ and the irreducible representations π_x of $C(X) \otimes M_n$ given by

$$\pi_x(f) = f(x). \tag{3.2}$$

Now let I be any index set; then

$$\prod_{i\in I} M_n \cong \ell^\infty(I) \otimes M_n \cong C(\beta I) \otimes M_n,$$

where βI denotes the Stone–Cech compactification of I. Recall that βI is identified with the set of all ultrafilters on I (here $I \subset \beta I$ corresponds to the principal ultrafilters on I in the obvious way). Under this identification and by (3.2), it follows that there is a one-to-one correspondence between the ultrafilters ω on I and the irreducible representations π_ω of $\prod_{i\in I} M_n$ given by

$$\pi_\omega((x_i)_{i\in I}) = \lim_{i\to\omega} x_i. \tag{3.3}$$

It is a well-known result of Krein that, for C*-algebras $A \subseteq B$, every pure state ϕ on A has a pure state extension to B. The following lemma is a consequence of this fact.

Lemma 3.5. *Let $A \subseteq B$ be C*-algebras. Let (π, H) be an irreducible representation of A. Then there is an irreducible representation (ρ, K) of B, such that $H \subseteq K$ and letting $p : K \to H$ be the orthogonal projection, we have*

$$p\rho(x)p = \pi(x) \quad \text{for every } x \in A.$$

In particular, if $A \subseteq B \cong C(X) \otimes M_n$ for some Hausdorff space X, and if π is an n-dimensional irreducible representation of A, then the extension ρ is necessarily n-dimensional by (3.2), and hence

$$\rho(x) = \pi(x) \quad \text{for every } x \in A. \tag{3.4}$$

We now combine all of this background material in the following lemma.

LEMMA 3.6. *Let $n \in \mathbb{N}$, let I be an index set and let $\phi : M_n \to \ell^\infty(I) \otimes M_n$ be a complete isometry. For every $\epsilon > 0$ there exists an index $i \in I$ such that ϕ_i is an injective complete contraction with $\|\phi_i^{-1}\|_{\mathrm{cb}} < 1 + \epsilon$.*

Proof. Let $W \subset \ell^\infty(I) \otimes M_n$ be the TRO generated by $\phi(M_n)$. By Theorem 3.4, there is a TRO homomorphism

$$T : W \to M_n \quad \text{such that } T = \phi^{-1} \text{ on } \phi(M_n). \tag{3.5}$$

Let $\pi_T : A(W) \to A(M_n) = M_{2n}$ be the *-homomorphism associated with T, as in Theorem 3.3. Since T is surjective, so is π_T, and hence it is irreducible. We extend π_T by (3.4) (still call it π_T) to an irreducible representation of $\ell^\infty(I) \otimes M_{2n}$. By (3.3), there exists an ultrafilter ω on I such that $\pi_T = \pi_\omega$.

Now let $x \in M_n \otimes M_n$ be arbitrary. Then

$$\begin{aligned} \|x\|_{M_n \otimes M_n} &= \left\| \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right\|_{A(M_n \otimes M_n)} = \left\| \begin{bmatrix} 0 & (\phi^{(n)})^{-1} \circ \phi^{(n)}(x) \\ 0 & 0 \end{bmatrix} \right\|_{A(M_n \otimes M_n)} \\ &= \left\| \begin{bmatrix} 0 & T^{(n)} \circ \phi^{(n)}(x) \\ 0 & 0 \end{bmatrix} \right\|_{A(M_n \otimes M_n)} = \left\| \pi_\omega^{(n)} \left(\begin{bmatrix} 0 & \phi^{(n)}(x) \\ 0 & 0 \end{bmatrix} \right) \right\|_{A(M_n \otimes M_n)} \\ &= \lim_{i \to \omega} \|\phi_i^{(n)}(x)\|_{M_n \otimes M_n}. \end{aligned} \tag{3.6}$$

Let $\delta > 0$ and let F be a finite δ-net for the unit sphere of $M_n \otimes M_n$. From (3.6) it follows that

$$\bigcap_{x \in F} \{ i \in \omega : \|\phi_i^{(n)}(x)\| \geqslant 1 - \delta \} \in \omega.$$

In particular this set is non-empty. Let $i' \in I$ be in the above set. Then, for every x in the unit sphere of $M_n \otimes M_n$, it follows that $\|\phi_{i'}^{(n)}(x)\| \geqslant 1 - 2\delta$. By Theorem 2.9, it follows that $\|\phi_{i'}^{-1}\|_{\mathrm{cb}} \leqslant (1 - 2\delta)^{-1}$. As δ was arbitrary, this proves the lemma. □

COROLLARY 3.7. *Let $\epsilon > 0$ and $n \in \mathbb{N}$. There exists a $\delta = \delta(\epsilon, n) > 0$ such that, for any index set I, if $\phi : M_n \to \ell^\infty(I) \otimes M_n$ is an injective complete contraction such that $\|\phi^{-1}\|_{\mathrm{cb}} < 1 + \delta$, then there is an index $i \in I$ such that the ith coordinate map $\phi_i : M_n \to M_n$ is an injective complete contraction with $\|\phi_i\|_{\mathrm{cb}} < 1 + \epsilon$.*

Proof. Suppose not. Then there is an $\epsilon > 0$, $n \in \mathbb{N}$, a sequence of index sets I_k and injective complete contractions

$$\phi^k : M_n \longrightarrow \ell^\infty(I_k) \otimes M_n$$

such that

$$\|(\phi^k)^{-1}\|_{\mathrm{cb}} < 1 + 1/k, \text{ but } \|(\phi_i^k)^{-1}\|_{\mathrm{cb}} > 1 + \epsilon \text{ for every } k \in \mathbb{N}, \text{ and } i \in I_k. \tag{3.7}$$

It follows that the map

$$\phi = \bigoplus_{k \in \mathbb{N}} \phi^k : M_n \longrightarrow \ell^\infty \left(\bigsqcup_{k=1}^{\infty} I_k \right) \otimes M_n$$

is a complete isometry; but by Lemma 3.6, one of the coordinate maps of ϕ must be injective with $\|\phi_i^{-1}\|_{\mathrm{cb}} < 1 + \epsilon$. This contradicts (3.7). □

Now let $m \leqslant n$ be positive integers. Set

$$I(m, n) = \{ p \in M_n : p \text{ is a rank } m \text{ projection} \}. \tag{3.8}$$

Let $x \in M_m \otimes M_n$ be norm 1.

Let $\xi = (\xi_1, \ldots, \xi_m), \eta = (\eta_1, \ldots, \eta_m) \in \ell^2(m) \otimes \ell^2(n)$ be norm 1 such that $x\xi = \eta$. Let $p_\xi, p_\eta \in M_n$ denote the projections onto $\operatorname{span}\{\xi_1, \ldots, \xi_m\}$ and $\operatorname{span}\{\eta_1, \ldots, \eta_m\}$, respectively. It is then clear that $\|x\| = \|(1_m \otimes p_\eta)x(1_m \otimes p_\xi)\|$. From which we deduce that the map $P_{m,n} : M_n \to \ell^\infty(I(m,n)^2) \otimes M_m$ defined by

$$P_{m,n}(x) = \bigoplus_{(p,q)\in I(m,n)^2} pxq \tag{3.9}$$

is a complete contraction, such that $P_{m,n}^{(m)}$ is an isometry.

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Set $\delta = \delta(\epsilon, n)$ from Corollary 3.7. Let $\phi : M_n \to \mathcal{B}(I, (n_i))$ be an injective complete contraction with $\|\phi^{-1}\|_{\mathrm{cb}} < 1 + \delta$. Without loss of generality, suppose that $n_i \geqslant n$ for each $i \in I$ (if not simply embed $M_{n_i} \hookrightarrow M_{n_i} \oplus M_{n-n_i} \subset M_n$).

Now consider the map $P : \mathcal{B}(I, (n_i)) \to \bigoplus_{i\in I} \bigoplus_{(p,q)\in I(n,n_i)^2} M_n$ defined by (recall (3.9))

$$P((x_i)_{i\in I}) = \bigoplus_{i\in I} P_{n,n_i}(x_i).$$

Then P is a complete contraction and $P^{(n)}$ is an isometry. It follows from Theorem 2.9 that $P \circ \phi$ is an injective complete contraction with $\|(P \circ \phi)^{-1}\|_{\mathrm{cb}} < 1 + \delta$. By Corollary 3.7, there is an index $i \in I$ and rank n projections $p, q \in M_{n_i}$ such that $x \mapsto p\phi_i(x)q$ is an injective complete contraction with an inverse cb-norm bound by $1 + \epsilon$. Hence ϕ_i is injective with $\|\phi_i^{-1}\|_{\mathrm{cb}} < 1 + \epsilon$. □

The following lemma is a variation of [**9**, Lemma 2.2]

Lemma 3.8. *Suppose that A is a unital* C*-*algebra with $\mathcal{OL}_\infty(A) = 1$. Then, for every finite subset $F \subset A$ and every $\delta > 0$, there exists a finite-dimensional* C*-*algebra B and maps $\phi : B \to A$ and $\psi : A \to B$ such that we have the following properties:*

(i) *ϕ and ψ are UCP;*
(ii) *$\operatorname{dist}(x, \psi(B)) < \delta$ for all $x \in F$;*
(iii) *$\|\psi\phi - \mathrm{id}_B\| < \delta$;*
(iv) *$\|\psi(xy) - \psi(x)\psi(y)\| \leqslant \delta\|x\|\|y\|$ for all $x, y \in F$.*

Proof. Obtain B, ψ and the unital self-adjoint map ϕ as in [**9**, Lemma 2.2]; moreover, assume that ψ satisfies (iv) by the proof of [**12**, Theorem 3.2]. Since A is nuclear, there is a matrix algebra M_n and UCP maps $\alpha : A \to M_n$ and $\beta : M_n \to A$ such that $\|\beta\alpha|_{\phi(B)} - \mathrm{id}_{\phi(B)}\|_{\mathrm{cb}} < \delta$. By [**21**, Proposition 1.19] there is a UCP map $T : B \to M_n$ such that $\|T - \alpha\phi\|_{\mathrm{cb}} < \delta$. Then $\widetilde{\phi} = \beta T$ and ψ are our desired maps. □

3.2. *Applications*

Definition 3.9. Let Prim(A) denote the primitive ideal space of the C*-algebra A and let $J \in \operatorname{Prim}(A)$. We say J is a *GCR ideal* if there is an irreducible representation (π, H) of A with $\ker(\pi) = J$ and $\pi(A) \cap K(H) \neq \{0\}$ (hence $K(H) \subset \pi(A)$).

Theorem 3.10. *Let A be a* C*-*algebra with $\mathcal{OL}_\infty(A) = 1$. Suppose that $J_1, \ldots, J_n$ are primitive ideals of A such that $J_1 \cap \ldots \cap J_n = \{0\}$. Then A is a strong NF algebra. In particular, all primitive* C*-*algebras with $\mathcal{OL}_\infty(A) = 1$ are strong NF.*

Proof. Without loss of generality, suppose that $\{J_1,\ldots,J_n\}$ is a minimal element (with respect to set inclusion) of the set of all finite subsets $F\subset \mathrm{Prim}(A)$ with $\ker(F)=\{0\}$. In particular, $J_i\neq J_k$ if $i\neq k$ and

$$\bigcap_{k\neq i} J_k \neq \{0\} \quad \text{for every } i=1,\ldots,n. \tag{3.10}$$

We also order the J_i as $J_1,\ldots,J_r,J_{r+1},\ldots,J_n$ so that $J_1,\ldots,J_r$ are all GCR ideals (Definition 3.9) and $J_{r+1},\ldots,J_n$ are not GCR ideals.

Let $(\pi_1,H_1),\ldots,(\pi_n,H_n)$ be irreducible representations of A with $\ker(\pi_i)=J_i$ and let

$$(\pi,H)=\bigoplus_i(\pi_i,H_i).$$

Since all of the J_i are different, it follows from [**17**, Theorem 3.8.11] that

$$\pi(A)''=\prod_{i=1}^n \pi_i(A)''=\prod_{i=1}^n B(H_i). \tag{3.11}$$

We will show that A is inner quasidiagonal, which combined with [**2**, Theorem 4.5] shows that A is a strong NF algebra. By (3.11) we must show that, for each finite set $F\subset A$ and $\epsilon>0$, there is a finite rank projection $p\in\prod_{i=1}^n B(H_i)$ such that

$$\max_{x\in F}\|[x,p]\|\leqslant\epsilon \quad\text{and}\quad \min_{x\in F}\|p\pi(x)p\|\geqslant 1-\epsilon. \tag{3.12}$$

To keep the notation within reason, we first prove the following lemma. □

Lemma 3.11. *We have that π_i is a quasidiagonal representation for each $i=1,\ldots,r$ (recall these are the GCR representations).*

Proof. Of course, it is enough to prove this for $i=1$. Set $I=J_2\cap\ldots\cap J_n$. By (3.10) we have $I\neq\{0\}$. Since $J_1=\ker(\pi_1)$ and $J_1\cap I=\{0\}$, it follows that

$$\pi_1|_I \ \text{ is a *-monomorphism.} \tag{3.13}$$

Moreover, since $K(H_1)\subseteq\pi_1(A)$ and $\pi_1(I)$ is a non-zero ideal of $\pi_1(A)$, we have $\pi_1(I)\cap K(H_1)\neq 0$ from which it follows that

$$K(H_1)\subseteq \pi_1(I). \tag{3.14}$$

Let $\epsilon>0$ and let $\pi_1(x_1),\ldots,\pi_1(x_s)\in\pi_1(A)$ be norm 1 such that $x_1,\ldots,x_s\in A$ are also norm 1. Let $r\in K(H_1)$ be any finite rank projection and let $p\in K(H_1)$ be a finite rank projection such that

$$\min_{i=1,\ldots,s}\|p\pi_1(x_i)p\|\geqslant 1-\epsilon/2 \quad\text{and}\quad r\leqslant p. \tag{3.15}$$

Set $N=\mathrm{rank}(p)$ and let $(e_{ij})_{i,j=1}^N$ be matrix units for $pB(H_1)p$. By (3.13) and (3.14) we are able to define

$$f_{ij}=(\pi_1|_I)^{-1}(e_{ij})\in I \quad\text{for } 1\leqslant i,j\leqslant N. \tag{3.16}$$

Set $F=\{x_1,\ldots,x_s\}\cup\{f_{i,j}: i,j=1,\ldots,N\}\subset A$. Choose $\delta>0$ satisfying

$$\frac{1}{1+\delta}-316\delta^{1/8}>0 \quad\text{and}\quad N\cdot 90\delta^{1/512}<\frac{\epsilon}{2}. \tag{3.17}$$

Now choose $\eta>0$ such that $\eta<\delta$ and $2\eta<\delta(\delta,N)$ (as defined in Theorem 3.2). Apply Lemma 3.8 to obtain a finite-dimensional C*-algebra B and a UCP map $\phi:B\to A$ such that (without loss of generality)

$$F\subset\phi(B) \quad\text{and}\quad \|\phi^{-1}\|_{\mathrm{cb}}<1+\eta<1+\delta, \tag{3.18}$$

where 'without loss of generality' pertains to taking $F \subset \phi(B)$ instead of $\phi(B)$ close to F.

Our first step is to obtain a minimal central projection $p_\ell \in B$ such that $\pi_1\phi|_{p_\ell B}$ can be perturbed to a complete order embedding. In turn, this will produce a finite rank projection satisfying (3.15).

Use Wittstock's extension Theorem to obtain a completely bounded extension $\psi : B(H_1) \oplus \ldots \oplus B(H_n) \to B$ of ϕ^{-1} with $\|\psi\|_{\mathrm{cb}} \leqslant 1+\eta$. Then ψ is unital and by replacing ψ with $(\psi + \psi^*)/2$ if necessary, we may assume that ψ is self-adjoint. We use [**21**, Proposition 1.19] applied to ψ to obtain a UCP map as follows:

$$T : B(H_1) \oplus \ldots \oplus B(H_n) \longrightarrow B \quad \text{with } \|T\pi\phi - \mathrm{id}_B\|_{\mathrm{cb}} \leqslant \eta. \tag{3.19}$$

By the definition of I, we have

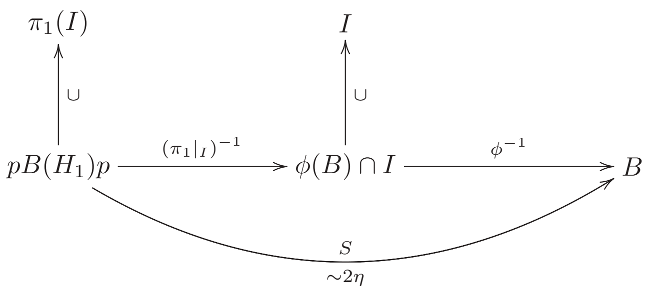
$$pB(H_1)p \subset \pi_1(\phi(B) \cap I) \oplus 0 \oplus \ldots \oplus 0 = \pi(\phi(B) \cap I). \tag{3.20}$$

Let S be the restriction of T to $pB(H_1)p$. It follows from (3.19) and (3.20) that

$$S \text{ is an injective CPC}, \quad \text{with } \|S^{-1}\|_{\mathrm{cb}} \leqslant (1-\eta)^{-1} < 1+2\eta. \tag{3.21}$$

We combine (3.16)–(3.21) in the following 'approximately' commutative diagram:

(3.22)



Since B is finite-dimensional, there are integers $n_1, \ldots, n_k$ such that

$$B \cong \bigoplus_{i=1}^{k} M_{n_i} \qquad \text{with minimal central projections} \qquad p_1, \ldots, p_k \in B.$$

Recall that $2\eta < \delta(\delta, N)$, and so by (3.21) and Theorem 3.2, there is an index $1 \leqslant \ell \leqslant k$ such that S_{p_ℓ} is injective with

$$\|S_{p_\ell}^{-1}\|_{\mathrm{cb}} \leqslant 1+\delta. \tag{3.23}$$

Recall $f_{11}, \ldots, f_{NN}$ as defined in (3.16). By Corollary 2.28, there exist rank 1 projections $r_1, \ldots, r_N \in B$ such that $\|\phi(r_i) - f_{ii}\| \leqslant 315\delta^{1/8}$, where $i = 1, \ldots, N$. We have

$$\begin{aligned} \|p_\ell r_i - p_\ell S\pi_1(f_{ii})\| &\leqslant \|r_i - S\pi_1(f_{ii})\| \\ &\leqslant \|T\pi(\phi(r_i)) - T\pi(f_{ii})\| + \delta \quad (\text{by } (3.19)) \\ &\leqslant \|\phi(r_i) - f_{ii}\| + \delta \leqslant 315\delta^{1/8} + \delta. \end{aligned} \tag{3.24}$$

By (3.24), (3.23), and (3.17), for $i = 1, \ldots, N$, we have

$$\|p_\ell r_i\| \geqslant \|S_{p_\ell}\pi_1(f_{ii})\| - 316\delta^{1/8} \geqslant \frac{1}{1+\delta} - 316\delta^{1/8} > 0.$$

Since each r_i is a rank 1 projection, it follows that

$$r_i \in M_{n_\ell} \qquad \text{for } i = 1, \ldots, N. \tag{3.25}$$

At this point, the map S is no longer useful to us; we only needed it to prove (3.25). We will now show that $\pi_1\phi|_{p_\ell B}$ can be perturbed to a complete order embedding. To this end, first note that

$$\|(\pi_2 \oplus \ldots \oplus \pi_n)(\phi(r_1))\| \leqslant \|(\pi_2 \oplus \ldots \oplus \pi_n)(f_{11})\| + 315\delta^{1/8} = 315\delta^{1/8}.$$

Since $\|(\pi\phi)^{-1}\|_{\text{cb}} < 1 + \delta$, by [**9**, Corollary 2.7] it follows that $\pi_1\phi|_{p_\ell B}$ is injective with $\|(\pi_1\phi|_{p_\ell B})^{-1}\|_{\text{cb}} \leqslant 1 + 3\delta^{1/3}$. Since $e_{11} = \pi_1(f_{11})$ is a rank 1 projection and $\|\pi_1\phi(r_1) - e_{11}\| \leqslant \|\phi(r_1) - f_{11}\| \leqslant 315\delta^{1/8}$, we apply Theorem 2.27 to obtain a complete order embedding $\psi : p_\ell B \cong M_{n_\ell} \to B(H_1)$ such that

$$\|\psi - \pi_1\phi|_{p_\ell B}\|_{\text{cb}} \leqslant 1360(315\delta^{1/8})^{1/32} \leqslant 2000\delta^{1/256}.$$

Moreover, by Theorem 2.11 we may assume that there is a rank n_ℓ projection $q \in B(H_1)$ such that $\psi = \psi_q + \psi_{(1-q)}$ with ψ_q a non-zero *-homomorphism.

To finish this portion of the proof, we will show that, for every $x \in F$, we have

$$\|[\pi_1(x), q]\| \leqslant \epsilon \quad \text{and} \quad \|q\pi_1(x)q\| \geqslant 1 - \epsilon.$$

Since F is contained in $\phi(B)$, the first inequality follows from (2.21). For the second inequality, let $1 \leqslant i \leqslant N$; then

$$\begin{aligned}\|e_{ii}qe_{ii}\| = \|qe_{ii}q\| &\geqslant \|q\pi\phi(r_i)q\| - 315\delta^{1/8}\\ &\geqslant \|q\psi(r_i)q\| - 2001\delta^{1/256} = 1 - 2001\delta^{1/256}.\end{aligned}$$

Since e_{ii} is a rank 1 projection, it follows that

$$\|e_{ii} - e_{ii}q\|^2 = \|e_{ii} - e_{ii}qe_{ii}\| \leqslant 2001\delta^{1/256}. \tag{3.26}$$

Recall from (3.15) that $p = \sum_{i=1}^N e_{ii}$. By applying (3.26), we have

$$\|p - pq\| \leqslant N \cdot 45\delta^{1/512}.$$

Finally, by (3.15), for every $x \in F$ we have

$$\|q\pi_1(x)q\| \geqslant \|pq\pi_1(x)qp\| \geqslant \|p\pi_1(x)p\| - N \cdot 90\delta^{1/512} \geqslant 1 - \epsilon/2 - N \cdot 90\delta^{1/512} \geqslant 1 - \epsilon.$$

From (3.15) recall that p dominates the arbitrary finite rank projection r. By [**5**, Proposition 3.6] it follows that π_1 is a quasidiagonal representation of A. This completes the proof of Lemma 3.11.

We now return to the proof of the Theorem. All variables defined in the proof of Lemma 3.11 are now free.

Define $\pi_{\text{GCR}} = \pi_1 \oplus \ldots \oplus \pi_r$. Let F be a finite subset of the unit sphere of A and let $\epsilon > 0$. If all of the J_i are GCR ideals, then A is inner quasidiagonal by Lemma 3.11. If not, then assume that there is an $x \in F$ such that $\|\pi_{\text{GCR}}(x)\| < \|x\|$. Let

$$G := \{y \in F : \|y\| > \|\pi_{\text{GCR}}(y)\|\} \quad \text{and} \quad \gamma := \min\{\|y\| - \|\pi_{\text{GCR}}(y)\| : y \in G\} > 0.$$

Choose $\delta > 0$ such that

$$\delta < \left(\frac{\epsilon}{|F|^{3/2}213}\right)^{12n} \quad \text{and} \quad \frac{1}{1 + 3\delta^{2/3n}} - 3\delta > 1 - \gamma. \tag{3.27}$$

Use Lemma 3.8 to obtain a finite-dimensional C*-algebra B and a UCP map $\phi : B \to A$ such that $\|\phi^{-1}\|_{\text{cb}} < 1 + \delta$ and (without loss of generality) $F \cup \{yy^* : y \in F\} \subset \phi(B)$.

Since B is finite-dimensional, there are integers $n_1, \ldots, n_k$ such that

$$B \cong \bigoplus_{i=1}^k M_{n_i} \quad \text{with minimal central projections } p_1, \ldots, p_k \in B.$$

For each $y \in G$ choose an index $1 \leqslant i_y \leqslant k$ such that $\|p_{i_y}\phi^{-1}(y)\| = \|\phi^{-1}(y)\|$ and let

$$M = \bigcup_{y \in G} \{i_y\} \subset \{1, \ldots, k\}.$$

For $i \in M$ and $1 \leqslant j \leqslant n$ define the maps $\phi_{i,j} : p_iB \to B(H_j)$ as

$$\phi_{i,j}(x) = \pi_j\phi(x) \quad \text{for all} \quad x \in p_iB.$$

By repeated use of [**9**, Corollary 2.7], for each $i \in M$ there is an index $1 \leqslant j(i) \leqslant n$ such that $\phi_{i,j(i)}$ is injective with

$$\|\phi^{-1}_{i,j(i)}\|_{\mathrm{cb}} \leqslant 1 + 3\delta^{2/3n}. \tag{3.28}$$

We now show that $j(i) > r$ for $i \in M$. By [**21**, Proposition 1.19] there is a UCP map $T : A \to B$ such that $\|T|_{\phi(B)} - \phi^{-1}\|_{\mathrm{cb}} \leqslant \delta$. Let $i \in M$ and $y \in G$ such that $i = i_y$. By (3.28) we have

$$\begin{aligned}
\frac{1}{1+3\delta^{2/3n}} &\leqslant \frac{\|\phi^{-1}(y)\|^2}{1+3\delta^{2/3n}} = \frac{\|p_i\phi^{-1}(y)\phi^{-1}(y)^*\|}{1+3\delta^{2/3n}} \\
&\leqslant \|\phi_{i,j(i)}(p_i\phi^{-1}(y)\phi^{-1}(y)^*)\| \\
&= \|\pi_{j(i)}\phi(p_i\phi^{-1}(y)\phi^{-1}(y)^*)\| \\
&\leqslant \|\pi_{j(i)}\phi(\phi^{-1}(y)\phi^{-1}(y)^*)\| \\
&\leqslant \|\pi_{j(i)}\phi(T(y)T(y)^*)\| + 2\delta \\
&\leqslant \|\pi_{j(i)}\phi(T(yy^*))\| + 2\delta \\
&\leqslant \|\pi_{j(i)}\phi(\phi^{-1}(yy^*))\| + 3\delta \\
&\leqslant \|\pi_{j(i)}(yy^*)\| + 3\delta.
\end{aligned}$$

Hence $\|\pi_{j(i)}(yy^*)\| > 1 - \gamma \geqslant \|\pi_{\mathrm{GCR}}(yy^*)\|$ by (3.27). Thus $j(i) > r$. Therefore $\pi_{j(i)}$ is not a GCR representation and so $\pi_{j(i)}(A) \cap K(H_{j(i)}) = \{0\}$ for each $i \in M$. We now apply Theorem 2.22 to obtain complete order embeddings $\psi^i : M_{n_i} \to B(H_{j(i)})$ with

$$\|\phi_{i,j(i)} - \psi^i\|_{\mathrm{cb}} \leqslant 273\delta^{1/6n} \quad \text{for each } i \in M.$$

By Theorem 2.11 we may assume that, for each $i \in M$, there is a rank n_i projection $q_i \in B(H_{j(i)})$ such that

$$\psi^i = \psi^i_{q_i} + \psi^i_{(1-q_i)} \quad \text{with } \psi^i_{q_i} \text{ a non-zero *-homomorphism.} \tag{3.29}$$

Let $y \in G$ and let $i_y = i \in M$. Then

$$\begin{aligned}
\|q_i\pi_{j(i)}(y)q_i\| &= \|q_i\pi_{j(i)}(\phi(p_i\phi^{-1}(y))q_i + q_i\pi(\phi((1-p_i)\phi^{-1}(y))q_i\| \\
&\geqslant \|q_i\phi_{i,j(i)}(p_i\phi^{-1}(y))q_i\| - 8(273)\delta^{1/6n} \quad \text{(by (2.20))} \\
&\geqslant \|q_i\psi^i(p_i\phi^{-1}(y))q_i\| - 9(273)\delta^{1/6n} \\
&= \|p_i\phi^{-1}(y)\| - 9(273)\delta^{1/6n} \\
&\geqslant 1 - 9(273)\delta^{1/6n} \geqslant 1 - 50\delta^{1/12n}.
\end{aligned} \tag{3.30}$$

Furthermore, for any $x \in F$ by (2.21) we have that

$$\|[\pi(x), q_i]\| \leqslant 8(273\delta^{1/6n})^{1/2} \leqslant 133\delta^{1/12n}. \tag{3.31}$$

Now we show that the sum of the q_i is almost a projection as follows:

$$\begin{aligned}
\left\|\sum_{i\in M} q_i\right\| &= \left\|\sum_{i\in M} q_i\psi^i(p_i)q_i\right\| \leqslant \left\|\sum_{i\in M}\psi^i(p_i)\right\| \leqslant \left\|\sum_{i\in M}\phi_{i,j(i)}(p_i)\right\| + |M|273\delta^{1/6n} \\
&\leqslant \left\|\sum_{i\in M}\phi(p_i)\right\| + |M|273\delta^{1/6n} \leqslant 1 + |M|273\delta^{1/6n}.
\end{aligned}$$

Then, for each $i \in M$, we have

$$\begin{aligned}\left\| q_i \left(\sum_{j \neq i} q_j \right) \right\| &\leqslant \left\| q_i \left(\sum_{j \neq i} q_j \right)^{1/2} \right\| (1 + |M| 273 \delta^{1/6n})^{1/2} \\ &= \left\| q_i \left(\sum_{j \neq i} q_j \right) q_i \right\|^{1/2} (1 + |M| 273 \delta^{1/6n})^{1/2} \\ &\leqslant (|M| 273 \delta^{1/6n})^{1/2} (1 + |M| 273 \delta^{1/6n})^{1/2} \leqslant 40 |M|^{1/2} \delta^{1/12n},\end{aligned}$$

from which it follows that

$$\left\| \sum_{i \in M} q_i - \left(\sum_{i \in M} q_i \right)^2 \right\| \leqslant 40 |M|^{3/2} \delta^{1/12n}.$$

Hence, basic spectral theory produces a projection $q \in B(H_{r+1}) \oplus \ldots \oplus B(H_n)$ such that

$$\left\| q - \sum_{i \in M} q_i \right\| \leqslant 80 |M|^{3/2} \delta^{1/12n}. \tag{3.32}$$

Combining (3.30) and (3.32), it follows that

$$\|q \pi_{j(i)}(y) q\| \geqslant 1 - |M|^{3/2} 130 \delta^{1/12n} \geqslant 1 - \epsilon \tag{3.33}$$

and then by (3.31) and (3.32), it follows that

$$\|[\pi(x), q]\| \leqslant 213 |M|^{3/2} \delta^{1/12n} \leqslant \epsilon \quad \text{for all} \quad x \in F. \tag{3.34}$$

We use Lemma 3.11 to obtain a finite rank projection $p \in B(H_1) \oplus \ldots \oplus B(H_r)$ such that

$$\max_{x \in F} \|[p, \pi(x)]\| \leqslant \epsilon \quad \text{and} \quad \min_{x \in F \setminus G} \|p \pi(x) p\| \geqslant 1 - \epsilon. \tag{3.35}$$

Finally, combining, (3.33)–(3.35), it follows that

$$\max_{x \in F} \|[p + q, \pi(x)]\| \leqslant \epsilon \quad \text{and} \quad \min_{x \in F} \|(p + q) \pi(x) (p + q)\| \geqslant 1 - \epsilon.$$

Since p and q are orthogonal, it follows that A is inner quasidiagonal. □

3.3. *Behaviour of $\mathcal{OL}_\infty$*

Theorem 3.10 also provides somewhat surprising results about the behaviour of the invariant $\mathcal{OL}_\infty$ that we now describe. From [**9**, Proposition 4.4] recall that, for any nuclear C*-algebras A and B, we have

$$\mathcal{OL}_\infty(A \otimes B) \leqslant \mathcal{OL}_\infty(A) \mathcal{OL}_\infty(B). \tag{3.36}$$

This inequality can be strict in general. Consider the Cuntz algebra $\mathcal{O}_2$. By [**12**, Theorem 3.4], it follows that $\mathcal{OL}_\infty(\mathcal{O}_2) > \sqrt{(1 + \sqrt{5})/2} > 1$. By a result of George Elliott (see [**19**, Chapter 5]) we have $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, showing that (3.36) can be strict.

It is at least reasonable to think that equality in (3.36) will hold if one of the algebras is strong NF. We now show that (3.36) can be strict even when one of the algebras is AF.

As pointed out in [**2**], there is a nuclear primitive quasidiagonal C*-algebra $\mathcal{B}$ (defined in [**4**]) that is not inner quasidiagonal, and hence not a strong NF algebra. By Theorem 3.10, it follows that $\mathcal{OL}_\infty(\mathcal{B}) > 1$. Let U be any UHF algebra. Then U is a strong NF algebra and

$\mathcal{B} \otimes U$ is primitive, quasidiagonal and antiliminal. By [**2**, Corollary 2.6] it follows that $\mathcal{B} \otimes U$ is a strong NF algebra. Therefore we have

$$\mathcal{OL}_\infty(\mathcal{B} \otimes U) = 1 < \mathcal{OL}_\infty(\mathcal{B}).$$

The fact that U is antiliminal played a key role in this example. Therefore, it would be interesting to know what happens for the Type I building blocks. In particular, we have the following question.

QUESTION 3.12. Let B be the compact operators or a commutative C*-algebra and let A be a nuclear C*-algebra. Do we have

$$\mathcal{OL}_\infty(A \otimes B) = \mathcal{OL}_\infty(B)?$$

We now show, using the same example, that $\mathcal{OL}_\infty$ is not continuous with respect to inductive limits. We need the following result.

THEOREM 3.13. *Let A be a unital C*-algebra with $\mathcal{OL}_\infty(A) = 1$ and let $p \in A$ be a projection. Then $\mathcal{OL}_\infty(pAp) = 1$.*

Proof. Let $p \in F \subset pAp$ be a finite subset and let $\delta > 0$. Then obtain a finite-dimensional C*-algebra B and the maps $\phi : B \to A$ and $\psi : A \to B$ that satisfy Lemma 3.8 for F and $\delta > 0$. Since ψ is δ-multiplicative on F, it follows that there is a projection $q \in B$ such that $\|\psi(p) - q\| \leqslant 2\delta$. Then, for every $x \in (qBq)^+$ of norm 1, we have $\phi(x) \leqslant \phi(q)$. From this it follows that, for every $x \in qBq$, we have

$$\|p\phi(x)p - \phi(x)\| \leqslant 12\delta.$$

Finally, we apply [**18**, Lemma 2.13.2] to obtain a map $\alpha : qBq \to pAp$ with

$$\|\alpha\|_{\mathrm{cb}}\|\alpha^{-1}\|_{\mathrm{cb}} \leqslant \frac{1+12\delta}{1-12\delta} \quad \text{and} \quad F \subset \alpha(qBq). \qquad \square$$

COROLLARY 3.14. *Let A be a unital C*-algebra with let $\mathcal{OL}_\infty(A) > 1$. Then there exists a constant $r > 1$ such that*

$$\inf_{n \in \mathbb{N}} \mathcal{OL}_\infty(M_n \otimes A) \geqslant r. \tag{3.37}$$

Proof. If not, then we could apply the proof of Theorem 3.13 to the sequence of algebras $e_{11} \otimes A \subseteq M_n \otimes A$ and deduce that $\mathcal{OL}_\infty(A) = 1$. $\square$

It is obvious from the definition of $\mathcal{OL}_\infty$ that if A is the inductive limit of the nuclear C*-algebras A_n, then

$$\limsup_{n\to\infty} \mathcal{OL}_\infty(A_n) \geqslant \mathcal{OL}_\infty(A).$$

We show that this inequality can be strict. Consider the algebras $\mathcal{B}$ and U as above. Since U is the inductive limit of the matrix algebras $(M_{n_k})_{k=1}^\infty$, it follows that $U \otimes \mathcal{B}$ is the inductive limit of the algebras $M_{n_k} \otimes \mathcal{B}$. By Corollary 3.14, we have

$$\limsup_{k\to\infty} \mathcal{OL}_\infty(M_{n_k} \otimes \mathcal{B}) > 1 = \mathcal{OL}_\infty(U \otimes \mathcal{B}).$$

4. *Concluding remarks*

Recall from Section 1 the following question.

QUESTION 4.1. Let A be a C*-algebra with $\mathcal{OL}_\infty(A) = 1$. Is A a strong NF algebra?

This question is still open, but we note the following necessary conditions.

THEOREM 4.2. *Suppose that A is a* C*-*algebra with* $\mathcal{OL}_\infty(A) = 1$, *but A is not a strong NF algebra. Then we have the following properties:*
(i) *A does not have a separating family of irreducible quasidiagonal representations;*
(ii) *A is quasidiagonal;*
(iii) *A has a separating family of irreducible stably finite representations;*
(iv) *for every finite subset $F \subseteq \mathrm{Prim}(A)$, we have $\cap_{J\in F} J \neq \{0\}$.*

Proof. The conclusions follow from [**3**, Corollary 1.3; **12**, Theorem 3.2; **9**, Theorem 5.4] and Theorem 3.10, respectively. □

We do not have to look very far to find a nuclear C*-algebra that satisfies (i)–(iv) in Theorem 4.2. Recall the C*-algebra $\mathcal{B}$ from above. Then $C[0,1] \otimes \mathcal{B}$ satisfies (i)–(iv) in Theorem 4.2. Therefore, it would certainly be interesting to know $\mathcal{OL}_\infty(C[0,1] \otimes \mathcal{B})$. Unfortunately, we have been unable to decide this question.

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