A PROOF OF SAGEEV'S THEOREM ON HYPERPLANES IN CAT(0) CUBICAL COMPLEXES

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ABSTRACT. We prove that any hyperplane H in a CAT(0) cubical complex X has no self-intersections and separates X into two convex complementary components. These facts were originally proved by Sageev. Our argument shows that his theorem (or this direction of his theorem) is a corollary of Gromov's link condition.

We also give new arguments establishing some combinatorial properties of hyperplanes. We show that these properties are sufficient to prove that the 0-skeleton of any CAT(0) cubical complex is a discrete median algebra, a fact that was previously proved by Roller.

1. Introduction

Two theorems are central in the theory of CAT(0) cubical complexes. The first is Gromov's well-known link condition. A complete statement and proof appear in [1]. The second theorem was proved by Sageev in [13]. He showed that a group G semisplits over a subgroup H if and only if G acts on a CAT(0) cubical complex X and there is a hyperplane $J \subseteq X$ such that: i) the action of G is essential relative to J, and ii) the stabilizer of J (as a set) is H. We refer the reader to [13] for details and definitions. Sageev's result extends the Bass-Serre theory of groups acting on trees, which says that a group G splits over H if and only if G acts without inversion on a tree T, in which the stabilizer subgroup of some edge e is H. Moreover, just as Bass-Serre theory gives a construction of the tree T from the splitting of G over H, Sageev gives a construction of the CAT(0) cubical complex X from the semisplitting of G over H. Both theories are also alike in that they explicitly describe the algebraic splittings or semisplittings using their geometric hypotheses.

Both the forward and the reverse directions of Sageev's theorem have significant applications. The forward direction (from algebra to geometry) is used in [9] and [14], among others. The proof of the reverse direction uses several properties of hyperplanes in CAT(0) cubical complexes (also established in [13]). Many of these properties are useful in their own right. For instance, Sageev showed that a hyperplane in a CAT(0) cubical complex X has no self-intersections and separates X into two convex complementary components [13]. This fact is essential in the proof that groups acting properly, isometrically, and cellularly on CAT(0) cubical complexes have the Haagerup property [10]. Sageev establishes the geometric properties of hyperplanes in CAT(0) cubical complexes using his own system of Reidemeister-style moves.

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The main purpose of this note is to offer a new (and, we believe, simpler) proof of the following theorem, which we hereafter call "Sageev's Theorem" for the sake of brevity:

Theorem 1.1. [13] *A hyperplane H in a CAT(0) cubical complex X has no self-intersections and separates X into two open convex complementary components.*

Our proof avoids using Sageev's Reidemeister moves. The main tool is a block complex $\mathcal{B}(X)$, which is endowed with a natural projection $\pi_{\mathcal{B}} : \mathcal{B}(X) \to X$. We apply a criterion, due to Crisp and Wiest [4], for showing that a map between cubical complexes is an isometric embedding. The criterion is a generalized form of Gromov's link condition. We are thus able to conclude that the restriction of $\pi_{\mathcal{B}}$ to each connected component of $\mathcal{B}(X)$ is an isometric embedding. The full statement of Theorem 1.1 then follows from the definition of $\mathcal{B}(X)$ after a little more work.

We also give new proofs of some of Sageev's secondary results – see Subsection 5.2, especially Propositions 5.5 and 5.8. Sageev's original proofs used his Reidemeister moves. Our proofs use techniques from the theory of CAT(0) spaces, including (especially) projection maps onto closed convex subspaces.

The paper concludes with some applications. We sketch a proof of the theorem that every group G acting properly, isometrically, and cellularly on a CAT(0) cubical complex has the Haagerup property. (The first proof appeared in [10].) We also show that the 0-skeleton of a CAT(0) cubical complex is a discrete median algebra under the "geodesic interval" operation. It appears that no proof of the discrete median algebra property has been published before, although Martin Roller produced a proof in his unpublished Habilitation Thesis [12]. Our argument is intended to highlight the utility of the combinatorial lemmas collected in Subsection 5.1, and, in particular, to suggest that the latter lemmas are a sufficient basis for establishing all of the combinatorial properties of CAT(0) cubical complexes. (Indeed, "discrete median algebra" and "CAT(0) cubical complex" are equivalent ideas, by [12] and [11].) We refer the reader to [2] for elegant characterizations of the Haagerup property and property T in terms of median algebras.

We note one limitation of the general methods of this paper: our methods apply only to locally finite-dimensional cubical complexes satisfying Gromov's link condition. We need our complexes to be locally finite-dimensional so that their metrics will be complete (see [1], Exercise 7.62, page 123). In fact, the CAT(0) property has been established only for locally finite-dimensional cubical complexes satisfying the link condition – see the passage after Lemma 2.7 in [6] for a useful discussion of this point. Although our argument is therefore slightly less general than the original one of Sageev, it still covers the cases that are most commonly encountered in practice.

Section 2 contains a description of the block complex. Section 3 describes the analogue of Gromov's theorem we need from [4]. Section 4 contains a proof of Sageev's theorem, Theorem 1.1. Section 5 collects some essential combinatorial lemmas. Finally, Section 6 contains applications of the main ideas, including proofs that every CAT(0) cubical complex is a set with walls and that the 0-skeleton of every CAT(0) cubical complex is a discrete median algebra.

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2. The Block Complex

Definition 2.1. A cubical complex X is *locally finite-dimensional* if the link of each vertex is a finite-dimensional simplicial complex.

Throughout the paper, "CAT(0) cubical complex" means locally finite-dimensional CAT(0) cubical complex.

Definition 2.2. Let $C \subseteq X$ be a cube of dimension at least one. A *marking* of C is an equivalence class of directed edges $e \subseteq C$. Two such directed edges e', e'' are said to be *equivalent*, i.e., to define the same marking, if there is a sequence of directed edges $e' = e_0, \ldots, e_k = e''$ such that, for $i \in \{0, \ldots, k-1\}$, e_i and e_{i+1} are opposite sides of a 2-cell $C_i \subseteq C$ and both point in the same direction. A *marked cube* is a cube (of dimension at least one) with a marking.

Example 2.3. Let $X = [0,1]^3$, with the usual cubical structure. We let $C = X$. There are six markings of C. They are represented by the directed edges $[(0,0,0),(1,0,0)]$, $[(0,0,0),(0,1,0)]$, $[(0,0,0),(0,0,1)]$, and by the three corresponding edges with the opposite directions.

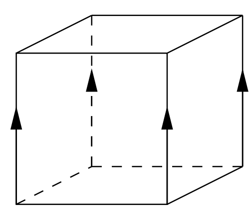

Figure 1. The directed edge $[(0,0,0),(0,0,1)]$ determines the marking of the cube. The x-axis is horizontal, and the coordinate system is a right-handed one.

It is fairly clear from the example that a cube of dimension n has exactly $2n$ markings. Note that not every face of a marked cube is itself marked. In Figure 1, the top and bottom faces are unmarked.

Definition 2.4. Let X be a CAT(0) cubical complex. We let $\mathcal{M}(X)$ denote the space of marked cubes of X, which is defined to be the disjoint union of all marked cubes of X. More formally, $\mathcal{M}(X)$ is the space of triples $(x, C, [e])$, where C is a cube in X, $[e]$ is a marking of C, and $x \in C$. For fixed C and $[e]$, the set

$$C_{[e]} = \{(x, C, [e]) \mid x \in C\}$$

is an isometric copy of C, and $\mathcal{M}(X)$ is the disjoint union of all such sets $C_{[e]}$. There is a natural map $\pi_{\mathcal{M}} : \mathcal{M}(X) \to X$, defined by sending $(x, C, [e])$ to x.

Example 2.5. If $X = [0,1]^3$, then $\mathcal{M}(X)$ is a disjoint union of 24 marked edges, 24 marked squares, and 6 marked three-dimensional cubes.

Definition 2.6. Let $(x_1, C_1, [e_1])$, $(x_2, C_2, [e_2]) \in \mathcal{M}(X)$. We write $(x_1, C_1, [e_1]) \sim (x_2, C_2, [e_2])$ if:

(1) $x_1 = x_2$, and
(2) there is a directed edge $e \in [e_1] \cap [e_2]$.

Lemma 2.7. *The relation* $\sim$ *is an equivalence relation on* $\mathcal{M}(X)$*.*

Proof. It is already clear that $\sim$ is reflexive and symmetric.

We prove that $\sim$ is transitive. Thus, we suppose that $(x_1, C_1, [e_1]) \sim (x_2, C_2, [e_2])$ and $(x_2, C_2, [e_2]) \sim (x_3, C_3, [e_3])$. Clearly, $x_1 = x_2 = x_3$. We can express C_2 as $C_2' \times [0,1]$, where C_2' is a cube of dimension one less than the dimension of C_2, and the second factor $[0,1]$ is the marked one. Since $C_1 \cap C_2$ is a marked face of C_2 (because of the condition $[e_1] \cap [e_2] \neq \emptyset$), we must have $C_1 \cap C_2 = \widehat{C} \times [0,1]$, for some non-empty face $\widehat{C} \subseteq C_2'$. Similarly, $C_2 \cap C_3 = \widetilde{C} \times [0,1]$, for some non-empty face $\widetilde{C} \subseteq C_2'$. Now $C_1 \cap C_2 \cap C_3 \neq \emptyset$, since $x_1 \in C_1 \cap C_2 \cap C_3$. It follows that $C_1 \cap C_2 \cap C_3 = (\widehat{C} \times \widetilde{C}) \times [0,1]$, where $\widehat{C} \times \widetilde{C}$ is a non-empty face of C_2'.

Let us suppose that the marking $[e_2]$ of C_2 is determined by the directed edge $e_2 = [(v,0),(v,1)]$, where v is a vertex of C_2'. It follows easily from the conditions $[e_1] \cap [e_2] \neq \emptyset$ and $[e_2] \cap [e_3] \neq \emptyset$ that the directed edge $[(v',0),(v',1)] \subseteq C_2$ is in $[e_1]$ (respectively, $[e_3]$) if and only if $v' \in \widehat{C}$ (respectively, $\widetilde{C}$). Thus, if v is a vertex of $\widehat{C} \cap \widetilde{C}$, then $[(v,0),(v,1)] \in [e_1] \cap [e_3]$. Such a vertex exists since $\widehat{C} \cap \widetilde{C} \neq \emptyset$, and this completes the proof. □

Definition 2.8. The *block complex of* X, denoted $\mathcal{B}(X)$, is the quotient $\mathcal{M}(X)/\sim$.

Definition 2.9. [4] A map $f : X \to Y$ between cubical complexes is called *cubical* if each cube in X is mapped isometrically onto some cube in Y.

We record the following lemma, the proof of which is straightforward.

Lemma 2.10. *The space* $\mathcal{B}(X)$ *is a cubical complex with a natural cubical map* $\pi_{\mathcal{B}} : \mathcal{B}(X) \to X$*, defined by* $\pi(x, C, [e]) = x$*.* □

Example 2.11. We describe the cubical complex $\mathcal{B}(X)$ in a special case. Suppose that $X = \mathbb{R}^2$ with the standard cubulation. The complex $\mathcal{B}(X)$ consists of an infinite disjoint union of strips having either the form $[m, m+1] \times \mathbb{R}$ or $\mathbb{R} \times [n, n+1]$ $(m, n \in \mathbb{Z})$. The map $\pi_{\mathcal{B}} : \mathcal{B}(X) \to X$ is "inclusion". Note that there are two identical copies of each strip $[m, m+1] \times \mathbb{R}$ in $\mathcal{B}(X)$, since there are two different orientations for the edge $[m, m+1] \times \{0\}$. (There are also two copies of $\mathbb{R} \times [n, n+1]$ in $\mathcal{B}(X)$ for a similar reason.)

3. A geometric lemma

The main lemma of this section (Lemma 3.2) relies heavily on a theorem due to Crisp and Wiest.

Theorem 3.1. ([4]*, Theorem 1(2)) Let* X *and* Y *be locally finite-dimensional cubical complexes and* $\Phi : X \to Y$ *a cubical map. Suppose that* Y *is locally CAT(0). The map* Φ *is a local isometry if and only if, for every vertex* $x \in X$*, the simplicial map* $Lk(x, X) \to Lk(\Phi(x), Y)$ *induced by* Φ *is injective with image a full subcomplex of* $Lk(\Phi(x), Y)$*.*

Proof. This is exactly Theorem 1(2) from [4], except that we allow locally finite-dimensional cubical complexes, rather than only finite-dimensional ones. Since the hypotheses and conclusions are all local in nature, the proof is unchanged. □

Lemma 3.2. *Let* X *and* Y *be locally finite-dimensional cubical complexes, let* Y *be CAT(0), and assume that* $\Phi : X \to Y$ *is a cubical map with the property that,* *for every vertex $x \in X$, the simplicial map $Lk(x, X) \to Lk(\Phi(x), Y)$ induced by Φ is injective with image a full subcomplex of $Lk(\Phi(x), Y)$.*

For every component $C \subseteq X$, we have:

(1) C is a CAT(0) cubical complex, and

(2) $\Phi_{|C}$ is an isometric embedding.

Proof. The previous theorem shows that Φ is a local isometry. We note that both X and Y are complete metric spaces, since both are locally finite-dimensional cubical complexes (see Exercise 7.62 on page 123 of [1]). Since Y is non-positively curved and X is locally a length space, Proposition 4.14 from page 201 of [1] applies. It follows that X is non-positively curved, the homomorphism $\Phi_* : \pi_1(C) \to \pi_1(Y)$ is injective, and every continuous lifting $\widetilde{\Phi} : \widetilde{C} \to \widetilde{Y}$ is an isometric embedding. Since Φ_* is injective, C is simply connected, and therefore $C = \widetilde{C}$, $Y = \widetilde{Y}$, and $\widetilde{\Phi} = \Phi$. The lemma follows. □

4. The main theorem

4.1. **A preliminary version of Sageev's theorem.**

Theorem 4.1. *If X is a locally finite-dimensional cubical complex, then the map $\pi_{\mathcal{B}} : \mathcal{B}(X) \to X$ embeds each connected component of $\mathcal{B}(X)$ isometrically.*

Proof. By Lemma 3.2, we need only show that the simplicial map on links $Lk(v, \mathcal{B}(X)) \to Lk(\pi_{\mathcal{B}}(v), X)$ is injective with image a full subcomplex of $Lk(\pi_{\mathcal{B}}(v), X)$.

We choose a vertex $v \in \mathcal{B}(x)$. Such a vertex can be represented by a vertex $(x, C, [e])$ in $\mathcal{M}(X)$, where $x \in X^0$. There is a unique directed edge $e' \in [e]$ containing x. We let C' denote the (undirected) 1-cell determined by e'. It follows from the definition of $\sim$ that we can represent v by $(x, C', [e'])$.

We let $X_{C'}$ be the subcomplex of X consisting of all closed cubes C such that $C' \subseteq C$. A marked cube $C_{[e]} \subseteq \mathcal{B}(X)$ touches v if and only if $C' \subseteq C$ and $e' \in [e]$, by the definition of $\sim$. Now, for a given cube $C \subseteq X$ such that $C' \subseteq C$, there is a unique marking $[e]$ of C such that $e' \in [e]$. It follows that the closed cubes touching v in $\mathcal{B}(X)$ are in one-to-one correspondence with the closed cubes of $X_{C'}$ touching $\pi_{\mathcal{B}}(v)$. Moreover, given two marked cubes $D_{[e_1]}$ and $E_{[e_2]}$ such that $e' \in [e_1] \cap [e_2]$, the intersection $D_{[e_1]} \cap E_{[e_2]}$ is mapped isometrically to $D \cap E$ by $\pi_{\mathcal{B}}$, since $D_{[e_1]} \cap E_{[e_2]} = (D \cap E)_{[e_3]}$, where $[e_3]$ is the unique marking of $D \cap E$ determined by the property $[e_3] \subseteq [e_1] \cap [e_2]$. It follows that the union of all closed cubes in $\mathcal{B}(X)$ touching v is combinatorially identical to $X_{C'}$, and the map $\pi_{\mathcal{B}} : \mathcal{B}(X) \to X$ is locally just the inclusion $X_{C'} \to X$. Therefore, the map on links is injective.

We now consider the image in $Lk(\pi_{\mathcal{B}}(x), X)$. There is a vertex $v' \in Lk(\pi_{\mathcal{B}}(v), X)$ which is contributed by the 1-cell C'. The above description of $\pi_{\mathcal{B}}$ implies that the image of the link $Lk(v, \mathcal{B}(X))$ is the union of all simplices touching v' (i.e., the simplicial neighborhood of v'). Since $Lk(\pi_{\mathcal{B}}(v), X)$ is flag, this simplicial neighborhood is necessarily a full subcomplex. □

4.2. **Sageev's theorem.**

Definition 4.2. Fix a component B of the block complex $\mathcal{B}(X)$. For each marked cube C of B, choose an isometric characteristic map $c : [0, 1]^n \to C$ such that the directed edge $[c(0, 0, \ldots, 0), c(0, 0, \ldots, 0, 1)]$ represents a marking of C. If $x \in C$ satisfies $x = c(t_1, \ldots, t_n)$, then the *height* of x, denoted $h(x)$, is t_n. This height function on marked cubes is easily seen to be compatible overlaps, and induces a height function $h : B \to [0,1]$. We let $B_t = h^{-1}(t)$ for $t \in [0,1]$.

Lemma 4.3. *(1) For any component B of $\mathcal{B}(X)$ and for any $t \in [0,1]$, B_t is a closed convex subset of $\mathcal{B}(X)$. The space $\pi_{\mathcal{B}}(B_t)$ is a closed convex subset of X.*
(2) Each component B of $\mathcal{B}(X)$ factors isometrically as $B_0 \times [0,1]$.
(3) Each B_t ($t \in [0,1]$) is a CAT(0) cubical complex.

Proof. (1) It is clear that B_t is closed.

Suppose that $x, y \in B_t$. Let $p : [0, d_B(x,y)] \to B$ be a path connecting x to y. We can factor each marked cube $C \subseteq B$ of dimension n as $C' \times [0,1]$, where C' is a cube of dimension $n-1$ and the factor $[0,1]$ determines the marking. There is a natural projection $\pi_t : C \to C' \times \{t\}$, and this projection doesn't increase distances. Moreover, all such projections are compatible, so in particular there is a projection $\pi_t : B \to B_t$ which fixes B_t and doesn't increase distances. It follows that $\pi_t \circ p$ is a path in B_t which is no longer than p. By the uniqueness of geodesics in CAT(0) spaces, it follows that any geodesic connecting x to y lies in B_t. Therefore, B_t is a closed convex subset of $\mathcal{B}(X)$. Since $\pi_{\mathcal{B}|B}$ is an isometric embedding, $\pi_{\mathcal{B}}(B_t)$ is a closed convex subset of X.

(2) There is a natural map $f : B \to B_0 \times [0,1]$, where $f(x) = (\pi_0(x), h(x))$ and $\pi_0 : B \to B_0$ is the usual projection onto the closed convex subspace B_0 (see Proposition 2.4 on page 176 of [1]).

Assume that $x, y \in B$. We need to show that

$$d_B(x,y) = \sqrt{[d_{B_0}(\pi_0(x), \pi_0(y))]^2 + |h(x) - h(y)|^2}.$$

This is clear if $\pi_0(x) = \pi_0(y)$. If $\pi_0(x) \neq \pi_0(y)$, then we consider the quadrilateral formed by the geodesic segments $[\pi_0(x), \pi_0(y)]$, $[\pi_0(x), \pi_1(x)]$, $[\pi_1(x), \pi_1(y)]$, and $[\pi_1(y), \pi_0(y)]$.

By Proposition 2.4(3) of [1], each of the four resulting Alexandrov angles measures at least $\pi/2$. It therefore follows from the Flat Quadrilateral Theorem (2.11 from page 181 of [1]) that all of the angles in the above quadrilateral measure exactly $\pi/2$, and that the convex hull of $\pi_0(x)$, $\pi_0(y)$, $\pi_1(x)$ and $\pi_1(y)$ in B is isometric to a rectangle in Euclidean space. The desired equality now follows from the definition of the metric in Euclidean space.

(3) It is sufficient to prove this for B_0. Since $B = B_0 \times [0,1]$ is CAT(0), it must be that each factor is CAT(0) (Exercise 1.16, page 168 of [1]). The space B_0 is a cubical complex because the identifications in the definition of B are height-preserving.

□

Theorem 4.4. *Each hyperplane $\pi_{\mathcal{B}}(B_t)$ ($0 < t < 1$) separates X into two open convex complementary half-spaces.*

Proof. We recall that $\pi_{\mathcal{B}}(B)$ is a closed convex subspace of X. We let $\pi : X \to \pi_{\mathcal{B}}(B)$ be the projection. By a slight abuse of notation, we let $h : \pi_{\mathcal{B}}(B) \to [0,1]$ denote the height function from Definition 4.2.

Consider the function $h \circ \pi : X \to [0,1]$. We claim

(1) if $[x, y]$ is any geodesic in X, then $(h \circ \pi)_{|[x,y]}$ must assume its maximum and minimum values at the endpoints, and
(2) if $h(\pi(x)) \in (0, 1)$, then $x = \pi(x)$.

We prove (2) first. Note that, if $h(\pi(x)) \in (0, 1)$, then $\pi(x)$ is an interior point of $\pi_{\mathcal{B}}(B)$. This is only possible if $\pi(x) = x$.

We now prove (1). We assume the contrary. Assume that $h \circ \pi$ attains its maximum value on the geodesic $[x, y]$ at neither of the endpoints. (The case in which $h \circ \pi$ attains its minimum value at neither of the endpoints is handled in an analogous way.) We assume that $h \circ \pi$ attains its maximum value at $z \in [x, y]$, $z \notin \{x, y\}$. It follows that there is some $t \in (0, 1)$ such that

$$\max\{(h \circ \pi)(x), (h \circ \pi)(y)\} < t < (h \circ \pi)(z).$$

This implies, by the Intermediate Value Theorem, that there are points x', y' such that $(h \circ \pi)(x') = t = (h \circ \pi)(y')$, where x' lies between x and z on $[x, y]$, and y' lies between y and z. It now follows, from (2), that $x', y' \in B_t$. Since $z \in [x', y'] \subseteq B_t$, $(h \circ \pi)(z) = t$, a contradiction. This proves (1).

We now prove the theorem. Consider the sets $(h \circ \pi)^{-1}([0, t)) = B_t^-$ and $(h \circ \pi)^{-1}((t, 1]) = B_t^+$. For any $x, y \in B_t^-$, the geodesic $[x, y]$ is clearly contained in B_t^- by (1). It follows that B_t^- is convex and (therefore) connected. By similar reasoning B_t^+ is convex and connected. Both B_t^- and B_t^+ are obviously open, and they are disjoint. We note finally that $B_t^- \cup B_t^+ = X - \pi_{\mathcal{B}}(B_t)$ (since $\pi_{\mathcal{B}}(B_t) = (h \circ \pi)^{-1}(t)$, by (2)), completing the proof. □

Definition 4.5. A *hyperplane* H in a CAT(0) cubical complex X is the image $\pi_{\mathcal{B}}(B_{1/2})$, where B is a connected component of $\mathcal{B}(X)$. We sometimes denote the complementary halfspaces H^+ and H^-.

Note 4.6. In what follows, we typically identify $\pi_{\mathcal{B}}(B)$ with B, and $\pi_{\mathcal{B}}(B_t)$ with B_t, for the sake of convenience in notation.

5. Combinatorics of Hyperplanes

Definition 5.1. Let X be a complete CAT(0) space. If C is a closed convex subset of X, then $\pi_{(X,C)}$ denotes the projection from X to C. If x_1, x_2, and x_3 are points in X, then $\angle_{x_2}^X(x_1, x_3)$ denotes the Alexandrov (or upper) angle in X between the geodesics $[x_2, x_1]$ and $[x_2, x_3]$. We refer the reader to [1] for the definitions, which appear on pages 176 and 9, respectively.

5.1. Three Lemmas.

Lemma 5.2. *Let H_1, H_2 be hyperplanes in X, and assume that $H_1 \cap H_2 \neq \emptyset$. The projections $\pi_{(X,H_i)} : X \to H_i$ and $\pi_{(X,H_i \cap H_j)} : X \to H_i \cap H_j$ agree on H_j, where $\{i, j\} = \{1, 2\}$.*

Proof. For the sake of simplicity, we let $j = 1$ and $i = 2$. Choose a point $x \in H_1$. We consider the block B containing H_1, and the projection $\pi_{(B,B \cap H_2)} : B \to B \cap H_2$. We denote the latter projection by π. Let C be a marked cube of B containing $\pi(x)$. We note that C must be at least two-dimensional, since C meets at least two hyperplanes. We write $C = C' \times [0, 1] \times [0, 1]$, where $C' \times \{1/2\} \times [0, 1] = H_2 \cap C$ and $C' \times [0, 1] \times \{1/2\} = H_1 \cap C$.

We claim that $\pi(x) \in H_1$ (i.e., $\pi(x) \in H_1 \cap H_2$, since $\pi(x) \in H_2$ by definition). Express $\pi(x)$ as $(y, 1/2, t) \in C' \times [0,1] \times [0,1] = C$. Now since $x \in B_{1/2} = H_1$, we have, by the product decomposition of B (Lemma 4.3(2)),

$$d(x, \pi(x)) = \sqrt{D^2 + |t - 1/2|^2},$$

where D is the distance from x to $(y, 1/2, 1/2)$. Since $(y, 1/2, 1/2) \in H_2 \cap B$ and $\pi(x)$ is the point of $B \cap H_2$ closest to B, we must have $t = 1/2$. That is, $\pi(x) = (y, 1/2, 1/2)$, so $\pi(x) \in H_1$, as claimed.

Next, we claim that $\pi(x) = \pi_{(X,H_2)}(x)$. The proof of this fact uses the following characterization of the projection: if X is a complete CAT(0) space, C is a closed convex subset of X, and $x \in X - C$, then $\pi_{(X,C)}(x)$ is the unique element of C with the property that $\angle^X_{\pi_{(X,C)}(x)}(x, z) \geq \pi/2$ for all $z \in C - \pi_{(X,C)}(x)$. We choose $z \in H_2 - \{\pi(x)\}$. Since $\pi(x)$ is in the interior of B, there is some $z' \in [\pi(x), z]$, $z' \neq \pi(x)$, such that $z' \in B \cap H_2$. By the definition of $\pi(x) = \pi_{(B, B\cap H_2)}(x)$, $\angle^B_{\pi(x)}(x, z) \geq \pi/2$. Since B is a convex subset of X, $\angle^B_{\pi(x)}(x, z) = \angle^X_{\pi(x)}(x, z')$. It now follows that

$$\angle^X_{\pi(x)}(x, z) = \angle^X_{\pi(x)}(x, z') \geq \pi/2,$$

so $\pi(x) = \pi_{(X,H_2)}(x)$.

Now we argue that $\pi(x) = \pi_{(X,H_1\cap H_2)}(x)$. If not, then there is $y \in H_1 \cap H_2$ such that $d_X(x, y) < d_X(x, \pi(x))$. This is impossible, however, since $\pi(x)$ is the closest point in H_2 to x. □

Lemma 5.3. *Assume that H_1 and H_2 are hyperplanes, $H_1 \neq H_2$, and $H_1 \cap H_2 \neq \emptyset$. If e is a marked edge perpendicular to H_1, then $d_{H_2|_e}$ is constant.*

Proof. Suppose that e is perpendicular to H_1. Let B denote the block containing the hyperplane H_1. Consider the midpoint of e; call it x. We let π denote the projection from X onto H_2. Let C be a closed marked cube of B which contains $\pi(x)$. As in the proof of Lemma 5.2, we write $C = C' \times [0,1] \times [0,1]$, where $H_1 \cap C = C' \times [0,1] \times \{1/2\}$ and $H_2 \cap C = C' \times \{1/2\} \times [0,1]$.

Since $\pi(x) \in H_1 \cap H_2$ by Lemma 5.2, one has that $[x, \pi(x)] \subseteq B_{1/2} = B_0 \times \{1/2\}$. We can express $[x, \pi(x)]$ as $[\pi_0(x), \pi_0(\pi(x))] \times \{1/2\}$, where π_0 denotes the projection from B to B_0. If y is some other point on e, then $[\pi_0(x), \pi_0(\pi(x))] \times \{h(y)\}$ is a geodesic connecting y to a point in H_2. It follows that $d_{H_2}(y) \leq d_{H_2}(x)$, for all $y \in e$.

One argues that equality always holds by the convexity of the function d_{H_2} (see Corollary 2.5 on page 178 of [1]). Indeed, suppose that y_1, $y_2 \in e$, where $h(y_1) < h(x) < h(y_2)$, and $d_{H_2}(y_i) < d_{H_2}(x)$ for at least one index $i \in \{1, 2\}$. The function d_{H_2} is concave up (i.e., convex) and non-constant on the geodesic $[y_1, y_2]$, and attains a maximum value of $d_{H_2}(x)$ at the interior point x. This is a contradiction. □

Lemma 5.4. *([5], Lemma 2.6(4)) If H_1 and H_2 are hyperplanes, $H_1^+ \cap H_2^+$, $H_1^- \cap H_2^+$, $H_1^+ \cap H_2^-$, and $H_1^- \cap H_2^-$ are all non-empty, then $H_1 \cap H_2 \neq \emptyset$.*

Proof. Assume that the four intersections in the hypothesis are all non-empty and $H_1 \cap H_2 = \emptyset$. It follows that $\{H_1^+ \cup H_2^+, H_1^- \cup H_2^-\}$ is an open cover of X. Each of the half-spaces H_1^+, H_1^-, H_2^+, and H_2^- is a convex subspace of the CAT(0) space X, and therefore contractible. Each of the four intersections in the hypothesis is contractible for the same reason.

It follows that the sets $X^+ = H_1^+ \cup H_2^+$ and $X^- = H_1^- \cup H_2^-$ are simply connected, since each is the union of two open contractible sets which intersect in an open contractible set. The intersection $X^+ \cap X^-$ is the disjoint union of two open contractible sets: $H_1^+ \cap H_2^-$ and $H_2^+ \cap H_1^-$. Let c be an arc contained in X^+, connecting $H_1^+ \cap H_2^-$ to $H_2^+ \cap H_1^-$, and meeting each in an open segment.

We apply van Kampen's theorem to the pieces $X^- \cup c$ and X^+. The first piece $X^- \cup c$ satisfies $\pi_1(X^- \cup c) \cong \mathbb{Z}$, while the second is simply connected. The intersection of these two pieces is the simply connected set $(H_1^+ \cap H_2^-) \cup (H_2^+ \cap H_1^-) \cup c$. It follows that $\pi_1(X^- \cup X^+) = \pi_1(X)$ is isomorphic to $\mathbb{Z}$. Since X is CAT(0), it must be contractible. This is a contradiction. □

5.2. **Sageev's Combinatorial Results.** We cover only some basic combinatorial results in this subsection. A more advanced treatment of the combinatorial properties of hyperplanes appears in an appendix to [8].

Proposition 5.5. [13] *An edge-path p in X^1 is geodesic if and only if p crosses any given hyperplane H at most once.*

Proof. We first prove the forward direction. Suppose, on the contrary, that a certain geodesic edge-path crosses some hyperplane more than once. We consider a shortest geodesic edge-path p which crosses some hyperplane multiple times. We write $p = (e_1, \ldots, e_n)$, and let $H_1, \ldots, H_n$ denote the hyperplanes crossed by the edges $e_1, \ldots, e_n$ (respectively). Since p is the shortest edge-path with the given property, we must have $H_1 = H_n$, but there are no other repetitions in the list $H_1, \ldots, H_n$ (i.e., a total of $n-1$ distinct hyperplanes are crossed by p). We let H_1^- denote the component of $X - H_1$ containing $\iota(e_1)$ and $\tau(e_n)$. Clearly the other component of $X - H_1$, denoted H_1^+, contains the edge-path $(e_2, \ldots, e_{n-1})$. We adopt the convention that $\iota(e_j) \in H_j^-$ and $\tau(e_j) \in H_j^+$, for $j \in \{2, \ldots, n-1\}$.

Consider an edge e_j, $j \in \{2, \ldots, n-1\}$. Note that $\iota(e_1) \in H_1^- \cap H_j^-$, $\iota(e_j) \in H_1^+ \cap H_j^-$, $\tau(e_j) \in H_1^+ \cap H_j^+$, and $\tau(e_n) \in H_1^- \cap H_j^+$. It follows that the hyperplanes H_1 and H_j intersect, for $j \in \{2, \ldots, n-1\}$, by Lemma 5.4.

We now apply Lemma 5.3. Since $d(\iota(e_2), H_1) = 1/2$ and d_{H_1} is constant on e_2, we must have $d_{H_2}(x) = 1/2$ for all x in e_2. We can inductively conclude that $d_{H_1}(x) = 1/2$ for all x in $(e_2, \ldots, e_{n-1})$.

It follows that the entire edge-path $p = (e_1, \ldots, e_n)$ is contained in the block B containing H_1. The edges $e_2, \ldots, e_{n-1}$ are all unmarked edges in the block $B = B_0 \times [0, 1]$. We identify $\iota(e_2)$ with a vertex $(v', 1) \in B$ and $\tau(e_{n-1})$ with a vertex $(v'', 1) \in B$. It follows that $\iota(e_1) = (v', 0)$ and $\tau(e_n) = (v'', 0)$. The edge-path $(e_2, \ldots, e_{n-1})$ connects $(v', 1)$ to $(v'', 1)$. There is a corresponding edge-path $(e_2', \ldots, e_{n-1}')$ connecting $(v', 0)$ to $(v'', 0)$. This contradicts the fact that p is geodesic.

Now suppose that p crosses any given hyperplane H at most once. It follows that the endpoints $\iota(p), \tau(p)$ of p are separated by all of the hyperplanes crossed by p. If we assume that there are n such hyperplanes in all (and so p has length n), then any edge-path q from $\iota(p)$ to $\tau(p)$ must cross all n of these hyperplanes, so the length of q is at least n. It follows that p is geodesic. □

Definition 5.6. Suppose that (e_1, e_2) is an edge-path in a CAT(0) cubical complex X such that e_1 and e_2 are perpendicular sides of a square C in X. We let e_i' denote the side of C opposite e_i, for $i = 1, 2$. The operation of replacing (e_1, e_2) by the edge-path (e_2', e_1') is called a *corner move*. Note that the edge-paths (e_1, e_2) and (e_2', e_1') have the same endpoints.

Proposition 5.7. *If (e_1, e_2) is an edge-path in X, e_i crosses the hyperplane H_i $(i = 1, 2)$, $H_1 \neq H_2$, and $H_1 \cap H_2 \neq \emptyset$, then the edges e_1 and e_2 are perpendicular sides of a square C in X.*

Proof. Let B denote the block containing the hyperplane H_1. We write $B = B_0 \times [0, 1]$, and assume that $\iota(e_1) = (v, 0)$, for some vertex $v \in B_0$. It follows that $\tau(e_1) = (v, 1)$. Since $H_2 \cap H_1 \neq \emptyset$ and $H_1 \neq H_2$, we have that d_{H_1} is constant on e_2, by Lemma 5.3. In particular, $d_{H_1}(x) = 1/2$, for any x on the edge e_2, since $d(\iota(e_2), H_1) = 1/2$. It follows that e_2 has the form $[(v, 1), (v', 1)]$, where $[v, v']$ is an edge in B_0. Therefore the edge-path (e_1, e_2) forms one half of the boundary of the square $(v, v') \times [0, 1] \subseteq B$, as desired. □

Proposition 5.8. [13] *If $H_1, \ldots, H_n$ satisfy $H_i \cap H_j \neq \emptyset$ for any $i, j \in \{1, \ldots, n\}$, then $H_1 \cap \ldots \cap H_n \neq \emptyset$.*

Proof. The proof is by induction on n. The conclusion is obvious if $n = 2$. We suppose that $n > 2$. By induction, $H_1 \cap \ldots \cap H_{n-1} \neq \emptyset$, so we take $x \in H_1 \cap \ldots \cap H_{n-1}$. By Lemma 5.2, $\pi_{(X,H_n)}(x) = \pi_{(X,H_j \cap H_n)}(x)$ for $j \in \{1, \ldots, n-1\}$. It follows that $\pi_{(X,H_n)}(x) \in H_1 \cap \ldots \cap H_n$. □

6. Applications

6.1. The set-with-walls property.

Definition 6.1. (first defined in [7]) Let S be a set. A *wall* W in S is a partition $\{W^-, W^+\}$ of S. Two points $x, y \in S$ are *separated* by the wall W if $x \in W^-$ and $y \in W^+$ (or vice versa). We say that $(S, \mathcal{W})$ is a *set with walls* if $\mathcal{W}$ is a collection of walls in S such that, for any $x, y \in S$, at most finitely many walls $W \in \mathcal{W}$ separate x from y.

If G is a group and S is a G-set, then $(S, \mathcal{W})$ is a *G-set with walls* if the natural action of G permutes the set $\mathcal{W}$.

Definition 6.2. If $(S, \mathcal{W})$ is a set with walls, then the *wall pseudometric* $d_{(S,\mathcal{W})} : S \times S \to \mathbb{R}^+$ is defined by

$$d_{(S,\mathcal{W})}(x, y) = |\{W \in \mathcal{W} \mid W \textit{ separates } x \textit{ from } y\}|.$$

If $(S, \mathcal{W})$ is a G-set with walls, then we say that G acts *properly* on $(S, \mathcal{W})$ if, for any $r > 0$ and $x \in S$, the set

$$\{g \in G \mid d_{(S,\mathcal{W})}(x, gx) < r\}$$

is finite.

Remark 6.3. It is straightforward to check that $d_{(S,\mathcal{W})}$ is symmetric and satisfies the triangle inequality, and that G acts isometrically on $(S, \mathcal{W})$ if the latter is a G-set with walls.

Theorem 6.4. *If X is a CAT(0) cubical complex, then $(X^0, \mathcal{W})$ is a set with walls, where $\mathcal{W} = \{\{H^+ \cap X^0, H^- \cap X^0\} \mid H \text{ is a hyperplane in } X\}$. If G acts cellularly and by isometries on X, then $(X^0, \mathcal{W})$ is a G-set with walls. If G acts properly on X, then G acts properly on $(X^0, \mathcal{W})$.*

Proof. (Sketch) The fact that $\{H^+ \cap X^0, H^- \cap X^0\}$ is a wall follows from Theorem 4.4; the fact that two vertices x, y are separated by at most finitely many walls $W_H = \{H^+ \cap X^0, H^- \cap X^0\} \in \mathcal{W}$ follows from the fact that a wall W_H separates x from y if and only if a geodesic edge-path from x to y crosses H. The remaining statements are similarly straightforward to check. □

We note that [3] contains a proof of the converse: there is a construction of a CAT(0) cubical complex associated to any space with walls.

Definition 6.5. A discrete group G has the *Haagerup property* if there is a proper affine isometric action of G on a Hilbert space $\mathcal{H}$. Here "proper" means metrically proper: if $v \in \mathcal{H}$ and $r > 0$ are given, then $|\{g \in G \mid d(v, g \cdot v) < r\}| < \infty$.

Theorem 6.6. [10] *If G acts properly, cellularly, and by isometries on a CAT(0) cubical complex X, then G has the Haagerup property.*

Proof. (Sketch) One chooses a basepoint $v \in X^0$ and orientations for all hyperplanes $H \subseteq X$. Let $\mathcal{W}^{or}$ denote the set of oriented hyperplanes. The group G acts as (infinite) signed permutation matrices on the Hilbert space $\ell^2(\mathcal{W}^{or})$. For $g \in G$, we let

$$\delta(g) = \sum \pm H,$$

where the sum is over all hyperplanes separating v from gv. Here H is counted with the plus sign if one crosses H in the direction of its given orientation when moving from v to gv, and it is counted with the minus sign otherwise.

The action $\alpha : G \times \ell^2(\mathcal{W}^{or}) \to \ell^2(\mathcal{W}^{or})$ given by $\alpha(g, v) = g \cdot v + \delta(g)$ has the desired properties. □

6.2. The median algebra property.

Let $\mathcal{P}(S)$ denote the power set of S.

Definition 6.7. A *median algebra* is a set S together with an interval operation $[,] : S \times S \to \mathcal{P}(S)$ such that

1. $[x, x] = \{x\}$ for $x \in S$;
2. $[x, y] = [y, x]$ for $x, y \in S$;
3. If $z \in [x, y]$, then $[x, z] \subseteq [x, y]$;
4. For any $x, y, z \in S$, $[x, y] \cap [y, z] \cap [x, z]$ is a singleton set. The unique element of this singleton set, denoted $m(x, y, z)$, is called the *median* of x, y, z.

A median algebra is *discrete* if each set $[x, y]$ is finite.

Definition 6.8. Assume that X is a CAT(0) cubical complex. If $x, y \in X^0$, then the *geodesic interval* $[x, y]$ is the set of all vertices $z \in X^0$ that lie on some geodesic edge-path connecting x to y.

Remark 6.9. Note, for instance, that the geodesic interval between two integral points (a, b) and (c, d) ($a \leq c$ and $b \leq d$) in $\mathbb{R}^2$ is $\{(x, y) \mid a \leq x \leq c;\ b \leq y \leq d;\ x, y \in \mathbb{Z}\}$.

Theorem 6.10. *Let X be a CAT(0) cubical complex. The set of vertices X^0 is a discrete median algebra, where the interval operation $[,] : X^0 \times X^0 \to \mathcal{P}(X^0)$ is the geodesic interval.*

Proof. Properties (1) and (2) are clear.

We now prove (3). Let $z \in [x, y]$. This means that there is a geodesic edge-path p connecting x to y and passing through z. We can express p as $p_1 \cup p_2$, where p_1 is a geodesic edge-path connecting x to z and p_2 is a geodesic edge-path connecting z to y. If $w \in [x, z]$, then there is a geodesic edge-path p'_1 connecting x to z and passing through w. Since p'_1 and p_1 have the same length, $p'_1 \cup p_2$ is also a geodesic edge-path connecting x to z, and it passes through w. Therefore $w \in [x, y]$. It follows that $[x, z] \subseteq [x, y]$, proving (3).

We now prove that $[x, y]$ is always finite. If $H_1, \ldots, H_n$ are the hyperplanes separating x from y, then, by Proposition 5.5, an edge-path p is a geodesic edge-path connecting x to y if and only if p begins at x and crosses exactly the hyperplanes $H_1, \ldots, H_n$. However, such an edge-path is uniquely determined by the order in which the hyperplanes $H_1, \ldots, H_n$ are crossed. It follows that there are at most $n!$ geodesic edge-paths, each of which passes through only finitely many points, so $|[x, y]| < \infty$.

We now prove (4). Fix $x, y, z \in X^0$. We first show that $[x, y] \cap [y, z] \cap [x, z]$ contains at most one element. Suppose $v, w \in [x, y] \cap [y, z] \cap [x, z]$ and $v \neq w$. There is a hyperplane H separating v from w. It must be that two (or more) elements of $\{x, y, z\}$ lie in one of the complementary components of $X - H$. It follows without loss of generality (i.e., up to relabelling) that v is separated from both x and y by H. Since $v \in [x, y]$ by our assumption, there is a geodesic edge-path p from x to y passing through v. The geodesic edge-path p would necessarily cross H twice, however. This is a contradiction.

We now need to show that $[x, y] \cap [y, z] \cap [x, z]$ is non-empty. We do this by induction on $d(x, y) + d(y, z) + d(x, z)$, where d denotes the edge-path (or combinatorial) distance. The base case is trivial. For the inductive step, we need a definition. If a hyperplane H separates both x and y from z, then we say that H is an $\{x, y\}$-*hyperplane*. We can similarly define $\{x, z\}$- and $\{y, z\}$-hyperplanes. Note that any hyperplane crossed by an edge-path geodesic between any two points of $\{x, y, z\}$ must be a $\{a, b\}$-hyperplane, where $\{a, b\} \subseteq \{x, y, z\}$. If $z \in [x, y]$, $x \in [y, z]$, or $y \in [x, z]$, then the desired conclusion is clear, so we assume that none of x, y, and z is contained in the interval of the other two. We choose geodesic edge-paths p_x, p_y connecting z to x and y, respectively.

We claim that there is some $\{x, y\}$-hyperplane H that is crossed by both p_x and p_y. Indeed, p_x crosses only $\{x, y\}$- and $\{y, z\}$-hyperplanes by definition, and p_y crosses only $\{x, y\}$- and $\{x, z\}$-hyperplanes. Thus, if no $\{x, y\}$-hyperplane is crossed by both p_x and p_y, then $p_x^{-1} p_y$ crosses no hyperplane more than once, and is therefore geodesic. Since $p_x^{-1} p_y$ passes through z, we have $z \in [x, y]$, a contradiction.

Next, we claim that there are geodesic edge-paths p'_x and p'_y from z to x and y with the property that p'_x and p'_y cross all $\{x, y\}$-hyperplanes before crossing any $\{x, z\}$- or $\{y, z\}$-hyperplanes. We prove only that there is such a p'_x, since the proof that there is such a p'_y is similar. To establish the existence of the desired p'_x, it is sufficient to show that, whenever p_x crosses a $\{y, z\}$-hyperplane H' before an $\{x, y\}$-hyperplane H'', $H' \cap H'' \neq \emptyset$, for then we can use corner moves to change p_x into the desired p'_x. We assume the convention that $z \in (H')^- \cap (H'')^-$. If e' is the (unique) edge of p_x crossing H', then $\tau(e') \in (H')^+ \cap (H'')^-$. If e'' is the edge of

p_x crossing H'', then $\tau(e'') \in (H')^+ \cap (H'')^+$. Now note that $y \in (H')^- \cap (H'')^+$. We now have $H' \cap H'' \neq \emptyset$, by Lemma 5.4. This proves the claim.

We therefore have p'_x and p'_y (as above). Let H_1 be the first hyperplane crossed by p'_x. It is, of course, an $\{x, y\}$-hyperplane. We claim that we can alter p'_y to obtain a new geodesic edge-path p''_y connecting z to y, such that p''_y crosses H_1 first. (We note that p'_y must cross H_1, since H_1 separates z from y by definition.) It is enough to show that if the hyperplane $\{x, y\}$-hyperplane H_2 is crossed by p'_y before H_1, then $H_1 \cap H_2 \neq \emptyset$, for then we can alter p'_y by corner moves in order to arrive at the desired p''_y. We assume the convention that $z \in (H_1)^- \cap (H_2)^-$. If e_2 is the edge of p'_y crossing H_2, then $\tau(e_2) \in (H_1)^- \cap (H_2)^+$. If e_1 is the edge of p'_y crossing H_1, then $\tau(e_1) \in (H_1)^+ \cap (H_2)^+$. If e_x is the edge of p'_x crossing H_1 then $\tau(e_x) \in (H_1)^+ \cap (H_2)^-$. It follows from Lemma 5.4 that $H_1 \cap H_2 \neq \emptyset$. This proves the claim.

We've now shown that there are geodesic edge-paths p'_x, p''_y connecting z to x and y (respectively), and having the same initial edge $\hat{e}$. We assume $z = \iota(\hat{e})$. By the induction hypothesis $[\tau(\hat{e}), y] \cap [x, \tau(\hat{e})] \cap [x, y]$ is non-empty. Since

$$[\tau(\hat{e}), y] \cap [x, \tau(\hat{e})] \cap [x, y] \subseteq [z, y] \cap [x, z] \cap [x, y]$$

by (3), the induction is complete. □

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