Math 10B - Calculus by Hughes-Hallett, et al.
Chapter 6 - Constructing antiderivatives
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1 Antiderivatives graphically and numerically

Definition 1.1. The antiderivative of a function f is a function F such that $F' = f$.

Example 1.2. Let $f(x) = 3x^2$. The *one* antiderivative for f would be x^3 since the derivative of x^3 is $3x^2$. But also $x^3 + 1$ would be an antiderivative, as would $x^3 + 2$. In fact, for any constant C,

$$\frac{d}{dx}(x^3 + C) = 3x^2.$$

Graph f and the family of functions F.

Example 1.3. Given $F = x^2$, graph the function $f = F'$ on $[0, 2]$ Since $F' = 2x$, then f is a linear function.

Remark 1.4. What about the other direction? How does knowing the graph of f give us an idea of the shape of the graph F? Since $F' = f$, then the y-coordinates of f represent slopes of the tangent lines. So, just as we 'think backwards' to get the function F for f, so too can we do this graphically.

Example 1.5. Exercise 4 on page 303: f is the line through $(-1, 0)$ and $(2, 1)$. Draw F such that $F(0) = 1$. We have a function that switches, at $(0, 1)$, from having tangent lines with negative slopes (decreasing) to having positive slopes (increasing). Moreover, the tangent line slopes (better: their absolute values) are greatest at the points given on this region. At $(0, 1)$, the tangent line of F has zero slope and so that is a critical point. Thus, F is a parabola.

Remark 1.6. We can also transfer from f to F by using the Fundamental theorem. Recall, if $F' = f$ then

$$\int_a^b f(x)\ dx = F(b) - F(a).$$

Thus, $F(b) - F(a)$ is the area under the graph of f from a to b.

Example 1.7. Let f be the isosceles triangular region from $[0, 4]$ with $y = x$ for $0 < x < 2$ and $y = 4 - x$ for $2 < x < 4$. Graph F given $F(0) = 0$.
The area under f from 0 to 1 is $\frac{1}{2}$. Thus,

$$\frac{1}{2} = \int_0^1 f(x)\ dx = F(1) - F(0) = F(1).$$

So $F(1) = \frac{1}{2}$.
We repeat from 0 to 2. Now the area under f is 1, so

$$1 = \int_0^2 f(x)\ dx = F(2) - F(0) = F(2).$$

Thus, $F(2) = 1$.
Continuing in this way we get $F(3) = \frac{3}{2}$ and $F(4) = 2$. This gives us an idea of the shape of the graph F.

2 Constructing antiderivatives analytically

Remark 2.1. Suppose $f(x) = 0$, what are the antiderivatives F? The condition $F' = f$ implies that every tangent line has zero slope, i.e., is horizontal. This means that F is a constant function.

Proposition 2.2. If $F'(x) = 0$ on an interval, then $F(x) = C$ on this interval, for some constant C.

Remark 2.3. Here is a related comment. We know that if F is an antiderivative of f then so is $F + C$ for any constant. Are there any other antiderivatives?
Let G be another antiderivative of f. Then

$$(F - G)' = F' - G' = f - f = 0.$$

Now by the previous proposition, this implies that $(F - G) = C$. In other words, $F = G + C$.

Proposition 2.4. If F and G are both antiderivatives of f on an interval, then $F(x) = G(x) + C$.

Definition 2.5. The *indefinite integral* of f is

$$\int f(x)\ dx = F(x) + C,$$

where F is an antiderivative of f.

Remark 2.6. The indefinite integral is a way of expressing the family of antiderivatives for f. Thus, the '$+C$' is **very** important.

Remark 2.7. Let k be a constant and let $f(x) = k$. What is $\int k\ dx$? We know *one* antiderivative of f is $F(x) = kx$. By the previous proposition, we now know all of them

$$\int k\ dx = kx + C.$$

Thus, the problem of finding *all* antiderivatives is now just a problem of finding *one* antiderivative.

Example 2.8. Compute the indefinite integral for $x^2, x^3, \ldots, x^n$.

$$\int x^n = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

Remark 2.9. Recall that $\frac{d}{dx} \ln x = \frac{1}{x}$. Thus, we would like to say, $\int \frac{1}{x}\ dx = \ln x$ but there is a problem, $\ln x$ is only defined for $x > 0$ (there is no point in considering $x = 0$). Thus, if $x < 0$ we consider

$$\frac{d}{dx} \ln(-x) = (-1)\frac{1}{-x} = \frac{1}{x}.$$

Thus, for $x < 0$, $\int \frac{1}{x}\ dx = \ln x$. We can sum this up by saying an antiderivative of $\frac{1}{x}$ is $\ln |x|$, or

$$\int \frac{1}{x}\ dx = \ln |x| + C.$$

Remark 2.10. Some other handy antiderivatives:

$$\int e^x\ dx = e^x + C$$

$$\int \cos x\ dx = \sin x + C$$

$$\int \sin x\ dx = -\cos x + C$$

Theorem 2.11. Let f and g be continuous functions and c a constant. Then,

$$(1) \int (f(x) \pm g(x)) = \int f(x)\ dx + \int g(x)\ dx$$
$$(2) \int cf(x)\ dx = c \int f(x)\ dx.$$

Example 2.12. Compute the integral

$$\int 3x^3 + 5x^2 + x\ dx$$

We have

$$\begin{aligned}\int 3x^3 + 5x^2 + x\ dx &= 3\int x^3\ dx + 5\int x^2\ dx + \int x\ dx \\ &= 3(\frac{x^4}{4}) + 5(\frac{x^3}{3}) + \frac{x^2}{2}.\end{aligned}$$

Example 2.13. Evaluate the integral

$$\int_0^2 4x^3\ dx.$$

We know that $F = x^4$ is an antiderivative of 4^3. Thus, by the FTC,

$$\int_0^2 4x^3\ dx = F(2) - F(0) = 16.$$

We abbreviate this notation in the following way:

$$\int_0^2 4x^3\ dx = [x^4]_0^2 = 2^4 - 0^4 = 16 - 0 = 16.$$

3 Differential Equations

Remark 3.1. A differential equation is an equation involving a derivative.

Example 3.2. Suppose a car has constant velocity of 40 mph. Write an equation for the distance traveled by the car.
Since distance = rate x time, then $s = 40t$ (we write s for distance for reasons that will become clear). Since the velocity is 40, then

$$\frac{ds}{dt} = 40.$$

This is a type of differential equation. We call $\frac{ds}{dt}$ the *differential*. The solution to this equation is $s = 40t+C$. Note that, when $t = 0$, then $s = C$ and so C represents the *initial distance* s_0 (value) of the function s.

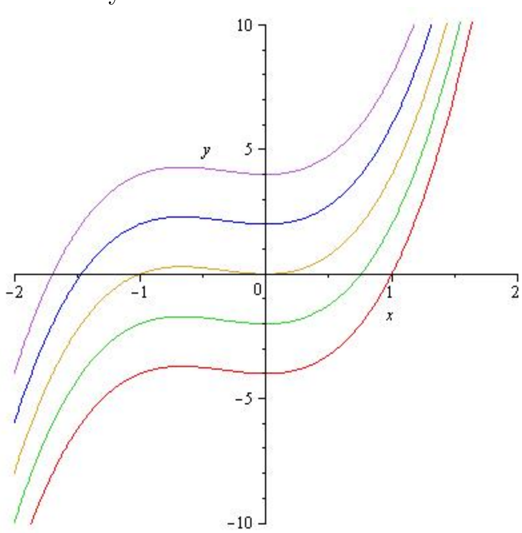
Example 3.3. Suppose we have the differential equation

$$\frac{dy}{dx} = 6x^2 + 4x.$$

We can find the general solution by finding the antiderivative of the function on the right. This is,

$$y = 2x^3 + 2x^2 + C.$$

The family of solutions looks like



How do we determine the exact solution? We need additional information. If, say, we knew a point on the graph, then we could determine exactly which solution is the precise one for this problem.

Remark 3.4. An *initial value problem* is a differential equation with initial conditions provided. This allows us to determine the precise value of C.

Example 3.5. Solve the initial value problem $\frac{dy}{dx} = 2x + 1$, $y(1) = 5$.
The general solution is

$$y = x^2 + x + C.$$

However, we have

$$5 = y(1) = 1 + 1 + C = 2 + C.$$

Thus, $C = 3$ and so the solution is

$$y = x^2 + x + 3.$$

4 Second fundamental theorem of calculus

Theorem 4.1 (Construction Theorem for Antiderivatives)**.** (Second fundamental theorem of Calculus) If f is a continuous function on an interval and if a is an y number in that interval, the the function F defined on the interval as follows is an antiderivative of f:

$$F(x) = \int_a^x f(t)\ dt.$$

Remark 4.2. Let's think about what this theorem means. Suppose $f(x) = \ln x$. This is not an *elementary function*, in the sense that we don't yet have a good way of taking the antiderivative. Recall from the FTC that

$$F(b) - F(a) = \int_a^b \ln x\ dx,$$

where F is an antiderivative of f. Suppose we choose F such that $F(1) = 0$. (Think: Is this always possible?). Then we can rewrite the above as

$$F(x) = \int_1^x \ln t\ dt.$$

Now we have represented the antiderivative F as an integral, dependent on the variable x.

Example 4.3. Find the derivative

$$\frac{d}{dx} \int_0^x \cos(t^2)\ dt.$$

Let F be an antiderivative of $f(x) = \cos(x^2)$ such that $F(0) = 0$. Thus, $F'(x) = f(x) = \cos(x^2)$ By the Construction Theorem,

$$F(x) = \int_0^x \cos(t^2)\ dt.$$

Thus,

$$\cos(x^2) = F'(x) = \frac{d}{dx} \int_0^x \cos(t^2)\ dt.$$

Here are some computations that are helpful for the homework:

t	1	2	3
$\int_0^x \cos(t^2)\ dt$	.747	.882	.886

Example 4.4. Find the value of $F(1)$ where $F'(x) = e^{-x^2}$ and $F(0) = 2$.
By the construction theorem,

$$F(x) - F(0) = \int_0^x e^{-t^2}\ dt.$$

We compute $\int_0^x e^{-t^2}\ dt$ numerically for given values:

x	-3	-2	-1
int	-.702	-.461	-.904

Thus,

$$F(1) = 2 + .747 = 2.747.$$

5 The equations of motion

Remark 5.1. Acceleration due to gravity is

$$\text{Acceleration} = \frac{dv}{dt} = -g.$$

The g is constant (9.8 m/sec^2). The negative sign indicates that a body is falling. The v is velocity of the falling body. What is an equation for velocity? To get this, we solve the differential equation:

$$v = -gt + v_0,$$

where v_0 is the initial velocity. What about position? We have

$$\frac{ds}{dt} = v = -gt + v_0.$$

Solving this differential equation gives,

$$s = -\frac{gt^2}{2} + v_0 t + s_0,$$

where s_0 is inital position.

Example 5.2. An object is dropped from a 400-foot tower. When does it hit the ground and how fast is it going at the time of the impact?
This tells us that the initial position $s(0) = s_0 = 400$. We have no initial velocity, so $v(0) = v_0 = 0$. Since gravity is $32ft/sec$, then our equation becomes,

$$s = -16t^2 + 400.$$

At what time is $s(t) = 0$? Solving for t gives,

$$t^2 = 25 \Rightarrow t = 5.$$