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On the isometry groups of certain CAT(0) spaces and trees

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Abstract

We show that the automorphism group of a locally finite tree is discrete, or pro-finite, or not the inverse limit of an inverse system of discrete groups, and provide necessary and sufficient conditions for each of these possibilities to occur. More generally, we demonstrate that for certain proper CAT(0) spaces X, the group of isometries of X is not an inverse limit of Lie groups. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is a classical theorem [2] that a locally compact Hausdorff topological group G is the inverse limit of an inverse system in the category of Lie groups provided the space of components of G is compact. For instance, if G is totally disconnected then G is pro-finite provided G is compact. But without the hypothesis of compactness it is not so clear when G is the inverse limit of an inverse system of Lie groups.

One case of this occurs when $G = \mathrm{Isom}(X)$, the group of isometries of a locally compact metric space (X, d). This is a locally compact Hausdorff group, by the Arzela–Ascoli Theorem, and if X is a Riemannian manifold then $\mathrm{Isom}(X, d)$ is actually a Lie group. In recent years there has been substantial interest in the extension of the classical differential geometry of Riemannian manifolds of non-positive sectional curvature to the much more general situation of proper CAT(0) spaces (defined below). The question naturally arises as to whether $\mathrm{Isom}(X, d)$ is the inverse limit of an inverse system of Lie groups when (X, d) is proper CAT(0). In Theorem 1 we show how to build examples where the answer is negative.

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We can do better in case X is a locally finite tree, i.e., a locally finite 1-dimensional simply connected simplicial complex. An *automorphism* of X is a simplicial homeomorphism from X to itself. The group of all such homeomorphisms, $\mathrm{Aut}(X)$, is a locally compact Hausdorff totally disconnected topological group when given the compact open topology. In Theorem 2 we can say exactly when $\mathrm{Aut}(X)$ is and is not the inverse limit of Lie groups (necessarily 0-dimensional, i.e., discrete).

Recall that a metric space is *proper* if every closed ball is compact. A *geodesic* in X is an isometric embedding $f:[0,d]\to X$; a *geodesic segment* is the image of such an embedding. A metric space X is *geodesic* if any two points of X may be joined by a geodesic. A *geodesic triangle* Δ in a metric space consists of 3 points x_1, x_2, and x_3, and a choice of 3 geodesic segments connecting them. For any such Δ there is a *comparison triangle* $\bar{\Delta}$ in $\mathbb{R}^2$, defined by choosing points $\bar{x}_1, \bar{x}_2$, and $\bar{x}_3$ so that $d(x_i, x_j) = d(\bar{x}_i, \bar{x}_j)$, for $i, j = 1, 2, 3$. There is an obvious, isometric correspondence between the sides of Δ and the sides of $\bar{\Delta}$. Given a point $x \in \Delta$, the *comparison point* for x is the corresponding point $\bar{x}$ in $\bar{\Delta}$. A geodesic metric space X is a CAT(0) *space* if for any geodesic triangle Δ in X and any two points $x_1, x_2 \in \Delta$, $d(x_1, x_2) \leqslant d(\bar{x}_1, \bar{x}_2)$, where $\bar{x}_1$ and $\bar{x}_2$ are comparison points for x_1 and x_2. Thus, roughly speaking, a geodesic metric space is CAT(0) if all of its triangles are no fatter than Euclidean triangles. We say that a geodesic metric space X is *non-positively curved* if, for every $x \in X$, there is an open neighborhood U of x so that U is a CAT(0) space with respect to the subspace metric. Given a covering space $p:\widetilde{X}\to X$, where X is a geodesic metric space, a natural way to define the length of a path α in $\widetilde{X}$ is to set $\ell(\alpha) = \ell(p(\alpha))$. We then get a metric d on $\widetilde{X}$, defined as follows: $d(x, y) = \inf\{\ell(\alpha)\colon \alpha \text{ is a path connecting } x \text{ to } y\}$. This is the *length metric* on the covering space $\widetilde{X}$. Under appropriate hypotheses, which are satisfied in our case, the universal cover of a non-positively curved space with its length metric is CAT(0) (see [1]).

The examples of our Theorem 1 are built using a theorem which involves putting a metric on the quotient of two metric spaces. Let X be a metric space with an equivalence relation $\sim$, let $\overline{X}$ be the set of all equivalence classes of X under $\sim$ and let $p:X\to\overline{X}$ be the natural projection. Define $\bar{d}:\overline{X}\times\overline{X}\to\mathbb{R}$ by the formula:

$$\bar{d}(\bar{x}, \bar{y}) = \inf \sum_i d(x_i, y_i),$$

where the infimum is taken over all sequences $C = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ of points of X such that $x_1 \in \bar{x}$, $y_i \sim x_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $y_n \in \bar{y}$. It is easy to show that $\bar{d}$ is symmetric and satisfies the triangle inequality, but in general $\bar{d}(\bar{x}, \bar{y})$ might be zero even when $\bar{x} \neq \bar{y}$. However, there is the following from [1]:

Proposition 1. *Let X_i be compact, non-positively curved* (*i.e., locally* CAT(0)) *spaces, and let $A_i \subset X_i$ be closed, connected, locally convex subspaces, $i = 1, 2$. Let $j : A_1 \to A_2$ be a bijective local isometry, and let $X = X_1 \cup_j X_2$. Then $\bar{d}$ is a metric and $(X, \bar{d})$ is non-positively curved.*

The authors of [1] add, in a remark, that the inclusion of X_i into X is a local isometry, for $i = 1, 2$.

Note that, under the hypotheses of the previous theorem, $\widetilde{X}$ with its length metric is a CAT(0) space. With this background we can state our Theorem 1:

Theorem 2. *Let X, X_i, A_i, and j be as in Proposition 1, $i = 1, 2$. If there is an isometry $g : X_1 \to X_1$, $g \neq 1$, which is the identity on A_1, and there is some $\gamma \in \pi_1(X_2)$ which is not conjugate to any element in the image of $i_* : \pi_1(A_2) \to \pi_1(X_2)$, then $\mathrm{Isom}(\widetilde{X}, \tilde{d})$ is not the inverse limit of an inverse system of Lie groups, where $\tilde{d}$ is the length metric.*

The statement of Theorem 2 requires some definitions. Tits [3] proved that any "hyperbolic" (fixed-point free) automorphism γ of a locally finite tree X acts on some line ℓ in X by translation. This line is called a *translation axis* for γ. An edge which lies on a translation axis will be called a *translation edge*. G_e denotes the stabilizer in G of the edge e. A group $G \subseteq \mathrm{Aut}(X)$ is *edge independent* provided whenever $\gamma \in G_e$ (e an edge), we have also $\gamma' \in G$, where γ' is the identity on one of the connected components of $X - e$ and equal to γ everywhere else. Note in particular that $\mathrm{Aut}(X)$ is edge independent for any tree X.

Theorem 3. *Let X be a locally finite tree, and let $G \leqslant \mathrm{Aut}(X)$ be edge independent.*

(1) *Suppose G contains a fixed point free automorphism.*
   (a) *If G_e is trivial for some translation edge e, then G is discrete.*
   (b) *If G_e is non-trivial for some translation edge e, then G is not the inverse limit of an inverse system of discrete groups.*

(2) *Suppose G does not contain a fixed point free automorphism (equivalently, every automorphism in G has a fixed point).*
   (a) *If G has a bounded orbit, then G is pro-finite.*
   (b) *If every (equivalently, some) orbit is unbounded, then G is not the inverse limit of an inverse system of discrete groups.*

As a consequence of Theorem 2, we get the following:

Corollary 1. *If X is a locally finite tree, then $\mathrm{Aut}(X)$ is either discrete, or pro-finite, or not the inverse limit of an inverse system of discrete groups.*

2. Preliminaries on CAT(0) spaces

In this section we collect various results from the theory of CAT(0) spaces to be used in the proof of Theorem 1. All of the results and definitions here are from [1].

The map $\pi : X \to C$ described in the following proposition is called the *projection onto C*.

Proposition 2. *Let X be a CAT(0) space, and let C be a closed convex subset which is complete in the induced (i.e., subspace) metric. Then,*

(1) *for every* $x \in X$, *there exists a unique point* $\pi(x) \in X$ *such that* $d(x, \pi(x)) = d(x, C) := \inf_{y \in C} d(x, y)$;
(2) *the map* $x \mapsto \pi(x)$ *is a retraction of* X *onto* C *which does not increase distances. The map* $H : X \times [0,1] \to X$ *associating to* (x,t) *the point at distance* t *from* x *on the geodesic segment* $[x, \pi(x)]$ *is a continuous homotopy from the identity map of* X *to* π.

Note that, letting C be a single point, this proposition shows that CAT(0) spaces are contractible.

A function $c : [0,1] \to X$, with X a metric space, is a *linearly reparametrized geodesic* if there is a constant λ such that for any $s, t \in [0,1]$, $\lambda|s - t| = d(c(s), c(t))$. A function $f : X \to \mathbb{R}$ defined on a geodesic metric space is *convex* if for any linearly reparametrized geodesic c we have $f(c(t)) \leqslant (1-t) f(c(0)) + t f(c(1))$, for each $t \in [0,1]$.

The following corollary is a consequence of the previous proposition and the convexity of the distance function in a CAT(0) space.

Corollary 3. *Let* C *be a complete convex subset in a* CAT(0) *space* X. *The distance function* d_C *is convex.*

The next theorem requires some definitions. A *geodesic line* in a CAT(0) space X is a distance preserving map from the real line into X. Two lines c and c' are *parallel* (or *asymptotic*) if there is a constant $K > 0$ such that $d(c(t), c'(t)) < K$, for any t in $\mathbb{R}$.

Theorem 4. *Let* X *be a* CAT(0) *space, and let* $c : \mathbb{R} \to X$ *and* $c' : \mathbb{R} \to X$ *be geodesic lines in* X. *If* c *and* c' *are asymptotic, then the convex hull of* $c(\mathbb{R}) \cup c'(\mathbb{R})$ *is isometric to a flat strip* $\mathbb{R} \times [0, D] \subset \mathbb{E}^2$.

Let (X, d) be a metric space. The *length metric* $\bar{d}$ associated to d is defined by letting $\bar{d}(x, y)$ be the infimum of the lengths of the paths joining x to y. A *closed local geodesic* in a metric space (X, d) is a map $c : (S^1, \bar{d}) \to (X, d)$ which is locally distance preserving, where $\bar{d}$ is the length metric associated to the metric S^1 inherits as a subset of $\mathbb{R}^2$, with its usual metric.

Proposition 3. *If* X *is a compact, locally simply connected, geodesic space, then every closed loop* $c : S^1 \to X$ *is homotopic to a closed local geodesic.*

Since CAT(0) spaces are contractible, and non-positively curved spaces are locally CAT(0), the previous proposition applies to the X of our Theorem 1.

3. Proof of Theorem 1

Let $X = X_1 \cup_j X_2$. Let $p : \widetilde{X} \to X$ be the universal cover of X. Choose a basepoint $\tilde{*}$ in $\widetilde{X}$ so that $p(\tilde{*}) = *$ is in A_1. The space $\widetilde{X}$ contains a copy of $\widetilde{X}_1$ and a copy of $\widetilde{X}_2$ which intersect in a copy of $\widetilde{A}_1$ containing the base point $\tilde{*}$. Whenever $\widetilde{X}_1$, $\widetilde{X}_2$, or $\widetilde{A}_1$ is mentioned, we refer to these copies. We will also refer to $\widetilde{A}_1$ as $\widetilde{A}_2$ when we especially want to think of it as a subset of $\widetilde{X}_2$. The $\widetilde{X}_1$ *direction* is $\{x \in \widetilde{X}\colon$ the geodesic joining x to $\tilde{*}$ meets $\widetilde{X}_1 - \widetilde{A}_1\}$.

Let g be the isometry of X_1 fixing A_1. The map f from X to itself which agrees with g on X_1 and with the identity on X_2 is an isometry of X, since it preserves the lengths of paths. The pointed map $fp : (\widetilde{X}, \tilde{*}) \to (X, *)$ lifts to a pointed isometry $\tilde{f} : (\widetilde{X}, \tilde{*}) \to (\widetilde{X}, \tilde{*})$. Set F equal to $\tilde{f}$ in the $\widetilde{X}_1$ direction, and the identity elsewhere. This new function is an isometry, since it preserves the lengths of paths.

We next show that F is the identity on balls of arbitrarily large radius. Let $\gamma \in \pi_1(X_2)$ have no conjugate in $i_*(\pi_1(A_2))$. The conjugacy class of the element γ in $\pi_1(X_2)$ can be represented by a closed local geodesic c in X_2, by Proposition 3, and, since the inclusion of X_2 into X is a local isometry, this c is also a closed local geodesic in X. Covering c is a geodesic line l in $\widetilde{X}$, and this l is contained in $\widetilde{X_2}$. Define $d_{\widetilde{A}_2} : \widetilde{X} \to \mathbb{R}$ by $d_{\widetilde{A}_2}(x) = \inf\{\tilde{d}(a, x)\colon a \in \widetilde{A}_2\}$. The function $d_{\widetilde{A}_2}$ is convex, by Corollary 2, and the restriction of $d_{\widetilde{A}_2}$ to l is a convex function defined on the entire real line; it follows that $d_{\widetilde{A}_2}$ is unbounded along l, or constant. Define $p_l : \widetilde{X} \to l$ and $p_{\widetilde{A}_2} : \widetilde{X} \to \widetilde{A}_2$ to be the projections onto the complete, closed, convex subspaces l and $\widetilde{A}_2$, respectively. Proposition 2 says that these projections do not increase distances. If $d_{\widetilde{A}_2}$ is constant on l, it is easy to see, from the definitions of p_l and $p_{\widetilde{A}_2}$, that $p_l p_{\widetilde{A}_2}$ is the identity on l. Since p_l does not increase distances, $p_{\widetilde{A}_2}$ maps l onto $p_{\widetilde{A}_2}(l)$ isometrically. It follows that l and $p_{\widetilde{A}_2}(l)$ are parallel lines in $\widetilde{X}$ and, by Theorem 3, their convex hull is a flat strip. In the quotient space X we get a cylinder with one component of the boundary in A_2 and the other component equal to c. This shows that c is freely homotopic to a loop in A_2, which contradicts the assumption about γ. This proves that $d_{\widetilde{A}_2}$ is not constant on l. Now the ball of radius $d_{\widetilde{A}_2}(x)$ centered at $x \in l$ is contained in the fixed set of F, and x may be chosen so that $d_{\widetilde{A}_2}(x)$ is arbitrarily large.

We now claim that F is in the kernel of any homomorphism $\phi : \mathrm{Isom}(\widetilde{X}) \to G$, where G is any Lie group. Let such a ϕ be given. It is well known that Lie groups have no small subgroups, that is, there is a neighborhood U of $1 \in G$ such that U contains no subgroup of G except the trivial group. Using continuity of ϕ, there is some compact C in $\widetilde{X}$ so that if $\beta \in \mathrm{Isom}(\widetilde{X})$ is the identity on C, then $\phi(\beta) \in U$. In fact, if this is the case, $\beta \in \mathrm{Ker}\,\phi$, since $\beta^n \in U$, for all $n \in \mathbb{Z}$, and thus $\phi(\langle\beta\rangle)$ is the trivial group. The group $\mathrm{Isom}(\widetilde{X})$ acts cocompactly on $\widetilde{X}$, so there is $\theta \in \mathrm{Isom}(\widetilde{X})$ such that $\theta(C)$ is contained in the fixed set of F, by the previous paragraph. It follows that $\theta^{-1}F\theta$ is the identity on C. This proves the claim.

An easy argument using the claim and the universal property of the inverse limit shows that $\mathrm{Isom}(\widetilde{X})$ is not an inverse limit of Lie groups.

Remark. The referee suggests an alternative hypothesis and proof. If we assume that neither map $\pi_1(A_i) \to \pi_1(X_i)$ $(i = 1, 2)$ is surjective, and remove the hypothesis that there is an element of $\pi_1(X_2)$ which is not conjugate to any element of $\pi_1(A_2)$, then the theorem is still true, and can be proved in the following way:

Using the structure of $\pi_1(X)$ as an amalgamated free product, we can find an element $\gamma \in \pi_1(X)$ which is not conjugate to any element of $\pi_1(X_1)$ or $\pi_1(X_2)$. The conjugacy class of γ can be represented by a closed local geodesic c and, by the assumption on γ, $c \cap (X_2 - A_2) \neq \emptyset$. A line $l : \mathbb{R} \to \widetilde{X}$ covering c may thus be chosen so that $l(0) \in \widetilde{A}_2$ and $l([0, \varepsilon)) \subset \widetilde{X}_2$. Now as $x \to \infty$, the distance between $l(x)$ and the $\widetilde{X}_1$ direction becomes arbitrarily large. Therefore, the isometry F of $\widetilde{X}$ is the identity on balls of arbitrarily large radius. The rest of the proof is unchanged.

4. Proof of Theorem 2

Proof. (1) (a) is trivial. In what follows, supp α, for an automorphism α, will denote the support of α in the ordinary sense, i.e., supp α is the closure of $\{x \in X\colon\ \alpha(x) \neq x\}$. If G_e is non-trivial for some translation edge e, then there is a hyperbolic automorphism γ_1 in G and an automorphism γ_2 in G which fixes some edge e along the translation axis ℓ of γ_1. After using edge independence, if necessary, we may assume supp γ_2 meets only one path component of $X - \mathring{e}$. Identify the real line $\mathbb{R}$ (considered as a simplicial complex with a vertex at each integer) with the translation axis of γ_1 in such a way that large-numbered vertices lie inside the path component of $X - \mathring{e}$ containing supp γ_2. Now either γ_1 or γ_1^{-1} acts on the translation axis of γ_1 by addition by a positive integer k; we may assume that γ_1 does.

For all integers n, define $X_{\geqslant n}$ to be the path component of n in $X - (n-1, n)$. It is clear that $\gamma_1 X_{\geqslant n} = X_{\geqslant n+k}$ and $X_{\geqslant m} \subseteq X_{\geqslant n}$, when $m > n$.

Now if $e = [m-1, m]$ then supp $\gamma_2 \subseteq X_{\geqslant m}$. Let C be a compact subset of X. For sufficiently large positive t, $C \cap X_{\geqslant t} = \emptyset$. Pick some $N \in \mathbb{Z}$ which is so large that $t < m + Nk \ldots$

$$\begin{aligned} \operatorname{supp} \gamma_1^N \gamma_2 \gamma_1^{-N} \cap C &= \gamma_1^N \operatorname{supp} \gamma_2 \cap C \\ &\subseteq \gamma_1^N X_{\geqslant m} \cap C = X_{\geqslant m+Nk} \cap C \\ &\subseteq X_{\geqslant t} \cap C = \emptyset. \end{aligned}$$

This shows that for any compact subset C of X, there is some member of the sequence $(\gamma_1^n \gamma_2 \gamma_1^{-n})$, say $\gamma_1^N \gamma_2 \gamma_1^{-N}$, so that supp $\gamma_1^N \gamma_2 \gamma_1^{-N} \cap C = \emptyset$; therefore, $(\gamma_1^n \gamma_2 \gamma_1^{-n}) \to 1$. It follows that γ_2 is in the kernel of any homomorphism from G to any discrete group. As in the proof of Theorem 1, this implies that G is not the inverse limit of any inverse system of discrete groups. This proves (1).

(2) If G has a bounded orbit then G is a compact Hausdorff totally disconnected topological group. According to [2], G is pro-finite, proving (a).

In case G has no bounded orbit, G has no global fixed point, so a theorem of Tits [3] says that there is a unique end ξ fixed by all of G. Using edge independence we can produce an automorphism α which has connected support. There is a unique ray r connecting supp α to ξ. Identify r with $[0, \infty)$ in such a way that supp $\alpha \cap r = \{0\}$.

Let C be a compact subset of X. After enlarging C, if necessary, we can assume that C is a finite subtree which meets r in some finite interval $[M, N]$, say. There exists some

$\gamma \in G$ so that $X^\gamma \cap r = [K, \infty)$, where $K > N$ and X^γ is the fixed set of γ. The geodesic joining C to $[K, \infty)$ is $[N, K]$. The geodesic joining $\gamma \operatorname{supp} \alpha$ to C is $\gamma[0, K] \cup [N, K]$; it follows that $\gamma \operatorname{supp} \alpha \cap C = \operatorname{supp} \gamma\alpha\gamma^{-1} \cap C = \emptyset$. It follows that $\{\gamma\alpha\gamma^{-1}\colon \gamma \in G\}$ has the identity as a limit point. This implies that α is in the kernel of any homomorphism from G to a discrete group. This proves (2). □

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