BRAIDED DIAGRAM GROUPS AND LOCAL SIMILARITY GROUPS

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Abstract. Hughes defined a class of groups that act as local similarities on compact ultrametric spaces. Guba and Sapir had previously defined braided diagram groups over semigroup presentations. The two classes of groups share some common characteristics: both act properly by isometries on CAT(0) cubical complexes, and certain groups in both classes have type F_∞, for instance.

Here we clarify the relationship between these families of groups: the braided diagram groups over tree-like semigroup presentations are precisely the groups that act on compact ultrametric spaces via small similarity structures. The proof can be considered a generalization of the proof that Thompson's group V is a braided diagram group over a tree-like semigroup presentation.

We also prove that certain additional groups, such as the Houghton groups H_n, and $QAut(T_{2,c})$, lie in both classes.

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1. Introduction

In [7], Hughes described a class of groups that act as homeomorphisms on compact ultrametric spaces. Fix a compact ultrametric space X. The essence of the idea was to associate to X a *finite similarity structure*, which is a function that associates to each ordered pair of balls $B_1, B_2 \subseteq X$ a finite set $\mathrm{Sim}_X(B_1, B_2)$ of surjective similarities from B_1 to B_2. (A *similarity* is a map that stretches or contracts distances by a fixed constant.) The finite sets $\mathrm{Sim}_X(B_1, B_2)$ are assumed to have certain desirable closure properties (such as closure under composition). A homeomorphism $h : X \to X$ is said to be *locally determined by* Sim_X if each $x \in X$ has a ball neighborhood B with the property that $h(B)$ is a ball and the restriction of h to B agrees with one of the local similarities $\sigma \in \mathrm{Sim}_X(B, h(B))$. The collection of all homeomorphisms that are locally determined by Sim_X forms

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a group under composition. We will call such a group an *FSS group* (finite similarity structure group) for short. Hughes [7] proved that each FSS group has the Haagerup property, and even acts properly on a CAT(0) cubical complex. In [5], the authors described a class of FSS groups that have type F_∞. That class includes Thompson's group V, and the main theorem of [5] is best understood as a generalization of [2], where Brown originally showed that V has type F_∞.

In earlier work, Guba and Sapir [6] had sketched a theory of braided diagram groups over semigroup presentations, and proved that Thompson's group V is a braided diagram group over the semigroup presentation $\langle x \mid x = x^2 \rangle$. Farley [4] showed that braided diagram groups over semigroup presentations act properly on CAT(0) cubical complexes.

The class $\mathcal{F}$ of FSS groups and the class $\mathcal{B}$ of braided diagram groups therefore have a common origin, as generalizations of Thompson's group V. Both classes also share other features in common (as noted above). It is therefore natural to wonder to what extent the two classes are the same. The main goal of this note is to prove Theorem 4.12, which says that the FSS groups determined by small similarity structures (Definition 4.6) are precisely the same as the braided diagram groups determined by tree-like semigroup presentations (Definition 4.1). It is even possible that Theorem 4.12 describes the precise extent of the overlap between $\mathcal{F}$ and $\mathcal{B}$, but we do not know how to prove this.

We include all relevant definitions, and our treatment is fairly self-contained as a result. A precise definition of braided diagram groups is given in Section 2, the precise definition of FSS groups appears in Section 3, and the main theorem is proved in Section 4. Along the way, we give additional examples in the class $\mathcal{F} \cap \mathcal{B}$, including the Houghton groups H_n and a certain group $QAut(T_{2,c})$ of quasi-automorphisms of the infinite binary tree. (These are Examples 4.3 and 4.4, respectively.)

This note has been adapted from the longer preprint [5]. The first part of the latter preprint (including roughly the first six sections) will be published elsewhere. The first author would like to thank the organizers of the Durham Symposium (August 2013) for the opportunity to speak. Example 4.3 first appeared as part of the first author's lecture. The idea of Example 4.4 occurred to the first author after listening to Collin Bleak's lecture at the Symposium.

2. Braided diagram groups

In this section, we will recall the definition of braided diagram groups over semigroup presentations. Note that the theory of braided diagram groups was first sketched by Guba and Sapir [6]. A more extended introduction to braided diagram groups appears in [4].

Definition 2.1. Let Σ be a set, called an *alphabet*. The *free semigroup on* Σ, denoted Σ^+, is the collection of all positive non-empty strings formed from Σ, i.e.,

$$\Sigma^+ = \{u_1 u_2 \ldots u_n \mid n \in \mathbb{N}, u_i \in \Sigma \text{ for } i \in \{1, \ldots, n\}\}.$$

The *free monoid on* Σ, denoted Σ^*, is the union $\Sigma^+ \cup \{1\}$, where 1 denotes the empty string. (Here we assume that $1 \notin \Sigma$ to avoid ambiguity.) The operations in Σ^+ and Σ^* are concatenation.

We write $w_1 \equiv w_2$ if w_1 and w_2 are equal as words in Σ^*.

Definition 2.2. A *semigroup presentation* $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ consists of an alphabet Σ and a set $\mathcal{R} \subseteq \Sigma^+ \times \Sigma^+$. The elements of $\mathcal{R}$ are called *relations*.

Remark 2.3. A relation $(w_1, w_2) \in \mathcal{R}$ can be viewed as an equality between the words w_1 and w_2. We use ordered pairs to describe these equalities because we will occasionally want to make a distinction between the left and right sides of a relation.

A semigroup presentation $\mathcal{P}$ determines a semigroup $S_{\mathcal{P}}$, just as a group presentation determines a group. We will, however, make essentially no use of this semigroup $S_{\mathcal{P}}$. Our interest is in braided diagrams over $\mathcal{P}$ (see below).

Definition 2.4. (Braided Semigroup Diagrams) A *frame* is a homeomorphic copy of $\partial([0,1]^2) = (\{0,1\} \times [0,1]) \cup ([0,1] \times \{0,1\})$. A frame has a *top* side, $(0,1) \times \{1\}$, a *bottom* side, $(0,1) \times \{0\}$, and *left* and *right* sides, $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$, respectively. The top and bottom of a frame have obvious left to right orderings.

A *transistor* is a homeomorphic copy of $[0,1]^2$. A transistor has top, bottom, left, and right sides, just as a frame does. The top and bottom of a transistor also have obvious left to right orderings.

A *wire* is a homeomorphic copy of $[0,1]$. Each wire has a bottom 0 and a top 1.

Let $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ be a semigroup presentation. Let $\mathcal{T}(\Delta)$ be a finite (possibly empty) set of transistors. Let $\mathcal{W}(\Delta)$ be a finite, nonempty set of wires. We let $F(\Delta) = \partial([0,1]^2)$ be a frame. We let $\ell_\Delta : \mathcal{W}(\Delta) \to \Sigma$ be an arbitrary function, called the *labelling function.*

For each wire $W \in \mathcal{W}(\Delta)$, we choose a point $t(W)$ on the bottom of a transistor, or on the top of the frame, and a point $b(W)$ on the top of a transistor, or on the bottom of the frame. The points $t(W)$ and $b(W)$ are called the *top* and *bottom contacts* of W, respectively.

We attach the top of each wire W to $t(W)$ and the bottom of W to $b(W)$. The resulting topological space Δ is called a *braided diagram over* $\mathcal{P}$ if the following additional conditions are satisfied:

(1) If W_i, $W_j \in \mathcal{W}(\Delta)$, $t(W_i) = t(W_j)$ only if $W_i = W_j$, and $b(W_i) = b(W_j)$ only if $W_i = W_j$. In other words, the disjoint union of all of the wires maps injectively into the quotient.

(2) We consider the top of some transistor $T \in \mathcal{T}(\Delta)$. Reading from left to right, we find contacts
$$b(W_{i_1}), b(W_{i_2}), \ldots, b(W_{i_n}),$$
where $n \geq 0$. The word $\ell_t(T) = \ell(W_{i_1})\ell(W_{i_2})\ldots\ell(W_{i_n})$ is called the *top label of* T. Similarly, reading from left to right along the bottom of T, we find contacts
$$t(W_{j_1}), t(W_{j_2}), \ldots, t(W_{j_m}),$$
where $m \geq 0$. The word $\ell_b(T) = \ell(W_{j_1})\ell(W_{j_2})\ldots\ell(W_{j_m})$ is called the *bottom label of* T. We require that, for any $T \in \mathcal{T}(\Delta)$, either $(\ell_t(T), \ell_b(T)) \in \mathcal{R}$ or $(\ell_b(T), \ell_t(T)) \in \mathcal{R}$. (We emphasize that it is not sufficient for $\ell_t(T)$ to be equivalent to $\ell_b(T)$ modulo the relation $\sim$ determined by $\mathcal{R}$. Note also that this condition implies that $\ell_b(T)$ and $\ell_t(T)$ are both non-empty, since $\mathcal{P}$ is a semigroup presentation. In particular, each transistor has wires attached to its top and bottom faces.)

(3) We define a relation $\preceq$ on $\mathcal{T}(\Delta)$ as follows. Write $T_1 \preceq T_2$ if there is some wire W such that $t(W) \in T_2$ and $b(W) \in T_1$. We require that the transitive closure $\dot{\preceq}$ of $\preceq$ be a strict partial order on $\mathcal{T}(\Delta)$.

Definition 2.5. Let Δ be a braided diagram over $\mathcal{P}$. Reading from left to right across the top of the frame $F(\Delta)$, we find contacts

$$t(W_{i_1}), t(W_{i_2}), \ldots, t(W_{i_n}),$$

for some $n \geq 1$. The word $\ell(W_{i_1})\ell(W_{i_2})\ldots\ell(W_{i_n}) = \ell_t(\Delta)$ is called the *top label of* Δ. We can similarly define the *bottom label of* Δ, $\ell_b(\Delta)$. We say that Δ is a *braided* $(\ell_t(\Delta), \ell_b(\Delta))$*-diagram over* $\mathcal{P}$.

Remark 2.6. One should note that braided diagrams, despite the name, are not truly braided. In fact, two braided diagrams are equivalent (see Definition 2.10) if there is a certain type of marked homeomorphism between them. Equivalence therefore does not depend on any embedding into a larger space. Braided diagram groups (as defined in Theorem 2.13) also seem to have little in common with Artin's braid groups.

Example 2.7. Let $\mathcal{P} = \langle a, b, c \mid ab = ba, ac = ca, bc = cb\rangle$. Figure 1 shows an example of a braided $(aabc, acba)$-diagram over the semigroup presentation $\mathcal{P}$. The frame is the box formed by the dashed line. The wires that appear to cross in the figure do not really touch, and it is unnecessary to specify which wire passes over the other one. See Remark 2.6.

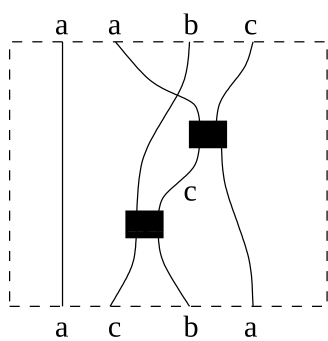

FIGURE 1. A braided $(aabc, acba)$-diagram over the semigroup presentation $\mathcal{P} = \langle a, b, c \mid ac = ca, ab = ba, bc = cb\rangle$.

Definition 2.8. (Concatenation of braided diagrams) Let Δ_1 and Δ_2 be braided diagrams over $\mathcal{P}$. We suppose that Δ_1 is a (w_1, w_2)-diagram and Δ_2 is a (w_2, w_3)-diagram. We can multiply Δ_1 and Δ_2 by stacking them. More explicitly, we remove the bottom of the frame of Δ_1 and the top of the frame of Δ_2, and then glue together the wires in order from left to right. This gluing is compatible with the labeling of the wires, since the bottom label of Δ_1 is the same as the top label of Δ_2. The result is a braided diagram $\Delta_1 \circ \Delta_2$, called the *concatenation* of Δ_1 and Δ_2.

Definition 2.9. (Dipoles) Let Δ be a braided semigroup diagram over $\mathcal{P}$. We say that the transistors $T_1, T_2 \in \mathcal{T}(\Delta)$, $T_1 \preceq T_2$, form a *dipole* if:

(1) the bottom label of T_1 is the same as the top label of T_2, and

(2) there are wires $W_{i_1}, W_{i_2}, \ldots, W_{i_n} (n \geq 1)$ such that the bottom contacts T_2, read from left to right, are precisely

$$t(W_{i_1}), t(W_{i_2}), \ldots, t(W_{i_n})$$

and the top contacts of T_1, read from left to right, are precisely

$$b(W_{i_1}), b(W_{i_2}), \ldots, b(W_{i_n}).$$

Define a new braided diagram as follows. Remove the transistors T_1 and T_2 and all of the wires $W_{i_1}, \ldots, W_{i_n}$ connecting the top of T_1 to the bottom of T_2. Let $W_{j_1}, \ldots, W_{j_m}$ be the wires attached (in that order) to the top of T_2, and let $W_{k_1}, \ldots, W_{k_m}$ be the wires attached to the bottom of T_1. We glue the bottom of W_{j_ℓ} to the top of W_{k_ℓ}. There is a natural well-defined labelling function on the resulting wires, since $\ell(W_{j_\ell}) = \ell(W_{k_\ell})$ by our assumptions. We say that the new diagram Δ' is obtained from Δ by *reducing the dipole* (T_1, T_2). The inverse operation is called *inserting a dipole*.

Definition 2.10. (Equivalent Diagrams) We say that two diagrams Δ_1, Δ_2 are *equivalent* if there is a homeomorphism $\phi : \Delta_1 \to \Delta_2$ that preserves the labels on the wires, restricts to a homeomorphism $\phi_| : F(\Delta_1) \to F(\Delta_2)$, preserves the tops and bottoms of the transistors and frame, and preserves the left to right orientations on the transistors and the frame. We write $\Delta_1 \equiv \Delta_2$.

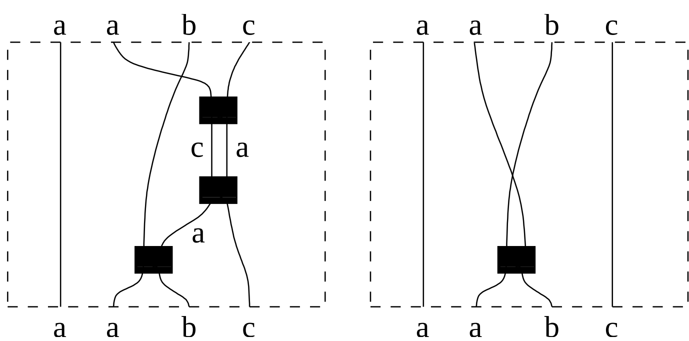

FIGURE 2. The diagram on the right is obtained from the one on the left by reduction of a dipole.

Definition 2.11. (Equivalent Modulo Dipoles; Reduced Diagram) We say that Δ and Δ' are *equivalent modulo dipoles* if there is a sequence $\Delta \equiv \Delta_1 \equiv \Delta_2 \equiv \ldots \equiv \Delta_n \equiv \Delta'$, where Δ_{i+1} is obtained from Δ_i by either inserting or removing a dipole, for $i \in \{1, \ldots, n-1\}$. We write $\Delta = \Delta'$. (The relation of equivalence modulo dipoles is indeed an equivalence relation – see [4].)

A braided diagram Δ over a semigroup presentation is called *reduced* if it contains no dipoles. Each equivalence class modulo dipoles contains a unique reduced diagram [4].

Example 2.12. In Figure 2, we have two braided diagrams over the semigroup presentation $\mathcal{P} = \langle a, b, c \mid ab = ba, ac = ca, bc = cb \rangle$. The two rightmost transistors in the diagram on the left form a dipole, and the diagram on the right is the result of reducing that dipole.

Theorem 2.13. [4] *Let* $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ *be a semigroup presentation, and let* $w \in \Sigma^+$. *We let* $D_b(\mathcal{P}, w)$ *denote the set of equivalence classes of braided* (w, w)*-diagrams modulo dipoles. The operation of concatenation induces a well-defined group operation on* $D_b(\mathcal{P}, w)$. *This group* $D_b(\mathcal{P}, w)$ *is called* the braided diagram group over $\mathcal{P}$ based at w.

3. Groups defined by finite similarity structures

3.1. **Review of ultrametric spaces and finite similarity structures.** We now give a quick review of finite similarity structures on compact ultrametric spaces, as defined in Hughes [7]. Most of this subsection is taken directly from [5].

Definition 3.1. An *ultrametric space* is a metric space (X, d) such that $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

Lemma 3.2. *Let* X *be an ultrametric space.*

(1) *Let* $B_r(x)$ *be an open metric ball in* X. *If* $y \in B_r(x)$, *then* $B_r(x) = B_r(y)$.
(2) *If* B_1 *and* B_2 *are open metric balls in* X, *then either the balls are disjoint, or one is contained in the other.*
(3) *Every open ball in* X *is a closed set, and every closed ball in* X *is an open set.*
(4) *If* X *is compact, then each open ball* B *is contained in at most finitely many distinct open balls of* X.
(5) *If* X *is compact, then each open ball in* X *is a closed ball (possibly of a different radius), and each closed ball is an open ball.*
(6) *If* X *is compact and* x *is not an isolated point, then each open ball* $B_r(x)$ *is partitioned by its maximal proper open subballs, which are finite in number.*

□

Convention 3.3. We assume for the rest of the section that X is a compact ultrametric space. By Lemma 3.2(5), open balls are closed balls, and closed balls are open balls, so we can refer to both simply as balls, and we will follow this practice from now on.

Definition 3.4. Let $f : X \to Y$ be a function between metric spaces. We say that f is a *similarity* if there is a constant $C > 0$ such that $d_Y(f(x_1), f(x_2)) = C d_X(x_1, x_2)$, for all x_1 and x_2 in X.

Definition 3.5. A *finite similarity structure for* X is a function Sim_X that assigns to each ordered pair B_1, B_2 of balls in X a (possibly empty) set $\mathrm{Sim}_X(B_1, B_2)$ of surjective similarities $B_1 \to B_2$ such that whenever B_1, B_2, B_3 are balls in X, the following properties hold:

(1) (Finiteness) $\mathrm{Sim}_X(B_1, B_2)$ is a finite set.
(2) (Identities) $\mathrm{id}_{B_1} \in \mathrm{Sim}_X(B_1, B_1)$.
(3) (Inverses) If $h \in \mathrm{Sim}_X(B_1, B_2)$, then $h^{-1} \in \mathrm{Sim}_X(B_2, B_1)$.
(4) (Compositions) If $h_1 \in \mathrm{Sim}_X(B_1, B_2)$ and $h_2 \in \mathrm{Sim}_X(B_2, B_3)$, then $h_2 h_1 \in \mathrm{Sim}_X(B_1, B_3)$.
(5) (Restrictions) If $h \in \mathrm{Sim}_X(B_1, B_2)$ and $B_3 \subseteq B_1$, then

$$h|B_3 \in \mathrm{Sim}_X(B_3, h(B_3)).$$

Definition 3.6. A homeomorphism $h\colon X \to X$ is *locally determined by* Sim_X provided that for every $x \in X$, there exists a ball B' in X such that $x \in B'$, $h(B')$ is a ball in X, and $h|B' \in \mathrm{Sim}(B', h(B'))$.

Definition 3.7. The *finite similarity structure* (*FSS*) *group* $\Gamma(\mathrm{Sim}_X)$ is the set of all homeomorphisms $h\colon X \to X$ such that h is locally determined by Sim_X.

Remark 3.8. The fact that $\Gamma(\mathrm{Sim}_X)$ is a group under composition is due to Hughes [7].

3.2. **A description of the homeomorphisms determined by a similarity structure.** In this subsection, we offer a somewhat simpler description of the elements in the groups $\Gamma(\mathrm{Sim}_X)$ (Proposition 3.11), which shows that elements $\gamma \in \Gamma(\mathrm{Sim}_X)$ can be described in a manner reminiscent of the tree pair representatives for elements in Thompson's group V (see [3]).

Definition 3.9. We define the *standard partitions* of X inductively as follows.

(1) $\{X\}$ is a standard partition.
(2) If $\mathcal{P} = \{\widehat{B}_1, \ldots, \widehat{B}_n\}$ is a standard partition, and $\{B_1, \ldots, B_m\}$ is the partition of $\widehat{B}_i$ into maximal proper subballs, then $(\mathcal{P} - \{\widehat{B}_i\}) \cup \{B_1, \ldots, B_m\}$ is also a standard partition.

Clearly, each standard partition is a partition of X into balls.

Lemma 3.10. *Every partition $\mathcal{P}$ of X into balls is standard.*

Proof. We prove this by induction on $|\mathcal{P}|$. It is clearly true if $|\mathcal{P}| = 1$. We note that compactness implies that each partition $\mathcal{P}$ of X into balls must be finite.

For an arbitrary ball $B \subseteq X$, we define the *depth* of B, denoted $d(B)$, to be the number of distinct balls of X that contain B. (This definition is similar to Definition 3.19 from [5].) We note that $d(B)$ is a positive integer by Lemma 3.2(4), and $d(X) = 1$.

Now we suppose that a partition $\mathcal{P}$ is given to us. We assume inductively that all partitions with smaller numbers of balls are standard. By finiteness of $\mathcal{P}$, there is some ball B having maximum depth m, where we can assume that $m \geq 2$. Let $\widehat{B}$ denote the ball containing B as a maximal proper subball. Clearly, $d(\widehat{B}) = m - 1$. We let $\{B_0, \ldots, B_k\}$ be the collection of maximal proper subballs of $\widehat{B}$, where $B = B_0$ and $k \geq 1$.

We claim that $\{B_0, B_1, B_2, \ldots, B_k\} \subseteq \mathcal{P}$. Choose $x \in B_i$. Our assumptions imply that x is in some ball B' of $\mathcal{P}$ such that $d(B') \leq m$. The only such balls are B_i, $\widehat{B}$, and any balls that contain $\widehat{B}$. (This uses an appeal to Lemma 3.2(2).) Since $\widehat{B} \cap B_0 \neq \emptyset$ and $B_0 = B \in \mathcal{P}$, the only possibility is that $B' = B_i$, since $\mathcal{P}$ is a partition. This proves that $\{B_0, \ldots, B_k\} \subseteq \mathcal{P}$.

Now we consider the partition $\mathcal{P}' = (\mathcal{P} - \{B_0, \ldots, B_k\}) \cup \{\widehat{B}\}$. This partition is standard by the induction hypothesis, and it follows directly that $\mathcal{P}$ itself is standard. □

Proposition 3.11. *Let Sim_X be a finite similarity structure on X, and let $\gamma \in \Gamma(\mathrm{Sim}_X)$. There exist standard partitions $\mathcal{P}_1 = \{B_1, \ldots, B_n\}$ and $\mathcal{P}_2$ of X, a bijection $\phi : \mathcal{P}_1 \to \mathcal{P}_2$, and elements $\sigma_i \in \mathrm{Sim}(B_i, \phi(B_i))$ such that $\gamma_{|B_i} = \sigma_i$, for $i = 1, \ldots, n$.*

Moreover, we can arrange that the balls B_i are maximal in the sense that if $B \subseteq X$ and $\gamma(B)$ are balls such that $\gamma_{|B} \in \mathrm{Sim}_X(B, \gamma(B))$, then $B \subseteq B_i$, for some $i \in \{1, \ldots, n\}$.

Proof. Since γ is locally determined by Sim_X, we can find an open cover of X by balls such that the restriction of γ to each ball is a local similarity in the Sim_X-structure. By compactness of X, we can pass to a finite subcover. An application of Lemma 3.2(2) allows us to pass to a subcover that is also a partition. We call this partition $\mathcal{P}_1$. We can then set $\mathcal{P}_2 = \gamma(\mathcal{P}_1)$. Both partitions are standard by Lemma 3.10.

The final statement is essentially Lemma 3.7 from [7]. □

4. Braided diagram groups and groups determined by finite similarity structures

4.1. Braided diagram groups over tree-like semigroup presentations.

Definition 4.1. A semigroup presentation $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ is *tree-like* if,

(1) every relation $(w_1, w_2) \in \mathcal{R}$ satisfies $|w_1| = 1$ and $|w_2| > 1$;
(2) if $(a, w_1), (a, w_2) \in \mathcal{R}$, then $w_1 \equiv w_2$.

Example 4.2. The generalized Thompson's groups V_d are isomorphic to the braided diagram groups $D_b(\mathcal{P}, x)$, where $\mathcal{P} = \langle x \mid (x, x^d) \rangle$ is a tree-like semigroup presentation. This fact was already proved in [6] and [4], and it is also a consequence of Theorem 4.12.

Example 4.3. Consider the graph G_n made up from a disjoint union of n rays: $G_n = \{1, \ldots, n\} \times [0, \infty)$. We assume that each ray is given the standard CW-complex structure with a vertex at each integer. We define the *Houghton group* H_n to be the set of bijections h of the vertices G_n^0 such that

(1) h preserves adjacency, with at most finitely many exceptions, and
(2) h preserves ends: that is, for each $i \in \{1, \ldots, n\}$, there are $i_1, i_2 \in \mathbb{N}$ such that $h(\{i\} \times [i_1, \infty)) = \{i\} \times [i_2, \infty)$.

Ken Brown [2] showed that H_n is a group of type F_{n-1} but not of type F_n.

We will sketch a proof that each Houghton group H_n is a braided diagram group over a tree-like semigroup presentation. (It follows, in particular, that each of these groups is $\Gamma(\mathrm{Sim}_X)$, for an appropriate compact ultrametric space X and finite similarity structure Sim_X, by Theorem 4.12.) For $n \geq 2$, consider the semigroup presentation

$$\mathcal{P}_n = \langle a, r, x_1, \ldots, x_n \mid (r, x_1 x_2 x_3 \ldots x_n), (x_1, a x_1), (x_2, a x_2), \ldots, (x_n, a x_n) \rangle.$$

Similarly, we can define $\mathcal{P}_1 = \langle a, r \mid (r, ar) \rangle$. We claim that $D_b(\mathcal{P}_n, r)$ is isomorphic to H_n. We sketch the proof for $n \geq 2$; the proof in case $n = 1$ is very similar.

The elements of $D_b(\mathcal{P}_n, r)$ can be expressed in the form $\Delta_2 \circ \Delta_1^{-1}$, where each transistor in Δ_i is "positive"; i.e., the top label of the transistor is the left side of a relation in $\mathcal{P}_n$, and the bottom label is the right side. (This is proved as part of the proof of Theorem 4.12.) We can think of each diagram Δ_i as a recipe for separating G_n into connected components. The wires running between Δ_1 and Δ_2 in the concatenation $\Delta_2 \circ \Delta_1^{-1}$ describe how these connected components should be matched by the bijection $h \in H_n$. To put it more explicitly, the relations represent the following operations:

(1) the relation $(r, x_1x_2\dots x_n)$ describes the initial configuration G_n of n disjoint rays. The letters x_i ($i \in \{1,\dots,n\}$) represent the isomorphism types of the different rays. The different subscripts prevent different ends of G_n from being permuted nontrivially. (If we wish to remove the end-preserving condition above, we can simply replace the n distinct symbols x_1, x_2, ..., x_n by the single symbol x.)
(2) the relation (x_i, ax_i) (for $i \in \{1,\dots,n\}$) represents the action of breaking the initial vertex away from the ray of isomorphism type x_i. The initial vertex of the ray gets the label a, and the new ray retains the label x_i (since it is of the same combinatorial type as the original ray, and the new ray is a permissible target for the original ray under the action of the Houghton group). The letter a thus represents a single floating vertex. Any two such vertices can be matched by an element of the Houghton group, which is why we use a single label for all of these vertices.

We illustrate how an (r,r)-diagram over $\mathcal{P}_2$ represents an element of H_2. Figure 3 depicts an (r,r)-diagram Δ_h over $\mathcal{P}_2$. This diagram Δ_h represents the element $h \in H_2$ that sends: $(2,n)$ to $(2,n-1)$, for each $n \geq 1$, $(1,n)$ to $(1,n+1)$ (for all n), and $(2,0)$ to $(1,0)$. Note that the bottom portion of the diagram represents a subdivision of the domain, and the top portion represents a subdivision of the range. It is straightforward to check that the indicated function does not change if we insert or remove dipoles.

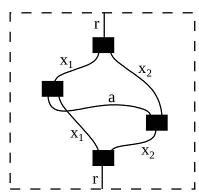

FIGURE 3. The given (r,r)-diagram over $\mathcal{P}_2$ represents an element of Houghton's group H_2.

Example 4.4. The same principle can be used to exhibit the group $\Gamma = \mathrm{QAut}(\mathcal{T}_{2,c})$ as a braided diagram group over a tree-like semigroup presentation. Here Γ (as defined in [1]) is the group of self-bijections h of the vertices of the infinite ordered rooted binary tree T such that

(1) h preserves adjacency, with at most finitely many exceptions, and
(2) h preserves the left-right ordering of the edges incident with and below a given vertex, again with at most finitely many exceptions.

Consider the semigroup presentation $\mathcal{P} = \langle a, x \mid (x, xax)\rangle$. We claim that $D_b(\mathcal{P}, x)$ is isomorphic to Γ. Much of the discussion from the previous example carries over identically. We will simply indicate how the single relation allows us to simulate breaking the binary tree into pieces. (Such a dissection of T would be represented by a positive diagram, as above. The wires connecting the bottoms of the positive diagrams Δ_1 and Δ_2 would again represent how the resulting pieces are matched by a bijection.)

The letter x represents the isomorphism type of the binary tree T. The relation represents breaking the tree at the root. The result of this operation yields a floating vertex (represented by the letter a), and two new rooted binary trees (both represented by x). The first x in xax represents the left branch, and the second x represents the right. The description of the isomorphism of $D_b(\mathcal{P}, x)$ with Γ now follows the general pattern of the previous example.

Remark 4.5. Example 4.3 shows that the F_∞ result of [5] cannot be extended to all groups determined by finite similarity structures (as defined in Section 3).

All of the groups in the above examples act properly on CAT(0) cubical complexes by a construction of [4].

We note also that the representation of the above groups as braided diagram groups suggests a method for producing embeddings into other groups, such as (perhaps most notably) Thompson's group V. For instance, the group from Example 4.4 can be embedded into V as follows. Given a braided diagram Δ over $\mathcal{P}$, systematically replace each a label with an x. The result is a braided diagram over the semigroup presentation $\mathcal{P}' = \langle x \mid (x, x^3) \rangle$. The indicated function $\phi : D_b(\mathcal{P}, x) \to D_b(\mathcal{P}', x)$ is easily seen to be a homomorphism, and ϕ is injective since it sends reduced diagrams to reduced diagrams. We can now appeal to the fact that $D_b(\mathcal{P}', x) \cong V_3$ (the 3-ary version of Thompson's group V), and the latter group embeds in V itself.

The group from Example 4.4 was previously known to embed in V by a result of [1].

4.2. **Groups determined by small similarity structures.**

Definition 4.6. Let X be a compact ultrametric space. We say that the finite similarity structure Sim_X is *small* if, for every pair of balls B_1, B_2 in X, $|\mathrm{Sim}_X(B_1, B_2)| \leq 1$.

Definition 4.7. Let X be a compact ultrametric space endowed with a small similarity structure Sim_X. If $B \subseteq X$ is a ball in X that is not an isolated point, then a *local ball order* at B is an assignment of a linear order $<$ to the set $\{\widehat{B}_1, \ldots, \widehat{B}_n\}$ of maximal proper subballs of B. A *ball order* on X is an assignment of such a linear order to each ball $B \subseteq X$ that is not a singleton. The ball order is *compatible* with Sim_X if each $h \in \mathrm{Sim}_X(B_1, B_2)$ induces an order-preserving bijection of the maximal proper subballs of B_1 and B_2, for all choices of B_1 and B_2.

Lemma 4.8. *Let X be a compact ultrametric space endowed with a small similarity structure. There exists a ball order on X that is compatible with* Sim_X.

Proof. We recall a definition from [5]. Let $B \subseteq X$ be a metric ball. Let $[B] = \{B' \subseteq X \mid \mathrm{Sim}_X(B, B') \neq \emptyset\}$; $[B]$ is called the Sim_X*-class* of B.

From a given Sim_X-class of balls, choose a ball B. If B is not a singleton, then there exists a collection of maximal proper subballs B_1, …, B_n of B. Choose a linear order on this collection of balls; without loss of generality, $B_1 < B_2 < \ldots < B_n$. If B' is another ball in $[B]$, then we can let h denote the unique element of $\mathrm{Sim}_X(B, B')$. This h carries the maximal proper subballs of B into maximal proper subballs of B', and thereby induces an order $h(B_1) < h(B_2) < \ldots < h(B_n)$ on the maximal proper subballs of B'. This procedure gives a local ball order to each ball $B' \in [B]$.

We repeat this procedure for each Sim_X-class of balls. The result is a ball order on X that is compatible with Sim_X. □

Remark 4.9. A ball order on X also determines a linear order on any given collection of pairwise disjoint balls in X. For let $\mathcal{C}$ be such a collection, and let $B_1, B_2 \in \mathcal{C}$. There is a unique smallest ball $B \subseteq X$ that contains both B_1 and B_2, by Lemma 3.2(4). Let $\{\widehat{B}_1, \ldots, \widehat{B}_n\}$ be the collection of maximal proper subballs of B. By minimality of B, we must have that B_1 and B_2 are contained in distinct maximal proper subballs of B; say $B_1 \subseteq \widehat{B}_1$ and $B_2 \subseteq \widehat{B}_2$. We write $B_1 < B_2$ if and only if $\widehat{B}_1 < \widehat{B}_2$. This defines a linear order on $\mathcal{C}$. The verification is straightforward.

Definition 4.10. Let X be a compact ultrametric space with a small similarity structure Sim_X and a compatible ball order. Define a semigroup presentation $\mathcal{P}_{\mathrm{Sim}_X} = \langle \Sigma \mid \mathcal{R} \rangle$ as follows. Let

$$\Sigma = \{[B] \mid B \text{ is a ball in } X\}.$$

If $B \subseteq X$ is a ball, let $B_1, \ldots, B_n$ be the maximal proper subballs of B, listed in order. If B is a point, then $n = 0$. We set

$$\mathcal{R} = \{([B], [B_1][B_2] \ldots [B_n]) \mid n \geq 1, B \text{ is a ball in } X\}.$$

Remark 4.11. We note that $\mathcal{P}_{\mathrm{Sim}_X}$ will always be a tree-like semigroup presentation, for any choice of compact ultrametric space X, small similarity structure Sim_X, and compatible ball order.

4.3. **The main theorem.**

Theorem 4.12. *If X is a compact ultrametric space with a small similarity structure Sim_X and compatible ball order, then*

$$\Gamma(\mathrm{Sim}_X) \cong D_b(\mathcal{P}_{\mathrm{Sim}_X}, [X]).$$

Conversely, if $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ is a tree-like semigroup presentation, and $x \in \Sigma$, then there is a compact ultrametric space $X_{\mathcal{P}}$, a small finite similarity structure $\mathrm{Sim}_{X_{\mathcal{P}}}$, and a compatible ball order such that

$$D_b(\mathcal{P}, x) \cong \Gamma(\mathrm{Sim}_{X_{\mathcal{P}}}).$$

Proof. If $\gamma \in \Gamma(\mathrm{Sim}_X)$, then, by Proposition 3.11, there are standard partitions $\mathcal{P}_1$, $\mathcal{P}_2$ of X into balls, and a bijection $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ such that, for any $B \in \mathcal{P}_1$, $\gamma(B) = \phi(B)$ and $\gamma|_B \in \mathrm{Sim}_X(B, \gamma(B))$. Since $|\mathrm{Sim}_X(B, \gamma(B))| \leq 1$, the triple $(\mathcal{P}_1, \mathcal{P}_2, \phi)$ determines γ without ambiguity. We call $(\mathcal{P}_1, \mathcal{P}_2, \phi)$ a *defining triple* for γ. Note that a given γ will usually have many defining triples. Let $\mathcal{D}$ be the set of all defining triples, for γ running over all of $\Gamma(\mathrm{Sim}_X)$.

We will now define a map $\psi : \mathcal{D} \to D_b(\mathcal{P}_{\mathrm{Sim}_X}, [X])$. To a partition $\mathcal{P}$ of X into balls, we first assign a braided diagram $\Delta_{\mathcal{P}}$ over $\mathcal{P}_{\mathrm{Sim}_X}$. There is a transistor $T_B \in \mathcal{T}(\Delta_{\mathcal{P}})$ for each ball B which properly contains some ball of $\mathcal{P}$. There is a wire $W_B \in \mathcal{W}(\Delta_{\mathcal{P}})$ for each ball B which contains a ball of $\mathcal{P}$. The wires are attached as follows:

(1) If $B = X$, then we attach the top of W_B to the top of the frame. If $B \neq X$, then the top of the wire W_B is attached to the bottom of the transistor $T_{\widehat{B}}$, where $\widehat{B}$ is the (unique) ball that contains B as a maximal proper subball.

Moreover, we attach the wires in an "order-respecting" fashion. Thus, if $\widehat{B}$ is a ball properly containing balls of $\mathcal{P}$, we let $B_1, B_2, \ldots, B_n$ be the collection of maximal proper subballs of $\widehat{B}$, listed in order. We attach the wires $W_{B_1}, W_{B_2}, \ldots, W_{B_n}$ so that $t(W_{B_i})$ is to the left of $t(W_{B_j})$ on the bottom of $T_{\widehat{B}}$ if $i < j$.

(2) The bottom of the wire W_B is attached to the top of T_B if B properly contains a ball of $\mathcal{P}$. If not (i.e., if $B \in \mathcal{P}$), then we attach the bottom of W_B to the bottom of the frame. We can arrange, moreover, that the wires are attached in an order-respecting manner to the bottom of the frame. (Thus, if $B_1 < B_2$ ($B_1, B_2 \in \mathcal{P}$), we have that $b(W_{B_1})$ is to the left of $b(W_{B_2})$.)

The labelling function $\ell : \mathcal{W}(\Delta_{\mathcal{P}}) \to \Sigma$ sends W_B to $[B]$. It is straightforward to check that the resulting $\Delta_{\mathcal{P}}$ is a braided diagram over $\mathcal{P}_{\mathrm{Sim}_X}$. The top label of $\Delta_{\mathcal{P}}$ is $[X]$.

Given a bijection $\phi : \mathcal{P}_1 \to \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of X into balls and $[B] = [\phi(B)]$, we can define a braided diagram Δ_ϕ over $\mathcal{P}_{\mathrm{Sim}_X}$ as follows. We let $\mathcal{T}(\Delta_\phi) = \emptyset$, and $\mathcal{W}(\Delta_\phi) = \{W_B \mid B \in \mathcal{P}_1\}$. We attach the top of each wire to the frame in such a way that $t(W_{B_1})$ is to the left of $t(W_{B_2})$ if $B_1 < B_2$. (Here $<$ refers to the ordering from Remark 4.9.) We attach the bottom of each wire to the bottom of the frame in such a way that $b(W_{B_1})$ is to the left of $b(W_{B_2})$ if $\phi(B_1) < \phi(B_2)$.

Now, for a defining triple $(\mathcal{P}_1, \mathcal{P}_2, \phi) \in \mathcal{D}$, we set $\psi((\mathcal{P}_1, \mathcal{P}_2, \phi)) = \Delta_{\mathcal{P}_2} \circ \Delta_{\phi^{-1}} \circ \Delta_{\mathcal{P}_1}^{-1} \in \mathcal{D}_b(\mathcal{P}_{\mathrm{Sim}_X}, [X])$.

We claim that any two defining triples $(\mathcal{P}_1, \mathcal{P}_2, \phi)$, $(\mathcal{P}_1', \mathcal{P}_2', \phi')$ for a given $\gamma \in \Gamma(\mathrm{Sim}_X)$ have the same image in $D_b(\mathcal{P}_{\mathrm{Sim}_X}, [X])$, modulo dipoles. We begin by proving an intermediate statement. Let $(\mathcal{P}_1, \mathcal{P}_2, \phi)$ be a defining triple. Let $B \in \mathcal{P}_1$, and let $\widehat{B}_1, \ldots, \widehat{B}_n$ be the collection of maximal proper subballs of B, listed in order. We let $B' = \phi(B)$ and let $\widehat{B}_1', \ldots, \widehat{B}_n'$ be the collection of maximal proper subballs of B'. (Note that $[B'] = [B]$ by our assumptions, so both have the same number of maximal proper subballs, and in fact $[\widehat{B}_i] = [\widehat{B}_i']$ for $i = 1, \ldots, n$, since $\gamma|_B \in \mathrm{Sim}_X(B, B')$ and the elements of $\mathrm{Sim}_X(B, B')$ preserve order.) We set $\widehat{\mathcal{P}}_1 = (\mathcal{P}_1 - \{B\}) \cup \{\widehat{B}_1, \ldots, \widehat{B}_n\}$, $\widehat{\mathcal{P}}_2 = (\mathcal{P}_2 - \{B'\}) \cup \{\widehat{B}_1', \ldots, \widehat{B}_n'\}$, and $\widehat{\phi}|_{\mathcal{P}_1 - \{B\}} = \phi|_{\mathcal{P}_1 - \{B\}}$, $\widehat{\phi}(\widehat{B}_i) = \widehat{B}_i'$. We say that $(\widehat{\mathcal{P}}_1, \widehat{\mathcal{P}}_2, \widehat{\phi})$ is obtained from $(\mathcal{P}_1, \mathcal{P}_2, \phi)$ by *subdivision* at (B, B'). A straightforward argument shows that $\psi((\widehat{\mathcal{P}}_1, \widehat{\mathcal{P}}_2, \widehat{\phi}))$ is in fact obtained from $\psi((\mathcal{P}_1, \mathcal{P}_2, \phi))$ by inserting a dipole. We omit the details, which rely on the fact that each element of the Sim_X-structure preserves the local ball order.

Now suppose that $(\mathcal{P}_1, \mathcal{P}_2, \phi)$ and $(\mathcal{P}_1', \mathcal{P}_2', \phi')$ are defining triples for the same element $\gamma \in \Gamma(\mathrm{Sim}_X)$. We can find a common refinement $\mathcal{P}_1''$ of $\mathcal{P}_1$ and $\mathcal{P}_1'$. Using the fact that all partitions of X into balls are standard (Lemma 3.10), we can pass from $(\mathcal{P}_1, \mathcal{P}_2, \phi)$ to $(\mathcal{P}_1'', \widehat{\mathcal{P}}_2, \widehat{\phi})$ by repeated subdivision (for some partition $\widehat{\mathcal{P}}_2$ of X into balls and some bijection $\widehat{\phi} : \mathcal{P}_1'' \to \widehat{\mathcal{P}}_2$). Since subdivision does not change the values of ψ modulo dipoles, $\psi((\mathcal{P}_1, \mathcal{P}_2, \phi)) = \psi((\mathcal{P}_1'', \widehat{\mathcal{P}}_2, \widehat{\phi}))$ modulo dipoles. Similarly, we can subdivide $(\mathcal{P}_1', \mathcal{P}_2', \phi')$ repeatedly in order to obtain $(\mathcal{P}_1'', \widehat{\mathcal{P}}_2', \widehat{\phi}')$, where $\psi((\mathcal{P}_1', \mathcal{P}_2', \phi')) = \psi((\mathcal{P}_1'', \widehat{\mathcal{P}}_2', \widehat{\phi}'))$ modulo dipoles. Both $(\mathcal{P}_1'', \widehat{\mathcal{P}}_2', \widehat{\phi}')$ and $(\mathcal{P}_1'', \widehat{\mathcal{P}}_2, \widehat{\phi})$ are defining triples for γ, so we are forced to have $\widehat{\phi} = \widehat{\phi}'$ and $\widehat{\mathcal{P}}_2 = \widehat{\mathcal{P}}_2'$. It follows that $\psi((\mathcal{P}_1, \mathcal{P}_2, \phi)) = \psi((\mathcal{P}_1', \mathcal{P}_2', \phi'))$, so ψ induces a function from $\Gamma(\mathrm{Sim}_X)$ to $D_b(\mathcal{P}_{\mathrm{Sim}_X}, [X])$. We will call this function $\widehat{\psi}$.

Now we will show that $\widehat{\psi} : \Gamma(\mathrm{Sim}_X) \to D_b(\mathcal{P}_{\mathrm{Sim}_X}, [X])$ is a homomorphism. Let $\gamma, \gamma' \in \Gamma(\mathrm{Sim}_X)$. After subdividing as necessary, we can choose defining triples $(\mathcal{P}_1, \mathcal{P}_2, \phi)$ and $(\mathcal{P}_1', \mathcal{P}_2', \phi')$ for γ and γ' (respectively) in such a way that $\mathcal{P}_2 = \mathcal{P}_1'$. It follows easily that $(\mathcal{P}_1, \mathcal{P}_2', \phi'\phi)$ is a defining triple for $\gamma'\gamma$. Therefore, $\widehat{\psi}(\gamma'\gamma) = \Delta_{\mathcal{P}_2'} \circ \Delta_{(\phi'\phi)^{-1}} \circ \Delta_{\mathcal{P}_1}^{-1}$. Now

$$\begin{aligned} \widehat{\psi}(\gamma') \circ \widehat{\psi}(\gamma) &= \Delta_{\mathcal{P}_2'} \circ \Delta_{(\phi')^{-1}} \circ \Delta_{\mathcal{P}_1'}^{-1} \circ \Delta_{\mathcal{P}_2} \circ \Delta_{\phi^{-1}} \circ \Delta_{\mathcal{P}_1}^{-1} \\ &= \Delta_{\mathcal{P}_2'} \circ \Delta_{(\phi')^{-1}} \circ \Delta_{\phi^{-1}} \circ \Delta_{\mathcal{P}_1}^{-1} \\ &= \Delta_{\mathcal{P}_2'} \circ \Delta_{(\phi'\phi)^{-1}} \circ \Delta_{\mathcal{P}_1}^{-1} \end{aligned}$$

Therefore, $\widehat{\psi}$ is a homomorphism.

We now show that $\widehat{\psi} : \Gamma(\mathrm{Sim}_X) \to D_b(\mathcal{P}_{\mathrm{Sim}_X}, [X])$ is injective. Suppose that $\widehat{\psi}(\gamma) = 1$. Using the final statement of Proposition 3.11, we choose a defining triple $(\mathcal{P}_1, \mathcal{P}_2, \phi)$ for γ with the property that, if $B \subseteq X$ is a ball, $\gamma(B)$ is a ball, and $\gamma|_B \in \mathrm{Sim}_X(B, \gamma(B))$, then B is contained in some ball of $\mathcal{P}_1$. We claim that $\psi((\mathcal{P}_1, \mathcal{P}_2, \phi))$ is a reduced diagram. If there were a dipole (T_1, T_2), then we would have $T_1 \in \mathcal{T}(\Delta_{\mathcal{P}_1}^{-1})$ and $T_2 \in \mathcal{T}(\Delta_{\mathcal{P}_2})$, since it is impossible for $\Delta_{\mathcal{P}}$ to contain any dipoles, for any partition $\mathcal{P}$ of X into balls. Thus $T_1 = T_{B_1}$ and $T_2 = T_{B_2}$, where $[B_1] = [B_2]$ and the wires from the bottom of T_{B_2} attach to the top of T_{B_1}, in order. This means that, if $\widehat{B}_1, \ldots, \widehat{B}_n$ are the maximal proper subballs of B_1, and $\widehat{B}_1', \ldots, \widehat{B}_n'$ are the maximal proper subballs of B_2, then $\gamma(\widehat{B}_i) = \widehat{B}_i'$, where the latter is a ball, and $\gamma|_{\widehat{B}_i} \in \mathrm{Sim}_X(\widehat{B}_i, \widehat{B}_i')$.

Now, since $[B_1] = [B_2]$, there is $h \in \mathrm{Sim}_X(B_1, B_2)$. Since Sim_X is closed under restrictions and h preserves order, we have $h_i \in \mathrm{Sim}_X(\widehat{B}_i, \widehat{B}_i')$ for $i = 1, \ldots, n$, where $h_i = h|_{\widehat{B}_i}$. It follows that $\gamma|_{\widehat{B}_i} = h_i$, so, in particular, $\gamma|_{B_1} = h$. Since B_1 properly contains some ball in $\mathcal{P}_1$, this is a contradiction. Thus, $\psi((\mathcal{P}_1, \mathcal{P}_2, \phi))$ is reduced.

We claim that $\psi((\mathcal{P}_1, \mathcal{P}_2, \phi))$ contains no transistors (due to the condition $\widehat{\psi}(\gamma) = 1$). We have shown that $\psi((\mathcal{P}_1, \mathcal{P}_2, \phi))$ is a reduced diagram in the same class as the identity $1 \in D_b(\mathcal{P}_{\mathrm{Sim}_X}, [X])$. The identity can be represented as the (unique) $([X], [X])$-diagram Δ_1 with only a single wire, W_X, and no transistors. We must have $\psi((\mathcal{P}_1, \mathcal{P}_2, \phi)) \equiv \Delta_1$. Thus, there is no ball that properly contains a ball of $\mathcal{P}_1$. It can only be that $\mathcal{P}_1 = \{X\}$, so we must have $\gamma \in \mathrm{Sim}_X(X, X)$. This forces $\gamma = 1$, so $\widehat{\psi}$ is injective.

Finally we must show that $\widehat{\psi} : \Gamma(\mathrm{Sim}_X) \to D_b(\mathcal{P}_{\mathrm{Sim}_X}, [X])$ is surjective. Let Δ be a reduced $([X], [X])$-diagram over $\mathcal{P}_{\mathrm{Sim}_X}$. A transistor $T \in \mathcal{T}(\Delta)$ is called *positive* if its top label is the left side of a relation in $\mathcal{P}_{\mathrm{Sim}_X}$, otherwise (i.e., if the top label is the right side of a relation in $\mathcal{P}_{\mathrm{Sim}_X}$) the transistor T is *negative*. It is easy to see that the sets of positive and negative transistors partition $\mathcal{T}(\Delta)$. We claim that, if Δ is reduced, then we cannot have $T_1 \dot{\preceq} T_2$ when T_1 is positive and T_2 is negative. If we had such $T_1 \dot{\preceq} T_2$, then we could find $T_1' \preceq T_2'$, where T_1' is positive and T_2' is negative. Since T_1' is positive, there is only one wire W attached to the top of T_1'. This wire must be attached to the bottom of T_2', since $T_1' \preceq T_2'$, and it must be the only wire attached to the bottom of T_2', since T_2' is negative and $\mathcal{P}_{\mathrm{Sim}_X}$ is a tree-like semigroup presentation by Remark 4.11. Suppose that $\ell(w) = [B]$. By the definition of $\mathcal{P}_{\mathrm{Sim}_X}$, $[B]$ is the left side of exactly one relation, namely $([B], [B_1][B_2]\ldots[B_n])$, where the B_i are maximal proper subballs of B, listed in order. It follows that the bottom label of T_1' is $[B_1][B_2]\ldots[B_n]$ and the top label of T_2' is $[B_1][B_2]\ldots[B_n]$. Therefore (T_1', T_2') is a dipole. This proves the claim.

A diagram over $\mathcal{P}_{\mathrm{Sim}_X}$ is *positive* if all of its transistors are positive, and *negative* if all of its transistors are negative. We note that Δ is positive if and only if Δ^{-1} is negative, by the description of inverses in the proof of Theorem 2.13. The above reasoning shows that any reduced $([X],[X])$-diagram over $\mathcal{P}_{\mathrm{Sim}_X}$ can be written $\Delta = \Delta_1^+ \circ \left(\Delta_2^+\right)^{-1}$, where Δ_i^+ is a positive diagram for $i = 1, 2$.

We claim that any positive diagram Δ over $\mathcal{P}_{\mathrm{Sim}_X}$ with top label $[X]$ is $\Delta_{\mathcal{P}}$ (up to a reordering of the bottom contacts), where $\mathcal{P}$ is some partition of X. There is a unique wire $W \in \mathcal{W}(\Delta)$ making a top contact with the frame. We call this wire W_X. Note that its label is $[X]$ by our assumptions. The bottom contact of W_X lies either on the bottom of the frame, or on top of some transistor. In the first case, we have $\Delta = \Delta_{\mathcal{P}}$ for $\mathcal{P} = \{X\}$ and we are done. In the second, the bottom contact of W_X lies on top of some transistor T, which we call T_X. Since the top label of T_X is $[X]$, the bottom label must be $[B_1]\ldots[B_k]$, where $B_1, \ldots, B_k$ are the maximal proper subballs of X. Thus there are wires $W_1, \ldots, W_k$ attached to the bottom of T_X, and we have $\ell(W_i) = [B_i]$, for $i = 1, \ldots, k$. We relabel each of the wires $W_{B_1}, \ldots, W_{B_k}$, respectively. Note that $\{B_1, \ldots, B_k\}$ is a partition of X into balls. We can continue in this way, inductively labelling each wire with a ball $B \subseteq X$. If we let $\overline{B}_1, \ldots, \overline{B}_m$ be the resulting labels of the wires which make bottom contacts with the frame, then $\{\overline{B}_1, \ldots, \overline{B}_m\} = \mathcal{P}$ is a partition of X into balls, and $\Delta = \Delta_{\mathcal{P}}$ by construction, up to a reordering of the bottom contacts.

We can now prove surjectivity of $\widehat{\psi}$. Let $\Delta \in D_b(\mathcal{P}_{\mathrm{Sim}_X}, [X])$ be reduced. We can write $\Delta = \Delta_2^+ \circ \left(\Delta_1^+\right)^{-1}$, where Δ_i^+ is positive, for $i = 1, 2$. It follows that $\Delta_i^+ = \Delta_{\mathcal{P}_i} \circ \sigma_i$, for $i = 1, 2$, where $\mathcal{P}_i$ is a partition of X into balls and σ_i is diagram containing no transistors. Thus, $\Delta = \Delta_{\mathcal{P}_2} \circ \sigma_2 \circ \sigma_1^{-1} \circ \Delta_{\mathcal{P}_1}^{-1} = \psi((\mathcal{P}_1, \mathcal{P}_2, \phi))$, where $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ is a bijection determined by $\sigma_2 \circ \sigma_1^{-1}$. Therefore, $\widehat{\psi}$ is surjective.

Now we must show that if $\mathcal{P} = \langle \Sigma \mid \mathcal{R} \rangle$ is a tree-like semigroup presentation, $x \in \Sigma$, then there is a compact ultrametric space $X_{\mathcal{P}}$, a small similarity structure $\mathrm{Sim}_{X_{\mathcal{P}}}$, and a compatible ball order, such that $D_b(\mathcal{P}, x) \cong \Gamma(\mathrm{Sim}_{X_{\mathcal{P}}})$. Construct a labelled ordered simplicial tree $T_{(\mathcal{P},x)}$ as follows. Begin with a vertex $*$, the root, labelled by $x \in \Sigma$. By the definition of tree-like semigroup presentation (Definition 4.1), there is at most one relation in $\mathcal{R}$ having the word x as its left side. Let us suppose first that $(x, x_1x_2\ldots x_k) \in \mathcal{R}$, where $k \geq 2$. We introduce k children of the root, labelled $x_1, \ldots, x_k$ (respectively), each connected to the root by an edge. The children are ordered from left to right in such a way that we read the word $x_1x_2\ldots x_k$ as we read the labels of the children from left to right. If, on the other hand, x is not the left side of any relation in $\mathcal{R}$, then the tree terminates – there is only the root. We continue similarly: if x_i is the left side of some relation $(x_i, y_1y_2\ldots y_m) \in \mathcal{R}$ $(m \geq 2)$, then this relation is unique and we introduce a labelled ordered collection of children, as above. If x_i is not the left side of any relation in $\mathcal{R}$, then x_i has no children. This builds a labelled ordered tree $T_{(\mathcal{P},x)}$. We note that if a vertex $v \in T_{(\mathcal{P},x)}$ is labelled by $y \in \Sigma$, then the subcomplex $T_v \leq T_{(\mathcal{P},x)}$ spanned by v and all of its descendants is isomorphic to $T_{(\mathcal{P},y)}$, by a simplicial isomorphism which preserves the labelling and the order.

We let $\mathrm{Ends}(T_{(\mathcal{P},x)})$ denote the set of all edge-paths p in $T_{(\mathcal{P},x)}$ such that: i) p is without backtracking; ii) p begins at the root; iii) p is either infinite, or p terminates at a vertex without children. We define a metric on $\mathrm{Ends}(T_{(\mathcal{P},x)})$ as follows. If $p, p' \in \mathrm{Ends}(T_{(\mathcal{P},x)})$ and p, p' have exactly m edges in common, then we set $d(p,p') = e^{-m}$. This metric makes $\mathrm{Ends}(T_{(\mathcal{P},x)})$ a compact ultrametric space, and a ball order is given by the ordering of the tree. We can describe the balls in $\mathrm{Ends}(T_{(\mathcal{P},x)})$ explicitly. Let v be a vertex of $T_{(\mathcal{P},x)}$. We set $B_v = \{p \in \mathrm{Ends}(T_{(\mathcal{P},x)}) \mid v \text{ lies on } p\}$. Every such set is a ball, and every ball in $\mathrm{Ends}(T_{(\mathcal{P},x)})$ has this form. We can now describe a finite similarity structure $\mathrm{Sim}_{X_\mathcal{P}}$ on $\mathrm{Ends}(T_{(\mathcal{P},x)})$. Let B_v and $B_{v'}$ be the balls corresponding to the vertices $v, v' \in T_{(\mathcal{P},x)}$. If v and v' have different labels, then we set $\mathrm{Sim}_{X_\mathcal{P}}(B_v, B_{v'}) = \emptyset$. If v and v' have the same label, say $y \in \Sigma$, then there is label- and order-preserving simplicial isomorphism $\psi : T_v \to T_{v'}$. Suppose that p_v is the unique edge-path without backtracking connecting the root to v. Any point in B_v can be expressed in the form $p_v q$, where q is an edge-path without backtracking in T_v. We let $\widehat{\psi} : B_v \to B_{v'}$ be defined by the rule $\widehat{\psi}(p_v q) = p_{v'}\psi(q)$. The map $\widehat{\psi}$ is easily seen to be a surjective similarity. We set $\mathrm{Sim}_{X_\mathcal{P}}(B_v, B_{v'}) = \{\widehat{\psi}\}$. The resulting assignments give a small similarity structure $\mathrm{Sim}_{X_\mathcal{P}}$ on the compact ultrametric space $\mathrm{Ends}(T_{(\mathcal{P},x)})$ that is compatible with the ball order.

Now we can apply the first part of the theorem: setting $X_{(\mathcal{P},x)} = \mathrm{Ends}(T_{(\mathcal{P},x)})$, we have $\Gamma(\mathrm{Sim}_{X_{(\mathcal{P},x)}}) \cong D_b(\mathcal{P}_{\mathrm{Sim}_{X_{(\mathcal{P},x)}}}, [X_{(\mathcal{P},x)}]) \cong D_b(\mathcal{P}, x)$, where the first isomorphism follows from the forward direction of the theorem, and the second isomorphism follows from the canonical identification of the semigroup presentation $\mathcal{P}_{\mathrm{Sim}_{X_{(\mathcal{P},x)}}}$ with $\mathcal{P}$. □

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