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Proper Isometric Actions of Thompson's Groups on Hilbert Space

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1 Introduction

In 1965, Thompson defined the groups F, T, and V [8]. Thompson's group V is the group of right-continuous bijections v of $[0,1]$ that map dyadic rational numbers to dyadic rational numbers, that are differentiable except at finitely many dyadic rational numbers, and such that, on each interval on which v is differentiable, v is affine with derivative a power of 2. The group F is the subgroup of V consisting of homeomorphisms. The group T is the subgroup of V consisting of those elements which induce homeomorphisms of the circle, where the circle is regarded as $[0,1]$ with 0 and 1 identified.

It is a long-standing open question to determine whether F is amenable. The main theorem of this paper establishes that the groups F, T, and V all have the weaker property of a-T-menability. A theorem of Higson and Kasparov [4] states that every a-T-menable group satisfies the Baum-Connes conjecture with arbitrary coefficients, so Thompson's groups F, T, and V satisfy the Baum-Connes conjecture as well.

An isometric action of a discrete group G on a metric space X is *proper* if, for any $x \in X$ and any bounded subset U of X, there are only finitely many elements of g that translate x inside U. A function $f : \mathcal{V}_1 \to \mathcal{V}_2$ between two vector spaces is *affine* if it is the composition of a linear map followed by a translation.

Definition 1.1. A discrete group G has the *Haagerup property* (or is *a-T-menable*, to use Gromov's term) if there exists a proper affine isometric action of G on a Hilbert space.

Every amenable group, for example, has the Haagerup property [1].

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Definition 1.2. A discrete group G has *property (T)* if every affine isometric action of G on any Hilbert space has a global fixed point.

By the main theorem, T and V do not have property (T). (It is clear that F does not have property (T) since $F/[F,F] \cong \mathbb{Z} \times \mathbb{Z}$, and thus F acts affinely and isometrically without a global fixed point on the Hilbert space $\mathbb{R}^2$.)

Some consequences of the main theorem were already known. The group F was known to act properly on a CAT(0) cubical complex [3], so this implies that F has the Haagerup property (see [6]). The fact that T does not have property (T) was first proved by Reznikov (see [7, page 527]). In 2002, Navas [5] proved that a group Γ of orientation-preserving diffeomorphisms of the circle of class C^{α}, $\alpha > 3/2$, has property (T) only if it is a finite cyclic group. By a result of Ghys and Sergiescu, quoted in [2], Thompson's group T is conjugate to a group of orientation-preserving C^{∞} diffeomorphisms of the circle, so Navas's theorem also implies that T does not have property (T).

2 Proof that V is a-T-menable

If G is a discrete group, $\mathcal{H}$ is a Hilbert space, and $\rho : G \to \text{Isom}(\mathcal{H})$ is a linear isometric representation, then one can produce an affine action on $\mathcal{H}$ having ρ as its linear part given a suitable assignment $\pi : G \to \mathcal{H}$ of translations to the elements of G. To determine which assignments are suitable, suppose that $\alpha : G \times \mathcal{H} \to \mathcal{H}$ is an action, where $\alpha(g)(v) = \rho(g)(v) + \pi(g)$. If $g_1, g_2 \in G$ and $v \in \mathcal{H}$, then

$$\begin{aligned} \rho(g_1 g_2)(v) + \pi(g_1 g_2) &= \rho(g_1)\big(\rho(g_2)(v) + \pi(g_2)\big) + \pi(g_1) \\ &= \rho(g_1 g_2)(v) + \rho(g_1)\pi(g_2) + \pi(g_1) \\ &\Longrightarrow \pi(g_1 g_2) = \rho(g_1)\pi(g_2) + \pi(g_1). \end{aligned} \tag{2.1}$$

Thus, π must be a 1-cocycle associated to ρ, or derivation. The action α will be proper if and only if π is proper, that is, if, for any $r > 0$, $\|\pi(g)\| < r$ for only finitely many $g \in G$.

Let $V_{[0,1/2)}$ be the subgroup of V that acts as the identity on $[0, 1/2)$. Let $X = \{gV_{[0,1/2)} : g \text{ is affine on } [0, 1/2)\}$. Define $\pi : V \to \ell^{\infty}(V/V_{[0,1/2)})$ by the formula $\pi(v) = (v - 1)\chi_X$, where χ_X denotes the characteristic function of X. It is easy to show that π satisfies the cocycle condition:

$$\begin{aligned} v_1\pi(v_2) + \pi(v_1) &= v_1(v_2 - 1)\chi_X + (v_1 - 1)\chi_X \\ &= (v_1 v_2 - 1)\chi_X \\ &= \pi(v_1 v_2). \end{aligned} \tag{2.2}$$

It follows that V is a-T-menable if (i) $\pi(v)$ lies in the Hilbert space $\ell^2(V/V_{[0,1/2)})$ and (ii) $\pi : V \to \ell^2(V/V_{[0,1/2)})$ is a proper map.

The proofs of (i) and (ii) require some background from [2]. A *standard dyadic interval* is an interval of the form $[a/2^k, (a+1)/2^k]$, where $k \geq 0, 0 \leq a \leq 2^k - 1$, and both k and a are integers; a standard dyadic interval may also be open or half open. A *standard dyadic partition* is a finite subset $\{x_0, \ldots, x_n\}$ of $[0,1]$ such that $0 = x_0 < x_1 < \cdots < x_n = 1$, and $[x_i, x_{i+1}]$ is a standard dyadic interval for $0 \leq i \leq n-1$. The *intervals of a partition* $0 = x_0 < \cdots < x_n = 1$ are the half-open intervals $[x_i, x_{i+1})$, where $0 \leq i \leq n-1$.

Proposition 2.1 (cf. [2, Lemma 2.2, page 220])**.** Let $g \in V$. There exists a standard dyadic partition $0 = x_0 < \cdots < x_n = 1$ such that g is affine on each interval of this partition and $\{g(x_0), \ldots, g(x_n)\}$ is a standard dyadic partition. □

Proof (following [2]). Choose a partition $\mathcal{P}$ of $[0,1]$ whose partition points are dyadic rational numbers such that f is affine on every interval of $\mathcal{P}$. Let $[a,b)$ be an interval of $\mathcal{P}$. Suppose that the derivative of f on $[a,b)$ is 2^{-k}. Let m be an integer such that $m \geq 0$, $m + k \geq 0$, $2^m a \in \mathbb{Z}$, $2^m b \in \mathbb{Z}$, $2^{m+k} f(a) \in \mathbb{Z}$, and $2^{m+k} f(b) \in \mathbb{Z}$. Then $a < a + 1/2^m < a + 2/2^m < a + 3/2^m < \cdots < b$ partitions $[a,b]$ into standard dyadic intervals, and $f(a) < f(a) + 1/2^{m+k} < f(a) + 2/2^{m+k} < f(a) + 3/2^{m+k} < \cdots < f(a) + (b-a)/2^k$ partitions the closure of $f([a,b))$ into standard dyadic intervals. Repeat this procedure on each interval of $\mathcal{P}$. ■

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two standard dyadic partitions of the unit interval and let ϕ be a bijection between the set of intervals of $\mathcal{P}_1$ and the set of intervals of $\mathcal{P}_2$. The triplet $T = (\mathcal{P}_1, \mathcal{P}_2, \phi)$, called a *representative triplet*, determines an element g_T of V in a natural way: g_T maps each interval I of the partition $\mathcal{P}_1$ onto the interval $\phi(I)$ by an affine homeomorphism. The previous proposition implies that every element of V is determined by a representative triplet.

Lemma 2.2 (cf. [2, remarks before Lemma 2.2, page 220])**.** Let S be a standard dyadic partition.

(1) If $(a/2^n, (a+1)/2^n) \cap S \neq \varnothing$, where $(a/2^n, (a+1)/2^n)$ is a standard dyadic interval, then $a/2^n, (a+1)/2^n \in S$.

(2) If $b/2^k \in S$ and b is odd, then $S \cap (0,1)$ contains at least k elements.

(3) If the standard dyadic interval $(c/2^l, (c+1)/2^l)$ contains an element of S, then $(2c+1)/2^{l+1} \in S$. □

Proof. (1) Suppose that $x_i = b/2^m$ is the smallest member of $(a/2^n, (a+1)/2^n) \cap S$, where $b/2^m$ is a reduced fraction, and let x_{i-1} be the element of the partition S immediately before x_i. Since S is a standard dyadic partition, then $[x_{i-1}, x_i] = [c/2^l, (c+1)/2^l]$ for some integers c and l. According to the assumptions, $c/2^l \le a/2^n < (c+1)/2^l$. It follows from this that either $c/2^l = a/2^n$ or $n > l$. Similarly, $a/2^n < b/2^m < (a+1)/2^n$ implies that $m > n$. If $m > l$, then $(c+1)2^{m-l}/2^m = b/2^m$, so $(c+1)2^{m-l} = b$. This is not possible since b is odd. The only remaining possibility is that $x_{i-1} = a/2^n$. One argues in a similar fashion that $(a+1)/2^n \in S$.

(2) The proof is by induction on k. The statement is obvious for $k = 1$, so assume that $k \ge 2$. If $b/2^k \in S$, where $b = 2c + 1$, then $c/2^{k-1}, (c+1)/2^{k-1} \in S$. It follows easily from part (1) that $S \backslash (c/2^{k-1}, (c+1)/2^{k-1})$ is a standard dyadic partition, and either c or $c+1$ must be odd. Thus $[S \backslash (c/2^{k-1}, (c+1)/2^{k-1})] \cap (0,1)$ contains at least $k-1$ elements, and therefore $S \cap (0,1)$ contains at least k elements.

(3) Choose some $x_j \in S \cap (c/2^l, (c+1)/2^l)$. If $x_j \neq (2c+1)/2^{l+1}$, then $x_j \in (c/2^l, (2c+1)/2^{l+1})$ or $((2c+1)/2^{l+1}, (c+1)/2^l)$. It follows from part (1) that $(2c+1)/2^{l+1} \in S$. ■

Proposition 2.3 (cf. [2, page 221])**.** Any $g \in V$ is determined by a unique representative triplet $T = (\mathcal{P}_1, \mathcal{P}_2, \phi)$ having the following property:

($*$) if $I = [a/2^k, (a+1)/2^k)$ is any standard dyadic interval, then the restriction of g to I is affine if and only if the interior of I contains no member of $\mathcal{P}_1$. □

Proof. Suppose that there is some interval $I = [a/2^n, (a+1)/2^n)$ such that g is affine on I and I contains some member of $\mathcal{P}_1$ in its interior. By Lemma 2.2(1), $a/2^n, (a+1)/2^n \in \mathcal{P}_1$, so the interval I is a union of intervals $I_1, \ldots, I_k$ of the partition $\mathcal{P}_1$ and therefore $g(I)$ is the union of the intervals $g(I_1), \ldots, g(I_k)$. Obtain a new standard dyadic partition $\mathcal{P}'_1$ by removing the points of $\mathcal{P}_1$ that lie in the interior of I, and obtain $\mathcal{P}'_2$ by removing the points of $\mathcal{P}_2$ inside $g(I)$. Let ϕ' be the bijection that sends I to $g(I)$ and agrees with ϕ on all intervals common to $\mathcal{P}_1$ and $\mathcal{P}'_1$. Call the process of replacing the triplet T by $T' = (\mathcal{P}'_1, \mathcal{P}'_2, \phi')$ a *reduction*.

After a sequence of at most finitely many reductions, one arrives at a representative triplet $\overline{T} = (\overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2, \overline{\phi})$ for g to which no more reductions may be applied. It follows that, whenever g is affine on an interval $I = [a/2^n, (a+1)/2^n)$, there is no partition point of $\overline{\mathcal{P}}_1$ in the interior of I. Conversely, it is clear that if I contains no member of $\overline{\mathcal{P}}_1$, g is affine on I.

It is not difficult to see that a representative triplet satisfying ($*$) must be unique. ■

Theorem 2.4. For any $v \in V$, $\pi(v) \in \ell^2(V/V_{[0,1/2)})$. The function $\pi : V \to \ell^2(V/V_{[0,1/2)})$ is a proper cocycle for the canonical left action of V on $\ell^2(V/V_{[0,1/2)})$. In particular, V (and therefore each of F and T) is a-T-menable. □

Proof. Let $\nu \in V$ and let $T = (\mathcal{P}_1, \mathcal{P}_2, \phi)$ be the representative triplet for ν satisfying Proposition 2.3(∗). For $i = 1, 2$, let $X_{\mathcal{P}_i} = \{gV_{[0,1/2)} : g$ maps $[0, 1/2)$ affinely and $g((0, 1/2))$ contains no element of $\mathcal{P}_i\}$.

If $gV_{[0,1/2)} \in X_{\mathcal{P}_1}$, then g maps $[0, 1/2)$ affinely and $g((0, 1/2))$ contains no member of the partition $\mathcal{P}_1$. It follows that νg also maps $[0, 1/2)$ affinely and $\nu g((0, 1/2))$ contains no member of $\mathcal{P}_2$, that is, $\nu gV_{[0,1/2)} \in X_{\mathcal{P}_2}$. Similarly, if $gV_{[0,1/2)} \in X_{\mathcal{P}_2}$, then $\nu^{-1} gV_{[0,1/2)} \in X_{\mathcal{P}_1}$. Thus, the bijection $\nu : V/V_{[0,1/2)} \to V/V_{[0,1/2)}$ induced by left multiplication by ν sends $X_{\mathcal{P}_1}$ bijectively onto $X_{\mathcal{P}_2}$. This implies that $\nu \cdot \chi_{X_{\mathcal{P}_1}} - \chi_{X_{\mathcal{P}_2}} = \chi_{\nu\cdot(X_{\mathcal{P}_1})} - \chi_{X_{\mathcal{P}_2}} = 0$.

Next consider the cocycle $\pi(\nu)$:

$$\begin{aligned}\pi(\nu) &= (\nu - 1) \cdot \chi_X \\ &= \left(\chi_{\nu\cdot(X - X_{\mathcal{P}_1})} + \chi_{\nu\cdot(X_{\mathcal{P}_1})}\right) - \left(\chi_{X - X_{\mathcal{P}_2}} + \chi_{X_{\mathcal{P}_2}}\right) \\ &= \chi_{\nu\cdot(X - X_{\mathcal{P}_1})} - \chi_{X - X_{\mathcal{P}_2}}.\end{aligned} \tag{2.3}$$

An element $gV_{[0,1/2)}$ of X is determined by the interval $g([0, 1/2))$. For $i = 1, 2$, define functions $f_i : X - X_{\mathcal{P}_i} \to \mathcal{P}_i - \{0, 1/2, 1\}$ by sending an element $gV_{[0,1/2)}$ of $X - X_{\mathcal{P}_i}$ to the midpoint of $g([0, 1/2))$. These functions are well defined, since $g([0, 1/2))$ contains some member of $\mathcal{P}_i$ in its interior, by the definition of $X - X_{\mathcal{P}_i}$, and therefore, by Lemma 2.2(3), the midpoint of $g([0, 1/2))$ is in $\mathcal{P}_i$; it is clear also that this midpoint is not 0, 1, or, by the bijectivity of g, 1/2. Moreover, each f_i is a bijection: the injectivity of f_i follows because a standard dyadic interval is determined by its midpoint and the surjectivity follows because every dyadic rational other than 0, 1/2, or 1 is the midpoint of some standard dyadic interval $I = [a/2^j, (a + 1)/2^j)$ which is the image of $h([0, 1/2))$, for some h which maps $[0, 1/2)$ affinely. This shows that $\pi(\nu)$ is finitely supported and, in particular, $\pi(\nu) \in \ell^2(V/V_{[0,1/2)})$.

Now if $gV_{[0,1/2)} \in X - X_{\mathcal{P}_1}$, then g maps $[0, 1/2)$ affinely and $g([0, 1/2))$ contains a member of $\mathcal{P}_1$ in its interior. It follows from (∗) that $\nu gV_{[0,1/2)} \notin X$, so $\nu \cdot (X - X_{\mathcal{P}_1}) \cap (X - X_{\mathcal{P}_2}) = \varnothing$. Thus

$$\begin{aligned}\|\pi(\nu)\|_2 &= \sqrt{|X - X_{\mathcal{P}_1}| + |X - X_{\mathcal{P}_2}|} \\ &= \sqrt{\left|\mathcal{P}_1 - \left\{0, \frac{1}{2}, 1\right\}\right| + \left|\mathcal{P}_2 - \left\{0, \frac{1}{2}, 1\right\}\right|}.\end{aligned} \tag{2.4}$$

Finally, a standard dyadic partition containing $k + 2$ or fewer elements must, by Lemma 2.2(2), consist entirely of dyadic rational numbers of the form $a/2^l$, where $l \le k$. This implies that there are only finitely many standard dyadic partitions of $[0, 1]$ of any given cardinality, which implies that, for any $r \in \mathbb{R}$, there are only finitely many representative triplets $(\mathcal{P}_1, \mathcal{P}_2, \phi)$ such that $\sqrt{|\mathcal{P}_1 - \{0, 1/2, 1\}| + |\mathcal{P}_2 - \{0, 1/2, 1\}|} < r$. It follows that $\pi : V \to \ell^2(V/V_{[0,1/2)})$ is proper. ■

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