A PROOF THAT THOMPSON'S GROUPS HAVE INFINITELY MANY RELATIVE ENDS

DANIEL FARLEY

ABSTRACT. We show that each of Thompson's groups F, T, and V has infinitely many ends relative to the groups $F_{[0,1/2]}$, $T_{[0,1/2]}$, and $V_{[0,1/2)}$ (respectively).

As an application, we simplify the proof, due to Napier and Ramachandran, that F, T, and V are not Kähler groups.

We also show, following an earlier argument of Ken Brown, that Thompson's groups T and V have Serre's property FA.

1. Introduction

Thompson's group F is the group of piecewise linear homeomorphisms h of the unit interval such that: i) each of the finitely many places at which h fails to be differentiable is a dyadic rational number, and ii) at every other point $x \in [0,1]$, $h'(x) \in \{2^i \mid i \in \mathbb{Z}\}$. Thompson's groups T and V have analogous definitions. The group T is a collection of homeomorphisms of the circle, and V can be viewed as a group of homeomorphisms of the Cantor set. A good introduction to all of these groups is [2].

The main purpose of this note is to prove the following theorem (definitions appear in Section 2):

Theorem 1.1. *The pairs $(F, F_{[0,1/2]})$, $(T, T_{[0,1/2]})$, and $(V, V_{[0,1/2)})$ all have infinitely many ends, where $G_S = \{g \in G \mid g \text{ is the identity on } S\}$, for $G \in \{F, T, V\}$ and $S = [0, 1/2]$ or $[0, 1/2)$.*

I became interested in proving this theorem because of a connection, established by Napier and Ramachandran, between Kähler groups and the theory of ends of pairs of groups.

Ross Geoghegan posed the problem of determining whether the group F is Kähler (see the introduction of [1]). A finitely presented group is called a *Kähler group* if it is the fundamental group of a compact Kähler manifold. The most important examples of Kähler groups (and perhaps the only ones) are the fundamental groups of smooth complex projective varieties.

Napier and Ramachandran showed that F, T, and V are not Kähler groups [9]. In [8] they proved that if every proper quotient of the group G is abelian, and G is a strictly ascending HNN extension of a finitely generated base group, then G is not Kähler. This theorem implies that F is not Kähler. Their proofs in [9] that T and V are not Kähler had two components. First, they showed that T and

2000 *Mathematics Subject Classification.* Primary: 20F69; Secondary: 20J05.

Key words and phrases. Thompson's groups, ends of group pairs, filtered ends of group pairs, property FA.

V have infinitely many filtered ends relative to the subgroups $T_{[0,1/2]}$ and $V_{[0,1/2)}$. (See [6] or [5] for a definition of filtered ends.) Second, they appealed to the main theorem from [9], which implies that if a Kähler group G has at least 3 filtered ends relative to some subgroup H, then H must have a quotient that is isomorphic to a hyperbolic surface group. Since the groups $T_{[0,1/2]}$ and $V_{[0,1/2)}$ do not have such quotients, it follows that T and V cannot be Kähler.

Earlier work of Napier and Ramachandran [7] showed that if a Kähler group G has at least 3 ends relative to a subgroup H, then H must have a quotient that is isomorphic to a hyperbolic surface group. Our main result therefore shows that we can systematically remove the word "filtered" from the discussion of the previous paragraph. The result is a simpler proof that T and V are not Kähler. (Note that there are examples of pairs (G, H), where $H \leq G$, such that (G, H) has finitely many ends, but infinitely many filtered ends. See [5], Examples 13.5.13 and 14.5.7.)

A second purpose of this note is to show that Thompson's groups T and V both have Serre's property FA, i.e., if T or V acts on a simplicial tree by automorphisms, then the action has a fixed point. The proof of this fact in Section 4 is due to Ken Brown.

Our results are relevant to a question posed by Mohan Ramachandran, who asked whether the following conditions are equivalent for a finitely presented group G: (A) G has a finite index subgroup admitting a fixed-point-free action on a simplicial tree, and (B) the pair (G, H) has two or more ends, for some subgroup H. This question appears on the problem list maintained by Mladen Bestvina. We show that property (A) fails for T and V, although T and V have multiple (indeed, an infinite number) of ends relative to certain subgroups, and thus satisfy (B).

I would like to thank Mohan Ramachandran for encouraging me to publish a proof of Theorem 1.1. After reading an earlier version of this paper, Mohan Ramachandran told me that Ken Brown had proved that T and V have property FA, and suggested the relevance of this fact to the above question. I thank Ken Brown for his notes (dating from the 1980s), which were the source of the argument in Section 4. Portions of this paper were written while I was visiting the Max Planck Institute for Mathematics. I thank the Institute for its hospitality and for the excellent working conditions during my stay.

2. Ends of Groups

A *graph* is a 1-dimensional CW complex. Let Γ be a locally finite graph. If $C \subseteq \Gamma$ is compact, then let $\mathrm{Comp}_\infty(\Gamma - C)$ denote the set of *unbounded* components of $\Gamma - C$, i.e., the components having non-compact closure. The *number of ends of* Γ, denoted $e(\Gamma)$, is

$$\sup_C \{|\mathrm{Comp}_\infty(\Gamma - C)|\}.$$

If G is a finitely generated group and S is a finite generating set, then $\Gamma_S(G)$, the *Cayley graph of G with respect to S*, is the graph having the group G as its vertex set, and an edge $e(g, s)$ connecting g to gs for each $g \in G$ and $s \in S$. The *coset graph of $H \backslash G$ with respect to S*, denoted $\Gamma_S(H\backslash G)$, is the quotient of $\Gamma_S(G)$ by the natural left action of H.

If G is a finitely generated group, then the *number of ends of G*, denoted $e(G)$, is the number of ends of its Cayley graph $\Gamma_S(G)$, where S is some finite generating set. This definition doesn't depend on the choice of finite generating set, so we will often simply leave off the subscript S, and say that $e(G)$ is the number of ends of $\Gamma(G)$. In a similar way, we define the *number of ends of the pair* (G, H), denoted $e(G, H)$, by the equation $e(G, H) = e(\Gamma(H\backslash G))$.

2.1. **Almost Invariant Subsets.** In the main argument of this paper, we will need a criterion, due to Sageev, for the pair (G, H) to have multiple ends. If G acts on a set S, then a subset T of S is said to be *almost invariant* if the symmetric difference $|gT \triangle T|$ is finite for any $g \in G$.

Theorem 2.1. [10] *Let G be a finitely generated group; let $H \leq G$. Consider the left (or right) action of G on the set G/H of left (or right) cosets of H. If there is a subset A of G/H such that*

(1) A is almost invariant, and
(2) each of A, A^c is infinite,

then $e(G, H) \geq 2$. Conversely, given a pair (G, H) such that $e(G, H) \geq 2$, there exists such an almost invariant set A in G/H.

Proof. ($\Rightarrow$) We prove the theorem in the case of the right action of G on the collection of right cosets $H\backslash G$. Begin by choosing a finite generating set S for G, and building the coset graph $\Gamma_S(H\backslash G)$. Suppose there is a set A as in the statement of the theorem.

The elements of A can naturally be identified with vertices of $\Gamma_S(H\backslash G)$. Consider the collection of all edges e which connect an element of A with an element in A^c. We claim that the union K of all such edges is a finite subgraph of $\Gamma_S(H\backslash G)$. If not, then, by the finiteness of S and without loss of generality, there must be infinitely many disjoint directed edges of K, each labelled by the same generator $s \in S$, and each running from an element of A^c to an element of A. It follows from this that each of the (infinitely many) terminal vertices of such edges are in A, but not in As. This implies that $A \triangle As$ is an infinite set, which contradicts the fact that A is almost invariant.

Since $A - K$ and $A^c - K$ are both infinite sets, and they are clearly separated by K, it follows that $\Gamma_S(G, H)$ has at least two ends.

($\Leftarrow$) We won't need to use this implication, so we leave the (easy) proof as an exercise. The idea is to choose a compact subgraph K which divides Γ into at least two unbounded components, and then use the vertices of one of these components as A. ☐

2.2. **Group Pairs with Multiple Ends.** We will apply the following Proposition in the proof of Theorem 1.1.

Proposition 2.2. *If G is a group, $H \leq G$ has infinite index in its normalizer $N_G(H)$, and $e(G, H) > 1$, then $e(G, H) = 2$ or ∞. If the pair (G, H) has two ends then the quotient*

$$N_G(H)/H$$

has an infinite cyclic subgroup of finite index.

Proof. It is enough to collect some results from [5]. First, Theorem 13.5.21 says that if H has infinite index in its normalizer in G, then the number of ends of (G, H) is 1, 2, or ∞. This directly implies the first statement.

Now we suppose that (G, H) has two ends. It follows that the coset graph $\Gamma = \Gamma_S(H/G)$ has two ends. The group $N_G(H)/H$ acts freely on Γ. We can now apply the result of Exercise 7, on page 308 of [5], which says that if an infinite group G acts freely on a path connected locally finite graph Y with two ends, then G has two ends, and G acts cocompactly on Y. (Note: We've also used the result of Proposition 10.1.12 [5] here, in order to avoid defining "strongly locally finite".) We conclude that the quotient $N_G(H)/H$ is finitely generated and has two ends.

Finally, Theorem 13.5.9 from [5] says that finitely generated groups with two ends have infinite cyclic subgroups of finite index, and, conversely, any group with an infinite cyclic subgroup of finite index is finitely generated and has two ends. □

3. A proof that Thompson's groups have infinitely many relative ends

Lemma 3.1. *Let G_1 and G_2 be finitely generated groups. Suppose that $G_1 \leq G_2$, $H_1 \leq G_1$, $H_1 \leq H_2$, and $H_2 \leq G_2$.*

If the natural map $\phi : G_1/H_1 \to G_2/H_2$ is injective and $A \subseteq G_2/H_2$ is an almost invariant subset (under the left action of G_2), then $\phi^{-1}(A)$ is almost invariant under the left action of G_1.

Proof. Let A be an almost invariant subset of G_2/H_2 under the left action of G_2. We consider the inverse image $\phi^{-1}(A)$; let $g \in G_1$. We have that

$$\phi(\phi^{-1}(A) \triangle g\phi^{-1}(A)) \subseteq A \triangle gA.$$

Since ϕ is injective, it directly follows that $\phi^{-1}(A) \triangle g\phi^{-1}(A)$ is finite. □

Proposition 3.2. *Let $V_{[0,1/2)}$ denote the subgroup of Thompson's group V which acts as the identity on $[0, 1/2)$.*

1. *The set $A = \{gV_{[0,1/2)} \mid g_{|[0,1/2)} \text{ is } affine \}$ is almost invariant under the action of V on $V/V_{[0,1/2)}$. Both A and its complement are infinite.*
2. *The quotient group $N\left(V_{[0,1/2)}\right)/V_{[0,1/2)}$ has no cyclic subgroup of finite index.*

In particular, $e(V, V_{[0,1/2)}) = \infty$.

Proof. (1) The statement that A is almost invariant is essentially the content of [3]. The main argument of [3] shows that $(v-1)\cdot\chi(A)$ (where $\chi : P(V/V_{[0,1/2)}) \to \mathbb{Z}$ is the characteristic function) is a finite sum for any element $v \in V$. This clearly means that A is almost invariant.

For suitable selections of elements g_i (i a positive integer), g_i is affine on $[0, 1/2)$ and $g_i \cdot [0, 1/2)$ is the dyadic interval $[0, 2^{-i})$. For instance, we can let $g_i = x_0^i$, where x_0 is one of the standard generators of $F \subseteq V$. As a piecewise linear homeomorphism of $[0, 1]$, x_0 is defined as follows:

$$x_0(t) = \begin{cases} \frac{1}{2}t & 0 \leq t \leq 1/2 \\ t - \frac{1}{4} & 1/2 \leq t \leq 3/4 \\ 2t - 1 & 3/4 \leq t \leq 1 \end{cases}$$

The cosets $g_iV_{[0,1/2)}$ are easily seen to be distinct, so A is infinite.

Any two distinct elements of the infinite subgroup $V_{[1/2,1)}$ represent distinct left cosets of $V_{[0,1/2)}$, and only one of these left cosets (containing the identity) lies in A. It follows that A^c is infinite. This proves (1).

(2) Each element of $V_{[1/2,1)}$ normalizes $V_{[0,1/2)}$, and any two elements in $V_{[1/2,1)}$ represent distinct left cosets of $V_{[0,1/2)}$. It follows that the group $V_{[1/2,1)}$ embeds in

the quotient from the statement of the proposition. But $V_{[1/2,1)}$ is isomorphic to V itself, and V has no cyclic subgroup of finite index. This proves (2).

The final statement now follows from (1), (2), Theorem 2.1, and Proposition 2.2. □

Proposition 3.3. $e(T, T_{[0,1/2]}) = e(F, F_{[0,1/2]}) = \infty$.

Proof. We argue that $e(F, F_{[0,1/2]}) = \infty$. The case of the group T is similar.

We first check the hypotheses of Lemma 3.1. It is clear that there are inclusions $F \to V$, $F_{[0,1/2]} \to V_{[0,1/2)}$. We next have to show that the induced map

$$\phi : F/F_{[0,1/2]} \to V/V_{[0,1/2)}$$

is injective. Suppose that $g_1F_{[0,1/2]}$ and $g_2F_{[0,1/2]}$ both have the same image under ϕ. It follows that $g_2g_1^{-1} \in V_{[0,1/2)}$. Now clearly $g_2g_1^{-1} \in F$, and it then follows from continuity that $g_2g_1^{-1} \in F_{[0,1/2]}$, so $g_1F_{[0,1/2]} = g_2F_{[0,1/2]}$. Therefore ϕ is injective. This implies that $\phi^{-1}(A) = \{gF_{[0,1/2]} \mid g_{|_{[0,1/2]}} \textit{ is linear}\}$ is an almost invariant set.

We can prove that $\phi^{-1}(A)$ and $\phi^{-1}(A)^c$ are both infinite sets as in the proof of Proposition 3.2. Indeed, as before, the cosets $x_0^iF_{[0,1/2]}$ are all distinct (proving that $\phi^{-1}(A)$ is infinite) and any two distinct elements of $F_{[1/2,1]}$ define distinct cosets of $F_{[0,1/2]}$, and exactly one of these cosets ($F_{[0,1/2]}$ itself) is in $\phi^{-1}(A)$. This proves that $\phi^{-1}(A)^c$ is also an infinite set. It follows that $e(F, F_{[0,1/2]}) = 2$ or ∞.

As in Proposition 3.2, there is an embedding of $F_{[1/2,1]}$ into the quotient group

$$N(F_{[0,1/2]})/F_{[0,1/2]},$$

and $F_{[1/2,1]} \cong F$. Since F has no infinite cyclic subgroup of finite index, it follows that $e(F, F_{[0,1/2]}) = \infty$. □

4. Proof that T and V have Serre's property FA

Suppose that G acts simplicially on the simplicial tree Γ. We say that G acts *without inversions* if, whenever $g \in G$ leaves an edge e invariant, g acts as the identity on e. We will assume (after barycentrically subdividing the tree, if necessary) that any simplicial action of G on a tree is an action without inversions. We say that G has *property FA* if every simplicial action of G on a tree has a fixed point, i.e., $G \cdot v = v$, for some $v \in \Gamma$.

We will need some standard facts about automorphisms of trees. If $g \in G$, then $Fix(g) = \{x \in \Gamma \mid g \cdot x = x\}$. The set $Fix(g)$ is a subtree of Γ if it is non-empty. If $Fix(g) \neq \emptyset$, then g is called *elliptic*; otherwise, g is *hyperbolic*.

Lemma 4.1. *Let G be a group acting on a simplicial tree Γ by automorphisms.*

1. *Let $g \in G$. Either g acts on a unique simplicial line in Γ by translation (called an* axis *for g), or $Fix(g) \neq \emptyset$.*
2. *If the fixed sets $Fix(g_1), Fix(g_2)$ are non-empty and disjoint, then $Fix(g_1g_2) = \emptyset$.*
3. *If g_1 and g_2 are elliptic and $g_1 \cdot Fix(g_2) = Fix(g_2)$, then $Fix(g_1) \cap Fix(g_2) \neq \emptyset$.*
4. *If G is generated by a finite set of elements $s_1, \ldots, s_m$ such that the s_j and the s_is_j have fixed points, then G has a fixed point.*

Proof. (1) is a consequence of Proposition 24 (page 63) from [11]. Our proof of (2) uses Corollary 1 from page 64 of [11], which says that if $abc = 1$ and each of a, b, c is elliptic, then a, b, and c have a common fixed point. If we read this corollary with $g_1 = a$, $g_2 = b$, and $g_2^{-1}g_1^{-1} = c$, then the assumption that g_1g_2 is elliptic leads to a contradiction, since $Fix(g_1) \cap Fix(g_2) = \emptyset$. To prove (3), we use Lemma 9 from page 61 of [11], which asserts that, if Γ_1 and Γ_2 are disjoint subtrees of Γ, then there is a unique minimal geodesic segment ℓ connecting Γ_1 to Γ_2. That is, if ℓ' connects a vertex of Γ_1 with a vertex in Γ_2, then $\ell \subseteq \ell'$. It is fairly clear that ℓ meets each of the subtrees Γ_1 and Γ_2 in exactly one point. Now suppose that g_1 and g_2 are elliptic; we assume that their fixed sets are disjoint. Let $\ell = [x, y]$ connect $x \in Fix(g_1)$ with $y \in Fix(g_2)$. We assume that ℓ is the minimal geodesic connecting these fixed sets. Our assumptions imply that $[x, y] \cup g_1 \cdot [x, y]$ is a geodesic segment connecting two points in $Fix(g_2)$, namely y and $g_1 \cdot y$. Since $Fix(g_2)$ is a tree, $[x, y] \cup g_1 \cdot [x, y] \subseteq Fix(g_2)$. This implies the contradiction $Fix(g_1) \cap Fix(g_2) \neq \emptyset$, proving (3).

Statement (4) is Corollary 2 from [11], page 64. □

Throughout the rest of this section, we assume that $G = T$ or V. Let $g \in G$. We say that g is *small* if g is the identity on some *standard dyadic subinterval of* $[0, 1]$ (i.e., some subinterval of the form $[\frac{i}{2^n}, \frac{i+1}{2^n}]$, where n is a non-negative integer and $0 \leq i < 2^n$). If $g \in G$, then the *support* of g is $supp(g) = \{x \in [0, 1] \mid gx \neq x\}$. The support of $H \leq G$ is $supp(H) = \bigcup_{h \in H} supp(h)$.

Lemma 4.2. *If G acts on a tree Γ, and $g \in G$ is small, then g has a fixed point.*

Proof. Let $g \in G$. Suppose, for a contradiction, that g is hyperbolic and acts by translation on the geodesic line ℓ. Let I be a standard dyadic subinterval of $[0, 1]$ such that $g_{|I} = id_I$. We consider the maximal subgroup H satisfying $supp(H) \subseteq I$. (This group is isomorphic either to F or to V, depending on whether G is T or V, respectively.) For any $h \in H$, $hgh^{-1} = g$, so

$$g \cdot h\ell = hgh^{-1} \cdot h\ell = h \cdot g\ell = h \cdot \ell.$$

It follows from the uniqueness of the axis ℓ that $h\ell = \ell$. Thus, the entire group H leaves the line ℓ invariant, so there is a homomorphism $\phi : H \to D_\infty$.

The kernel of ϕ is large. If $G = T$ (and H is isomorphic to F), then $Ker\phi$ will contain the commutator subgroup $[H, H]$, since every proper quotient of F is abelian by Theorem 4.3 from [2]. If $G = V$, then $Ker\phi = H$, since $H \cong V$ is simple. If $G = T$ and $H \cong F$, then $[H, H]$ consists of the subgroup of H which acts as the identity in neighborhoods of the left- and right-endpoints of I, by Theorem 4.1 from [2]. Thus, in either case, if I' is a standard dyadic interval whose closure is contained in the interior I, then any element of G supported in I' will fix the entire line ℓ.

We choose $k \in G$ so that $k \cdot supp(g) \subseteq I'$. We have $supp(kgk^{-1}) \subseteq I'$. It follows that $kgk^{-1} \in Ker\phi$, so kgk^{-1} is elliptic, and therefore g is elliptic as well. This is a contradiction. □

Theorem 4.3. *T has Serre's property FA.*

Proof. Identify $[0, 1]/\sim$ with the standard unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ by the quotient map $f : [0, 1] \to S^1$, where $f(t) = (\cos(2\pi t), \sin(2\pi t))$. This identification induces an action of T on S^1. We consider four subgroups of T:

$$\begin{aligned} T_L &= \{g \in T \mid g \cdot (x,y) = (x,y) \ if \ x \geq 0\} \\ T_R &= \{g \in T \mid g \cdot (x,y) = (x,y) \ if \ x \leq 0\} \\ T_U &= \{g \in T \mid g \cdot (x,y) = (x,y) \ if \ y \leq 0\} \\ T_D &= \{g \in T \mid g \cdot (x,y) = (x,y) \ if \ y \geq 0\} \end{aligned}$$

Each of these groups is isomorphic to F, and so can be generated by two elements, which are necessarily small as elements of T. Moreover $T = \langle T_L, T_R, T_U, T_D\rangle$. It follows that T is generated by 8 elements, each of which is small, such that the product of any two generators is also small. This implies that T fixes a point, by Lemma 4.1, (4). □

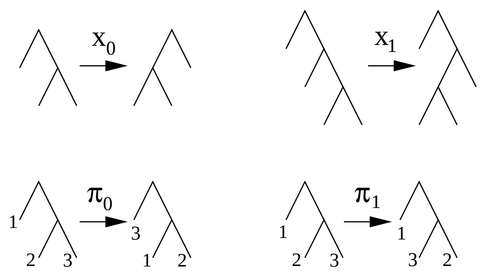

FIGURE 1. A generating set for Thompson's group V. The elements x_0, x_1, and π_0 generate Thompson's group T.

Theorem 4.4. *The group V has Serre's property FA.*

Proof. We recall from [2] that $T = \langle x_0, x_1, \pi_0\rangle$ and $V = \langle T, \pi_1\rangle$, where x_0, x_1, π_0, and π_1 appear in Figure 1.

Let V act on a tree Γ. By the previous theorem, we know that T has a fixed point. Thus, the elements x_0, x_1, and π_0 all have fixed points, and any two of these elements have a common fixed point. The element π_1 has finite order, so it must be elliptic. It will be sufficient (by Lemma 4.1 (4)) to show that each product $\pi_1\pi_0$, $\pi_1 x_0$, and $\pi_1 x_1$ is elliptic.

First, we note that $\pi_1\pi_0$ is an element of finite order, so it must be elliptic. Next, we note that $\pi_1 x_1$ is small, so it must be elliptic.

It is routine to check that $x_0^{-1}\pi_1 x_0 \pi_1$ is small, and therefore elliptic. It follows from Lemma 4.1(2) that $Fix(x_0^{-1}\pi_1 x_0) \cap Fix(\pi_1) \neq \emptyset$. Now

$$\begin{aligned} Fix(x_0^{-1}\pi_1 x_0) \cap Fix(\pi_1) \neq \emptyset &\Rightarrow x_0^{-1}(Fix(\pi_1)) \cap Fix(\pi_1) \neq \emptyset \\ &\Rightarrow Fix(x_0^{-1}) \cap Fix(\pi_1) \neq \emptyset. \end{aligned}$$

This implies that $\pi_1 x_0$ is elliptic. □

REFERENCES

[1] Kenneth S. Brown. The homology of Richard Thompson's group F. In *Topological and asymptotic aspects of group theory*, volume 394 of *Contemp. Math.*, pages 47–59. Amer. Math. Soc., Providence, RI, 2006.

[2] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.

[3] Daniel S. Farley. Proper isometric actions of Thompson's groups on Hilbert space. *Int. Math. Res. Not.*, (45):2409–2414, 2003.
[4] Daniel S. Farley. Actions of picture groups on CAT(0) cubical complexes. *Geom. Dedicata*, 110:221–242, 2005.
[5] Ross Geoghegan. *Topological methods in group theory*, volume 243 of *Graduate Texts in Mathematics*. Springer, New York, 2008.
[6] P. H. Kropholler and M. A. Roller. Relative ends and duality groups. *J. Pure Appl. Algebra*, 61(2):197–210, 1989.
[7] T. Napier and M. Ramachandran. Structure theorems for complete Kähler manifolds and applications to Lefschetz type theorems. *Geom. Funct. Anal.*, 5(5):809–851, 1995.
[8] Terrence Napier and Mohan Ramachandran. Thompson's group F is not Kähler. In *Topological and asymptotic aspects of group theory*, volume 394 of *Contemp. Math.*, pages 197–201. Amer. Math. Soc., Providence, RI, 2006.
[9] Terrence Napier and Mohan Ramachandran. Filtered ends, proper holomorphic mappings of Kähler manifolds to Riemann surfaces, and Kähler groups. *Geom. Funct. Anal.*, 17(5):1621–1654, 2008.
[10] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3)*, 71(3):585–617, 1995.
[11] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

Miami University of Ohio, Oxford, OH 45056
E-mail address: `farleyds@muohio.edu`