

CLASSIFICATION OF C*-ALGEBRAS GENERATED BY REPRESENTATIONS OF THE UNITRIANGULAR GROUP $UT(4, \mathbb{Z})$

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ABSTRACT. It was recently shown that each C*-algebra generated by a faithful irreducible representation of a finitely generated, torsion free nilpotent group is classified by its ordered K-theory. For the three step nilpotent group $UT(4, \mathbb{Z})$ we calculate the ordered K-theory of each C*-algebra generated by a faithful irreducible representation of $UT(4, \mathbb{Z})$ and see that they are all simple AT algebras. We also point out that there are many simple *non* AT algebras generated by irreducible representations of nilpotent groups.

1. INTRODUCTION

The last few years witnessed several breakthroughs in the theory of simple, nuclear C*-algebras. The hands of Rørdam, H. Lin, Z. Niu, Winter, Matui and Sato joined to show that if a C*-algebra satisfies several abstract properties (see Theorem 2.1) then it necessarily has the concrete property of being an approximately subhomogenous algebra and is moreover classified¹ by its Elliott invariant.

In other words, if a C*-algebra A satisfies Theorem 2.1 then it is an inductive limit of subhomogeneous C*-algebras—but knowledge of the Elliott invariant of A provides (in theory) an explicit decomposition of A as a limit of subhomogenous C*-algebras and therefore a wealth of information about its structure. This provides an entire new avenue of study for those important classes of C*-algebras that do not have an obvious inductive limit structure, e.g. C*-algebras produced by dynamical systems or group representations. We travel this new avenue of study by calculating the Elliott invariant of C*-algebras generated by faithful irreducible representations of the three step nilpotent group $UT(4, \mathbb{Z})$.

The odd-numbered authors showed that if Γ is a finitely generated torsion free nilpotent group and π is a faithful irreducible representation of Γ , then $C^*(\pi(\Gamma))$ satisfies Theorem 2.1. Therefore our present results combined with Theorem 2.3 determine the structure of C*-algebras generated by faithful irreducible representations of $UT(4, \mathbb{Z})$.

Our current program was carried out (without the aid of Theorem 2.1) for many two-step nilpotent groups. Perhaps the best known example is Elliott and Evans's work on the irrational rotation algebras in [10]. They showed that the C*-algebras generated by faithful irreducible representations of the discrete Heisenberg group \mathbb{H}_3 are AT algebras. The Elliott invariant had long been known for these algebras by the work of Rieffel and Pimsner and Voiculescu [22, 23, 25].

¹See Definition 2.2 for our working definition of “classified”

It follows from the main theorem in Phillips’s preprint [21] that the C^* -algebra generated by a faithful irreducible representation of a finitely generated two-step nilpotent group is an AT algebra. This is essentially the reason that we focus on the three step nilpotent group $UT(4, \mathbb{Z})$ as it is the least complicated, most natural group covered by Theorem 2.3, but not by Phillips’s theorem.

We briefly explain the technical aspects of our calculations. The C^* -algebras generated by faithful irreducible representations of $UT(4, \mathbb{Z})$ are parameterized by the irrational numbers in $(0, 1)$. For $\alpha \in (0, 1)$ and irrational let us denote by B_α the C^* -algebra generated by this representation. The Pimsner-Voiculescu six term exact sequence and a straightforward application of a theorem of Packer and Raeburn [19] show that $K_i(B_\alpha) \cong \mathbb{Z}^{10}$ for $i = 0, 1$. The bulk of our work is then devoted to divining the order structure on $K_0(B_\alpha)$. We do this by locating a well-behaved (with respect to order structure) finite index subgroup $G \leq UT(4, \mathbb{Z})$ and applying Pimsner’s [24] to show that the order on $K_0(B_\alpha)$ is determined by the representation restricted to G . In particular we show the order on $K_0(B_\alpha)$ is given by the hyperplane with normal vector $(1, \alpha, \alpha^2, 0, \dots, 0)$.

With the K -theory of each B_α in hand, in Section 6, we address when $B_\alpha \cong B_\beta$. It is fairly easy to see that if α is transcendental or algebraic with minimal polynomial of degree greater than or equal to five, we have $B_\alpha \cong B_\beta$ if and only if $\alpha = \pm\beta \pmod{\mathbb{Z}}$. On the other hand, if the degree of the minimal polynomial for α is less than or equal to four the situation is much more interesting (see Theorem 6.2 and following examples) and contrasts with the case of the irrational rotation algebras. An extremely crude summation of the works [22, 23, 25] is “two irrational rotation algebras are isomorphic if and only if they are obviously isomorphic.” Theorem 6.2 shows cases with $B_\alpha \cong B_\beta$ that are not obviously isomorphic, i.e. the classification theorem (Theorem 2.1) is essential.

As mentioned above, Phillips showed that every C^* -algebra generated by an irreducible representation of a finitely generated two-step nilpotent group is an AT algebra. All of the algebras considered here also turn out to be AT algebras. For the sake of completeness we finish the paper with Section 7 by pointing out that there are many C^* -algebras generated by faithful irreducible representations of three-step nilpotent groups that are not AT algebras.

2. PRELIMINARIES

In this section we define our objects of study and recall the necessary C^* -algebraic background. For information on the properties of AT algebras we refer the reader to Rørdam’s monograph [27]. The following major theorem is crucial to our investigations.

Theorem 2.1 (Rørdam, H. Lin, Z. Niu, Winter, Matui and Sato [13–16, 28, 33]). *Let A and B be unital, separable, simple, nuclear, quasidiagonal C^* -algebras with unique tracial states and finite nuclear dimension that satisfy the universal coefficient theorem. Then A is an approximately subhomogeneous C^* -algebra. Moreover, if $(K_0(A), K_0(A)^+, [1_A], K_1(A)) \cong (K_0(B), K_0(B)^+, [1_B], K_1(B))$, then $A \cong B$.*

Definition 2.2. In this paper, when we say a C^* -algebra is **classifiable** we mean that it satisfies the hypotheses of Theorem 2.1. In reality, the term “classifiable” refers to a much larger class of C^* -algebras. We sacrificed generality for clarity in our definition. See the recent preprint [11] for a much more general definition of classifiable.

Theorem 2.3 (See [5,6]). *Let Γ be a torsion free finitely generated nilpotent group and π a faithful irreducible representation of Γ . Then $C^*(\pi(\Gamma))$ is classifiable (Definition 2.2).*

Definition 2.4. For a group Γ , let $Z(\Gamma) \leq \Gamma$ denote the center of Γ . For $a, b \in \Gamma$, set $[a, b] = aba^{-1}b^{-1}$. The **unitriangular subgroup** of $GL(4, \mathbb{Z})$ is defined as

$$(2.1) \quad UT(4, \mathbb{Z}) = \left\{ a = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ & 1 & a_{23} & a_{24} \\ & & 1 & a_{34} \\ & & & 1 \end{pmatrix} : a_{ij} \in \mathbb{Z} \right\}.$$

The following subgroup of $UT(4, \mathbb{Z})$ plays a key role in our calculations,

$$\mathbb{H}_4 = \{a \in UT(4, \mathbb{Z}) : a_{23} = 0\}.$$

For each $1 < i < j \leq 4$, let $e_{ij} \in M_4(\mathbb{Z})$ be the (i, j) -matrix unit. One easily verifies the following commutation relations:

$$(2.2) \quad [1 + e_{ij}, 1 + e_{kl}] = 1 + \delta_{jk}e_{il} - \delta_{il}e_{kj}.$$

Notice that

$$(2.3) \quad Z(UT(4, \mathbb{Z})) = \left\{ \begin{pmatrix} 1 & 0 & 0 & a \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} : a \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

2.1. Representation-theoretic description.

Definition 2.5. Let $\theta \in \mathbb{R}$. Define the trace on $UT(4, \mathbb{Z})$ as follows:

$$\tau_\theta(x) = \begin{cases} e^{2\pi i x \theta} & \text{if } x \in Z(UT(4, \mathbb{Z})) \\ 0 & \text{if } x \notin Z(UT(4, \mathbb{Z})) \end{cases}.$$

Let π_θ denote the GNS representation of $UT(4, \mathbb{Z})$ associated with τ_θ .

Let π be an irreducible, faithful, unitary representation of $UT(4, \mathbb{Z})$. It is well known (see [3, 12, 17] or the introduction of [5]) that there is an irrational $\theta \in \mathbb{R}$ such that $C^*(\pi(UT(4, \mathbb{Z}))) \cong C^*(\pi_\theta(UT(4, \mathbb{Z})))$.

Definition 2.6. For each $\theta \in \mathbb{R}$ we define

$$B_\theta = C^*(\pi_\theta(UT(4, \mathbb{Z}))).$$

2.2. Crossed product construction. In order to describe the order structure on $K_0(B_\theta)$ we describe B_θ as a crossed product. Recall the definition of \mathbb{H}_4 in Definition 2.4.

Lemma 2.7. *Let θ be irrational and A_θ denote the irrational rotation algebra associated with θ . Then $C^*(\pi_\theta(\mathbb{H}_4)) \cong A_\theta \otimes A_\theta$ (π_θ is the representation from Definition 2.5).*

Proof. By the relations in (2.2), one sees that $C^*(\pi_\theta(1+e_{12}), \pi_\theta(1+e_{24}))$ and $C^*(\pi_\theta(1+e_{13}), \pi_\theta(1+e_{34}))$ are two commuting copies of A_θ . The conclusion then follows by the simplicity of A_θ , the nuclearity of A_θ and Takesaki's theorem on the simplicity of simple minimal tensor products (see [32, Corollary IV.4.21]). \square

We now describe the conjugation action of $1 + e_{23}$ on $A_\theta \otimes A_\theta$. Define the automorphism $\beta : \mathbb{H}_4 \rightarrow \mathbb{H}_4$ by

$$(2.4) \quad \beta(x) = (1 + e_{23})x(1 + e_{23})^{-1} = [1 + e_{23}, x]x.$$

This combined with the relations in (2.2) produces

$$\begin{aligned} \beta(1 + e_{12}) &= 1 + e_{12} - e_{13}, \\ \beta(1 + e_{13}) &= 1 + e_{13}, \\ \beta(1 + e_{34}) &= 1 + e_{24} + e_{34}, \\ \beta(1 + e_{24}) &= 1 + e_{24}. \end{aligned}$$

Summarizing the above discussion we obtain

Theorem 2.8. *Let u, v be standard generators of A_θ . Then $B_\theta \cong (A_\theta \otimes A_\theta) \rtimes_\beta \mathbb{Z}$ where*

$$\begin{aligned} \beta(u \otimes 1) &= u \otimes u^{-1}, \\ \beta(v \otimes 1) &= v \otimes 1, \\ \beta(1 \otimes u) &= 1 \otimes u, \\ \beta(1 \otimes v) &= v \otimes v. \end{aligned}$$

2.3. Twisted group C*-algebra. In order to calculate the K-groups of B_θ we describe B_θ as a twisted group C*-algebra. Everything in this section will be well-known to experts in twisted group C*-algebras. We recall the necessary definitions (see for example Section 1 of [18]) for the non-experts. Let Γ be a discrete group and $\sigma : \Gamma \times \Gamma \rightarrow \mathbb{T}$ a 2-cocycle (also referred to as a *multiplier*). The involutive Banach algebra $\ell^1(\Gamma, \sigma)$ is formed with the following multiplication and involution

$$(2.5) \quad f * g(s) := \sum_{t \in \Gamma} f(t)g(t^{-1}s)\sigma(t, t^{-1}s), \quad f^*(s) = \overline{\sigma(s^{-1}, s)f(s^{-1})}.$$

The left regular representation of $\ell^1(\Gamma, \sigma)$ is defined on $B(\ell^2(\Gamma))$ as

$$\lambda(f)(g)(t) = \sum_{s \in \Gamma} \sigma(s, s^{-1}t)f(s)g(s^{-1}t).$$

The *reduced twisted group C*-algebra* is defined as $C_r^*(\Gamma, \sigma) = C^*(\lambda(\ell^1(\Gamma, \sigma)))$.

Let Z be the center of $UT(4, \mathbb{Z})$ and $C = \{x \in UT(4, \mathbb{Z}) : x_{14} = 0\} \subseteq UT(4, \mathbb{Z})$.

Notice that C is a complete choice of coset representatives for $UT(4, \mathbb{Z})/Z$. Let $c : UT(4, \mathbb{Z})/Z \rightarrow C$ be the unique lifting of the quotient map. Following [20] for each $\theta \in \mathbb{R}$, we define the 2-cocycle $\omega_\theta : UT(4, \mathbb{Z})/Z \times UT(4, \mathbb{Z})/Z \rightarrow \mathbb{T}$ by

$$\omega_\theta(xZ, yZ) = \tau_\theta(c(xZ)c(yZ)c(xyZ)^{-1}).$$

Proposition 2.9. *Let ω_θ be the cocycle from above. Then B_θ is isomorphic to the reduced twisted group C^* -algebra $C_r^*(UT(4, \mathbb{Z})/Z, \omega_\theta)$.*

Proof. Let $L^2(UT(4, \mathbb{Z}), \tau_\theta)$ denote the Hilbert space associated to the GNS representation of τ_θ .

It was shown in [5, Lemma 2.4], that $\{\delta_x : x \in C\}$ is an orthonormal basis for $L^2(UT(4, \mathbb{Z}), \tau_\theta)$ and $W : L^2(UT(4, \mathbb{Z}), \tau_\theta) \rightarrow \ell^2(UT(4, \mathbb{Z})/Z)$, given by $W\delta_x = \delta_{xZ}$, is unitary. Moreover [5, Lemma 2.4] shows that B_θ is generated by $\{\pi_\theta(x) : x \in C\}$. Very easy calculations and [5, Lemma 2.4] show that for each $x \in C$ we have

$$W\pi_\theta(x) = \lambda(xZ)W.$$

It then follows that $B_\theta \cong C_r^*(UT(4, \mathbb{Z})/Z, \omega_\theta)$. \square

3. COMPUTATION OF K_*

We now use the fact that each B_θ is a twisted group C^* -algebra to calculate their unordered K -groups. Throughout this section we set

$$\Gamma = UT(4, \mathbb{Z})/Z(UT(4, \mathbb{Z})).$$

Lemma 3.1. *We have*

$$K_0(C^*(\Gamma)) = K_1(C^*(\Gamma)) = \mathbb{Z}^{10}.$$

Proof. Note that $\Gamma \cong \mathbb{Z}^4 \rtimes_\alpha \mathbb{Z}$ where $\langle e_{12}, e_{13}, e_{24}, e_{34} \rangle / Z(UT(4, \mathbb{Z})) \cong \mathbb{Z}^4$ and the automorphism α is implemented by conjugation of $e_{23} \bmod Z(UT(4, \mathbb{Z}))$.

In other words, let x_1, x_2, x_3, x_4 be a free basis of \mathbb{Z}^4 and define

$$(3.1) \quad \alpha(x_1) = x_1 - x_2, \quad \alpha(x_2) = x_2, \quad \alpha(x_3) = x_3, \quad \alpha(x_4) = x_3 + x_4,$$

then $\Gamma \cong \mathbb{Z}^4 \rtimes_\alpha \mathbb{Z}$.

It is well known (see for example [8, 2.1] or [2]) that the graded ring $K_*(C(\mathbb{T}^4)) = K_0(C(\mathbb{T}^4)) \oplus K_1(C(\mathbb{T}^4))$ can be identified with the exterior algebra of 4 generators e_1, e_2, e_3, e_4 over \mathbb{Z} . Furthermore, K_0 is identified with those terms of even degree and the standard coordinate functions in $C(\mathbb{T}^4)$ correspond to e_1, e_2, e_3, e_4 . The induced action of α on $K_*(C(\mathbb{T}^4))$ is a ring automorphism. This fact combined with (3.1) shows that $\alpha_*(x) = x$ for

$$x \in \{1, e_2, e_3, e_1 \wedge e_2, e_2 \wedge e_3, e_3 \wedge e_4, e_1 \wedge e_2 \wedge e_3, e_2 \wedge e_3 \wedge e_4, e_1 \wedge e_2 \wedge e_3 \wedge e_4\}.$$

Furthermore,

$$\begin{aligned} \alpha_*(e_1) &= e_1 - e_2, & \alpha_*(e_1 \wedge e_3 \wedge e_4) &= e_1 \wedge e_3 \wedge e_4 - e_2 \wedge e_3 \wedge e_4, \\ \alpha_*(e_4) &= e_3 + e_4, & \alpha_*(e_1 \wedge e_2 \wedge e_4) &= e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_2 \wedge e_4, \end{aligned}$$

which determines the homomorphism $\alpha_* : K_1(C(\mathbb{T}^4)) \rightarrow K_1(C(\mathbb{T}^4))$. It follows that

$$(3.2) \quad K_1(C(\mathbb{T}^4))/(\text{id} - \alpha_*)(K_1(C(\mathbb{T}^4))) \cong \mathbb{Z}^4.$$

Similarly, the following calculations determine the K_0 counterpart:

$$\begin{aligned}\alpha_*(e_1 \wedge e_3) &= e_1 \wedge e_3 - e_2 \wedge e_3, & \alpha_*(e_2 \wedge e_4) &= e_2 \wedge e_3 + e_2 \wedge e_4, \\ \alpha_*(e_1 \wedge e_4) &= e_1 \wedge e_3 + e_1 \wedge e_4 - e_2 \wedge e_3 - e_2 \wedge e_4.\end{aligned}$$

It follows that

$$(3.3) \quad K_0(C(\mathbb{T}^4))/(\text{id} - \alpha_*)(K_0(C(\mathbb{T}^4))) \cong \mathbb{Z}^2.$$

As was implied above, we have $K_0(C(\mathbb{T}^4)) \cong K_1(C(\mathbb{T}^4)) \cong \mathbb{Z}^8$. This fact combines with (3.2), (3.3) and the Pimsner-Voiculescu six term exact sequence for crossed products [23, Theorem 2.4] to prove the claim. \square

Corollary 3.2. *Let $\theta \in \mathbb{R}$. Then*

$$K_i(B_\theta) \cong \mathbb{Z}^{10} \quad \text{for } i = 0, 1.$$

Proof. By Proposition 2.9, we have $B_\theta \cong C^*(\Gamma, \omega_\theta)$. When $\theta = 0$, B_θ is isomorphic to the group C*-algebra $C^*(\Gamma)$. For any two θ_1, θ_2 , the cocycles ω_{θ_1} and ω_{θ_2} are homotopic. Indeed, define $r(t) := t\theta_1 + (1-t)\theta_2$, and a homotopy of 2-cocycles $\tilde{\omega} : \Gamma \times \Gamma \times [0, 1] \rightarrow \mathbb{T}$ by $\tilde{\omega}(\cdot, \cdot, t) := \omega_{r(t)}(\cdot, \cdot)$.

Since Γ is discrete in the simply connected (contractible actually) nilpotent Lie group $UT(4, \mathbb{R})/Z(UT(4, \mathbb{R}))$, it follows from Theorem 4.2 of [19] that $K_*(B_\theta) \cong K_*(C^*(\Gamma))$. \square

4. RANGE OF THE TRACE

Let A be a unital C*-algebra and τ a tracial state on A . We also denote by τ the state on $K_0(A)$. In general it is difficult to calculate the range $\tau(K_0(A)) \subset \mathbb{R}$.

Pimsner showed in [24] that in the case of crossed products by free groups an examination of the de la Harpe-Skandalis determinant [4] can sometimes lead to a satisfying description of the range of the trace of the crossed product. Pimsner's ideas work particularly well for the cases at hand.

We very briefly recall the necessary background from [4] and [24] (see also [2]) and refer the reader to [24] for more information and proofs of the claims made below.

Let $U(A)$ denote the unitary group of a C*-algebra. Let $U_\infty(A)$ be the inductive limit of $U(M_n(A))$ in the usual way. Let $U_\infty(A)_0$ denote the connected component of the identity in $U_\infty(A)$.

For a piecewise differentiable path $\xi : [0, 1] \rightarrow U_\infty$ one defines

$$\Delta_\tau(\xi) = \frac{1}{2\pi i} \int_0^1 \tau(\xi'(t)\xi(t)^{-1}) dt.$$

The map Δ_τ is constant on homotopy classes with fixed endpoints. Let ξ be a differentiable path of unitaries and $n \geq 1$ an integer. From the following easily derivable formula,

$$\frac{d}{dt} \xi^n(t) = n\xi'(t)\xi^{n-1}(t) - \sum_{i=1}^{n-1} [\xi'(t), \xi^{n-i}(t)]\xi(t)^{i-1},$$

one deduces that

$$(4.1) \quad \Delta_\tau(\xi^n) = n\Delta_\tau(\xi).$$

Lemma 4.1. *Let $u, v \in U_n(A)$ be unitaries. Then for any piecewise differentiable path ξ from 1 to $uvu^{-1}v^{-1}$ we have $\Delta_\tau(\xi) = 0$.*

Proof. Let $w(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ \sin(\pi t/2) & -\cos(\pi t/2) \end{pmatrix}$ and define

$$\xi(t) = \begin{pmatrix} u & \\ & 1 \end{pmatrix} w(t) \begin{pmatrix} v & \\ & 1 \end{pmatrix} w(t) \begin{pmatrix} u^{-1} & \\ & 1 \end{pmatrix} w(t) \begin{pmatrix} v^{-1} & \\ & 1 \end{pmatrix} w(t).$$

It is then a straightforward calculation to see that $\Delta_\tau(\xi) = 0$. \square

Under the Bott isomorphism $K_0(A) \cong K_1(SA)$, one sees that Δ_τ restricted to $K_0(A)$ is just τ . Let $q : \mathbb{R} \rightarrow \mathbb{R}/\tau(K_0(A))$ be the quotient map. Then for any unitary $u \in U_\infty(A)_0$ and any piecewise differentiable path ξ from 1 to u the map

$$\underline{\Delta}_\tau(u) = q \circ \Delta_\tau(\xi)$$

is a well-defined group homomorphism.

Suppose now that α is an automorphism of A and that $\tau = \tau \circ \alpha$. Define the group homomorphism $\underline{\Delta}_\tau^\alpha : \ker(K_1(\text{id}) - K_1(\alpha)) \rightarrow \mathbb{R}/\tau(K_0(A))$ by

$$\underline{\Delta}_\tau^\alpha([u]_1) = \underline{\Delta}_\tau(u\alpha^{-1}(u^{-1})).$$

Finally, we have

Theorem 4.2 (Pimsner [24]). *The following sequence is exact:*

$$0 \longrightarrow \tau(K_0(A)) \longrightarrow \tau(K_0(A \rtimes_\alpha \mathbb{Z})) \longrightarrow \underline{\Delta}_\tau^\alpha(\ker(K_1(\text{id}) - K_1(\alpha))) \longrightarrow 0,$$

where the first map is the inclusion and the second is the quotient $q : \mathbb{R} \rightarrow \mathbb{R}/\tau(K_0(A))$.

It is well known that A_θ arises as a crossed product $C(\mathbb{T}) \rtimes \mathbb{Z}$, hence we may apply the Künneth formula [30], which provides

$$K_0(A_\theta \otimes A_\theta) \cong \left(K_0(A_\theta) \otimes K_0(A_\theta) \right) \oplus \left(K_1(A_\theta) \otimes K_1(A_\theta) \right).$$

Let τ be the unique trace on $A_\theta \otimes A_\theta$. We also denote by τ the extension of this trace to B_θ . Pimsner and Voiculescu showed in [23] that the range of the trace on $K_0(A_\theta)$ is $\mathbb{Z} + \theta\mathbb{Z}$. From this it follows that

$$\mathbb{Z} + \theta\mathbb{Z} + \theta^2\mathbb{Z} \subseteq \tau(K_0(A_\theta \otimes A_\theta)).$$

Moreover from the description of the map in the Künneth formula one checks that

$$\ker(\tau) \supseteq K_1(A_\theta) \otimes K_1(A_\theta),$$

hence

$$(4.2) \quad \tau(K_0(A_\theta \otimes A_\theta)) = \mathbb{Z} + \theta\mathbb{Z} + \theta^2\mathbb{Z}.$$

4.1. A finite index subgroup. Eventually we will apply Pimsner's ideas of the previous section to compute the range of the trace for $B_\theta \cong A_\theta \otimes A_\theta \rtimes_\beta \mathbb{Z}$. As one might expect from Pimsner's Theorem 4.2, a good description of $\ker(K_1(\text{id}) - K_1(\beta))$ is helpful. In our case a complication arises because the automorphism β of $A_\theta \otimes A_\theta$ mixes up the tensor factors making it difficult to describe $\ker(K_1(\text{id}) - K_1(\beta))$. In this section we look at a subalgebra $A \subseteq B_\theta$ that is isomorphic to a crossed product of $A_{2\theta} \otimes A_{2\theta}$ by an automorphism that does factor as a tensor product of automorphisms of $A_{2\theta}$ (providing an easy path to the range of the trace calculation for A). The algebra A is "big enough" to then yield the range of trace calculation for B_θ .

Recall the Heisenberg group \mathbb{H}_4 defined in (2.1), and the automorphism β from (2.4). Then $\mathbb{H}_4/Z(\mathbb{H}_4) \cong \mathbb{Z}^4$. Since β fixes the center, it drops to an automorphism of \mathbb{Z}^4 (which we still denote by β). The matrix for $\beta \curvearrowright \mathbb{Z}^4$ with respect to the basis $\{1 + e_{12}, 1 + e_{13}, 1 + e_{24}, 1 + e_{34}\} \bmod Z(\mathbb{H}_4)$ is

$$\beta = \begin{pmatrix} 1 & 0 & & \\ -1 & 1 & & \\ & & 1 & 1 \\ & & 0 & 1 \end{pmatrix}.$$

Then the following two subgroups of \mathbb{Z}^4 are invariant under β and β^{-1} :

$$(4.3) \quad \tilde{X} = \left\langle \left(\begin{array}{c} 0 \\ -1 \\ 1 \\ 0 \end{array} \right), \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right) \right\rangle, \quad \tilde{Y} = \left\langle \left(\begin{array}{c} 0 \\ -1 \\ -1 \\ 0 \end{array} \right), \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ -1 \end{array} \right) \right\rangle.$$

Let e_1, e_2, e_3, e_4 be the standard basis for \mathbb{Z}^4 and $\pi : \mathbb{Z}^4 \rightarrow \mathbb{Z}^4/\langle \tilde{X}, \tilde{Y} \rangle$ the quotient map. Then $3\pi(e_1) = \pi(e_1) = \pi(e_4) \neq \pi(e_2) = \pi(e_3) = 3\pi(e_3)$, so

$$(4.4) \quad \mathbb{Z}^4/\langle \tilde{X}, \tilde{Y} \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Now set X (resp. Y) equal to the inverse image of \tilde{X} (resp. \tilde{Y}) under the quotient map $\mathbb{H}_4 \rightarrow \mathbb{H}_4/Z(\mathbb{H}_4)$.

Definition 4.3. Set $u_1 = \pi_\theta(1 - e_{13} + e_{24})$, $v_1 = \pi_\theta(1 + e_{12} + e_{34})$, $u_2 = \pi_\theta(1 - e_{13} - e_{24})$, $v_2 = \pi_\theta(1 + e_{12} - e_{34})$. Then $C^*(u_1, v_1) = C^*(\pi_\theta(X)) \cong A_{2\theta} \cong C^*(\pi_\theta(Y)) = C^*(u_2, v_2)$ and by similar reasoning as in Lemma 2.7 we have $C^*(\{u_i, v_i : i = 1, 2\}) \cong C^*(u_1, v_1) \otimes C^*(u_2, v_2) \cong A_{2\theta} \otimes A_{2\theta}$.

Furthermore we have we have $\beta|_{A_{2\theta} \otimes A_{2\theta}} = \beta_1 \otimes \beta_2$ where

$$(4.5) \quad \beta_j(u_j) = u_j, \quad \text{and} \quad \beta_j(v_j) = \exp((-1)^j 2\pi i \theta) u_j v_j, \quad \text{for } j = 1, 2.$$

The above information combines with [22, Corollary 2.5] to provide

Lemma 4.4. *For $j = 1, 2$ we have $K_0(\beta_j) = \text{id}$ and $K_1(\beta_j)([u_j]_1) = [u_j]_1$ and $K_1(\beta_j)([v_j]_1) = [u_j v_j]_1$.*

Let $G \leq \mathbb{H}_4$ be the subgroup generated by X and Y . Then $|\mathbb{H}_4/G| = 4$ by (4.4). Let $e = x_0, x_1, x_2, x_3 \in \mathbb{H}_4$ be \mathbb{H}_4/G coset representatives. As in Definition 4.3, we have $A_{2\theta} \otimes A_{2\theta} \cong C^*(\pi_\theta(G))$.

Consider the unitary representation $\pi_\theta|_G$ of G . The unitary representation π_θ of \mathbb{H}_4 is just the induced representation $\text{Ind}_G^{\mathbb{H}_4}(\pi_\theta|_G)$. It then follows from the general theory of induced representations (see for example, [1, Appendix E]) and C*-algebras that there is an embedding $\sigma : A_\theta \otimes A_\theta \rightarrow M_{|\mathbb{H}_4/G|}(A_{2\theta} \otimes A_{2\theta}) = M_4(A_{2\theta} \otimes A_{2\theta})$ such that

$$\sigma(\pi_\theta(t)) = \begin{pmatrix} \pi_\theta(t) & & & \\ & \pi_\theta(x_1^{-1}tx_1) & & \\ & & \pi_\theta(x_2^{-1}tx_2) & \\ & & & \pi_\theta(x_3^{-1}tx_3) \end{pmatrix}$$

for all $t \in G$. Notice that each generator $u_1, u_2, v_1, v_2 \in C^*(\pi_\theta(G))$ is mapped to a scalar multiple of itself under the automorphisms $\text{Ad}(\pi_\theta(x_i))$ for $i = 1, 2, 3$. In particular each of the automorphisms $\text{Ad}(\pi_\theta(x_i))$ is homotopic to the identity. We have shown the following

Proposition 4.5. *Let $\iota : A_{2\theta} \otimes A_{2\theta} \rightarrow A_\theta \otimes A_\theta$ be the inclusion map given by $C^*(\pi_\theta(G)) \subseteq C^*(\pi_\theta(\mathbb{H}_4))$. Let σ be as above. Then $K_*(\sigma \circ \iota) = 4 \cdot \text{id}_{K_*(A_{2\theta} \otimes A_{2\theta})}$. By the functoriality of K_* , and the fact that all the K-groups of $A_\theta \otimes A_\theta$ and $A_{2\theta} \otimes A_{2\theta}$ are torsion free, we have $K_*(\iota)$ and $K_*(\sigma)$ are both injective.*

4.2. Range of trace. Let A and B be C*-algebras. In [30], Schochet describes a $\mathbb{Z}/2\mathbb{Z}$ graded pairing

$$(4.6) \quad \alpha : K_p(A) \otimes K_q(B) \rightarrow K_{p+q}(A \otimes B), \quad p, q \in \mathbb{Z}/2\mathbb{Z}$$

that is an isomorphism when A is in the ‘‘bootstrap class’’ and B has torsion free K-theory [30, Theorem 2.14]. Suppose now that A and B satisfy the conditions of the previous sentence.

Let $\beta = \beta_1 \otimes \beta_2$ be an automorphism of $A \otimes B$. It is clear that $K_*(\beta_1) \otimes K_*(\beta_2)$ induces a graded automorphism of $K_*(A) \otimes K_*(B)$. Moreover, by the description of α , a straightforward verification reveals that the following diagram commutes:

$$(4.7) \quad \begin{array}{ccc} K_*(A) \otimes K_*(B) & \xrightarrow{\alpha} & K_*(A \otimes B) \\ K_*(\beta_1) \otimes K_*(\beta_2) \downarrow & & \downarrow K_*(\beta_1 \otimes \beta_2) \\ K_*(A) \otimes K_*(B) & \xrightarrow{\alpha} & K_*(A \otimes B) \end{array} .$$

Since A_θ is isomorphic to a crossed product of the form $C(\mathbb{T}) \rtimes \mathbb{Z}$, it is in the bootstrap class by [30, Proposition 2.7]. Since A_θ has torsion free K-theory, we have an isomorphism in (4.6) when $A = B = A_\theta$.

Recall from [22] that $K_0(A_{2\theta}) \cong K_1(A_{2\theta}) \cong \mathbb{Z}^2$. Moreover for unitary generators w_1 and w_2 , we have $\{[w_1]_1, [w_2]_1\}$ a free basis for $K_1(A_{2\theta})$. By [30] we have

$$K_1(A_{2\theta} \otimes A_{2\theta}) \cong [K_0(A_{2\theta}) \otimes K_1(A_{2\theta})] \oplus [K_1(A_{2\theta}) \otimes K_0(A_{2\theta})] \cong \mathbb{Z}^4 \oplus \mathbb{Z}^4 \cong \mathbb{Z}^8.$$

Under this identification and the ordered free bases $\{[u_j]_1, [v_j]_1\}$ of $K_1(A_{2\theta})$, by Lemma 4.4, we have

$$K_1(\beta_1 \otimes \beta_2) = \left[\text{id}_{\mathbb{Z}^2} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right] \oplus \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \text{id}_{\mathbb{Z}^2} \right].$$

We therefore have the following

Lemma 4.6. *The subgroup $\ker(K_1(\text{id}) - K_1(\beta_1 \otimes \beta_2)) \leq K_1(A_{2\theta \otimes 2\theta})$ is generated by the set*

$$(4.8) \quad \{[p \otimes u_2 + (1-p) \otimes 1]_1, [u_1 \otimes p + 1 \otimes (1-p)]_1 : p \in A_{2\theta} \text{ is a projection}\}.$$

Theorem 4.7. *We have*

$$\tau(K_0(B_\theta)) = \tau(K_0((A_\theta \otimes A_\theta) \rtimes_\beta \mathbb{Z})) = \mathbb{Z} + \theta\mathbb{Z} + \theta^2\mathbb{Z}.$$

Proof. By (4.2) and Theorem 4.2 it suffices to show that for all $x \in \ker(K_1(\text{id}) - K_1(\beta))$ we have $\underline{\Delta}_\tau^\beta(x) = 0$. By Proposition 4.5, for any $x \in \ker(K_1(\text{id}_{A_\theta \otimes A_\theta}) - K_1(\beta))$ we have $4x \in \ker(K_1(\text{id}_{A_{2\theta} \otimes A_{2\theta}}) - K_1(\beta))$. Therefore by (4.1) we need to show that for any $[w]_1 \in \ker(K_1(\text{id}_{A_{2\theta} \otimes A_{2\theta}}) - K_1(\beta))$ and differentiable path ξ from the identity to $w\beta^{-1}(w)$ in $(U_\infty)_0$ we have $\Delta_\tau(\xi) = 0$.

By Lemma 4.6 we only need to show this for those w of the form in (4.8). We will only show it for w of the form $[p \otimes u_2 + (1-p) \otimes 1]_1$ (one proves it for $[u_1 \otimes p + 1 \otimes (1-p)]_1$ in exactly the same manner).

By a result of Rieffel [26, Corollary 2.5] the projections p and $\beta_1^{-1}(p)$ are unitarily equivalent. Since $\beta_2(u_2) = u_2$ it follows that $p \otimes u_2 + (1-p) \otimes 1$ is unitarily equivalent to $\beta(p \otimes u_2 + (1-p) \otimes 1)$. The conclusion now follows from Lemma 4.1. \square

5. ELLIOTT INVARIANTS

We now gather the information of the proceeding sections to describe the Elliott invariants of the algebras B_θ .

Theorem 5.1. *Let θ be irrational. Then B_θ is a simple AT algebra with unique trace and $K_i(B_\theta) \cong \mathbb{Z}^{10}$ for $i = 0, 1$. Let $\Theta = (1, \theta, \theta^2, 0, \dots, 0) \in \mathbb{R}^{10}$. Then*

$$(5.1) \quad K_0(B_\theta)^+ = \{0\} \cup \{x \in \mathbb{Z}^{10} : \langle x, \Theta \rangle > 0\}.$$

Proof. The K -groups were calculated in Corollary 3.2. Since B_θ is the C^* -algebra generated by an irreducible representation of a finitely generated nilpotent group, it is simple with a unique trace (this is well known, see e.g. the introduction of [5]).

It follows from [6, Theorems 2.9 & 4.4] that B_θ has strict comparison. Since B_θ has unique trace, this means the order structure on $K_0(B_\theta) = \mathbb{Z}^{10}$ is completely determined by the range of the trace, which is $\mathbb{Z} + \theta\mathbb{Z} + \theta^2\mathbb{Z}$ by Theorem 4.7. This shows (5.1).

This description of $(K_0(B_\theta), K_0(B_\theta)^+)$ shows it is a Riesz group, and therefore a dimension group by the Effros-Handelman-Shen theorem [7]. Therefore B_θ has the same Elliott invariant of an AT algebra by [9].

Since A_θ is an AT algebra by [10], it follows that $A_\theta \otimes A_\theta$ is an AT² algebra (it is actually an AT algebra see [27, Proposition 3.25]) and therefore satisfies the universal

coefficient theorem by [29]. Also by [29], it follows that $(A_\theta \otimes A_\theta) \rtimes_{\beta} \mathbb{Z}$ satisfies the universal coefficient theorem. By [5], B_θ is quasidiagonal and by [6] B_θ has finite nuclear dimension. By Theorem 2.1, B_θ is therefore isomorphic to an AT algebra. \square

6. AN ISOMORPHISM CRITERION

As mentioned in the introduction we intend to show there are irrational numbers θ, η such that $B_\theta \cong B_\eta$ but B_θ and B_η are not “obviously” isomorphic (i.e. $\theta \neq -\eta \pmod{\mathbb{Z}}$). This shows that one must use the classification theorem (Theorem 2.1) to classify the algebras $\{B_\theta : \theta \in (0, 1) \setminus \mathbb{Q}\}$ amongst themselves.

By the results of the preceding sections it follows that B_θ and B_η are isomorphic if and only if the ordered groups $(\mathbb{Z}^3, (1, 0, 0), P_\theta)$ and $(\mathbb{Z}^3, (1, 0, 0), P_\eta)$ with distinguished order unit are isomorphic, where $P_\theta = \{x \in \mathbb{Z}^3 \mid x_1 + x_2\theta + x_3\theta^2 > 0\} \cup \{0\}$.

An automorphism of \mathbb{Z}^3 to \mathbb{Z}^3 is implemented by some $A^t \in \text{GL}(3, \mathbb{Z})$ (t denotes transpose), and it is easy to see that A^t sends P_θ to P_η if and only if

$$(6.1) \quad A \begin{pmatrix} 1 \\ \theta \\ \theta^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \eta \\ \eta^2 \end{pmatrix}.$$

We will find below that for most pairs (θ, η) , (6.1) holds if and only if $\eta = \pm\theta \pmod{\mathbb{Z}}$. This reflects the situation with the irrational rotation algebras A_θ . However, for certain θ , there are more possibilities.

Fix an irrational θ . Our goal is to more easily describe the relation between θ and η given by (6.1). Equation (6.1) defines an equivalence relation $\theta \sim \eta$ which extends the equivalence relation $\theta = \pm\eta \pmod{\mathbb{Z}}$, since if $\eta = \pm\theta + k$ with $k \in \mathbb{Z}$, then

$$\begin{pmatrix} 1 & 0 & 0 \\ k & \pm 1 & 0 \\ k^2 & \pm 2k & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \theta \\ \theta^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \eta \\ \eta^2 \end{pmatrix}.$$

We will describe the relation \sim in several cases based on $\deg\theta$ (that is, the degree of the minimal polynomial of θ over \mathbb{Q}).

6.1. $\deg\theta > 2$. Throughout this subsection fix a θ with degree strictly bigger than two. Suppose that $\theta \sim \eta$. The degree restriction implies that the first row of A must be $(1 \ 0 \ 0)$. Also, since we are only counting the number of η 's which are distinct mod \mathbb{Z} and modulo the automorphism $\eta \mapsto -\eta$, we may assume that $a_{21} = 0$ and $\det A = 1$. Then we may write A in the form

$$(6.2) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ c & d & e \end{pmatrix}.$$

By (6.1),

$$(a\theta + b\theta^2)^2 = \eta^2 = c + d\theta + e\theta^2,$$

hence

$$(6.3) \quad b^2\theta^4 + 2ab\theta^3 + (a^2 - e)\theta^2 - d\theta - c = 0.$$

We now split into three subcases.

Case (a): θ has degree > 4 . Then the coefficients of the polynomial in (6.3) must be zero. In particular, $b = 0$ and $e = a^2$. Then A is lower triangular, and since $\det A = 1$, it follows that $a = e = 1$. But then $\eta = \theta$.

Case (b): θ has degree 4. Let $p(x) = x^4 + \lambda_3x^3 + \lambda_2x^2 + \lambda_1x + \lambda_0$, with $\lambda_i \in \mathbb{Q}$, be the minimal polynomial for θ . Then it follows that

$$\begin{aligned} 2ab &= b^2\lambda_3, \\ a^2 - e &= b^2\lambda_2, \\ -d &= b^2\lambda_1, \\ -c &= b^2\lambda_0. \end{aligned}$$

Moreover, $ae - bd = \det(A) = 1$. Note that if $b = 0$, then just as in case (a), we have $a = 1$ and hence $\eta = \theta$. Otherwise, we have

$$\begin{aligned} a &= \frac{1}{2}b\lambda_3, \\ e &= \frac{1}{4}b^2\lambda_3^2 - b^2\lambda_2, \\ d &= -b^2\lambda_1, \\ c &= -b^2\lambda_0, \end{aligned}$$

and hence $\mu b^3 = 1$, where

$$\mu = \frac{1}{8}\lambda_3^3 - \frac{1}{2}\lambda_3\lambda_2 + \lambda_1.$$

Case (c): θ has degree 3. Let $p(x) = x^3 + \lambda_2x^2 + \lambda_1x + \lambda_0$ be the minimal polynomial for θ . Then, we have

$$(2ab - b^2\lambda_2)\theta^3 + (a^2 - e - b^2\lambda_1)\theta^2 - (d + b^2\lambda_0)\theta - c = 0$$

in which case,

$$\begin{aligned} a^2 - e - b^2\lambda_1 &= (2ab - b^2\lambda_2)\lambda_2, \\ -(d + b^2\lambda_0) &= (2ab - b^2\lambda_2)\lambda_1, \\ -c &= (2ab - b^2\lambda_2)\lambda_0. \end{aligned}$$

The above equations allow us to express c , d and e in terms of a and b . Then, again using $\det A = ae - bd = 1$, we have

$$(6.4) \quad a^3 - 2\lambda_2a^2b + (\lambda_2^2 + \lambda_1)ab^2 + (\lambda_0 - \lambda_1\lambda_2)b^3 = \det A = 1.$$

The above equation is of the form $F(a, b) = 0$, where $F(x, y) = x^3 + px^2y + qxy^2 + ry^3 - 1$ is a cubic polynomial with rational coefficients. Its projectivization is $F(X, Y, Z) = X^3 + pX^2Y + qXY^2 + rY^3 - Z^3$.

Lemma 6.1. *The projective curve defined by $F(X, Y, Z) = 0$ is nonsingular, that is, there are no points $[X : Y : Z] \in \mathbb{P}^2$ for which $F, \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y},$ and $\frac{\partial F}{\partial Z}$ all vanish simultaneously.*

Proof. Since $\frac{\partial F}{\partial Z} = 3Z^2$, any nonsingular point must lie in \mathbb{R}^2 . So suppose $(x, y) \in \mathbb{R}^2$ satisfy $\frac{\partial F}{\partial x}(x, y) = \frac{\partial F}{\partial y}(x, y) = 0$. Writing $t = x + \frac{1}{3}py$, we have

$$\frac{\partial F}{\partial x} = 3t^2 + \left(q - \frac{1}{3}p^2\right)y^2 = 0$$

which implies $t^2 = \left(\frac{1}{9}p^2 - \frac{1}{3}q\right)y^2$. Let $k = \frac{1}{9}p^2 - \frac{1}{3}q$. Rewriting $\frac{\partial F}{\partial y}$ in terms of t , and using that $t^2 = ky^2$, we get

$$\frac{\partial F}{\partial y} = \left(2pk - 6k^{3/2} + 3r - \frac{1}{3}pq\right)y^2 = 0.$$

If $y = 0$, then $t = 0$ and hence $x = 0$. But then $F(x, y) = -1$. Hence we may assume instead that

$$\alpha = 2pk - 6k^{3/2} + 3r - \frac{1}{3}pq = 0.$$

Now, rewriting F in terms of t , and again using that $t^2 = ky^2$, we get

$$F(x, y) = \left(-2k^{3/2} - \frac{1}{27}p^3 + pk + r\right)y^3 - 1.$$

Writing β for the coefficient of y^3 above, we see that

$$3\beta = -6k^{3/2} - \frac{1}{9}p^3 + 3pk + 3r = \alpha + pk - \frac{1}{9}p^3 + \frac{1}{3}pq = \alpha.$$

Since $\alpha = 0$, it follows that $F(x, y) = -1$, and hence (x, y) does not lie on the curve. \square

Siegel's theorem ([31]) implies that the curve $F(x, y) = 0$ can only have finitely many integral points. It follows that there are only finitely many choices for A , and hence for η .

6.2. $\deg \theta = 2$. Here we take a different strategy for determining how many η may satisfy $\theta \sim \eta$, by studying the groups $G_\eta = \mathbb{Z} + \eta\mathbb{Z} + \eta^2\mathbb{Z}$. Note that if $\theta \sim \eta$, then $G_\theta = G_\eta$. So suppose this is true for some η . Since θ has degree 2, we may write

$$\theta = \frac{a + b\sqrt{k}}{c}$$

where $a, b, c \in \mathbb{Z}$ have no common factors, $k \in \mathbb{Q}$, and k can be written r/s where $r, s \in \mathbb{Z}$ have no common factors and are each square free. Since

$$\mathbb{Q} + \sqrt{k}\mathbb{Q} = \mathbb{Q} + \theta\mathbb{Q} + \theta^2\mathbb{Q} = \mathbb{Q} + \eta\mathbb{Q} + \eta^2\mathbb{Q},$$

it follows that $\deg \eta = 2$ as well, and moreover

$$\eta = \frac{x + y\sqrt{k}}{z}$$

for some $x, y, z \in \mathbb{Z}$ with no common factors. Now, we have

$$G_\theta = \mathbb{Z} + \left(\frac{a + b\sqrt{k}}{c} \right) \mathbb{Z} + \left(\frac{a^2 + b^2k + 2ab\sqrt{k}}{c^2} \right) \mathbb{Z} \subseteq \frac{1}{c^2s} (\mathbb{Z} + \sqrt{k}\mathbb{Z}).$$

Since $\eta \in G_\theta$, it follows that

$$(6.5) \quad z \mid c^2s.$$

Now consider the group $\mathbb{Q} + G_\theta$. We clearly have

$$\mathbb{Q} + G_\theta = \mathbb{Q} + \frac{b}{c}\sqrt{k} \left(\mathbb{Z} + \frac{2a}{c}\mathbb{Z} \right) = \mathbb{Q} + \frac{b}{cd}\sqrt{k}\mathbb{Z}$$

where d is the denominator of $\frac{2a}{c}$ when written in lowest form. Since $\mathbb{Q} + G_\theta = \mathbb{Q} + G_\eta$, we must have

$$(6.6) \quad \frac{b}{cd} = \frac{y}{zw}$$

where w is the denominator of $2x/z$ when written in lowest form.

Now by (6.5), there can be only finitely many choices for z . Moreover, w must divide z , so there are only finitely many choices for w . By equation (6.6), once z and w are fixed, there can only be finitely many choices for y . Finally, we only have finitely many choices (mod \mathbb{Z}) for x , and hence we can have only finitely many choices for η (mod \mathbb{Z}).

Summarizing the above, we have

Theorem 6.2. *Let θ be irrational.*

- If θ has degree > 4 , then for all η , $B_\theta \cong B_\eta$ if and only if $\theta = \pm\eta$ (mod \mathbb{Z}).
- Suppose θ has degree 4, with minimal polynomial $p(x) = x^4 + \lambda_3x^3 + \lambda_2x^2 + \lambda_1x + \lambda_0$. If, for some nonzero $k \in \mathbb{Z}$,

$$\frac{1}{8}\lambda_3^3 - \frac{1}{2}\lambda_3\lambda_2 + \lambda_1 = \frac{1}{k^3},$$

then $B_\theta \cong B_\eta$ if and only if $\eta = \pm\theta$ (mod \mathbb{Z}) or $\eta = \pm\zeta$ (mod \mathbb{Z}), where

$$\zeta = \frac{1}{2}k\lambda_3\theta + k\theta^2.$$

Otherwise, $B_\theta \cong B_\eta$ if and only if $\eta = \pm\theta$ (mod \mathbb{Z}).

- If θ has degree 2 or 3, then there are finitely many distinct values of η (mod \mathbb{Z}) such that $B_\theta \cong B_\eta$.

It is already apparent from Theorem 6.2 that in the case where the degree of θ is less than or equal to four, that $B_\theta \cong B_\eta$ will happen more often than $A_\theta \simeq A_\eta$. For the cases $\deg \theta = 2, 3$, we provide examples to illustrate that possibly even more can happen.

Example 6.3. Let θ be the real solution to $\theta^3 = \theta + 1$. Then the equation (6.4) from case (d) in this case is

$$a^3 - ab^2 - b^3 = 1$$

whose integer solutions are $(1, 0)$, $(0, -1)$, $(\pm 1, -1)$, and $(4, 3)$. These correspond to the values $\eta = \theta$, $-\theta^2$, $\pm\theta - \theta^2$, and $4\theta + 3\theta^2$. Since $\deg \theta > 2$, these values are all distinct mod \mathbb{Z} .

Finally, we provide a similar example in the case where $\deg \theta = 2$.

Example 6.4. Let $\theta = \frac{1+\sqrt{2}}{3}$ and $\eta = \frac{1+2\sqrt{2}}{3}$. Then (in the notation of case (d)) we have

$$G_\theta = \frac{1}{3}\mathbb{Z} + \frac{1}{9}\sqrt{2}\mathbb{Z} = G_\eta.$$

Therefore, $B_\theta \cong B_\eta$. On the other hand, θ is not equal to $\pm\eta + n$ for any integer n .

7. A C*-ALGEBRA GENERATED BY A FAITHFUL IRREDUCIBLE REPRESENTATION OF A FINITELY GENERATED TORSION FREE NILPOTENT GROUP THAT IS NOT AN AT ALGEBRA

So far all of the algebras we considered turned out to be AT algebras. By Chris Phillips' theorem [21] if G is any finitely generated two-step nilpotent group and π a faithful irreducible representation of G , then $C^*(\pi(G))$ is an AT algebra. Since a natural conjecture forms from the previous sentences we would like to point out that it is very easy to produce irreducible representations of finitely generated nilpotent groups (of nilpotency step necessarily larger than 2) that are not AT algebras.

To this end, let θ be irrational and u and v be generators of A_θ satisfying the commutation relation. Consider the automorphism β of A_θ defined by $\beta(u) = u$ and $\beta(v) = u^2v$. It is then clear from the Pimsner-Voiculescu six-term exact sequence that $K_1(A_\theta \rtimes_\beta \mathbb{Z})$ contains an element of order 2. Since all AT algebras have torsion free K_1 , it follows that $A_\theta \rtimes_\beta \mathbb{Z}$ is not an AT algebra. We claim that $A_\theta \rtimes_\beta \mathbb{Z}$ is isomorphic to the C*-algebra generated by an irreducible representation of a 3-step nilpotent group.

Indeed, let $\mathbb{H}_3 \leq GL(3, \mathbb{Z})$ be the Heisenberg group with generators a, b . Then $\beta(a) = a$ and $\beta(b) = a^2b$ defines an automorphism of \mathbb{H}_3 . Notice that β fixes $Z(\mathbb{H}_3)$. Moreover the induced action of β on $\mathbb{H}_3/Z(\mathbb{H}_3) \cong \mathbb{Z}^2$ is given by the unipotent matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, showing that $\mathbb{H}_3 \rtimes_\beta \mathbb{Z}$ is a three step nilpotent group. Let π_θ be the representation of $\mathbb{H}_3 \rtimes_\beta \mathbb{Z}$ induced from $\theta \in \mathbb{T} = \widehat{Z(\mathbb{H}_3)}$. One checks fairly easily that $C^*(\pi_\theta(\mathbb{H}_3 \rtimes_\beta \mathbb{Z})) \cong A_\theta \rtimes_\beta \mathbb{Z}$.

Remark 7.1. The above example can clearly be generalized (with minimal effort) in a variety of ways to produce all kinds of finitely generated K_1 groups.

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