2.4 - Subsets and Permutations
Some announcements

- Homework #3: Text (59-66): 2, 4, 5, 7, 10, 12, 14, 20, 25, 28, 29, 34, 36, 58, 63, 73 (this is posted on Sakai)
- Exam 1: next Monday (Feb. 15).
- Keep working on projects. Be sure to review guidelines. You’re welcome to show me your ballots ahead of time.
• Homework questions?
• Discuss classwork activity from Friday.
• Discuss subsets and permutations. Questions arising from our discussions on coalitions. (This material will appear again in our study of symmetry - Unit 3.)
In general, a coalition is any collection of voters in a system.

Different measures of power use different types of coalitions.

BPI counts critical players in winning coalitions (coalitions that meet or exceed the quota). In these coalitions, order does not matter.

SSPI counts pivotal players in sequential coalitions. A sequential coalition always consists of all players but order does matter.
A set is a collection of objects (called elements). A coalition of players in a weighted voting system is an example of a set.

A set may have no elements, finitely many elements, or infinitely many elements. A set with no elements it is called the empty set.

A subset is any combination of elements from a set. The empty set is a subset of any set.

In sets written with the notation \{,\}, order does not matter. The notation \langle,\rangle means order does matter.
Consider a set with one element: \{ A \}. How many subsets are there of this set?

There are two: the empty set \{ \}, and the set itself \{ A \}. 
Consider a set with two elements: \{A, B\}. How many subsets are there of this set? We want to reduce this problem to the previous one.

Say we remove \(B\) from the set. The sets that do not contain \(B\) are the same as the subsets of \{A\}. These are \{\}\ and \{A\}.

Next we find the subsets that do contain \(B\): \{B\} and \{A, B\}.

Thus, there are a total of 4 subsets of \{A, B\}:

\[
\{\}, \{A\}, \{B\}, \{A, B\}
\]
Consider a set with three elements: \( \{A, B, C\} \). How many subsets are there of this set?

The sets that *do not* contain \( C \) are the same as the subsets of \( \{A, B\} \):

\[
\{\}, \{A\}, \{B\}, \{A, B\}
\]

The subsets that *do* contain \( C \):

\[
\{C\}, \{A, C\}, \{B, C\}, \{A, B, C\}
\]

Thus, there are a total of 8 subsets of \( \{A, B, C\} \).
Let’s collect what we’ve done so far.

There is 1 set with zero elements.

There are 2 subsets of a set with one element.

There are 4 subsets of a set with two elements.

There are 8 subsets of a set with three elements.

How many subsets of a set with four elements? N elements?
A set with $N$ elements has $2^N$ subsets.

The reasoning behind this statement comes from a mathematical idea called *induction*.

Show the statement holds for the first case, then show that one can *step up* to the next case.
A weighted voting system with $N$ players has $2^N - 1$ coalitions.

We remove the empty set because it is not a coalition.

If a system does not have a dictator, then we can remove all subsets consisting of 1 player. Thus, there are $2^N - N - 1$ coalitions of 2 or more players.
We had a different formula before.

The number of subsets of \( k \) elements in a set of \( N \) elements

\[
\binom{N}{k} = \frac{N!}{k!(N - k)!}.
\]

In fact, what we’ve shown is that

\[
\binom{N}{0} + \binom{N}{1} + \cdots + \binom{N}{N - 1} + \binom{N}{N} = 2^N.
\]
Wait a minute! Does that really work?

Yes! One way to see it is with Pascal’s Triangle.

\[
\begin{array}{cccc}
N=0 & 1 & \text{Total } = 1 = 2^0 \\
N=1 & 1 & 1 & \text{Total } = 2 = 2^1 \\
N=2 & 1 & 2 & 1 & \text{Total } = 4 = 2^2 \\
N=3 & 1 & 3 & 3 & 1 & \text{Total } = 8 = 2^3 \\
N=4 & 1 & 4 & 6 & 4 & 1 & \text{Total } = 16 = 2^4 \\
\end{array}
\]

\[
(\binom{4}{0}) (\binom{4}{1}) (\binom{4}{2}) (\binom{4}{3}) (\binom{4}{4})
\]
Here is a fact we’ve discussed previously.

**There are \( N! \) sequential coalitions of \( N \) players.**

Mathematically one sees this in the study of *permutations*. 
A **permutation** of a set of objects is an ordered list of the objects. Different permutations correspond to different orders.

Given \( N \) objects, there are \( N \) choices for the first spot, \( N - 1 \) choices for the second spot, \( N - 2 \) choices for the second spot, etc. Thus, the total number of choices is

\[
N \times (N - 1) \times (N - 2) \times \cdots \times 2 \times 1 = N!.
\]