

FINITELY GENERATED NILPOTENT GROUP C*-ALGEBRAS HAVE FINITE NUCLEAR DIMENSION

CALEB ECKHARDT AND PAUL MCKENNEY

ABSTRACT. We show that group C*-algebras of finitely generated, nilpotent groups have finite nuclear dimension. It then follows, from a string of deep results, that the C*-algebra A generated by an irreducible representation of such a group has decomposition rank at most 3. If, in addition, A satisfies the universal coefficient theorem, another string of deep results shows it is classifiable by its Elliott invariant and is approximately subhomogeneous. We give a large class of irreducible representations of nilpotent groups (of arbitrarily large nilpotency class) that satisfy the universal coefficient theorem and therefore are classifiable and approximately subhomogeneous.

1. INTRODUCTION

The noncommutative dimension theories of Kirchberg and Winter (decomposition rank) and of Winter and Zacharias (nuclear dimension) play a prominent role in the theory of nuclear C*-algebras. This is especially apparent in Elliott's classification program where finite noncommutative dimension is essential for a satisfying classification theory. In [40], Winter and Zacharias express a hope that nuclear dimension will “shed new light on the role of dimension type conditions in other areas of noncommutative geometry.” We share this hope and this work aims to use the theory of nuclear dimension to shed new light on the representation theory of discrete nilpotent groups.

A *discrete* group is Type I (and therefore has a “tractable” representation theory) if and only if it has an abelian subgroup of finite index [37]. Therefore being Type I is a highly restrictive condition for discrete groups and therefore for most discrete groups, leaves many of the tools of classic representation theory out of reach. Recent breakthroughs of several mathematicians (H. Lin, Z. Niu, H. Matui, Y. Sato and W. Winter to name a few) gave birth to the possibility of classifying the C*-algebras generated by the irreducible representations of nilpotent groups by their ordered K-theory. A key missing ingredient was knowing whether or not the group C*-algebras of finitely generated nilpotent groups have finite nuclear dimension. Our main result (Theorem 4.4) supplies this ingredient. In particular we show that the nuclear dimension of $C^*(G)$ is bounded by $10^{h(G)-1} \cdot h(G)!$ where $h(G)$ is the Hirsch number of G (see Section 2.1.1).

C.E. was partially supported by NSF grant DMS-1262106.

Each finitely generated nilpotent group has an algebraic “basis” of sorts and the Hirsch number returns the size of this basis—it is therefore not surprising to see $h(G)$ appear in the nuclear dimension estimate.

Fix a finitely generated nilpotent group G and an irreducible representation π of G . The C^* -algebra generated by $\pi(G)$ is simple, nuclear, quasidiagonal with unique trace and, by a combination of Theorem 4.4 with many deep results (see Theorem 2.9 for a complete list), has finite decomposition rank. Therefore if $C^*(\pi(G))$ satisfies the universal coefficient theorem (see [35]), it is classified by its ordered K -theory and isomorphic to an approximately subhomogeneous C^* -algebra by [22, 25, 26] (see [26, Corollary 6.2])

In the case that G is a two-step nilpotent group, it is well-known that the C^* -algebras generated by irreducible representations of G are either finite dimensional or AT -algebras. Indeed Phillips showed in [31] that all simple higher dimensional non commutative tori (a class of C^* -algebras that include $C^*(\pi(G))$ when G is two-step and π is an irreducible, infinite dimensional representation) are AT algebras. In some sense this result forms the base case for our induction proof (see below for a more detailed description). Peeling back a couple layers, we mention that Phillips’ work relies on that of Elliott and Evans [12] and Kishimoto [20] (see also [3] and [23] for precursors to Phillips’ Theorem).

In Section 4 we provide specific examples of groups and representations that satisfy the universal coefficient theorem and are therefore classified by their ordered K -theory. The class of groups have arbitrarily large nilpotency class.

Since in general G is not Type I we are left with essentially no possibility of reasonably classifying its irreducible representations up to unitary equivalence. On the other hand if every primitive quotient of $C^*(G)$ satisfies the UCT, then one could classify the C^* -algebras generated by these representations by their ordered K -theory. This provides a dual viewpoint to the prevailing one of parametrizing irreducible representations by primitive ideals of $C^*(G)$ or by the space of characters of G (see [18])—we thank Nate Brown for sharing this nice observation with us.

Let us provide a broad outline of our proof. First we prefer to deal with torsion free groups. Since every finitely generated nilpotent group has a finite index torsion free subgroup we begin in Section 3 by showing that finite nuclear dimension is stable under finite extensions. We then focus on the torsion free case.

We proceed by induction on the Hirsch number (see Section 2.1.1) of the nilpotent group G . When dealing with representation theoretic objects, (like a group C^* -algebra) induction on the Hirsch number is sometimes more helpful than induction on, say, the nilpotency class for the simple reason that non-trivial quotients of G have Hirsch number strictly less than G while the nilpotency class of the quotient may be unchanged.

By [30] (see also Theorem 2.6 below) we can view $C^*(G)$ as a continuous field over the dual of its center, $Z(G)$. Since $Z(G)$ is a finitely generated abelian group, our task is to bound the nuclear dimension of the fibers as the base space is already controlled.

Since we are proceeding by induction we can more or less focus on those fibers (we call them $C^*(G, \tilde{\gamma})$) induced by characters $\gamma \in \widehat{Z(G)}$ that are faithful (as group homomorphisms) on $Z(G)$. In the case that G is a two step nilpotent group, then $C^*(G, \tilde{\gamma})$ is a simple higher dimensional noncommutative torus. Phillips showed in [31] that any such C*-algebra is an AT-algebra and therefore has nuclear dimension (decomposition rank in fact) bounded by 1. Phillips' result is crucial for us as it allows us to reduce to the case that the nilpotency class of G is at least 3 and provides enough "room" for the next step of the proof.

We then delete an element from G and have $G \cong N \rtimes \mathbb{Z}$ for N with strictly smaller Hirsch number. By induction we know $C^*(N)$ has finite nuclear dimension and so we analyze the action of \mathbb{Z} on fibers of $C^*(N)$. It turns out that there are two cases: In the first case, the fiber is simple and the action of \mathbb{Z} on the fiber is strongly outer and hence the crossed product (which is a fiber of $C^*(G)$) absorbs the Jiang-Su algebra by [27]. This in turn implies finite decomposition rank of the fiber by a string of deep results (see Theorem 2.9). If the fiber is not simple, we can no longer employ the results of [27], but the non-simplicity forces the action restricted to the center of the fiber to have finite Roklin dimension and so the crossed product has finite nuclear dimension by [14].

2. PRELIMINARIES

We assume the reader is familiar with the basics of group C*-algebras and discrete crossed products and refer them to Brown and Ozawa [5] for more information.

2.1. Facts about Nilpotent Groups.

2.1.1. *Group Theoretic Facts.* We refer the reader to Segal's book [36] for more information on polycyclic and nilpotent groups. Here we collect some facts about nilpotent groups that we will use frequently.

Let G be a group and define $Z_1(G) = Z(G)$ to be the center of G . Recursively define $Z_n(G) \leq G$ to satisfy $Z_n(G)/Z_{n-1}(G) := Z(G/Z_{n-1}(G))$. A group G is called **nilpotent** if $Z_n(G) = G$ for some n and is called **nilpotent of class n** if n is the least integer satisfying $Z_n(G) = G$.

A group G is **polycyclic** if it has a normal series

$$\{e\} \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n$$

such that each quotient G_{i+1}/G_i is cyclic. The number of times that G_{i+1}/G_i is infinite is called the **Hirsch number** of the group G and is denoted by $h(G)$. The Hirsch number is an invariant of the group. If G is polycyclic and N is a normal subgroup then both N and G/N are polycyclic with

$$h(G) = h(N) + h(G/N)$$

Every finitely generated nilpotent group is polycyclic.

Let now G be finitely generated and nilpotent. Let G_f denote the subgroup consisting of those elements with finite conjugacy class. By [1], the center $Z(G)$ has finite index in G_f . If in addition G is torsion free, then by [24] the quotient groups $G/Z_i(G)$ are also all torsion free. These facts combine to show that if G is torsion free, then every non-central conjugacy class is infinite.

2.1.2. Representation Theoretic Facts.

Definition 2.1. *Let G be a group and $\phi : G \rightarrow \mathbb{C}$ a positive definite function with $\phi(e) = 1$. If ϕ is constant on conjugacy classes, then we say ϕ is a **trace** on G . It is clear that any such ϕ defines a tracial state on $C^*(G)$. Following the representation theory literature we call an extreme trace a **character**. Every character gives rise to a factor representation and is therefore multiplicative on $Z(G)$.*

Let G be a finitely generated nilpotent group. Let J be a primitive ideal of $C^*(G)$ (i.e. the kernel of an irreducible representation). Moore and Rosenberg showed in [28] that J is actually a maximal ideal. Shortly after this result, Howe showed in [15] (see especially the introduction of [6]) that every primitive ideal is induced from a *unique* character, i.e. there is a unique character ϕ on G such that

$$J = \{x \in C^*(G) : \phi(x^*x) = 0\}.$$

Let G be a finitely generated nilpotent group and ϕ a character on G . Set $K(\phi) = \{g \in G : \phi(g) = 1\}$. By [15] and [6], G is **centrally inductive**, i.e. ϕ vanishes on the complement of $\{g \in G : gK(\phi) \in (G/K(\phi))_f\}$. We use the following well-known fact repeatedly,

Lemma 2.2. *Let G be a finitely generated torsion free nilpotent group and γ a faithful character on $Z(G)$. Then $\tilde{\gamma}$ is a character of G (see Definition 2.5)*

Proof. Since γ is a character and the extension $\tilde{\gamma}$ is a trace on G , by a standard extreme point argument there is a character ω on G that extends γ . Since G is nilpotent, every non-trivial normal subgroup intersects the center non-trivially. So $K(\tilde{\gamma}) \cap Z(G)$ is trivial we have $K(\tilde{\gamma})$ is also trivial. By the preceding section, $G_f = Z(G)$, i.e. ω vanishes off of $Z(G)$. But this means precisely that $\omega = \tilde{\gamma}$ \square

It is clear from the definition of nilpotent group that they are amenable and therefore by [21], their group C^* -algebras are nuclear. In summary, for every primitive ideal J of $C^*(G)$, the quotient $C^*(G)/J$ is simple and nuclear with a unique trace.

2.2. C^* -facts.

Definition 2.3 ([27]). *Let G be a group acting on a C^* -algebra A with unique trace τ . Since τ is unique the action of G leaves τ invariant and therefore extends to an automorphism of $\pi_\tau(A)''$, the von Neumann algebra generated by the GNS representation of τ . If for each $g \in G \setminus \{e\}$ the automorphism of $\pi_\tau(A)''$ corresponding to g is an outer automorphism, then we say that the action is **strongly outer**.*

The above definition can be modified to make sense even if A does not have a unique trace (see [27]), but we're only concerned with the unique trace case. The following Theorem provides a key step in our main result.

Theorem 2.4 ([27, Corollary 4.11]). *Let G be a discrete elementary amenable group (in particular this holds for G abelian) acting on a simple C*-algebra A with property (SI). If the action of G is strongly outer, then the crossed product $A \rtimes G$ is \mathcal{Z} -absorbing.*

The above theorem is actually given in greater generality in [27]. Another key idea for us is the fact that discrete amenable group C*-algebras decompose as continuous fields over their centers. First a definition,

Definition 2.5. *Let $N \leq G$ and ϕ a positive definite function on N .*

- (i) *Let $C^*(N, \phi)$ be the C*-algebra generated by the GNS representation of ϕ .*
- (ii) *Denote by $\tilde{\phi}$ the **trivial extension** of ϕ to G , i.e. $\tilde{\phi}(x) = 0$ if $x \notin N$.*

We recall the following special case of [30, Theorem 1.2]:

Theorem 2.6. *Let G be a discrete amenable group. Then $C^*(G)$ is a continuous field of C*-algebras over $\widehat{Z(G)}$, the dual group of $Z(G)$. Moreover, for each multiplicative character $\gamma \in \widehat{Z(G)}$ the fiber at γ is isomorphic to $C^*(G, \tilde{\gamma})$.*

Definition 2.7. *Let A be a C*-algebra. Denote by $\dim_{\text{nuc}}(A)$ the **nuclear dimension** of A (see [40] for the definition of nuclear dimension).*

It will be crucial for us to view $C^*(G)$ as a continuous field for our inductive step to work in the proof of our main theorem. The following allows us to control the nuclear dimension of continuous fields

Theorem 2.8 ([7, Lemma 3.1], [38, Lemma 5.1]). *Let A be a continuous field of C*-algebras over the finite dimensional compact space X . For each $x \in X$, let A_x denote the fiber of A at x . Then*

$$\dim_{\text{nuc}}(A) \leq (\dim(X) + 1)(\sup_{x \in X} \dim_{\text{nuc}}(A_x) + 1) - 1.$$

Throughout this work we never explicitly work with decomposition rank, nuclear dimension, property (SI) or the Jiang-Su algebra \mathcal{Z} (for example, we never actually build any approximating maps in proving finite nuclear dimension). For this reason we do not recall the lengthy definitions of these properties but simply refer the reader to [19, Definition 3.1] for decomposition rank, [40, Definition 2.1] for nuclear dimension, [13, 17] for the Jiang-Su algebra and its role in the classification program, and [25, Definition 4.1] for property (SI).

Finally, for easy reference we gather together several results into

Theorem 2.9 (Rørdam, Winter, Matui and Sato). *Let A be a unital, separable, simple, nuclear, quasidiagonal C*-algebra with a unique tracial state. If A has any of the following properties then it has all of them.*

- (i) *Finite nuclear dimension*
- (ii) *\mathcal{Z} -stability*
- (iii) *Strict comparison*
- (iv) *Property (SI) of Matui and Sato.*
- (v) *Decomposition rank at most 3*

In particular, if A is a primitive quotient of a nilpotent group C^* -algebra then it satisfies the hypotheses of this Theorem.

Proof. Winter showed (i) implies (ii) in [39]. Results of Matui and Sato [25] and Rørdam [33] show that (ii), (iii) and (iv) are all equivalent. Strict comparison and Matui and Sato's [26] shows (v). Finally (v) to (i) follows trivially from the definitions.

If G is a finitely generated nilpotent group, by [10], any (primitive) quotient of $C^*(G)$ is quasidiagonal and therefore satisfies the hypotheses of the theorem by the discussion in Section 2.9. \square

3. STABILITY UNDER FINITE EXTENSIONS

In this section we show that if a nilpotent group G has a finite index normal subgroup H such that $\dim_{\text{nuc}}(C^*(H)) < \infty$, then $\dim_{\text{nuc}}(C^*(G)) < \infty$. Perhaps surprisingly, this portion of the proof is the most involved and relies on several deep results of C^* -algebra theory. Moreover we lean heavily on the assumption that G is nilpotent.

This section exists because every finitely generated nilpotent group has a finite index subgroup that is torsion free. Reducing to this case allows the reader to have a clear idea of what is happening without getting bogged down in torsion. We begin with the following special case that isolates most of the technical details.

Theorem 3.1. *Let G be a finitely generated nilpotent group. Suppose H is a normal subgroup of finite index such that every primitive quotient of $C^*(H)$ has finite nuclear dimension. Then every primitive quotient of $C^*(G)$ has decomposition rank at most 3.*

Proof. We proceed by induction on $|G/H|$. Since G is nilpotent, so is G/H . In particular G/H has a cyclic group of prime order as a quotient. By our induction hypothesis we may therefore suppose that G/H is cyclic of prime order p .

Let $e, x, x^2, \dots, x^{p-1} \in G$ be coset representatives of G/H . Let α denote the action of G on $\ell^\infty(G/H)$ by left translation. It is well-known, and easy to prove, that $\ell^\infty(G/H) \rtimes_\alpha G \cong M_p \otimes C^*(H)$ and that under this inclusion $C^*(H) \subseteq C^*(G) \subseteq M_p \otimes C^*(H)$ we may realize this copy of $C^*(H)$ as the C^* -algebra generated by the diagonal matrices

$$(3.1) \quad (\lambda_h, \lambda_{xhx^{-1}}, \dots, \lambda_{x^{p-1}hx^{-(p-1)}}), \text{ with } h \in H.$$

Let (π, \mathcal{H}_π) be an irreducible representation of G . We show that $C^*(G)/\ker(\pi)$ has decomposition rank less than or equal to 3. Let τ be the unique character on G that induces $\ker(\pi)$ (see Section 2.1.2) and set $K(\tau) = \{x \in G : \tau(x) = 1\}$. By

assumption every primitive quotient of $H/H \cap K(\tau)$ has finite nuclear dimension. Since $(G/K(\tau))/(H/H \cap K(\tau))$ is a quotient of G/H it is either trivial, in which case we are done by assumption or it is isomorphic to G/H . We may therefore assume, without loss of generality, that

$$(3.2) \quad \text{the character } \tau \text{ that induces } \ker(\pi) \text{ is faithful on } Z(G).$$

By a well-known application of Stinespring's Theorem (see [29, Theorem 5.5.1]) there is an irreducible representation $\text{id}_p \otimes \sigma$ of $M_p \otimes C^*(H)$, such that $\mathcal{H}_\pi \subseteq \mathcal{H}_{\text{id}_p \otimes \sigma}$ and if $P : \mathcal{H}_{\text{id}_p \otimes \sigma} \rightarrow \mathcal{H}_\pi$ is the orthogonal projection, then

$$(3.3) \quad P((\text{id}_p \otimes \sigma)(x))P = \pi(x) \quad \text{for all } x \in C^*(G).$$

Let $J = \ker(\sigma) \subseteq C^*(H)$ and $J_G \subseteq C^*(G)$ be the ideal of $C^*(G)$ generated by J . By (3.3) we have $J_G \subseteq \ker(\pi)$.

We now consider two cases:

Case 1: J_G is a maximal ideal of $C^*(G)$, i.e. $J_G = \ker(\pi)$.

In this case we have

$$C^*(H)/J \subseteq C^*(G)/J_G \subseteq M_p \otimes (C^*(H)/J).$$

We would like to reiterate that in general the copy of $C^*(H)/J$, is *not* conjugate to the diagonal copy $1_p \otimes (C^*(H)/J)$, but rather the twisted copy of (3.1). If $C^*(H)/J$ is conjugate to the diagonal copy, then the proof is quite short (see the last paragraph of Case 1b), so most of the proof consists of overcoming this difficulty.

Let $\mathbb{Z}_p \cong G/H$ denote the cyclic group of order p . Define an action β of \mathbb{Z}_p on $\ell^\infty(G/H) \rtimes_\alpha G$ by $\beta_t(f)(s) = f(s - t)$ for all $f \in \ell^\infty(G/H)$ (i.e. β acts by left translation on $\ell^\infty(G/H)$) and $\beta(\lambda_g) = \lambda_g$ for all $g \in G$. Since the G -action α and \mathbb{Z}_p -action β commute with each other it is easy to see that β defines an action of \mathbb{Z}_p on $\ell^\infty(G/H) \rtimes_\alpha G$. Moreover

$$\beta_t(x) = x \quad \text{for all } t \in \mathbb{Z}/p\mathbb{Z} \text{ if and only if } x \in C^*(G).$$

From this it follows that β fixes J_G pointwise. Notice that $M_p \otimes J$ is generated by the β -invariant set $\{e_1 a_1 + \cdots + e_p a_p \mid a_i \in J\}$, where e_i denotes the i th minimal projection of M_p . Therefore, β leaves $M_p \otimes J$ invariant, and induces an automorphism of $M_p \otimes (C^*(H)/J)$. Moreover

$$\beta_t(x) = x \quad \text{for all } t \in \mathbb{Z}/p\mathbb{Z} \text{ if and only if } x \in C^*(G)/J_G.$$

We now split further into two sub cases based on the behavior of β .

Case 1a: The action $\beta \curvearrowright M_p \otimes (C^*(H)/J)$ is strongly outer.

By assumption, $M_p \otimes (C^*(H)/J)$ has finite nuclear dimension. By Theorem 2.9, $M_p \otimes (C^*(H)/J)$ then has property (SI). Since β is strongly outer, by [27, Corollary

4.11], the crossed product $M_p \otimes (C^*(H)/J) \rtimes_{\beta} \mathbb{Z}_p$ is \mathcal{Z} -stable. Since $M_p \otimes (C^*(H)/J)$ has a unique trace and $\ell^{\infty}(p) \rtimes \mathbb{Z}_p \cong M_p$, it follows that $[M_p \otimes (C^*(H)/J)] \rtimes_{\beta} \mathbb{Z}_p$ has unique trace. It follows from [10] that $[M_p \otimes (C^*(H)/J)] \rtimes_{\beta} \mathbb{Z}_p$ is quasidiagonal. Therefore by Theorem 2.9, $M_p \otimes (C^*(H)/J) \rtimes_{\beta} \mathbb{Z}_p$ has decomposition rank at most 3.

By [34], the fixed point algebra of β , i.e. $C^*(G)/J_G$ is isomorphic to a corner of $M_p \otimes (C^*(H)/J) \rtimes_{\beta} \mathbb{Z}_p$, which by Brown's isomorphism theorem [4] implies that $C^*(G)/J_G$ is stably isomorphic to $M_p \otimes (C^*(H)/J) \rtimes_{\beta} \mathbb{Z}_p$ and therefore $C^*(G)/J_G = C^*(G)/\ker(\pi)$ has decomposition rank at most 3 by [19, Corollary 3.9].

Case 1b: The action $\beta \curvearrowright M_p \otimes (C^*(H)/J)$ is not strongly outer.

Choose a generator t of \mathbb{Z}_p and set $\beta = \beta_t$ (note that every β_t is strongly outer or none of them are). We will first show that β is actually an inner automorphism of $M_p \otimes (C^*(H)/J)$.

The unique trace on $M_p \otimes (C^*(H)/J)$ restricts to the unique trace τ on $C^*(G)/J_G$. We will use the common letter τ for both of these traces.

Let $G_f \leq G$ be the subgroup consisting of those elements with finite conjugacy classes. By [1, Lemma 3], $G_f/Z(G)$ is finite. Let (π_{τ}, L^2) be the GNS representation of $\ell^{\infty}(G/H) \rtimes_{\alpha} G$ associated with τ .

Since τ is multiplicative on $Z(G)$ it easily follows that

$$\langle \lambda_x, \lambda_y \rangle_{\tau} = \tau(y^{-1}x) \in \mathbb{T}, \quad \text{for } x, y \in Z(G).$$

By the Cauchy-Schwarz inequality it follows that in L^2 we have

$$(3.4) \quad \lambda_x =_{L^2} \tau(y^{-1}x) \lambda_y \text{ for all } x, y \in Z(G).$$

Let $x_1, \dots, x_n \in G_f$ be coset representatives of $G_f/Z(G)$ and let $C \subseteq G$ be a set of coset representatives for G/G_f .

The preceding discussion shows that the following set spans a dense subset of L^2 :

$$(3.5) \quad \{f \lambda_{tx_i} : f \in \ell^{\infty}(G/H), t \in C, 1 \leq i \leq n\}.$$

Since τ is β -invariant for each minimal projection $e \in \ell^{\infty}(p)$ and $t \in G$ we have $\tau(e \lambda_t) = \tau(\beta(e) \lambda_t)$. So for each $f \in \ell^{\infty}(G/H)$ and $t \in G$ we have $\tau(f \lambda_t) = \tau(f) \tau(t)$.

Combining this with the fact that τ vanishes on infinite conjugacy classes produces

$$(3.6) \quad \langle f \lambda_t, g \lambda_s \rangle_{\tau} = 0 \text{ for all } f, g \in \ell^{\infty}(G/H), \text{ and } t, s \in C, t \neq s.$$

By assumption there is a unitary $W \in \pi_{\tau}(\ell^{\infty}(G/H) \rtimes_{\alpha} G)''$ such that

$$W \pi_{\tau}(x) W^* = \pi_{\tau}(\beta(x)) \quad \text{for all } x \in \ell^{\infty}(G/H) \rtimes_{\alpha} G.$$

Since β leaves $\pi_{\tau}(G)$ pointwise invariant, W must commute with $\pi_{\tau}(\lambda_t)$ for all $t \in G$. For $t \in G$, let $\text{Conj}(t) = \{sts^{-1} : s \in G\}$ be the conjugacy class of t .

Let $s \in G \setminus G_f$. Suppose first that $\text{Conj}(s)$ intersects infinitely many G/G_f cosets.

Let $(s_n)_{n=1}^\infty$ be a sequence from G such that the cosets $s_n s s_n^{-1} G_f$ are all mutually distinct. Let $f \in \ell^\infty(G/H)$. Since W commutes with the λ_t 's, for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \langle W, f \lambda_s \rangle_\tau &= \langle \lambda_{s_n^{-1}} W \lambda_{s_n}, f \lambda_s \rangle_\tau \\ &= \langle W, \alpha_{s_n}(f) \lambda_{s_n s s_n^{-1}} \rangle_\tau \end{aligned}$$

By (3.6), the vectors $\{\alpha_{s_n}(f) \lambda_{s_n s s_n^{-1}} : n \in \mathbb{N}\}$ form an orthogonal family of vectors, each with the same L^2 norm. Since W has L^2 norm equal to 1, this implies that $\langle W, f \lambda_s \rangle = 0$ for all $f \in \ell^\infty(G/H)$.

Suppose now that $\text{Conj}(s)$ intersects only finitely many G/G_f cosets. Since $\text{Conj}(s)$ is infinite, there is an $x \in G$ so $\text{Conj}(s) \cap x G_f$ is infinite. Since $Z(G)$ has finite index in G_f , there is a $y \in G$ such that $\text{Conj}(s) \cap y Z(G)$ is infinite. Let s_n be a sequence from G and t_n a sequence of distinct elements from $Z(G)$ so $s_n s s_n^{-1} = y t_n$ for all $n \in \mathbb{N}$.

Let $f \in \ell^\infty(G/H)$. Since the set $\{\alpha_g(f) : g \in G\}$ is finite we may, without loss of generality, suppose that $\alpha_{s_n}(f) = \alpha_{s_m}(f)$ for all $n, m \in \mathbb{N}$. By (3.4), we have $\lambda_{y t_n} =_{L^2} \tau(t_m^{-1} t_n) \lambda_{y t_m}$ for all $n, m \in \mathbb{N}$. We then have

$$\begin{aligned} \langle W, f \lambda_s \rangle_\tau &= \langle \lambda_{s_n^{-1}} W \lambda_{s_n}, f \lambda_s \rangle_\tau \\ &= \langle W, \alpha_{s_n}(f) \lambda_{s_n s s_n^{-1}} \rangle_\tau \\ &= \langle W, \alpha_{s_1}(f) \lambda_{y t_n} \rangle_\tau \\ &= \tau(t_1^{-1} t_n) \langle W, \alpha_{s_1}(f) \lambda_{y t_1} \rangle_\tau \end{aligned}$$

Since τ is faithful on $Z(G)$ (by (3.2)) all of the $\tau(t_1^{-1} t_n)$ are distinct and nonzero, so $\langle W, f \lambda_s \rangle_\tau = 0$.

We have shown that for all $s \in G \setminus G_f$ and $f \in \ell^\infty(G/H)$ we have $\langle W, f \lambda_s \rangle_\tau = 0$. By (3.5) it follows that

$$W \in \text{span}\{\pi_\tau(f \lambda_{x_i}) : f \in \ell^\infty(G/H), 1 \leq i \leq n\} \subseteq \pi_\tau(\ell^\infty(G/H) \rtimes G) \cong M_p \otimes (C^*(H)/J)$$

By the way that β acts on $\ell^\infty(G/H)$ there are $a_1, \dots, a_p \in C^*(H)/J$ so

$$W = e_{1p} \otimes a_1 + \sum_{i=2}^p e_{i,i-1} \otimes a_i$$

Set $U = \text{diag}(1, a_2^*, a_2^* a_3^*, \dots, a_2^* a_3^* \cdots a_p^*) \in M_p \otimes (C^*(H)/J)$. Since W commutes with $C^*(H)/J$, by (3.1) it follows that

$$(3.7) \quad U \left(C^*(H)/J \right) U^* = 1_{M_p} \otimes C^*(H)/J \subseteq U \left(C^*(G)/J_G \right) U^* \subseteq M_p \otimes (C^*(H)/J).$$

Let \mathcal{U} be a free ultrafilter on \mathbb{N} and for a C*-algebra A , let $A^\mathcal{U}$ denote the ultrapower of A . We think of $A \subset A^\mathcal{U}$ via the diagonal embedding and write $A' \cap A^\mathcal{U}$ for those elements of the ultrapower that commute with this diagonal embedding.

By Theorem 2.9, $C^*(H)/J$ is \mathcal{Z} -stable. By [32, Theorem 7.2.2], there is an embedding ϕ of \mathcal{Z} into $(C^*(H)/J)^\mathcal{U} \cap (C^*(H)/J)'$. By both inclusions of (3.7) it is clear

that $1_{M_p} \otimes \phi$ defines an embedding of \mathcal{Z} into $(C^*(G)/J_G)^{\mathcal{U}} \cap (C^*(G)/J_G)'$. Therefore, again by [32, Theorem 7.2.2] it follows that $C^*(G)/J_G$ is \mathcal{Z} -stable. By Theorem 2.9, $C^*(G)/J_G$ has decomposition rank at most 3.

Case 2: The ideal J_G is not maximal. Recall the definition of σ from the beginning of the proof.

For each $i = 0, \dots, p-1$ define the representations of H , $\sigma_i(h) = \sigma(x^i h x^{-i})$. By [10, Section 3] either all of σ_i are unitarily equivalent to each other or none of them are. We treat these cases separately.

Case 2a: All of the σ_i are unitarily equivalent to each other.

By the proof of Lemma 3.4 in [10] it follows that there is a unitary $U \in M_p \otimes C^*(H)/J$ such that $U(C^*(H)/J)U^* = 1_p \otimes (C^*(H)/J)$. Moreover from the same proof there is a projection $q \in M_p \otimes 1_{C^*(H)/J}$ that commutes with $U(\text{id}_{M_p} \otimes \sigma(C^*(G)))U^*$ so $q(U\text{id}_{M_p} \otimes \sigma(C^*(G))U^*) \cong C^*(G)/\ker(\pi)$. We therefore have a chain of inclusions

$$q \otimes C^*(H)/J \subseteq q(U\text{id}_{M_p} \otimes \sigma(C^*(G))U^*) \subseteq qM_p q \otimes C^*(H)/J.$$

We can now complete the proof as in the end of Case 1b (following nearly verbatim everything that follows (3.7)).

Case 2b. None of the σ_i are unitarily equivalent to each other.

From the proof of Lemma 3.5 in [10] there is a projection $q \in \ell^\infty(p) \otimes 1_{C^*(H)/J}$ that commutes with $(\text{id}_{M_p} \otimes \sigma)(C^*(G))$ so $q(\text{id}_{M_p} \otimes \sigma(C^*(G))) \cong C^*(G)/\ker(\pi)$. But since G acts transitively on G/H (and hence ergodically on $\ell^\infty(G/H)$) we must have $q = 1$. But this implies that $\ker(\text{id}_{M_p} \otimes \sigma|_{C^*(G)}) = \ker(\pi)$.

Recall the coset representatives $e, x, x^2, \dots, x^{p-1}$ of G/H . Each $x \in C^*(G)$ can be written uniquely as $\sum_{i=0}^{p-1} a_i \lambda_{x^i}$ for some $a_i \in C^*(H)$. Fix an $0 \leq i \leq p-1$ and consider $\lambda_{x^i} \in M_p \otimes C^*(H)$. If there is an index (k, ℓ) such that the (k, ℓ) -entry of λ_{x^i} is non-zero, then for any $j \neq i$ the (k, ℓ) -entry of λ_{x^j} must be 0. From this observation it follows that

$$\text{id}_p \otimes \sigma \left(\sum_{i=0}^{p-1} a_i \lambda_{x^i} \right) = 0, \text{ if and only if } (\text{id}_p \otimes \sigma)(a_i) = 0 \text{ for all } 1 \leq i \leq p$$

In other words $\ker(\pi) = \ker(\text{id}_{M_p} \otimes \sigma|_{C^*(G)}) \subseteq J_G$ and we are done by Case 1. \square

Lemma 3.2. *Let G be a finitely generated nilpotent group and N a finite index subgroup of $Z(G)$ and ϕ a faithful multiplicative character on N . Then there is a finite set \mathcal{F} of characters of G such that $\tilde{\phi}$ is in the convex hull of \mathcal{F} .*

Proof. This result follows from Thoma's work on characters [37] (see also the discussion on page 355 of [18]). For the convenience of the reader we outline a proof and

keep the notation of [18]. Let $G_f \leq G$ denote the group consisting of elements with finite conjugacy class. By [1] $Z(G)$ (and hence N) has finite index in G_f .

Let π be the GNS representation of G_f associated with $\tilde{\phi}|_{G_f}$. Since N has finite index in G_f and $\pi(N) \subseteq \mathbb{C}$ it follows that $\pi(G_f)$ generates a finite dimensional C*-algebra. Therefore there are finitely many characters $\omega_1, \dots, \omega_n$ of G_f that extend ϕ (trivially as $\pi(N) \subseteq \mathbb{C}$) and a sequence of positive scalars λ_i such that

$$(3.8) \quad \tilde{\phi}|_{G_f} = \sum_{i=1}^n \lambda_i \omega_i$$

The positive definite functions $\tilde{\omega}_i$ on G need not be traces, but this is easily remedied by the following averaging process (which we took from [18] and [37]).

Let $x \in G_f$. Then the centralizer of x in G , denoted $C_G(x)$, is finite. Let A_x be a complete set of coset representatives of $G/C_G(x)$. For each $1 \leq i \leq n$ and $x \in G_f$ define

$$(3.9) \quad \tilde{\omega}_i^G(x) = \frac{1}{[G : C_G(x)]} \sum_{a \in A_x} \tilde{\omega}_i(axa^{-1}).$$

Then each $\tilde{\omega}_i^G$ is extreme in the space of G -invariant traces on G_f (see [18, Page 355]). Since each $\tilde{\omega}_i^G$ is G -invariant, the trivial extension to G (which we still denote by $\tilde{\omega}_i^G$) is a trace on G . We show that the $\tilde{\omega}_i^G$ are actually characters on G .

By a standard convexity argument, there is a character ω on G which extends $\tilde{\omega}_i^G$. Let $K(\omega) = \{x \in G : \omega(x) = 1\}$. Then $K(\omega) \subseteq G_f$. Indeed if there is a $g \in K(\omega) \setminus G_f$, then g necessarily has infinite order (otherwise the torsion subgroup of G would be infinite). Since every finitely generated nilpotent group has a finite index torsion free subgroup and every nontrivial subgroup of a nilpotent group intersects the center non-trivially, this would force $K(\omega) \cap Z(G)$ to have non-zero Hirsch number, but by assumption $\phi = \omega|_N$ is faithful on a finite index subgroup of $Z(G)$.

Since G is finitely generated and nilpotent, it is centrally inductive (see [6]). This means that ω vanishes outside of $G_f(\omega) = \{x \in G : xK(\omega) \in (G/K(\omega))_f\}$. But since $K(\omega) \subseteq G_f$ and $\omega|_N = \phi$, it follows that $K(\omega)$ is finite. From this it follows that $G_f(\omega) = G_f$, i.e. ω must vanish outside of G_f . But this means precisely that $\omega = \tilde{\omega}_i^G$. By (3.8) and (3.9) for $x \in G_f$ we have

$$\begin{aligned} \sum_{i=1}^n \lambda_i \tilde{\omega}_i^G(x) &= \sum_{i=1}^n \lambda_i \left(\frac{1}{[G : C_G(x)]} \sum_{a \in A_x} \tilde{\omega}_i(axa^{-1}) \right) \\ &= \frac{1}{[G : C_G(x)]} \sum_{a \in A_x} \left(\sum_{i=1}^n \lambda_i \tilde{\omega}_i(axa^{-1}) \right) \\ &= \frac{1}{[G : C_G(x)]} \sum_{a \in A_x} \tilde{\phi}(x) \\ &= \tilde{\phi}(x). \end{aligned}$$

For $x \notin G_f$ we clearly have $\sum_{i=1}^n \lambda_i \tilde{\omega}_i^G(x) = 0 = \tilde{\phi}(x)$.

□

Lemma 3.3. *Set $f(n) = 10^{n-1}n!$. Let G be a finitely generated nilpotent group. Let $H \trianglelefteq G$ be normal of finite index. If $\dim_{\text{nuc}}(C^*(H/N)) \leq f(h(H/N))$ for every normal subgroup N of H , then $\dim_{\text{nuc}}(C^*(G/K)) \leq f(h(G/K))$ for every normal subgroup K of G .*

Proof. We proceed by induction on $h(G)$. If $h(G) = 0$, then G is finite and there is nothing to prove.

So assume that for every finitely generated nilpotent group A with $h(A) < h(G)$ that satisfies $\dim_{\text{nuc}}(C^*(A/N)) \leq f(h(A/N))$ for every normal subgroup N of A , we have $\dim_{\text{nuc}}(A'/N') \leq f(h(A'/N'))$ where A' is a finite normal extension of A and N' is a normal subgroup of A' .

Let now H be a finite index normal subgroup of G that satisfies the hypotheses. If $G/Z(G)$ is finite, then $\dim_{\text{nuc}}(C^*(G)) = h(G) \leq f(h(G))$ by Theorem 2.8. Suppose that $G/Z(G)$ is infinite, i.e. $h(Z(G)) < h(G)$.

Since for any quotient G/K of G , the group $H/(H \cap K)$ has finite index in G/K it suffices to show that $\dim_{\text{nuc}}(C^*(G)) \leq f(h(G))$.

We use Theorem 2.6 to view $C^*(G)$ as a continuous field over $\widehat{Z(G)}$. We estimate the nuclear dimension of the fibers. Let $\gamma \in \widehat{Z(G)}$.

Suppose first that $h(\ker(\gamma)) > 0$. The fiber $C^*(G, \tilde{\gamma})$ is a quotient of the group C^* -algebra $C^*(G/\ker(\gamma))$. By our induction hypothesis

$$\dim_{\text{nuc}}(C^*(G/\ker(\gamma))) \leq f(h(G) - 1).$$

Suppose now that $F = \ker(\gamma)$ is finite. If $x \in G$ with $[x, y] \in F$ for all $y \in G$, then $yx^{|F|}y^{-1} = x^{|F|}$ for all $y \in G$, i.e. $x^{|F|} \in Z(G)$. Hence $Z(G/F)/(Z(G)/F)$ is a finitely generated, nilpotent torsion group, hence is finite.

We therefore replace G with G/F and suppose that $Z(G)$ contains a finite index subgroup N such that γ is a faithful homomorphism on N . By Lemma 3.2, there are finitely many distinct characters $\omega_1, \dots, \omega_n$ on G such that $\tilde{\gamma}$ is a convex combination of the ω_i . The GNS representation associated with $\tilde{\gamma}$ is then the direct sum of the GNS representations associated with the ω_i . We can then view $C^*(G, \tilde{\gamma})$ as a continuous field over the discrete space $\{1, \dots, n\}$ where the i -th fiber is isomorphic to $C^*(G, \omega_i)$. Since each ω_i is a character on G , it follows by Theorem 3.1 that the decomposition rank of $C^*(G, \omega_i)$ is bounded by 3 which also bounds the decomposition rank of $C^*(G, \tilde{\gamma})$ by 3 by Theorem 2.8.

Since the nuclear dimension of all the fibers of $C^*(G)$ are bounded by $f(h(G) - 1)$, by Theorem 2.8 we have $\dim_{\text{nuc}}(C^*(G)) \leq 2h(G)f(h(G) - 1) \leq f(h(G))$ □

4. MAIN RESULT

The work of Matui and Sato on strongly outer actions (see Theorem 2.4) and of Hirshberg, Winter and Zacharias [14] on Roklin dimension are both crucial to the proof of our main result. In our case, their results turned extremely difficult problems into ones with more or less straightforward solutions.

Lemma 4.1. *Let α be an outer automorphism of a torsion free nilpotent group G . Then for every $a \in G$, the following set is infinite*

$$\{s^{-1}a\alpha(s) : s \in G\}.$$

Proof. Suppose that for some a , the above set is finite. Then for any $s \in G$ there are $0 \leq m < n$ so

$$s^{-m}a\alpha(s^m) = s^{-n}a\alpha(s^n) \quad \text{or} \quad a^{-1}s^{n-m}a = \alpha(s^{n-m})$$

Therefore

$$s^{n-m} = a\alpha(s^{n-m})a^{-1} = (a\alpha(s)a^{-1})^{n-m}$$

Since G is nilpotent and torsion free it has unique roots (see [24] or [2, Lemma 2.1]), i.e. $s = a\alpha(s)a^{-1}$, or α is an inner automorphism. \square

Lemma 4.2. *Let G be a torsion free nilpotent group of class $n \geq 3$. Suppose that $G = \langle N, x \rangle$ where $x \in G \setminus Z_{n-1}(G)$, $N \cap \langle x \rangle = \{e\}$ and $Z(G) = Z(N)$. Let ϕ be a trace on G that is multiplicative on $Z(G)$ and that vanishes off of $Z(G)$. Let α be the automorphism of $C^*(N, \phi|_N)$ induced by conjugation by x . Then α is strongly outer.*

Proof. We first show that α induces an outer automorphism of $N/Z(G)$. If not, then there is a $z \in N$ such that

$$xax^{-1}Z(G) = z^{-1}azZ(G) \quad \text{for all } a \in N.$$

In other words $zx \in Z_2(G)$. Since $x \notin Z_{n-1}(G)$ we also have $z \notin Z_{n-1}(G)$. Then $z^{-1}Z_{n-1}(G) = xZ_{n-1}(G)$, from which it follows that $z \notin N$.

Since G is torsion free so is N . Since ϕ is multiplicative on $Z(N)$ and vanishes outside of $Z(N)$ it follows that ϕ is a character (see Lemma 2.2) and therefore $C^*(N, \phi)$ has a unique trace (see Section 2.1). Let π_ϕ be the GNS representation associated with ϕ . By way of contradiction suppose there is a $W \in \pi_\phi(N)''$ such that $W\pi_\phi(g)W^* = \pi_\phi(\alpha(g))$ for all $g \in N$.

Let $(\pi_\phi, L^2(N, \phi))$ be the GNS representation associated with ϕ . For each $s \in N$ we let $\delta_s \in L^2(N, \phi)$ be its canonical image. Notice that if $aZ(N) \neq bZ(N)$, then δ_a and δ_b are orthogonal. In particular for any complete choice of coset representatives $C \subseteq N$ for $N/Z(G)$, the set $\{\delta_c : c \in C\}$ is an orthonormal basis for $L^2(N, \phi)$. Since W is in the weak closure of the GNS representation π_ϕ it is in $L^2(N, \phi)$ with norm 1. Therefore there is some $a \in N$ such that $\langle W, \delta_a \rangle \neq 0$.

We now have, for all $s \in N$,

$$\begin{aligned} \langle W, \delta_a \rangle &= \langle W\pi_\phi(s), \delta_{as} \rangle \\ &= \langle \pi_\phi(\alpha(s))W, \delta_{as} \rangle \\ &= \langle W, \delta_{\alpha(s)^{-1}as} \rangle \end{aligned}$$

Since α is outer on $N/Z(N)$, by Lemma 4.1 the set $\{\alpha(s)^{-1}asZ(N) : s \in N\}$ is infinite. Since distinct cosets provide orthogonal vectors of norm 1, this contradicts the fact that $W \in L^2(N, \phi)$.

Essentially the same argument shows that any power of α is also not inner on $\pi_\tau(N)''$, i.e. the action is strongly outer. \square

We refer the reader to the paper [14] for information on Roklin dimension of actions on C^* -algebras. For our purposes we do not even need to know what it is, simply that our actions have finite Roklin dimension. Therefore we omit the somewhat lengthy definition [14, Definition 2.3]. We do mention the following corollary to the definition of Roklin dimension: If α is an automorphism of a C^* -algebra A and there is an α -invariant, unital subalgebra $B \subseteq Z(A)$ such that the action of α on B has Roklin dimension bounded by d , then the action of α on A also has Roklin dimension bounded by d .

Lemma 4.3. *Let G be a finitely generated, torsion free nilpotent group. Suppose that $G = \langle N, x \rangle$ where $N \triangleleft G$, $N \cap \langle x \rangle = \{e\}$, $Z(G) \subseteq Z(N)$, and $Z(G) \neq Z(N)$. Let ϕ be a trace on G that is multiplicative on $Z(G)$ and vanishes off of $Z(G)$. Let α be the automorphism of $C^*(N, \phi|_N)$ defined by conjugation by x . Then the Roklin dimension of α is 1.*

Proof. Consider the action of α restricted to $Z(N)$. Since $Z(N) \neq Z(G)$, α is not the identity on $Z(N)$. Since $Z(N)$ is a free abelian group we have $\alpha \in GL(\mathbb{Z}, d)$ where d is the rank of $Z(N)$. Since G is nilpotent, so is the group $Z(N) \rtimes_\alpha \mathbb{Z}$. Therefore $(1 - \alpha)^d = 0$. In particular there is a $y \in Z(N)$ such that $(1 - \alpha)(y) \neq 0$ but $(1 - \alpha)^2(y) = 0$. From this we deduce that $\alpha(y) = y + z$ for some $z \in Z(G) \setminus \{0\}$.

Therefore the action of α on $C^*(\pi_\phi(y)) \cong C(\mathbb{T})$ is a rotation by $\phi(z)$. Since ϕ is faithful we have $\phi(z) = e^{2\pi i\theta}$ for some irrational θ . By [14, Theorem 6.1] irrational rotations of the circle have Roklin dimension 1. Since $C^*(\pi_\phi(y)) \subseteq Z(C^*(N, \phi))$, the remark preceding this lemma shows that the Roklin dimension of α acting on $C^*(N, \phi)$ is also equal to 1. \square

Theorem 4.4. *Define $f : \mathbb{N} \rightarrow \mathbb{N}$ by $f(n) = 10^{n-1}n!$. Let G be a finitely generated nilpotent group. Then $\dim_{\text{nuc}}(C^*(G)) \leq f(h(G))$.*

Proof. We proceed by induction on the Hirsch number of G . If $h(G) = 0$, there is nothing to prove. It is well-known that G contains a finite index torsion-free subgroup N (see section 2.1.1). Therefore by Lemma 3.3 we may assume that G is torsion free. We decompose $C^*(G)$ as a continuous field over $\widehat{Z(G)}$ as in Theorem 2.6. Let $\gamma \in \widehat{Z(G)}$. If γ is not faithful on $Z(G)$, then since G is torsion free we have $h(\ker(\gamma)) > 0$. By our induction hypothesis we then have $\dim_{\text{nuc}}(C^*(G, \tilde{\gamma})) \leq f(h(G) - 1)$.

Suppose now that γ is faithful on $Z(G)$. If G is a 2 step nilpotent group, then $C^*(G, \tilde{\gamma})$ is a simple higher dimensional noncommutative torus and therefore an AT algebra by [31]. AT algebras have nuclear dimension (decomposition rank in fact) bounded by 1 [19].

Suppose then that G is nilpotent of class $n \geq 3$ and let $Z_i(G)$ denote its upper central series. By [24] (see also [16, Theorem 1.2]), the group $G/Z_{n-1}(G)$ is a free

abelian group. Let $xZ_{n-1}(G), x_1Z_{n-1}(G), \dots, x_dZ_{n-1}(G)$ be a free basis for $G/Z_{n-1}(G)$. Let N be the group generated by $Z_{n-1}(G)$ and $\{x_1, \dots, x_d\}$. Then N is a normal subgroup of G with $h(N) = h(G) - 1$ and $G = N \rtimes_{\alpha} \mathbb{Z}$ where α is conjugation by x .

Suppose first that $Z(N) = Z(G)$. Since $h(N) = h(G) - 1$, the group C*-algebra $C^*(N)$ has finite nuclear dimension by our induction hypothesis. Assuming $Z(N) = Z(G)$ means that $\tilde{\gamma}$ is a character on N , i.e. $C^*(N, \tilde{\gamma})$ is primitive quotient of $C^*(N)$. It then enjoys all of the properties of Theorem 2.9.

By Lemma 4.2, the action of α on $C^*(N, \tilde{\gamma})$ is strongly outer. By [27, Corollary 4.11], the crossed product $C^*(N, \tilde{\gamma}) \rtimes_{\alpha} \mathbb{Z} \cong C^*(G, \tilde{\gamma})$ is \mathcal{Z} -stable, hence $C^*(G, \tilde{\gamma})$ has decomposition rank bounded by 3 by Theorem 2.9.

Suppose now that $Z(G)$ is strictly contained in $Z(N)$. By [14, Theorem 4.1] and Lemma 4.3 we have

$$\dim_{\text{nuc}}(C^*(G, \tilde{\gamma})) = \dim_{\text{nuc}}(C^*(N, \tilde{\gamma}) \rtimes_{\alpha} \mathbb{Z}) \leq 8(\dim_{\text{nuc}}(C^*(N, \tilde{\gamma})) + 1) \leq 9f(h(N)).$$

Therefore the nuclear dimension of every fiber of $C^*(G)$ is bounded by $9f(h(G) - 1)$ and the dimension of its center is at most $h(G) - 1$.

By Theorem 2.8, we have

$$\dim_{\text{nuc}}(C^*(G)) \leq 10h(G)f(h(G) - 1) = f(h(G)).$$

□

4.1. Application to the Classification Program. Combining Theorem 4.4 with results of Lin and Niu [22] and Matui and Sato [25, 26] (see especially Corollary 6.2 of [26]) we display the reach of Elliott's classification program

Theorem 4.5. *Let G be a finitely generated nilpotent group and J a primitive ideal of $C^*(G)$. If $C^*(G)/J$ satisfies the universal coefficient theorem, then $C^*(G)/J$ is classifiable by its Elliott invariant and is isomorphic to an approximately subhomogeneous C*-algebra.*

4.1.1. *Unitriangular groups.* We now consider a class of concrete examples covered by the above corollary. Let $d \geq 3$ be an integer. Consider the group of upper triangular matrices,

$$U_d = \left\{ A \in GL_d(\mathbb{Z}) : A_{ii} = 1 \text{ and } A_{ij} = 0 \text{ for } i > j \right\}.$$

The center $Z(U_d) \cong \mathbb{Z}$ is identified with those elements whose only non-zero, non-diagonal entry can occur in the $(1, d)$ matrix entry. U_d is a finitely generated $d-1$ -step nilpotent group. Fix an irrational θ and consider the character τ_{θ} on U_d induced from the multiplicative character $n \mapsto e^{2\pi n\theta i}$. Let $u_{ij} \in U_d$ be the matrix $1 + e_{ij}$ where e_{ij} is the ij matrix unit. It is easy to check that $C^*(\pi_{\tau_{\theta}}(u_{12}, u_{2d}))$ is isomorphic to the irrational rotation algebra A_{θ} . Since the UCT holds for abelian C*-algebras and is stable under \mathbb{Z} -actions (by [35]) A_{θ} satisfies the UCT. It is then a straightforward algebraic exercise to see that $C^*(U_d, \tau_{\theta})$ is isomorphic to an iterated crossed product of \mathbb{Z} -actions on C*-algebras that satisfy the UCT and therefore satisfies the UCT.

Therefore

Corollary 4.6. *For all $d \geq 3$ and irrational θ the C^* -algebras $C^*(U_d, \tau_\theta)$ are classified by their Elliott invariant and isomorphic to approximately subhomogeneous C^* -algebras.*

In the case of $d = 4$, we show in [11], together with Craig Kleski, that for continuum many θ the C^* -algebras $C^*(U_d, \tau_\theta)$ are mutually non-isomorphic.

5. QUESTIONS AND COMMENTS

Very broadly this section contains one question: Is there a group theoretic characterization of finitely generated groups whose group C^* -algebras have finite nuclear dimension? We parcel this into more manageable chunks.

Since finite decomposition rank implies strong quasidiagonality (see [19]) there are many examples of finitely generated group C^* -algebras with infinite decomposition rank [8]. On the other hand we are unaware of any such groups which have infinite nuclear dimension.

Question 5.1. *Are there any finitely generated amenable groups with infinite nuclear dimension? What if we restrict to the polycyclic groups?*

It certainly seems that the answer to the above general question must be no. On the other hand if we restrict to polycyclic groups—given the finite-dimensional feel of the Hirsch number and the role it played in the present work—we are not confident enough to guess either way. Note that in [9] there are numerous examples of polycyclic groups that are not strongly quasidiagonal and therefore have infinite decomposition rank.

The paper [9] also provides examples of non virtually nilpotent, polycyclic groups whose group C^* -algebras are strongly quasidiagonal. It seems that these groups could be a good starting point for a general investigation into nuclear dimension of polycyclic groups. The difficulty here is that these groups have trivial center and we therefore do not have a useful continuous field characterization of their group C^* -algebras as in Theorem 2.6.

Unfortunately we were unable to extend our results to the virtually nilpotent case, and thus are left with the

Question 5.2. *Do virtually nilpotent group C^* -algebras have finite nuclear dimension?*

Question 5.3. *If G is finitely generated and nilpotent, does $C^*(G)$ have finite decomposition rank?*

The careful reader will notice that the only part of our proof where we cannot deduce finite decomposition rank is in the second case of the proof of Theorem 4.4. There is definitely a need for both cases as there exist torsion free nilpotent groups G such that whenever $G/N \cong \mathbb{Z}$ with $Z(G) \leq N$, then $Z(N)$ is strictly bigger than $Z(G)$ (we thank the user YCor on mathoverflow.net for kindly pointing this out to us). In general if a C^* -algebra A has finite decomposition rank and an automorphism has finite Roklin dimension, one cannot deduce that the crossed product has finite

decomposition rank (for example, consider $\alpha \otimes \beta$ where α is a shift on a Cantor space and β is an irrational rotation of \mathbb{T}).

REFERENCES

- [1] Reinhold Baer. Finiteness properties of groups. *Duke Math. J.*, 15:1021–1032, 1948.
- [2] Gilbert Baumslag. *Lecture notes on nilpotent groups*. Regional Conference Series in Mathematics, No. 2. American Mathematical Society, Providence, R.I., 1971.
- [3] Florin P. Boca. The structure of higher-dimensional noncommutative tori and metric Diophantine approximation. *J. Reine Angew. Math.*, 492:179–219, 1997.
- [4] Lawrence G. Brown. Stable isomorphism of hereditary subalgebras of C^* -algebras. *Pacific J. Math.*, 71(2):335–348, 1977.
- [5] Nathaniel P. Brown and Narutaka Ozawa. *C^* -algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
- [6] A. L. Carey and W. Moran. Characters of nilpotent groups. *Math. Proc. Cambridge Philos. Soc.*, 96(1):123–137, 1984.
- [7] José R. Carrión. Classification of a class of crossed product C^* -algebras associated with residually finite groups. *J. Funct. Anal.*, 260(9):2815–2825, 2011.
- [8] José R. Carrión, Marius Dadarlat, and Caleb Eckhardt. On groups with quasidiagonal C^* -algebras. *J. Funct. Anal.*, 265(1):135–152, 2013.
- [9] Caleb Eckhardt. A note on strongly quasidiagonal groups. arXiv:1309.2205 [math.OA], to appear in *Journal of Operator Theory*, 2013.
- [10] Caleb Eckhardt. Quasidiagonal representations of nilpotent groups. *Adv. Math.*, 254:15–32, 2014.
- [11] Caleb Eckhardt, Craig Kleski, and Paul McKenney. K-theory associated with representations of unitriangular groups. In Preparation, 2014.
- [12] George A. Elliott and David E. Evans. The structure of the irrational rotation C^* -algebra. *Ann. of Math. (2)*, 138(3):477–501, 1993.
- [13] George A. Elliott and Andrew S. Toms. Regularity properties in the classification program for separable amenable C^* -algebras. *Bull. Amer. Math. Soc. (N.S.)*, 45(2):229–245, 2008.
- [14] Ilan Hirshberg, Wilhelm Winter, and Joachim Zacharias. Rokhlin dimension and C^* -dynamics. math.OA/1209.1618, 2012.
- [15] Roger E. Howe. On representations of discrete, finitely generated, torsion-free, nilpotent groups. *Pacific J. Math.*, 73(2):281–305, 1977.
- [16] S. A. Jennings. The group ring of a class of infinite nilpotent groups. *Canad. J. Math.*, 7:169–187, 1955.
- [17] Xinhui Jiang and Hongbing Su. On a simple unital projectionless C^* -algebra. *Amer. J. Math.*, 121(2):359–413, 1999.
- [18] Eberhard Kaniuth. Induced characters, Mackey analysis and primitive ideal spaces of nilpotent discrete groups. *J. Funct. Anal.*, 240(2):349–372, 2006.
- [19] Eberhard Kirchberg and Wilhelm Winter. Covering dimension and quasidiagonality. *Internat. J. Math.*, 15(1):63–85, 2004.
- [20] A. Kishimoto. Unbounded derivations in AT algebras. *J. Funct. Anal.*, 160(1):270–311, 1998.
- [21] Christopher Lance. On nuclear C^* -algebras. *J. Functional Analysis*, 12:157–176, 1973.
- [22] Huaxin Lin and Zhuang Niu. Lifting KK -elements, asymptotic unitary equivalence and classification of simple C^* -algebras. *Adv. Math.*, 219(5):1729–1769, 2008.
- [23] Qing Lin. Cut-down method in the inductive limit decomposition of non-commutative tori. III. A complete answer in 3-dimension. *Comm. Math. Phys.*, 179(3):555–575, 1996.
- [24] A. I. Mal'cev. Nilpotent torsion-free groups. *Izvestiya Akad. Nauk. SSSR. Ser. Mat.*, 13:201–212, 1949.

- [25] Hiroki Matui and Yasuhiko Sato. Strict comparison and \mathcal{Z} -absorption of nuclear C^* -algebras. *Acta Math.*, 209(1):179–196, 2012.
- [26] Hiroki Matui and Yasuhiko Sato. Decomposition rank of UHF-absorbing C^* -algebras. arXiv:1303.4371v2 [math.OA], 2013.
- [27] Hiroki Matui and Yasuhiko Sato. \mathcal{Z} -stability of crossed products by strongly outer actions II. arXiv:1205.1590v2 [math.OA], 2013.
- [28] Calvin C. Moore and Jonathan Rosenberg. Groups with T_1 primitive ideal spaces. *J. Functional Analysis*, 22(3):204–224, 1976.
- [29] Gerard J. Murphy. *C^* -algebras and operator theory*. Academic Press Inc., Boston, MA, 1990.
- [30] Judith A. Packer and Iain Raeburn. On the structure of twisted group C^* -algebras. *Trans. Amer. Math. Soc.*, 334(2):685–718, 1992.
- [31] N. Christopher Phillips. Every simple higher dimensional noncommutative torus is an AT algebra. arXiv:math/0609783 [math.OA], 2006.
- [32] M. Rørdam. Classification of nuclear, simple C^* -algebras. In *Classification of nuclear C^* -algebras. Entropy in operator algebras*, volume 126 of *Encyclopaedia Math. Sci.*, pages 1–145. Springer, Berlin, 2002.
- [33] Mikael Rørdam. The stable and the real rank of \mathcal{Z} -absorbing C^* -algebras. *Internat. J. Math.*, 15(10):1065–1084, 2004.
- [34] Jonathan Rosenberg. Appendix to: “Crossed products of UHF algebras by product type actions” [Duke Math. J. **46** (1979), no. 1, 1–23; MR 82a:46063 above] by O. Bratteli. *Duke Math. J.*, 46(1):25–26, 1979.
- [35] Jonathan Rosenberg and Claude Schochet. The Künneth theorem and the universal coefficient theorem for Kasparov’s generalized K -functor. *Duke Math. J.*, 55(2):431–474, 1987.
- [36] Daniel Segal. *Polycyclic groups*, volume 82 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1983.
- [37] Elmar Thoma. Über unitäre Darstellungen abzählbarer, diskreter Gruppen. *Math. Ann.*, 153:111–138, 1964.
- [38] Aaron Tikuisis and Wilhelm Winter. Decomposition rank of \mathcal{Z} -stable C^* -algebras. *Anal. PDE*, 7(3):673–700, 2014.
- [39] Wilhelm Winter. Nuclear dimension and \mathcal{Z} -stability of pure C^* -algebras. *Invent. Math.*, 187(2):259–342, 2012.
- [40] Wilhelm Winter and Joachim Zacharias. The nuclear dimension of C^* -algebras. *Adv. Math.*, 224(2):461–498, 2010.

DEPARTMENT OF MATHEMATICS, MIAMI UNIVERSITY, OXFORD, OH, 45056
E-mail address: eckharc@miamioh.edu

E-mail address: mckennp2@miamioh.edu