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Actions of Picture Groups on CAT(0) Cubical Complexes

DANIEL S. FARLEY
Mathematics Department, 405 McAllister Building, Pennsylvania State University, State College, PA 16801, USA

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Abstract. A class of groups, called picture groups, is defined. Richard Thompson's groups F, T, and V are all picture groups. Each picture group is shown to act properly and isometrically on a CAT(0) cubical complex. In particular, all picture groups are a-T-menable.

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1. Introduction

Richard Thompson first defined the groups F, T, and V in 1965. Thompson's group V may be defined as the group of all right-continuous piecewise linear bijections v of $[0, 1]$ that map dyadic rational numbers to dyadic rational numbers, that are differentiable except at finitely many dyadic rational numbers and such that, on each interval on which v is differentiable, v is affine, and has a derivative which is a power of 2. Thompson's group T is the subgroup of V consisting of those elements which induce a homeomorphism of the space $[0, 1]$ with 0 and 1 identified. Thompson's group F is the subgroup of V consisting of homeomorphisms.

Guba and Sapir [10] have shown that Thompson's group F is a diagram group. They developed a substantial theory of diagram groups in [10–13]. In [10], they sketched a theory of annular and braided diagram groups, and showed that T is an annular diagram group, and V is a braided diagram group.

In [7], the author proved that every (ordinary) diagram group acts properly and freely by isometries on a CAT(0) cubical complex. This implies (see [2, 19], or Section 5) that every diagram group is a-T-menable. A group G is *a-T-menable* (or has the *Haagerup property*) if it acts properly, affinely and isometrically on a Hilbert space, where *proper* in this context means that for any $r > 0$, there are at most finitely many elements $g \in G$ such that $||g \cdot 0|| < r$. The property of a-T-menability is useful for a number of reasons [5]. One important reason was given by Higson and Kasparov [15], who showed that every discrete a-T-menable group satisfies the Baum–Connes conjecture with arbitrary coefficients.

It is thus worthwhile to extend the theory of [7] to cover annular and braided diagram groups. That is done in this paper, but here braided and annular diagram groups are called *picture groups* instead (see the end of Section 2).

MAIN THEOREM 1.1. *Every picture group acts properly by isometries on a CAT(0) cubical complex. In particular, every picture group is a-T-menable and satisfies the Baum–Connes conjecture with arbitrary coefficients.*

It is a long-standing open problem to determine whether the group F is amenable [4]. Partial results have been obtained by Paul Jolissaint in [16], where he showed that the factors associated with F and some of its subgroups are *McDuff factors*, meaning that the factor M is isomorphic to its tensor product $M \otimes R$ with the hyperfinite type II_1 factor R. Corollary 7.2 of [6] implies that, for every countable amenable group G in which every conjugacy class is infinite or trivial, the associated factor $L(G)$ is isomorphic to R. Thus Jolissaint's result proves that a weak form of amenability holds for F. In earlier work [17], Jolissaint showed that F is *inner amenable*, which means that there is a mean on $\ell^{\infty}(F - \{1\})$ which is invariant under conjugation.

Since a-T-menability is another weak form of amenability [1], it may be interesting to understand an explicit action of V (and thus F) on Hilbert space. A formula for such an action is produced for V in the final section (see also [8]). The method that is used in deriving the formula shows how to associate a 1-cocycle with any action ρ of a group G on a CAT(0) cubical complex.

2. Picture Groups

Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation. A picture consists of a frame $\partial([0,1]^2)$, a finite, possibly empty, set of *transistors*, which are each homeomorphic to $[0,1]^2$, and a finite, non-empty set of *wires*, each homeomorphic to $[0,1]$. The *top* of the frame or a transistor is $(0,1) \times \{1\}$ and the *bottom* is $(0,1) \times \{0\}$. Each wire is attached at both of its ends to the set consisting of the disjoint union of the frame and the transistors. The initial end of any given wire is attached to the bottom of a transistor or to the top of the frame, and the terminal end is attached to the top of a transistor or to the bottom of the frame. This attaching must be done so that no two wires have a point of intersection in the adjunction space. Define a relation $>$ on the set of transistors, where $T_1 > T_2$ if there is a wire w such that $w(0)$ is a point on the bottom of T_1 and $w(1)$ is a point on the top of the transistor T_2. Let $>$ also denote the transitive closure of this relation. The last condition one imposes on the attaching maps of the wires is that this transitive closure $>$ must be a strict partial order on transistors. If these conditions are all satisfied, then the adjunction space is called a *picture*.

The picture Π is a *picture over* $\mathcal{P}$ if each wire is labelled by an element of Σ, and these labellings satisfy certain conditions. Call the endpoints of a wire in Π

contacts. Each contact inherits a labelling from the wire it touches. Each transistor T thus inherits a top label and a bottom label; the *top label* of T is the word in Σ^+, the free semigroup generated by Σ, obtained by reading the labels on the top contacts of T from left to right. The *bottom label* of T is defined in the same way. (The top and bottom labels of the frame are defined similarly.) The picture Π is a picture over $\mathcal{P}$ if, for any transistor T of Π having, say, the word $w_1 \in \Sigma^+$ as its top label and $w_2 \in \Sigma^+$ as its bottom label, $w_1 = w_2$ or $w_2 = w_1$ is in $\mathcal{R}$. [Note: and not merely in the relation generated by $\mathcal{R}$.] A picture Π over $\mathcal{P}$ is a (w_1, w_2)-*picture* if w_1 is the top label of the frame and w_2 is the bottom label.

It will sometimes be convenient to let $\mathcal{T}_\Pi$ denote the set of all transistors in Π. If $\mathcal{T}_\Pi = \emptyset$, then Π is a *permutation picture*.

From now on, fix a semigroup presentation $\mathcal{P}$. 'Picture' will now mean 'picture over $\mathcal{P}$'.

Two pictures Π_1 and Π_2 are *isomorphic* if there is a label-preserving homeomorphism $h: \Pi_1 \to \Pi_2$ which maps the top of each transistor of Π_1 to the top of a transistor of Π_2, and, similarly, maps the bottom of each transistor of Π_1 to the bottom of a transistor of Π_2. The map h should also send the top and bottom of the frame for Π_1 to the top and bottom (respectively) of the frame for Π_2, and preserve the orientations on both the transistors and the frame, where the orientation in all cases is the natural one obtained by regarding each as a subset of the plane. One writes that $\Pi_1 \equiv \Pi_2$.

Figure 1(a) is an immersed (abc, abc)-picture Π over the semigroup presentation $\mathcal{P} = \langle a, b, c \mid ab = ba, bc = cb, ac = ca \rangle$. The only double points of the immersion occur at the crossings of the wires. The frame is dotted. All of the defining information of Π can easily be read from the image of such an immersion, including the orientations of the frame and transistors. Figure 1(b) shows the images of two immersed pictures over $\mathcal{P} = \langle x \mid x = x^2 \rangle$. The label of each wire is x. Note that these pictures are homeomorphic but not isomorphic, since any homeomorphism between the two must reverse the implied orientation on one of the transistors in the bottom picture. Figure 1(c) shows two nonexamples of pictures. The top image

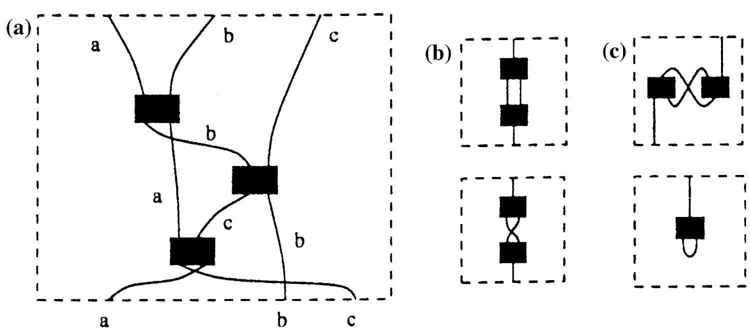

Figure 1.

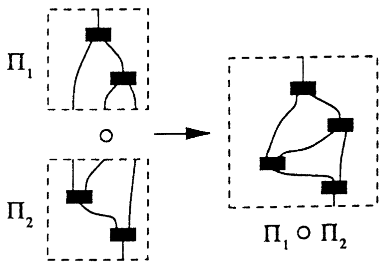

Figure 2.

violates the condition on the relation > and the bottom image has a wire connecting two points on the bottom of a transistor, which also is not allowed.

Suppose that Π_1 is a (w_1, w_2)-picture and Π_2 is a (w_2, w_3)-picture. Choose some orientation-preserving homeomorphism h from the bottom of the frame for Π_1 to the top of the frame for Π_2 which matches contacts. The concatenation $\Pi_1 \circ \Pi_2$ is formed by gluing Π_1 and Π_2 together by h, and then deleting the subset of the adjunction space corresponding to the bottom of the frame for Π_1 (excepting contacts).

Figure 2 illustrates the operation of concatenation in the case of two diagrams Π_1 and Π_2 over $\langle x \mid x = x^2 \rangle$.

Two transistors T_1 and T_2 form a *dipole* if the set of the bottom contacts of T_1 is joined *in order* by wires with the set of top contacts of T_2, and the top label of T_1 is equal to the bottom label of T_2 in the free semigroup Σ^+. Thus, for example, the two transistors in the top picture of Figure 1(b) form a dipole, but the transistors in the bottom picture of Figure 1(b) do not.

To *remove a dipole* one deletes the transistors T_1 and T_2 and all wires connecting them, and then glues in order the wires that formed top contacts of T_1 with those that formed bottom contacts of T_2. The result is still a picture. The inverse operation is called *inserting a dipole*. Two pictures Π_1 and Π_2 are *equal modulo dipoles* if one can be obtained from the other by repeatedly inserting and removing dipoles. One writes that $\Pi_1 = \Pi_2$.

Figure 3 illustrates the process of removing a dipole. The pictures here are over $\mathcal{P} = \langle a, b, c, d \mid cb = bc, ab = cd, ab = ba \rangle$. Note that the right-hand pair of transistors do not form a dipole, since the top label of the top transistor does not match the bottom label of the bottom transistor.

It will be convenient in what follows to use the language of rewrite systems. A *rewrite system* is a directed graph Γ. Write $a \to_\Gamma b$ if there is a directed edge starting at a and ending at b. Let $\dot{\to}_\Gamma$ denote the reflexive, transitive closure of the relation $\to_\Gamma$. Let $\dot{\leftrightarrow}_\Gamma$ denote the equivalence relation generated by $\to_\Gamma$. It is convenient to omit the Γ if doing so should cause no confusion. A rewrite system is called *terminating* if every sequence $a_1 \to a_2 \to \cdots \to a_n \to \cdots$ is finite. A rewrite system is *confluent* if, whenever $a \dot{\to} b$ and $a \dot{\to} c$, there is a vertex d such that $c \dot{\to} d$

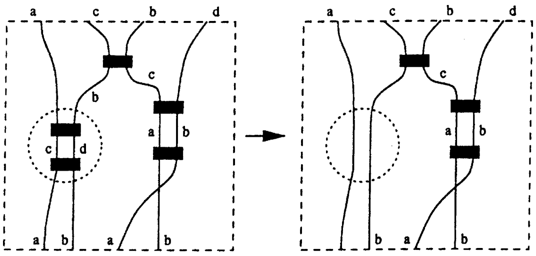

Figure 3.

and $b \dot{\to} d$. A rewrite system is *locally confluent* if, whenever $a \to b$ and $a \to c$, there is a vertex d such that $c \dot{\to} d$ and $b \dot{\to} d$.

LEMMA 2.1 ([18]). *Every terminating locally confluent rewrite system is confluent.*

A rewrite system is *complete* if it is both terminating and confluent. It is easy to argue that every equivalence class of $\dot{\leftrightarrow}$ in a complete rewrite system has a unique reduced element, i.e., a unique element which is not the initial vertex of any directed edge in Γ.

Let $\Gamma_b(\mathcal{P})$ be the rewrite system having the set of all isomorphism classes of pictures over $\mathcal{P}$ as its vertices, and having a directed edge $\Pi_1 \to \Pi_2$ if Π_2 is the result of removing a dipole in Π_1.

LEMMA 2.2. *For any semigroup presentation $\mathcal{P}$, the rewrite system $\Gamma_b(\mathcal{P})$ is complete.*

Proof. It is clear that $\Gamma_b(\mathcal{P})$ must be terminating, since each reduction of dipoles in Π decreases the number of transistors in Π by two, and there are only finitely many transistors in any picture.

It is thus sufficient, by the lemma, to show that $\Gamma_b(\mathcal{P})$ is locally confluent. Proving this amounts to showing that, given two dipoles D_1 and D_2 in some picture Π, the picture Π_1 obtained by removing the dipole D_1 followed by D_2 is isomorphic to the picture Π_2 obtained by removing the dipoles in the opposite order.

If the dipoles D_1 and D_2 have no transistors in common, then the operation of removing D_1 is disjoint from that of removing D_2, and thus these operations commute.

The only remaining case occurs when there are three transistors forming two dipoles. If this happens, it must be that the top and bottom transistors of the three have identical top and bottom labels, and the result of removing either of the two dipoles is the same (see Figure 4). □

If Π is any picture, let $r(\Pi)$ denote the unique reduced picture in its class modulo dipoles.

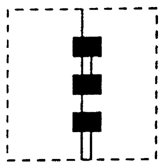

Figure 4.

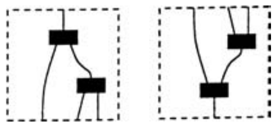

Figure 5.

The previous lemma shows that the operation of concatenation is well-defined even on classes modulo dipoles. The inverse of a given picture Π may be produced by reflecting Π across a horizontal line in the plane. See for example Figure 5, which portrays a (x, x^3)-picture over $\mathcal{P} = \langle x \mid x = x^2 \rangle$ and its inverse. One can therefore make the

DEFINITION 2.3. If $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ is any semigroup presentation and w is a word in Σ^+, then $\mathcal{D}_b(\mathcal{P}, w)$ is the group of isomorphism classes of (w, w)-pictures over $\mathcal{P}$ modulo dipoles, with the operation of concatenation.

The groups $\mathcal{D}_b(\mathcal{P}, w)$ are the same as the braided diagram groups defined in [10]. One can also consider the class of annular diagram groups, which were also defined in [10]. In terms of the formalism of this section, these may be defined as follows. Given a picture Π, identify the subsets $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$ of the frame by the equivalence relation $(0, t) \sim (1, t)$. Now delete the image of the subset $\{0\} \times (0, 1)$ in the adjunction space. The result $\overline{\Pi}$ resembles a picture, except that the frame has been replaced by two disjoint circles. See Figure 6 below. If the new space $\overline{\Pi}$ can be embedded in the plane in an orientation-preserving fashion, call the original picture Π *annular*. The annular (w, w)-pictures over $\mathcal{P}$ form a group denoted $\mathcal{D}_a(\mathcal{P}, w)$. When $\mathcal{P} = \langle x \mid x = x^2 \rangle$ and $w = x$, $\mathcal{D}_a(\mathcal{P}, w)$ is Thompson's group T (see Section 6 for a description of the isomorphism).

One can also define *ordinary pictures* to be the pictures Π which can be embedded in the plane. The ordinary (w, w)-pictures over $\mathcal{P}$ form a group $\mathcal{D}(\mathcal{P}, w)$; these groups are the diagram groups considered in [7, 10–13]. Thompson's group F is $\mathcal{D}(\mathcal{P}, w)$, where $\mathcal{P} = \langle x \mid x = x^2 \rangle$ and $w = x$.

All of the results in this paper are proved for the groups $\mathcal{D}_b(\mathcal{P}, w)$, but the proofs easily generalize to cover the groups $\mathcal{D}_a(\mathcal{P}, w)$ and $\mathcal{D}(\mathcal{P}, w)$. The changes amount just to using annular pictures or ordinary pictures (respectively) in place of pictures throughout the paper.

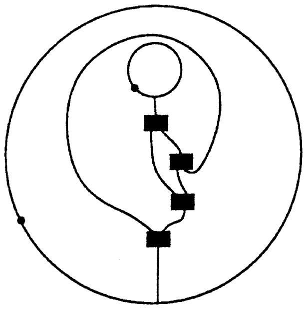

Figure 6.

3. Technicalities

3.1. A PARTIAL ORDER ON VERTICES

If Π_1 and Π_2 are reduced pictures, and there is some permutation picture Ψ such that $\Pi_1 \circ \Psi \equiv \Pi_2$ then write $\Pi_1 \approx \Pi_2$. The relation $\approx$ is an equivalence relation on reduced pictures. The equivalence classes, denoted in brackets, are called *formal vertices*, or simply *vertices* when this can cause no confusion. Write $[\Pi_1] \leqslant [\Pi_2]$ if there is some picture Δ such that $\Pi_1 \circ \Delta \equiv \Pi_2$. Note that this means $\Pi_1 \circ \Delta$ and Π_2 are isomorphic, and not merely equivalent modulo dipoles. The relation $\leqslant$ is a well-defined partial order on vertices.

Let Π be a reduced picture. An arc $c:[0,1] \to \Pi$ is *monotonic* if $c(0)$ is a member of the top of Π, the intersection of $c([0,1])$ with the sides of the frame is empty, and, whenever a wire w is contained in $c([0,1])$, say $c^{-1}(w) = [t_i, t_j]$, $c_{|[t_i,t_j]}: [t_i, t_j] \to w$ is orientation-preserving, i.e., $c(t_i)$ is the top of w and $c(t_j)$ is the bottom. Suppose $c:[0,1] \to \Pi$ is a monotonic arc in Π and $(t_1, u_1) \cup (t_2, u_2) \cup \ldots \cup (t_n, u_n) = c^{-1}(\coprod \mathrm{int}(w))$, where $t_1 < u_1 < t_2 < \cdots < t_n < u_n$ and the disjoint union is over the interiors of all wires. The *invariant of* c, denoted $\mathrm{Inv}(c)$, is the n-tuple $(m_1, m_2, \ldots, m_n) \in \mathbf{Z}^n$, where $c(t_1)$ is the m_1^{th} contact from the left on the top of the frame, and, for $2 \leqslant k \leqslant n$, $c(t_k)$ is the m_k^{th} contact from the left on the bottom of its transistor. For example, the invariant of the bold arc in Figure 7 is $(2, 1, 2)$. If Π is a picture, and T is a transistor of Π, then the *invariant of* T, denoted $\mathrm{Inv}(T)$, is $\{\mathrm{Inv}(c) \mid c \text{ is a monotonic arc and } c(1) \in T\}$.

SUBLEMMA 3.1. *If $\phi: \Pi_1 \to \Pi_2$ is an isomorphism of reduced pictures and T is a transistor of Π_1, then $\mathrm{Inv}_{\Pi_1}(T) = \mathrm{Inv}_{\Pi_2}(\phi(T))$. If T_1, T_2 are transistors of Π and $\mathrm{Inv}_\Pi(T_1) \cap \mathrm{Inv}_\Pi(T_2) \neq \emptyset$, then $T_1 = T_2$.*

Proof. This follows easily from the definition of isomorphism of diagrams. □

Let Π be a picture and let $\mathcal{T}_\Pi$ be the set of its transistors. A set $\mathcal{S} \subseteq \mathcal{T}_\Pi$ is an *initial subset of* $\mathcal{T}_\Pi$ if whenever $T_1 \geqslant T_2$ and $T_2 \in \mathcal{S}$, $T_1 \in \mathcal{S}$.

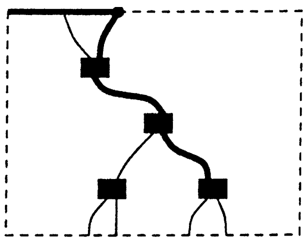

Figure 7.

LEMMA 3.2. *Let* Π *be a picture, and let* $\mathcal{T}_\Pi$ *be its set of transistors.*

(1) *There is a one-to-one correspondence* $F:\{\mathcal{S} \mid \mathcal{S}$ *is an initial subset of* $\mathcal{T}_\Pi\} \to \{[\Pi_1] \mid [\Pi_1] \leqslant [\Pi]\}$ *which is order-preserving and has an order-preserving inverse, i.e., if* $\mathcal{S}_1, \mathcal{S}_2$ *are initial subsets of* $\mathcal{T}_\Pi$*, then* $\mathcal{S}_1 \subseteq \mathcal{S}_2$ *if and only if* $F(\mathcal{S}_1) \leqslant F(\mathcal{S}_2)$ *in the partial order on formal vertices.*

(2) *If* Π_1*,* Π_2 *are pictures having the same top label, then the formal vertices* $[\Pi_1]$*,* $[\Pi_2]$ *have a greatest lower bound. If the vertices* $[\Pi_1]$*,* $[\Pi_2]$ *have an upper bound, then they have a least upper bound.*

Proof. (1) Let $\mathcal{S}$ be an initial subset of $\mathcal{T}_\Pi$. If w is a wire connecting either the top of Π or a transistor in $\mathcal{S}$ to either the bottom of Π or a transistor in $\mathcal{T}_\Pi - \mathcal{S}$, then introduce a vertex v_w at its midpoint. Next, introduce a line segment $\ell = [0, 1]$; glue 0 to the frame at $(0, \frac{1}{2})$, 1 to the frame at $(1, \frac{1}{2})$, and, for each v_w as above, choose a real number $r(v_w) \in (0, 1) \subseteq \ell$ in such a way that the function r is injective, and glue each v_w to ℓ by r. The resulting adjunction space $(\Pi \coprod \ell)/\sim$ contains two pictures as subsets: The first, denoted $\Pi_{\mathcal{S},r}$, has $(\{0\} \times [\frac{1}{2}, 1]) \cup ([0, 1] \times \{1\}) \cup (\{1\} \times [\frac{1}{2}, 1]) \cup \ell$ as its frame, and also consists of all transistors in the set $\mathcal{S}$, all wires connecting either the top of Π or a transistor of $\mathcal{S}$ to either ℓ or a transistor of $\mathcal{S}$. The second picture Δ consists of all remaining wires and transistors, together with the lower half of the original frame and ℓ.

Note that $\Pi \equiv \Pi_{\mathcal{S},r} \circ \Delta$, and the choice of r involved in the definition of $\Pi_{\mathcal{S},r}$ does not affect $[\Pi_{\mathcal{S},r}]$. Let $F(\mathcal{S}) = [\Pi_{\mathcal{S},r}]$. Clearly, $[\Pi_{\mathcal{S},r}] \leqslant [\Pi]$.

Now suppose that $\mathcal{S}_1, \mathcal{S}_2$ are initial subsets of $\mathcal{T}_\Pi$ and $F(\mathcal{S}_1) = F(\mathcal{S}_2)$. One can choose representatives $\Pi_{\mathcal{S}_1,r_1} \in F(\mathcal{S}_1)$ and $\Pi_{\mathcal{S}_2,r_2} \in F(\mathcal{S}_2)$ so that $\Pi_{\mathcal{S}_1,r_1} \equiv \Pi_{\mathcal{S}_2,r_2}$. Let $\mathrm{Inv}(\Pi_{\mathcal{S}_1,r_1}) = \{\mathrm{Inv}(T) \mid T \in \mathcal{T}_{\Pi_{\mathcal{S}_1,r_1}}\}$, and $\mathrm{Inv}(\Pi_{\mathcal{S}_2,r_2}) = \{\mathrm{Inv}(T) \mid T \in \mathcal{T}_{\Pi_{\mathcal{S}_2,r_2}}\}$. By the construction of $\Pi_{\mathcal{S}_i,r_i}$, $\mathrm{Inv}(\Pi_{\mathcal{S}_i,r_i}) = \{\mathrm{Inv}(T) \mid T \in \mathcal{S}_i\}$. Since $\Pi_{\mathcal{S}_1,r_1} \equiv \Pi_{\mathcal{S}_2,r_2}$, the sublemma implies $\{\mathrm{Inv}(T) \mid T \in \mathcal{S}_1\} = \{\mathrm{Inv}(T) \mid T \in \mathcal{S}_2\}$, and this implies that $\mathcal{S}_1 = \mathcal{S}_2$. It follows that F is one-to-one.

Suppose that $[\Pi_1] \leqslant [\Pi]$. This implies that $\Pi \equiv \Pi_1 \circ \Delta$ for some picture Δ. Consider the identification space $\Pi_1 \coprod \Delta/\sim$, where $\sim$ is the equivalence relation generated by a homeomorphism h which maps the top of Δ to the bottom of Π_1 and matches contacts. Thus $(\Pi_1 \circ \Delta) \cup \ell = \Pi_1 \coprod \Delta/\sim$ where ℓ forms the bottom of the frame for Π_1 and the top of the frame for Δ. If $\phi: \Pi_1 \circ \Delta \to \Pi$ is an equivalence, consider the adjunction space $\Pi \coprod_\phi (\Pi_1 \coprod \Delta/\sim)$. The image of Π_1 in this quotient exhibits $[\Pi_1]$ as $F(\mathcal{S})$, where $\mathcal{S}$ is the (necessarily initial) subset of transistors contained in the image of Π_1 under the quotient map. It follows that F is onto.

It is clear that F preserves order: If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ are initial subsets of $\mathcal{T}_\Pi$, then $\mathcal{S}_1$ is an initial subset of $\mathcal{T}_{\Pi_{\mathcal{S}_2,r_2}}$, so $[\Pi_{\mathcal{S}_1,r_1}] \leqslant [\Pi_{\mathcal{S}_2,r_2}]$; that is, $F(\mathcal{S}_1) \leqslant F(\mathcal{S}_2)$. The fact that F^{-1} is order-preserving follows easily from the sublemma.

(2) The two vertices $[\Pi_1]$, $[\Pi_2]$ have the lower bound $[\Psi]$, where Ψ is a permutation picture having the same top label as that of Π_1 and Π_2. Suppose that $[\Delta_1]$, $[\Delta_2]$ each are less than $[\Pi_1]$ and $[\Pi_2]$. This implies that there are isomorphisms $\phi_1 : \Delta_1 \circ \Theta_1 \to \Pi_1$, $\phi_2 : \Delta_1 \circ \Theta_2 \to \Pi_2$, $\phi_3 : \Delta_2 \circ \Theta_3 \to \Pi_1$, and $\phi_4 : \Delta_2 \circ \Theta_4 \to \Pi_2$ for appropriate pictures Θ_1, Θ_2, Θ_3, and Θ_4. Let $\phi_i(\Delta_j)$ denote the image of Δ_j-(the bottom of the frame for Δ_j) under ϕ_i. There are maps $\phi_2 \circ \phi_1^{-1} : \phi_1(\Delta_1) \to \phi_2(\Delta_1)$ and $\phi_4 \circ \phi_3^{-1} : \phi_3(\Delta_2) \to \phi_4(\Delta_2)$ which behave as equivalences: they are homeomorphisms which preserve the orders of contacts and the orientations on transistors.

From this and the sublemma it follows that $\phi_2 \circ \phi_1^{-1}$ and $\phi_4 \circ \phi_3^{-1}$ agree on $\phi_1(\Delta_1) \cap \phi_3(\Delta_2)$ up to isotopy, so, after altering the maps within their isotopy classes if necessary, there is an induced map $\Phi : \phi_1(\Delta_1) \cup \phi_3(\Delta_2) \to \phi_2(\Delta_1) \cup \phi_4(\Delta_2)$.

The union $\phi_1(\Delta_1) \cup \phi_3(\Delta_2)$ corresponds to a formal vertex, namely $\Pi_{\mathcal{S}_1}$, where $\mathcal{S}_1 = \{T \in \mathcal{T}_{\Pi_1} \mid T \subseteq \phi_1(\Delta_1) \cup \phi_3(\Delta_2)\}$ and $\phi_2(\Delta_1) \cup \phi_4(\Delta_2)$ similarly corresponds to a vertex $\Pi_{\mathcal{S}_2}$, where $\mathcal{S}_2 = \{t \in \mathcal{T}_{\Pi_2} \mid T \subseteq \phi_2(\Delta_1) \cup \phi_4(\Delta_2)\}$. The existence of the map Φ shows that $[\Pi_{\mathcal{S}_1}] = [\Pi_{\mathcal{S}_2}]$. Moreover $[\Pi_{\mathcal{S}_1}] \leqslant [\Pi_1]$, $[\Pi_{\mathcal{S}_2}] \leqslant [\Pi_2]$, $[\Delta_1] \leqslant [\Pi_{\mathcal{S}_1}]$, $[\Delta_2] \leqslant [\Pi_{\mathcal{S}_2}]$.

Thus the set $L = \{[\Delta] \mid [\Delta]$ is a lower bound of $\{[\Pi_1], [\Pi_2]\}\}$ contains an upper bound of any of its two-element subsets. Since L is finite, it must therefore contain an element that is larger than all of its other members. This proves the first part of (2).

Now suppose that two vertices $[\Pi_1]$, $[\Pi_2]$ have an upper bound $[\Pi]$. Let $U = \{[\Delta] \mid [\Delta]$ is an upper bound of $\{[\Pi_1], [\Pi_2]\}$ and $[\Delta] \leqslant [\Pi]\}$. The set U is finite, so it contains a minimal element $[\overline{\Pi}]$. If $[\Pi']$ is another upper bound of $\{[\Pi_1], [\Pi_2]\}$ (not necessarily one in U) then $[\overline{\Pi}]$ and $[\Pi']$ have a greatest lower bound $[\widetilde{\Pi}]$, which is necessarily greater than $[\Pi_1]$ and $[\Pi_2]$. Therefore $[\widetilde{\Pi}] \leqslant [\overline{\Pi}] \leqslant [\Pi]$, so $[\widetilde{\Pi}] \in U$, and $[\widetilde{\Pi}] = [\overline{\Pi}]$ by the minimality of $[\overline{\Pi}]$. It follows that $[\overline{\Pi}] \leqslant [\Pi']$, which makes $[\overline{\Pi}]$ the least upper bound of $\{[\Pi_1], [\Pi_2]\}$ since $[\Pi']$ was arbitrary. □

An interval $[[\Pi_1], [\Pi_2]]$ is *elementary* if $[\Pi_1]$ is $F(\mathcal{S})$ for some initial subset $\mathcal{S}$ of $\mathcal{T}_{\Pi_2}$, the set of transistors of Π_2, where $\mathcal{S}$ has the property that no two elements of $\mathcal{T}_{\Pi_2} - \mathcal{S}$ are comparable in the partial order on transistors.

LEMMA 3.3. *For any formal vertex* $[\Pi_2]$, *there is some formal vertex* $[\Pi_1]$ *such that* $[[\Pi_1], [\Pi_2]]$ *is an elementary interval and, for any other elementary interval of the form* $[[\Pi], [\Pi_2]]$, $[[\Pi], [\Pi_2]] \subseteq [[\Pi_1], [\Pi_2]]$.

Proof. Let $\mathcal{T}_{\Pi_1} = \{T \in \mathcal{T}_{\Pi_2} \mid T$ is not minimal in the partial order on the transistors$\}$; let $[\Pi_1]$ be $F(\mathcal{T}_{\Pi_1})$, where F is as defined in (1) of the previous lemma.

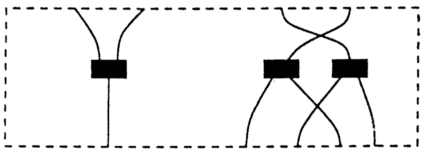

Figure 8.

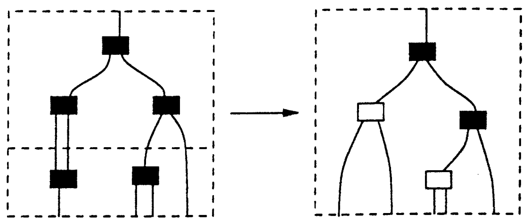

Figure 9.

Suppose that $[[\Pi], [\Pi_2]]$ is an elementary interval, and let $\mathcal{T}_\Pi$ be the set of transistors of some representative Π of $[\Pi]$. One needs to show that $\mathcal{T}_{\Pi_1} \subseteq \mathcal{T}_\Pi$. If $T \in \mathcal{T}_{\Pi_1} - \mathcal{T}_\Pi$, then T is not minimal in $\mathcal{T}_{\Pi_2}$, so there is $\overline{T} \in \mathcal{T}_{\Pi_2}$ such that $T > \overline{T}$ in the partial order on transistors. Since $\mathcal{T}_\Pi$ is an initial subset of $\mathcal{T}_{\Pi_2}$ and $T \notin \mathcal{T}_\Pi$, $\overline{T}$ is not in $\mathcal{T}_\Pi$ either. But this implies that there are two comparable elements of $\mathcal{T}_{\Pi_2} - \mathcal{T}_\Pi$, which contradicts the assumption that $[[\Pi], [\Pi_2]]$ is elementary. □

3.2. THIN PICTURES AND ELEMENTARY INTERVALS

Call a picture *thin* if no two of its transistors are comparable in the partial order on transistors. Figure 7 depicts the immersed image of a thin picture over the presentation $\langle x \,|\, x = x^2 \rangle$. Let $\mathcal{C}(\Pi, \Delta) = \{[r(\Pi \circ \Delta')] \,|\, [\Delta'] \leqslant [\Delta]\}$.

LEMMA 3.4 (The Elementary Interval Lemma). (i) *Let* Π *be a* (w_1, w_2)*-picture and let* Δ *be a thin* (w_2, w_3)*-picture. The set* $\mathcal{C}(\Pi, \Delta)$ *is an elementary interval* $[[\Pi_1], [\Pi_2]]$ *for some formal vertices* $[\Pi_1]$, $[\Pi_2]$.

(ii) *If* $[[\Pi_1], [\Pi_2]]$ *is an elementary interval and* $[\Pi] \in [[\Pi_1], [\Pi_2]]$, *then there is a thin picture* Δ *such that* $\mathcal{C}(\Pi, \Delta) = [[\Pi_1], [\Pi_2]]$.

Proof. (i) Let $\mathcal{T}_{\Pi_1} \subseteq \mathcal{T}_\Pi$ consist of those transistors that do not form dipoles with transistors of Δ in $\Pi \circ \Delta$. Let $\mathcal{T}_{\Pi_2} \subseteq \mathcal{T}_{\Pi \circ \Delta}$ consist of all transistors of $\mathcal{T}_{\Pi \circ \Delta}$ except those that form the bottom half of a dipole. It is not too difficult to see that $\mathcal{C}(\Pi, \Delta) = [[\Pi_1], [\Pi_2]]$.

Figure 9 illustrates the correspondence between the sets $\mathcal{C}(\Pi, \Delta)$ and elementary intervals in a simple case. The left half of the figure portrays the concatenation of Π and Δ (the dotted line in the center is retained for the sake of clarity). The right half is intended to represent the corresponding elementary interval $[[\Pi_1], [\Pi_2]]$. Here $[\Pi_1]$ is the vertex corresponding to the set of dark transistors and $[\Pi_2]$ is the vertex corresponding to the set of all of the transistors.

(ii) Suppose $[\Pi] \in [[\Pi_1], [\Pi_2]]$. Choose some representative Π_2 of $[\Pi_2]$. Since $[\Pi] \leqslant [\Pi_2]$ and $[[\Pi_1], [\Pi_2]]$ is elementary, $\Pi_2 \equiv \Pi \circ \Delta_1$, for some $\Pi \in [\Pi]$ and thin picture Δ_1, where the concatenation is reduced. Build a thin picture Δ_2 such that, in $\Pi_2 \circ \Delta_2$, every $T \in \mathcal{T}_{\Pi} - \mathcal{T}_{\Pi_1}$ forms a dipole with a transistor in Δ_2, and every Δ_2 is half of such a dipole. There is only one choice for Δ_2 up to right-multiplication by a permutation picture. Note that $\Delta_1 \circ \Delta_2$ is a thin picture, since the effect of right-multiplying Π_2 by Δ_2 is to attach a cancelling transistor to each transistor of $\mathcal{T}_{\Pi} - \mathcal{T}_{\Pi_1}$, and $(\mathcal{T}_{\Pi} - \mathcal{T}_{\Pi_1}) \cap (\mathcal{T}_{\Pi_2} - \mathcal{T}_{\Pi}) = \emptyset$.

It now follows by the construction in the proof of (i) that $\mathcal{C}(\Pi, \Delta_1 \circ \Delta_2)) = [[\Pi_1], [\Pi_2]]$. □

4. Cubical Complex

The following definition is taken from [7].

DEFINITION 4.1. The *standard abstract n-cube* is the set $\{0, 1\}^n$. For the present purposes, let the standard abstract 0-cube $\{0, 1\}^0$ be equal to the set $\{0\}$. A *face* of the standard abstract n-cube is a product $A_1 \times \cdots \times A_n$, where each A_n is a nonempty subset of $\{0, 1\}$. The standard abstract 0-cube is considered to be a face of itself.

An *abstract cubical complex* $K = (\mathcal{V}, \mathcal{C})$ consists of a nonempty set $\mathcal{V}$ of vertices and a set $\mathcal{C}$ of nonempty subsets of $\mathcal{V}$, called *cubes*, satisfying:

(i) $\mathcal{C}$ is a cover of $\mathcal{V}$.
(ii) For any $C_1, C_2 \in \mathcal{C}$, $C_1 \cap C_2 \in \mathcal{C}$ or $C_1 \cap C_2 = \emptyset$.
(iii) For any $C \in \mathcal{C}$ there is a bijection $\phi_C : C \to \{0, 1\}^n$, for some n, satisfying:
(∗) if $C_1 \subseteq C$, then $C_1 \in \mathcal{C}$ if and only if $\phi_C(C_1)$ is a face of $\{0, 1\}^n$.

The *first derived* K' of an abstract cubical complex $K = (\mathcal{V}, \mathcal{C})$ is the abstract simplicial complex having the set $\mathcal{C}$ as its vertices and the set of all finite nonempty ascending chains of cubes of K as simplices. The *realization* $|K|$ of an abstract cubical complex $K = (\mathcal{V}, \mathcal{C})$ is the realization of the abstract simplicial complex K' in the sense of [21]. An *automorphism* of an abstract cubical complex $K = (\mathcal{V}, \mathcal{C})$ is a bijection $\rho : \mathcal{V} \to \mathcal{V}$ which induces a bijection of $\mathcal{C}$.

The realization of an abstract cubical complex K has a natural cubical complex structure [7] in the sense of [3]. Every cubical complex admits a natural metric, called the *intrinsic metric* (see [3]). An automorphism of K induces an isometry of $|K|$ with respect to this metric [7].

DEFINITION 4.2. Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation, and let w be a word in Σ^+. Define $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)_{ab}$ to be the pair $(\mathcal{V}, \mathcal{C})$, where $\mathcal{V}$ is the set of all formal vertices over $\mathcal{P}$ having w as a top label and $\mathcal{C}$ is the set of all elementary intervals which are subsets of $\mathcal{V}$

THEOREM 4.3. *$\widetilde{\mathrm{K}}_b(\mathcal{P}, w)_{ab}$ is an abstract cubical complex.*

Proof. (i) is clear.

(ii) Suppose C_1, and C_2 are cubes; assume that $C_1 \cap C_2 \neq \emptyset$. Let $C_1 = [[\Pi_1], [\Pi_2]]$ and $C_2 = [[\Pi_3], [\Pi_4]]$. If $[\Delta] \in C_1 \cap C_2$, then $[\Delta] \geqslant [\Pi_1]$ and $[\Delta] \geqslant [\Pi_3]$, so $[\Delta] \geqslant \mathrm{lub}\{[\Pi_1], [\Pi_3]\}$. Similar reasoning shows that $[\Delta] \leqslant \mathrm{glb}\{[\Pi_2], [\Pi_4]\}$, so $[\mathrm{lub}\{[\Pi_1], [\Pi_2]\}, \mathrm{glb}\{[\Pi_2, \Pi_4]\}] \supseteq C_1 \cap C_2$. The converse inclusion is clear. The interval $[\mathrm{lub}\{[\Pi_1], [\Pi_3]\}, \mathrm{glb}\{[\Pi_2], [\Pi_4]\}]$ is a cube since subintervals of elementary intervals are elementary.

(iii) Suppose $I = [[\Pi_1], [\Pi_2]] \in \mathcal{C}$ and $[\Pi_1]$ is the vertex corresponding to the initial subset $\mathcal{S}$ of $\mathcal{T}_{\Pi_2}$. Since I is an elementary interval, $\mathcal{T}_{\Pi_2} - \mathcal{S}$ has the property that no two of its members are comparable in the partial order on transistors. Number the elements of $\mathcal{T}_{\Pi_2} - \mathcal{S} = \{T_1, \ldots, T_n\}$. Now, since no two of the elements of $\mathcal{T}_{\Pi_2} - \mathcal{S}$ are comparable, $\mathcal{S} \cup \mathcal{S}'$ is an initial subset of $\mathcal{T}_{\Pi_2}$, for any $\mathcal{S}' \subseteq \mathcal{T}_{\Pi_2} - \mathcal{S}$. Thus there is a one-to-one correspondence $\phi : \mathcal{P}(\mathcal{T}_{\Pi_2} - \mathcal{S}) \to I$, where $\phi(\mathcal{S}') = F(\mathcal{S} \cup \mathcal{S}')$. There is, moreover, a one-to-one correspondence $\psi : \{0, 1\}^n \to \mathcal{P}(\mathcal{T}_{\Pi_2} - \mathcal{S})$ which sends $(x_1, \ldots, x_n)$ to $\{T_i \mid x_i = 1\}$. The composition $\phi \circ \psi : \{0, 1\}^n \to I$ is an isomorphism of partially ordered sets, where $\{0, 1\}^n$ is given the order in which $(a_1, \ldots, a_n) \leqslant (b_1, \ldots, b_n)$ if and only if $a_i \leqslant b_i$ for all i. This directly implies that faces of $\{0, 1\}^n$ are mapped to elements of $\mathcal{C}$. □

Figure 10 shows a 2-cell in $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$, where $\mathcal{P} = \langle x \mid x = x^2 \rangle$ and $w = x$. Note that there are no bottom contacts in any of the pictures. This is because any bottom contacts would have no significance; a vertex is an equivalence class of pictures modulo right multiplication by a permutation picture.

Let $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$ denote the realization of $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)_{ab}$.

THEOREM 4.4. *$\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$ is contractible with respect to the weak topology.*

Proof. Let $f : S^n \to \widetilde{\mathrm{K}}_b(\mathcal{P}, w)$ be an arbitrary continuous map. Let L_f be the carrier of $f(S^n)$, i.e., the smallest subcomplex of $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$ which contains $f(S^n)$. Let $X_f^0 = \{[\Pi_1] \in \mathcal{V} \mid [\Pi_1] \leqslant [\Pi] \text{ for some vertex } [\Pi] \in L_f\}$. Let X_f be the largest subcomplex of $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$ having X_f^0 as its 0-skeleton. Note that X_f is a finite complex, by 32.

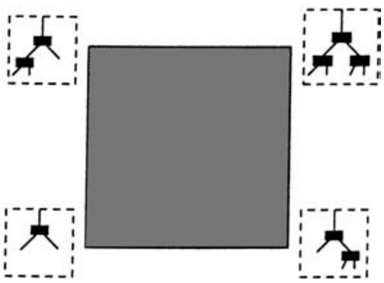

Figure 10.

Let $[\Pi_{\max}]$ be a maximal vertex in X_f. Let Y_f be the largest subcomplex of $\widetilde{K}_b(\mathcal{P}, w)$ having $X_f^0 - \{[\Pi_{\max}]\}$ as its 0-skeleton. Let $[\Pi']$ be such that $[[\Pi'], [\Pi_{\max}]]$ has the property described in 33. Let $|C'|$ denote the cube in X_f corresponding to the elementary interval $[[\Pi'], [\Pi_{\max}]]$. Let $|C|$ be some cube in X_f, corresponding to some elementary interval $[[\Pi_1], [\Pi_2]]$. If $[\Pi_2] \neq [\Pi_{\max}]$, then $|C| \subseteq Y_f$. If $[\Pi_2] = [\Pi_{\max}]$, then $[[\Pi_1], [\Pi_2]] \subseteq [[\Pi'], [\Pi_{\max}]]$ so $|C| \subseteq |C'|$.

Finally note that $Y_f \cap |C'|$ is the union of subfaces of $|C'|$ that do not contain $[\Pi_{\max}]$. As such, $Y_f \cap |C'|$ is topologically a ball of dimension $n-1$, where the dimension of $|C'|$ is n. Now X_f is topologically $Y_f \cup |C'|$ by the previous paragraph, and $Y_f \cap |C'| = B^{n-1}$, so that X_f collapses onto Y_f.

An argument by induction on the number of vertices shows that X_f is collapsible, so f is null homotopic. Every map of S^n is thus null homotopic for any $n \geqslant 0$, so $\widetilde{K}_b(\mathcal{P}, w)$ is contractible by Whitehead's Theorem. □

THEOREM 4.5. *$\mathcal{D}_b(\mathcal{P}, w)$ acts isometrically on $\widetilde{K}_b(\mathcal{P}, w)$.*

Proof. If $\Pi \in \mathcal{D}_b(\mathcal{P}, w)$, and $[\Pi_1] \in \widetilde{K}_b(\mathcal{P}, w)^0$, let $\Pi \cdot [\Pi_1] = [r(\Pi \circ \Pi_1)]$. One needs to show that $\cdot$ is an action by automorphisms on the abstract cubical complex $\widetilde{K}_b(\mathcal{P}, w)_{ab}$.

First, let $\Pi_1, \Pi_2 \in \mathcal{D}_b(\mathcal{P}, w)$ and let $[\Pi] \in \widetilde{K}_b(\mathcal{P}, w)^0$.

$$\Pi_1 \cdot (\Pi_2 \cdot [\Pi]) = \Pi_1 \cdot [r(\Pi_2 \circ \Pi)] = [r(\Pi_1 \circ r(\Pi_2 \circ \Pi))] = [r((\Pi_1 \circ \Pi_2) \circ \Pi)] = r(\Pi_1 \circ \Pi_2) \cdot [\Pi].$$

This shows that $\cdot$ is a group action on the set of vertices.

It needs to be shown that the group action on the set of vertices permutes cubes. Let $[[\Pi_1], [\Pi_2]]$ be an elementary interval. By the elementary interval lemma, $[[\Pi_1], [\Pi_2]] = \mathcal{C}(\overline{\Pi}, \overline{\Delta})$ for some appropriate picture $\overline{\Pi}$ and thin picture $\overline{\Delta}$.

Let $\Pi \in \mathcal{D}_b(\mathcal{P}, w)$.

$$\Pi \cdot \mathcal{C}(\overline{\Pi}, \overline{\Delta}) = \{[r(\Pi \circ r(\overline{\Pi} \circ \overline{\Delta}'))] \mid [\overline{\Delta}'] \leqslant [\overline{\Delta}]\} = \{[r(r(\Pi \circ \overline{\Pi}) \circ \overline{\Delta}')] \mid [\overline{\Delta}'] \leqslant [\overline{\Delta}]\} = \mathcal{C}(r(\Pi \circ \overline{\Pi}), \overline{\Delta}).$$

By the elementary interval lemma, this is a cube. □

DEFINITION 4.6. [7] The *link* of a vertex v in a cubical complex K is a simplicial complex which may be described topologically as the boundary of a small ϵ-neighborhood of v in K. The simplices of the link are the intersections of this boundary with the cubes containing v. The *abstract link* of v, denoted $\mathrm{lk}_{ab}(v)$, is the underlying abstract simplicial complex.

The abstract link $\mathrm{lk}_{ab}(v)$ of a vertex in the realization $|K|$ of an abstract cubical complex K may be described as follows:

(i) The vertex set of $\mathrm{lk}_{ab}(v)$ consists of all vertices of K which are adjacent to v;

(ii) A simplex S is a set of vertices such that some cube C of K contains $S \cup \{v\}$.

For complete details about the link, see [3].

PROPOSITION 4.7. *Let* $[\Pi]\in\mathcal{V}(\widetilde{\mathrm{K}}(\mathcal{P},w)_p)$. *Suppose that the bottom label of* Π *is* w_1. *The abstract link* $\mathrm{lk_{ab}}([\Pi])$ *is isomorphic to the abstract simplicial complex* (V_{w_1},S_{w_1}) *where*

$V_{w_1}=\{[\Delta]\mid\Delta$ *is a thin picture having* w_1 *as its top label, and only one transistor*$\}$,

and

$S_{w_1}=\{\{[\Delta]\in V_{w_1}\mid[\Delta]\leqslant[\Delta_1]\}\mid\Delta_1$ *is a thin picture having* w_1 *as its top label*$\}$.

Proof. Suppose $[\Pi]\in\mathcal{V}(\widetilde{\mathrm{K}}_b(\mathcal{P},w))$, and the bottom label of Π is w_1. The vertex set of $\mathrm{lk_{ab}}([\Pi])$ is the set of all vertices of $\widetilde{\mathrm{K}}_b(\mathcal{P},w)$ adjacent to $[\Pi]$. Thus $\mathcal{V}(\mathrm{lk}_{ab}([\Pi]))=\{[\Pi_1]\mid[\Pi]\leqslant[\Pi_1]$, or $[\Pi_1]\leqslant[\Pi]$, and $|\mathcal{T}_{\Pi_1}\triangle\mathcal{T}_{\Pi}|=1\}$. (Here $\triangle$ denotes the symmetric difference of sets.)

Define a map $\Phi_{[\Pi]}\colon V_{w_1}\to V(\mathrm{lk_{ab}}([\Pi]))$, sending $[\Delta]$ to $[r(\Pi\circ\Delta)]$. Now if $[\Pi_1]$ is adjacent to $[\Pi]$, then either $\Pi\circ\Delta\equiv\Pi_1$ or $\Pi_1\circ\Delta\equiv\Pi$, for some Δ, where Δ has just one transistor. It follows easily that $\Phi_{[\Pi]}$ is surjective. If $[r(\Pi\circ\Delta_1)]=[r(\Pi\circ\Delta_2)]$, then

$$\Pi\circ\Delta_1\circ\Psi=\Pi\circ\Delta_2\Rightarrow\Delta_1\circ\Psi=\Delta_2\Rightarrow[\Delta_1]=[\Delta_2].$$

Thus $\Phi_{[\Pi]}$ is one-to-one as well.

Now suppose that Δ_1 is a thin picture with top label w_1. The corresponding abstract simplex $\{[\Delta]\in V_{w_1}\mid[\Delta]\leqslant[\Delta_1]\}$ is mapped by Φ to $\{[r(\Pi\circ\Delta)]\mid[\Delta]\leqslant[\Delta_1],[\Delta]\in V_{w_1}\}$, which, by definition, is the set of all vertices of $\mathcal{C}(\Pi,\Delta_1)$ which are adjacent to $[\Pi]$. Since $\mathcal{C}(\Pi,\Delta_1)$ is an elementary interval by the elementary interval lemma, $\{r(\Pi\circ\Delta)\mid[\Delta]\leqslant[\Delta_1],[\Delta]\in V_{w_1}\}$ is a simplex of $\mathrm{lk}_{ab}([\Pi])$.

Suppose conversely that any simplex U of $\mathrm{lk}_{ab}([\Pi])$ is given. Such a simplex is $\mathcal{V}(\mathrm{lk}_{ab}([\Pi]))\cap[[\Pi_1],[\Pi_2]]$ for some elementary interval $[[\Pi_1],[\Pi_2]]$. By the elementary interval lemma, $[[\Pi_1],[\Pi_2]]=\mathcal{C}(\Pi,\Delta)$ for an appropriate thin picture Δ.

Thus $\mathcal{V}(\mathrm{lk}_{ab}([\Pi]))\cap[[\Pi_1],[\Pi_2]]=\mathcal{V}(\mathrm{lk}_{ab}([\Pi]))\cap\mathcal{C}(\Pi,\Delta)=\{[r(\Pi\circ\Delta')]\mid[\Delta']\leqslant[\Delta]$ and $[r(\Pi\circ\Delta')]\in\mathcal{V}(\mathrm{lk}_{ab}([\Pi]))\}$. This implies that $U=\Phi([\Delta])$. It follows that Φ is an isomorphism. □

The proof of the main theorem requires a few more words of background. An abstract simplicial complex is a *flag complex* if every finite set of vertices that is pairwise joined by edges is a simplex [3].

THEOREM 4.8 ([3, 9]). *If* K *is a locally finite, simply connected cubical complex and, for any* $v\in K^0$, $\mathrm{lk}_{ab}(v)$ *is a flag complex, then* K*, with its intrinsic metric, is a proper CAT(0) space.*

MAIN THEOREM 4.9. *If* $\mathcal{P}=\langle\Sigma\mid\mathcal{R}\rangle$ *is a finite semigroup presentation, then* $\widetilde{\mathrm{K}}_b(\mathcal{P},w)$ *is a proper CAT(0) space for any word* $w\in\Sigma^+$. *The action of* $\mathcal{D}_b(\mathcal{P},w)$ *is proper and by isometries. The action is cocompact if and only if* $\{w_1\mid$ *there is a* (w,w_1)*-picture over* $\mathcal{P}\}$ *is finite.*

Proof. Since $\mathcal{P}$ is a finite semigroup presentation, there are only finitely many vertices in V_w, for any $w \in \Sigma^+$, which implies that $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$ is locally finite, and therefore proper.

The action of $\mathcal{D}_b(\mathcal{P}, w)$ on $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$ will be proper if the stabilizer of any vertex is finite. Let $[\Pi]$ be any vertex, choose $\Pi \in [\Pi]$ and suppose that $\Pi_1 \cdot [\Pi] = [\Pi]$. This implies that $\Pi_1 \circ \Pi = \Pi \circ \Psi$, where Ψ is some permutation picture. Thus $\Pi_1 = \Pi \circ \Psi \circ \Pi^{-1}$. Since, up to isomorphism, there are only finitely many permutation pictures having any given top label, there are only finitely many possibilities for Π_1. Thus the action of $\mathcal{D}_b(\mathcal{P}, w)$ is proper.

The statement about cocompactness follows from the fact that two vertices $[\Pi_1]$, $[\Pi_2]$ are in the same orbit if and only if there are representatives Π_1' and Π_2' of $[\Pi_1]$ and $[\Pi_2]$, respectively, which are both (w, w_1)-pictures, for the same w_1. The necessity of this condition follows easily from the fact that the product of a (w, w)-picture with a (w, w_1)-picture is a (w, w_1)-picture. Conversely, if Π_1' and Π_2' are as above, then $(\Pi_2' \circ \Pi_1'^{-1}) \cdot [\Pi_1] = [\Pi_2]$.

It remains to be shown that (V_w, S_w) is a flag complex for every word $w \in \Sigma^+$. Suppose $[\Delta_1], \ldots, [\Delta_n] \in V_w$ and $\{[\Delta_i], [\Delta_j]\}$ span a 1-simplex of the link, for every $i, j \in \{1, \ldots, n\}$, $i \neq j$. This means that for every such pair there is a formal vertex $[\Delta_{i,j}]$ such that $[\Delta_i], [\Delta_j] \leqslant [\Delta_{i,j}]$.

For each $[\Delta_i]$, let $N_i \subseteq \{1, \ldots, k\}$ be such that the mth contact from the left on top of the frame is connected by a wire to the transistor of Δ_i if and only if $m \in N_i$. This definition of N_i does not depend on the representative Δ_i of $[\Delta_i]$. Since $[\Delta_i]$, $[\Delta_j]$ have a common upper bound for all $i, j \in \{1, \ldots, n\}$, $N_i \cap N_j = \emptyset$ if $i \neq j$, by 31.

Now form a new picture $\Delta_{1,\ldots,n}$ having transistors $T_1, \ldots, T_n$ where the T_i each have the same top and bottom labels as the transistor in Δ_i. Introduce contacts on the top of the frame for $\Delta_{1,\ldots,n}$, one for each letter in w. For each i, attach the top contacts of the transistor T_i by wires to the contacts corresponding to the subset $N_i \subseteq \{1, \ldots, n\}$, mimicking the attaching maps of the wires from Δ_i.

Since $N_i \cap N_j = \emptyset$ if $i \neq j$, all such transistors may be attached in this way simultaneously. Each contact c on the top of the frame that doesn't intersect with a wire at this stage may be assigned a new wire arbitrarily to have c as its top contact and an arbitrary point on the bottom of the frame for $\Delta_{1,\ldots,n}$ as its bottom contact.

Figure 11 illustrates the construction of the preceding paragraphs. From left to right, the pictures depicted are Δ_1, Δ_2, Δ_3, and $\Delta_{1,2,3}$. The $\vee$ denotes the operation of taking the least upper bound.

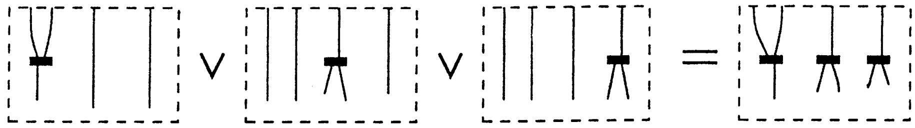

Figure 11.

It follows from the construction of $\Delta_{1,\ldots,n}$ that $[\Delta_i] \leqslant [\Delta_{1,\ldots,n}]$ for all $i \in \{1, \ldots, n\}$. By the definition of the simplices in (V_w, S_w), $\{[\Delta_1, \ldots, \Delta_n\}$ is a simplex, which implies that $\mathrm{lk}_{\mathrm{ab}}(w)$ is a flag complex. □

COROLLARY 4.10. *The group $\mathcal{D}_b(\mathcal{P}, w)$ is a-T-menable, for any countable semigroup presentation $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ and word $w \in \Sigma^+$.*

Proof. Since the property of a-T-menability is closed under taking unions, it is sufficient to prove this for finite semigroup presentations. The previous theorem shows that every picture group over a finite semigroup presentation is a-T-menable (see the next section and the references therein). □

5. G-sets-with-walls

Let G be a group. A *G-set-with-walls* $(S, \mathcal{W})$ [14] consists of a G-set S and a set of walls $\mathcal{W}$ which are permuted by the action of G. A *wall in S* is a partition of S into two (nonempty) subsets. The walls of $\mathcal{W}$ are required to have the additional property that, for any s_1, s_2 in S, there are only finitely many walls separating s_1 and s_2, where a wall $W = \{A, B\}$ *separates* s_1 and s_2 if $s_1 \in A$ and $s_2 \in B$ (or vice versa).

With every G-set-with-walls there is associated an affine isometric action of G on a Hilbert space. Each wall $\{A, B\}$ in $\mathcal{W}$ has two *orientations*, (A, B) and (B, A). For each wall in $\mathcal{W}$ choose an orientation; call the set of the choices $\mathcal{W}^{\mathrm{or}}$. For the rest of the paper, the element χ_s of $\ell^2(S)$, defined to be the function which sends s to 1 and all other elements of S to 0, will simply be denoted s. With this convention, the group G acts by unitary operators on $\ell^2(\mathcal{W}^{\mathrm{or}})$, where $g \cdot (A, B) = (gA, gB)$ and, by definition, $(A, B) = -(B, A)$.

One adds suitable translations $\pi_x(g)$ to get an affine, nonlinear action. First choose a basepoint $x \in S$. Let $\pi_x : G \to \ell^2(\mathcal{W}^{\mathrm{or}})$ be defined as follows. The coefficient of an oriented wall (A, B) in the sum $\pi_x(g)$ is 0 if x and gx are not separated by $\{A, B\}$, 1 if $x \in A$ and $gx \in B$, and -1 if $gx \in A$ and $x \in B$. It is not difficult to check that the function $\alpha_x : G \to \mathrm{Isom}(\ell^2(\mathcal{W}^{\mathrm{or}}))$, where $\alpha_x(g)(v) = gv + \pi_x(g)$ is a homomorphism. This amounts to checking that π_x satisfies the 'cocycle condition':

$$\pi_x(gh) = g \cdot \pi_x(h) + \pi_x(g).$$

A G-set-with-walls $(S, \mathcal{W})$ gives a natural pseudo-metric $d_{\mathcal{W}}$ on S, where $d_{\mathcal{W}}(s_1, s_2)$ is the number of walls separating s_1 and s_2. If the action of G on $(S, d_{\mathcal{W}})$ is proper, then the action α_x is proper as well.

Sageev [20] showed that a cellular G-action on a CAT(0) cubical complex X gives rise to a G-set-with-walls $(S, \mathcal{W})$ where $S = X^0$ and $\mathcal{W}$ is in one-to-one correspondence with the set of combinatorial hyperplanes in X. A *combinatorial hyperplane* in X [20] is an equivalence class of edges in X generated by the relation $\sim$,

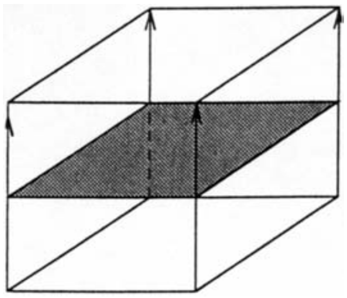

Figure 12.

where $e_1^1 \sim e_2^1$ if the edges e_1^1 and e_2^1 are parallel across some 2-cell of X. An *oriented combinatorial hyperplane* is an equivalence class of oriented edges, where the equivalence relation is defined in an analogous way.

A combinatorial hyperplane may be developed into a geometric hyperplane which divides X into two parts [20], and so defines a wall in X^0 in the above sense. The set $\mathcal{W}$ consists of all such walls. Figure 12 shows four oriented edges from the same oriented combinatorial hyperplane in a cube and the (geometric) hyperplane they generate.

6. An Example

Cannon *et al.* [4] described elements of Thompson's group V by pairs of finite rooted binary trees, as depicted in the upper left-hand corner of Figure 13. The *leaves*, i.e., vertices of degree one, correspond to subintervals of $[0, 1)$. The leaves of the (identical) trees in Figure 13, for example, correspond, from left to right, to $[0, \frac{1}{2})$, $[\frac{1}{2}, \frac{3}{4})$, and $[\frac{3}{4}, 1)$, respectively. The map C matches intervals according to the numbering of the leaves, so C maps $[0, \frac{1}{2})$ to $[\frac{3}{4}, 1)$, $[\frac{1}{2}, \frac{3}{4})$ to $[0, \frac{1}{2})$, and $[\frac{3}{4}, 1)$ to $[\frac{1}{2}, \frac{3}{4})$.

Each finite rooted binary tree corresponds naturally to an (x, x)-picture over the semigroup presentation $\langle x \mid x = x^2 \rangle$. The pair of pictures one gets from this correspondence may be combined together into a single element of $\mathcal{D}_b(\mathcal{P}, w)$ – see

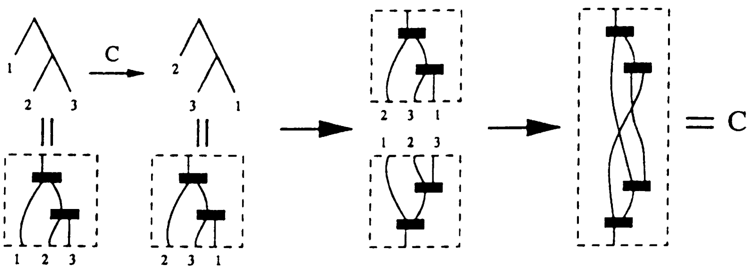

Figure 13.

Figure 13. Notice that the picture corresponding to the range goes on top. The correspondence, which is an isomorphism between V and $\mathcal{K}_b(\langle x \mid x = x^2\rangle, x)$, is illustrated again in Figure 14. The elements A, B, C, and Π_0 are generators of V [4].

For the rest of the section, let $\mathcal{P} = \langle x \mid x = x^2\rangle$ and $w = x$. To find an explicit formula for the cocycle associated to the action of $\mathcal{D}_b(\mathcal{P}, w)$ on $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$, one needs to understand the action of $\mathcal{D}_b(\mathcal{P}, w)$ on hyperplanes.

PROPOSITION 6.1. *There are exactly two orbits of oriented hyperplanes in* $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$. *Each of the hyperplanes in the first orbit consists of a single point. Each unoriented hyperplane in the second orbit is generated by a translate of the edge* $\mathcal{C}(\Pi', \Delta')$ *(see Figure 15). The action of* $\mathcal{D}_b(\mathcal{P}, w)$ *on* $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$ *does not reverse orientations on any hyperplanes.*

The stabilizer subgroup of any oriented hyperplane in the first orbit is trivial; the stabilizer subgroup of the oriented hyperplane generated by $\mathcal{C}(\Pi', \Delta')$ *is* $V_{[0,\frac{1}{2})}$, *the subgroup of* V *which acts as the identity on* $[0, \frac{1}{2})$.

Proof. The hyperplanes in the first orbit are the midpoints of the translates of the edge $\mathcal{C}(\Psi, \Delta)$, where Ψ is the unique (x, x) permutation picture and Δ is a thin (x, x^2)-picture. (This description defines Δ uniquely up to right multiplication by a permutation picture, so the choice of Δ does not matter.) It is easy to check that the stabilizer subgroup of this oriented edge is trivial.

Now suppose that $\mathcal{C}(\Pi, \Delta)$ is an arbitrary unoriented edge in $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$, where Π is a (x, x^n)-picture and $n \geqslant 2$. One can arrange that the top of the transistor in Δ is labelled by x, by replacing $\mathcal{C}(\Pi, \Delta)$ by $\mathcal{C}(r(\Pi \circ \Delta), \Delta^{-1})$ if necessary. One can also ensure that the first contact from the left on the top of the frame for Δ is attached by a wire to the top contact of the transistor in Δ, by replacing $\mathcal{C}(\Pi, \Delta)$ by $\mathcal{C}(\Pi \circ \Psi, \Psi^{-1} \circ \Delta)$ for some appropriate permutation diagram Ψ. Finally, since

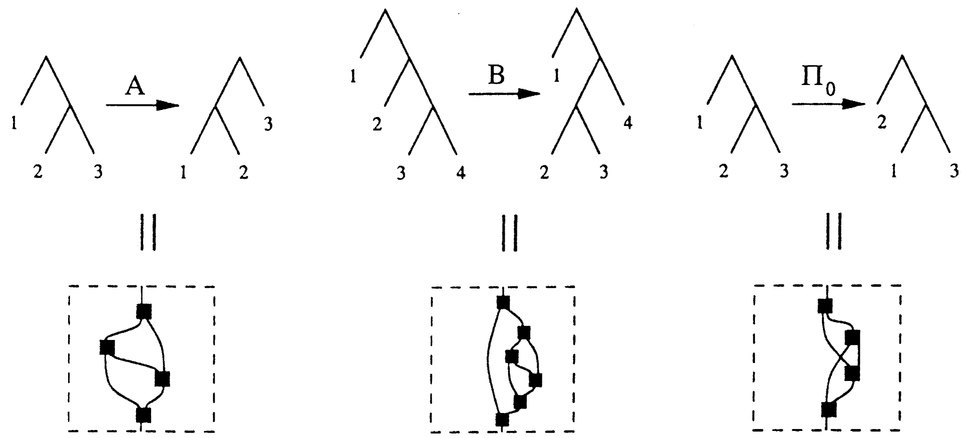

Figure 14.

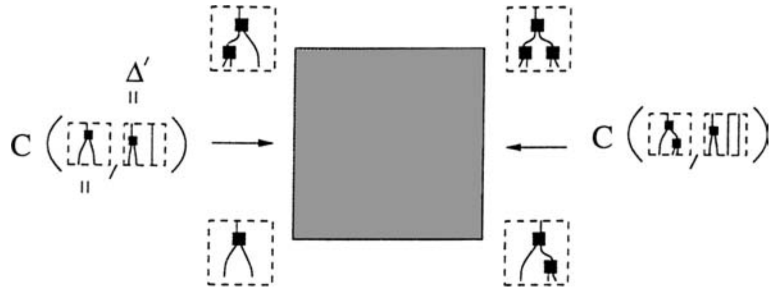

Figure 15.

$\mathcal{C}(\Pi, \Delta) = \mathcal{C}(\Pi, \Delta \circ \Psi)$ for any permutation diagram Ψ, one can assume that the two bottom contacts of the transistor in Δ are attached by wires to the first two contacts on the bottom of the frame for Δ, in a fashion that preserves the left-to-right order, and every wire which does not connect to the transistor is a vertical line segment.

Such a thin picture Δ over $\mathcal{P}$ is in *standard form*.

Let $\mathcal{C}(\Pi_1, \Delta_1)$, $\mathcal{C}(\Pi_2, \Delta_2)$ be edges of $\widetilde{K}_b(\mathcal{P}, w)$, where, for $i = 1, 2$, Δ_i is a (x^{n_i}, x^{n_i+1})-picture in standard form. These unoriented edges are in the same orbit if and only if $n_1 = n_2$, for if $n_1 = n_2$ then the concatenation $\Pi_2 \circ \Pi_1^{-1}$ is defined and

$$(\Pi_2 \circ \Pi_1^{-1}) \cdot \mathcal{C}(\Pi_1, \Delta_1) = \mathcal{C}(\Pi_2, \Delta_1) = \mathcal{C}(\Pi_2, \Delta_2),$$

where the last equality is true because Δ_1 and Δ_2 are both in standard form and have the same number of top and bottom contacts, and thus are isomorphic. The necessity of the condition $n_1 = n_2$ follows from the fact that the action of $\mathcal{D}_b(\mathcal{P}, w)$ on $\widetilde{K}_b(\mathcal{P}, w)$, for any $\mathcal{P}$ and w, sends a vertex having a representative with n bottom contacts to another such vertex.

It will thus follow that the action of $\mathcal{D}_b(\mathcal{P}, w)$ on $\widetilde{K}_b(\mathcal{P}, w)$ is transitive on the remaining unoriented combinatorial hyperplanes if each such hyperplane may be represented, for any $n \geqslant 2$, by an edge $\mathcal{C}(\Pi, \Delta)$, where Δ is an (x^n, x^{n+1})-picture.

Now consider the relation $\sim$ of elementary square equivalence between *oriented* edges. Make the convention that the initial vertex of an edge $\mathcal{C}(\Pi, \Delta)$ is $[\Pi]$. If $\mathcal{C}(\Pi, \Delta)$ and $\mathcal{C}(\overline{\Pi}, \overline{\Delta'})$ are edges and Δ and $\overline{\Delta'}$ are in standard form, then $\mathcal{C}(\overline{\Pi}, \overline{\Delta'}) \sim \mathcal{C}(\Pi, \Delta)$ if and only if $\mathcal{C}(\overline{\Pi}, \overline{\Delta'})$ may be obtained from $\mathcal{C}(\Pi, \Delta)$ by the following operation:

Clip the wires at one or two of the bottom of contacts of Π other than the leftmost one, attach those wires to the top of a new transistor T, which will have, respectively, one or two top contacts and two or one bottom contacts. Connect every bottom contact of T to the bottom of the frame for Π by a wire, fastening the bottom ends of the new wires onto the frame at some point to the *right* of the leftmost bottom contact of Π. The new picture is $\overline{\Pi}$. Alter Δ accordingly, to make a new picture $\overline{\Delta}$ in standard form with the correct number of top contacts.

It follows from the description of $\sim$ that, for example, the oriented combinatorial hyperplane generated by $\mathcal{C}(\Pi', \Delta')$ (pictured) contains all edges of the form pictured on the right half of Figure 16(i). The picture Π is any (x, x^n)-picture, where $n > 0$ is arbitrary. More generally, any unoriented combinatorial hyperplane may be represented by an edge $\mathcal{C}(\Pi, \Delta)$, where Δ is a (x^n, x^{n+1})-picture in standard form, by the description of $\sim$ from the previous paragraph. From this fact and the argument of the preceding paragraphs it follows that the orbit of the unoriented combinatorial hyperplane generated by $\mathcal{C}(\Pi', \Delta')$ contains all of the unoriented combinatorial hyperplanes that are not in the first orbit.

Now suppose that some element of $\mathcal{D}_b(\mathcal{P}, w)$ reversed the orientation on the hyperplane generated by the edge $\mathcal{C}(\Pi', \Delta')$. This implies that, for some $\Pi \in \mathcal{D}_b(\mathcal{P}, w)$, $\Pi \cdot [\Pi'] = [\overline{\Pi} \circ \overline{\Delta}]$, where the concatenation $\overline{\Pi} \circ \overline{\Delta}$ is reduced and $[[\overline{\Pi}], [\overline{\Pi} \circ \overline{\Delta}]]$ has the form depicted in the right half of Figure 16(i). This is not possible, since the representatives of $\Pi \cdot [\Pi']$ all have two bottom contacts, and the representatives of $[\overline{\Pi} \circ \overline{\Delta}]$ have at least three. The contradiction shows that the action $\mathcal{D}_b(\mathcal{P}, w)$ is orientation-preserving on the hyperplanes.

Finally, one needs to show that the oriented combinatorial hyperplane H^+, generated by the oriented edge $\mathcal{C}(\Pi', \Delta')$, has $V_{[0,\frac{1}{2})}$ as its stabilizer subgroup. Let $\overline{\Pi}$ be a (x, x)-picture such that $\overline{\Pi} \cdot [[\Pi'], [\Pi' \circ \Delta']] \in H^+$. This means that $\overline{\Pi} \cdot [[\Pi'], [\Pi' \circ \Delta']] = [[\Pi_1], [\Pi_2]]$, where $[[\Pi_1], [\Pi_2]]$ is the interval pictured on the right half of Figure 16(i), and the picture Π is some (x, x)-diagram. Since $\overline{\Pi}$ is orientation-preserving, $\overline{\Pi} \cdot [\Pi'] = [\Pi_1]$ and $\overline{\Pi} \cdot [\Pi' \circ \Delta'] = [\Pi_2]$. Thus one has the simultaneous equations $\overline{\Pi} = \Pi_1 \circ \Psi_1 \circ \Pi'^{-1}$ and $\overline{\Pi} = \Pi_2 \circ \Psi_2 \circ \Delta'^{-1} \circ \Pi'^{-1}$, where Ψ_i is a permutation diagram, for $i = 1, 2$, and Π_i are representatives of $[\Pi_i]$, for $i = 1, 2$. The only solution for $\overline{\Pi}$ is pictured in Figure 16(ii); and, using the isomorphism of $\widetilde{\mathrm{K}}_b(\mathcal{P}, w)$ with V described previously, one can easily check that this element is in $V_{[0,\frac{1}{2})}$. Thus the stabilizer subgroup of H^+ is a subset of $V_{[0,\frac{1}{2})}$. The reverse inclusion is straightforward; one checks that the picture in Figure 16(ii) fixes the oriented hyperplane generated by $\mathcal{C}(\Pi', \Delta')$. $\square$

PROPOSITION 6.2. *The map* $d \colon V \to \ell^2(V/V_{[0,\frac{1}{2})})$ *defined by:*

$$d(A) = (1 - AC^{-1}A^{-1})V_{[0,\frac{1}{2})}$$
$$d(B) = (C^{-1}A^{-1} + \Pi_0 - C^{-1}AB^{-1}C^{-1} - C^{-1}AB^{-1}A^{-1})V_{[0,\frac{1}{2})}$$
$$d(C) = (C^{-1}A^{-1} - A^{-1})V_{[0,\frac{1}{2})}$$
$$d(\Pi_0) = (C^{-1}A^{-1} - \Pi_0^{-1}CA^{-1})V_{[0,\frac{1}{2})}$$

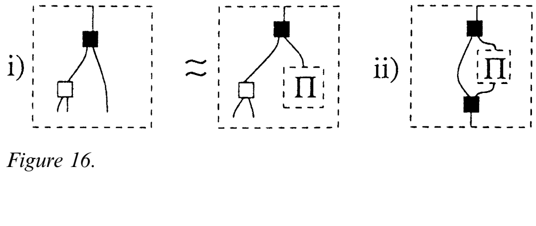

Figure 16.

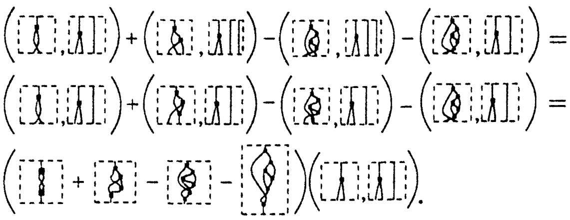

Figure 17.

is a proper cocycle for the action of V *on* $\ell^2(V/V_{[0,\frac{1}{2})})$.

Proof. The calculation of $d(B)$ will be given in detail; the other (easier) calculations are left to the reader.

Begin by connecting the vertex $[\Psi]$ to $B\cdot[\Psi]$ by an edge path, where Ψ is the permutation (x,x)-picture. One omits the edges which represent the hyperplane with a trivial stabilizer subgroup, since it can be shown that they contribute nothing interesting to the cocycle, although they do change the formulas.

In the first line of Figure 17, the four remaining edges are depicted, counted according to their orientations. (That is, the last two edges are crossed against their natural orientations as one passes from $[\Psi]$ to $B\cdot[\Psi]$.) The group elements on the last line of the figure are, respectively, $C^{-1}A^{-1}$, Π_0, $C^{-1}AB^{-1}C^{-1}$, and $C^{-1}AB^{-1}A^{-1}$. The orbit of the oriented hyperplane generated by $\mathcal{C}(\Pi',\Delta')$ may be identified with $V/V_{[0,\frac{1}{2})}$ by the previous proposition. The formula for $d(B)$ follows easily.

The properness of the cocycle follows directly from the properness of the action of $\mathcal{D}_b(\mathcal{P},w)$ on $\widetilde{\mathrm{K}}_b(\mathcal{P},w)$. □

It can be shown that the cocycle derived here differs from the one in [8] by only a minus sign; in fact, the calculation of this section was the source for [8].

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References

1. Bekka, M. E. B., Cherix, P. A. and Valette, A.: Proper affine isometric actions of amenable groups. *Novikov Conjecture, Index Theorems and Rigidity*, London Math. Soc. Lecture Notes Ser. 227, Cambridge University Press, 1995.
2. Bridson, M. R.: Remarks on a talk of Higson, unpublished note.
3. Bridson, M. R. and Haefliger, A.: *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, 1999.
4. Cannon, J. W., Floyd, W. J. and Parry, W. R.: Introductory notes on Richard Thompson's groups, *Enseign. Math. (2)* **42**(3-4) (1996), 215–256.

5. Cherix, P. A., Cowling, M., Jolissaint, P., Julg, P. and Valette, A.: *Groups with the Haagerup Property. Gromov's a-T-menability*, Progr. in Math. 197, Birkhäuser, Basel, 2001.
6. Connes, A.: Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$, *Ann. Math. (2)* (1)**104** (1976), 73–115.
7. Farley, D. S.: Finiteness and CAT(0) properties of diagram groups, *Topology* **42**(5) (2003), 1065–1082.
8. Farley, D. S.: Proper isometric actions of Thompson's groups on Hilbert space, *IMRN* **45** (2003), 2409–2414.
9. Gromov, M. Hyperbolic groups, in S. M. Gersten, (ed.), *Essays in Group Theory*, MSR1 Publ. 8, Springer, New York, 1987, pp. 75–263.
10. Guba, V. S. and Sapir, M. V.: Diagram groups, *Mem. Amer. Math. Soc.* **130**(620) (1997).
11. Guba, V. S. and Sapir, M. V.: On subgroups of R. Thompson's group F and other diagram groups (Russian), *Mat. Sb.* **190**(8) (1999), 3–60.
12. Guba, V. S. and Sapir, M. V.: Rigidity properties of diagram groups, *International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000)*, *Internat. J. Algebra Comput.* **12**(1-2) (2002), 9–17.
13. Guba, V. S. and Sapir, M. V.: Diagram groups and directed 2-complexes: homotopy and homology. Preprint at front.math.ucdavis.edu.
14. Higson, N.: Private communication.
15. Higson, N. and Kasparov, G. G.: E-Theory and KK-Theory for groups which act properly and isometrically on Hilbert space, *Invent. Math.* **144**(1) (2001), *23–74.*
16. Jolissaint, P.: Central sequences in the factor associated with Thompson's group F, *Ann. Inst. Fourier(Grenoble)* **48**(4) (1998), 1093–1106.
17. Jolissaint, P.: Moyennabilité intérieure du groupe F de Thompson, *C.R. Acad. Sci. Paris. Sér. I Math* **325**(1) (1997), 61–64.
18. Newman, M. H. A.: On theories with a combinatorial definition of equivalence, *Ann. Math.* **43** (1942), 223–243.
19. Niblo, G. A. and Reeves, L. D.: Groups acting on CAT(0) cube complexes, *Geom. Topol.* **1** (1997), approx. 7 pages (electronic).
20. Sageev, M.: Ends of group pairs and non-positively curved cubical complexes, *Proc. London Math. Soc. (3)* **71**(3) (1995), 585–617.
21. Spanier, E. H.: *Algebraic Topology*, McGraw-Hill, New York, 1966.
22. Thompson, R. J.: Handwritten notes.