1 Areas and volumes

Remark 1.1. We now move into several sections on using the definite integral. The first is applications to geometry, specifically, area and volume.

Example 1.2. Use integration to find the area of a right triangle with base 6 and height 3.
Clearly, we do not *need* integration to solve this problem. However, this will be a useful introduction into the topic so that we can apply it later to more exotic objects.
We begin by taking a vertical, rectangular slice of the triangle. We must choose whether to use right-hand or left-hand endpoints. The width of this slice we denote by Δx (we assume, as usual, that our slices are evenly spaced). The height will vary. Let h_i be the height of the ith slice. Using similar triangles we get

$$\frac{h_i}{x} = \frac{3}{6} \Rightarrow h_i = \frac{1}{2}x.$$

Thus, the area is approximated by

$$A \approx \sum_{i=1}^{n} h_i \Delta x = \frac{1}{2}\sum_{i=1}^{n} x\Delta x.$$

Taking the limit as $n \to \infty$ and $\Delta x \to 0$, we get

$$A = \frac{1}{2}\int_0^6 x\ dx = 9.$$

Example 1.3. Approximate the volume in problem 10.
Here we take cross-sections which are circles of width Δx Again, using similar triangles, we get $r_i = \frac{1}{3}x$. Thus, the volume of one slice is approximated as

$$V_i = \pi r_i^2 \cdot \Delta x = \frac{\pi}{9}x^2\Delta x.$$

Thus, the volume is approximated as

$$V \approx \sum_{i=1}^{n} V_i \Delta x,$$

and taking a limit gives

$$V = \frac{\pi}{9} \int_0^6 x^2 \, dx = 8\pi.$$

2 Applications to geometry

Remark 2.1. In this section, we determine the volume of objects by rotating them around the x or y-axis (initially, we will eventually rotate around more general lines).

Example 2.2. Suppose we are given the region bounded by $y = \sqrt{x+1}$, $y = 0$, $x = -1$ and $x = 1$. This region is rotated around the x-axis. What is its volume?
We will take vertical slices of equal width. Each slice is a cylinder. The radius of the cylinder is y and its width is Δx. The equation for this is then

$$V_s = \pi y^2 \Delta x = \pi(x+1)\Delta x.$$

Therefore, the total volume is approximately

$$T \approx \sum \pi y^2 \Delta x = \sum \pi(x+1)\Delta x.$$

Taking the limit as $\Delta x \to 0$ gives

$$T = \int_{-1}^{1} \pi(x+1)\ dx = \frac{pi}{2}\left[(x+1)^2\right]_{-1}^{1} = 2\pi.$$

Example 2.3. Suppose we are given the region bounded by $y = x^2$, $x = 0$, $y = 0$, and $y = 1$. This region is rotated around the y-axis. What is its volume?
We make a change of coordinates to $x = \sqrt{y}$. Each horizontal slice now has height Δy and width $\sqrt{y}$. The equation is then

$$V_s = \pi(\sqrt{y})^2 \Delta y.$$

Therefore, the total volume is approximately

$$T \approx \sum \pi y \Delta y = \sum \pi y \Delta y.$$

Taking the limit as $\Delta y \to 0$ gives

$$T = \int_{0}^{1} \pi y\ dy = \frac{pi}{2}\left[y^2\right]_{0}^{1} = \frac{\pi}{2}.$$

Remark 2.4. In the previous two examples, we used the techniques of section 8.1 to compute the volume of a surface rotated around an axis. What if it is not rotated around an axis but instead a different line?

Example 2.5. Find the volume of the region bounded by $y = \sqrt[3]{x}$, $x = 4y$, and $x = 0$ rotated about the axis $y = 3$.
These functions intersect at $x = 0$ and $x = 8$ (also $x = -8$). Rotating the two-dimensional figure gives a hollowed out figure. The outer radius is $r_{out} = 3 - \frac{1}{4}x$ and the inner radius is $r_{in} = 3 - \sqrt[3]{x}$. Then the volume of a slice is

$$V_s = \pi r_{out}^2 \Delta x - \pi r_{in}^2 \Delta x.$$

Therefore, the total volume is approximately

$$T \approx \sum \pi (r_{out}^2 - r_{in}^2)\Delta x.$$

Taking the limit as $\Delta x \to 0$ gives approximately 15.5.

Remark 2.6. We now shift gears and talk about another application of the definite integral: arc length.
Suppose we are given a curve $y = f(x)$ from $x = a$ to $x = b$. We can approximate the length of the curve on a small interval of length Δx by the pythagorean theorem:

$$\begin{aligned} \ell^2 &= (\Delta x)^2 + (\Delta y)^2 \\ &\approx (\Delta x)^2 + (f'(x)\Delta x)^2 \\ &= (1 + f'(x)^2)(\Delta x)^2. \end{aligned}$$

Thus, $\ell \approx \sqrt{(1 + (f'(x))^2)}\Delta x$. Therefore, we have

$$\text{Arc length} \approx \sum \sqrt{(1 + (f'(x))^2)}\Delta x.$$

Letting $\Delta x \to 0$, we get the formula,

$$\text{Arc length} = \int_a^b \sqrt{(1 + (f'(x))^2)}\, dx.$$

3 Polar coordinates

4 Density

5 Physics

6 Applications to economics

Remark 6.1. Exam next Wednesday, Nov 20. Bring blue book and sheet of notes. Covers chapters 7 and 8. Review on Monday.

Definition 6.2. The **future value**, B, of a payment, P, is the amount to which P would have grown if deposited in an interest bearing bank account.

Definition 6.3. The **present value**, P, of a future payment, B, is the amount which would have to be deposited in a bank account today to produce exactly B in the account at the relevant time in the future.

Remark 6.4. Let r be an interest rate, compounded annually, over a period of t years. A deposit of P dollars grows to a future balance of B dollars, where,

$$B = P(1+r)^t \text{ and } P = \frac{B}{(1+r)^t}.$$

If interest is compounded continuously, then

$$B = Pe^{rt} \text{ and } P = Be^{-rt}.$$

Example 6.5. Suppose you are to be paid either \$100 each year for the next 10 years or \$800 right now. Assume a 5% interest rate, compounded continuously, and ignoring taxes. Which option is best?

The present value of \$800 lump sum payment is \$800. The future value is

$$800e^{.05(9)} = 1254.65.$$

The present value of the first annual payment is \$100. Since the second payment is made 1 year from now, it's present value is

$$100e^{-.05(1)} \approx 95.12.$$

Doing this for each we find

$$PV = \sum_{i=0}^{9} 100e^{-.05i} \approx 806.78.$$

On the other hand, the future value of the first payment is

$$100e^{.05(9)} \approx 156.83.$$

The total future value is then

$$FV = \sum_{i=0}^{9} 100e^{.05i} \approx 1265.28.$$

Remark 6.6. Hopefully the above comment is familiar. Let's think about a related problem. Consider money deposited into account (or payments made) continuously, rather than discretely. We want to determine the future and present value of this *income stream*.

Remark 6.7. Suppose the income stream pays in money at a rate of $P(t)$ dollars per year. We are interested in the present value of the income stream from now until M years in the future. We divide the interval $0 \leq t \leq M$ into subintervals of length Δt. Between t and $t + \Delta t$, we assume the rate to be constant, thus, on one subinterval,

$$\text{Amount deposited} \approx (Rate \times Time) \approx P(t)\Delta t \text{ dollars.}$$

Thus, the present value of money deposited in the interval t to $t + \Delta t$ is approximately $P(t)\Delta t e^{-rt}$. Summing over all such subintervals gives

$$\text{Total present value} \approx \sum P(t)e^{-rt}\Delta t \text{ dollars.}$$

Taking the limit as $\Delta t \to 0$ gives

$$\text{Present value} = \int_0^M P(t)e^{-rt}\, dt \text{ dollars.}$$

Similarly, we can compute

$$\text{Future value} = \int_0^M P(t)e^{r(M-t)}\, dt \text{ dollars.}$$

Example 6.8. Find the present and future values of an income stream of \$ 3000 per year over a 15-year period, assuming a 6% annual interest rate compounded continuously.

$$\begin{aligned} PV &= \int_0^{15} 3000e^{-.06t}\, dt = 3000\left[-\frac{1}{.06}e^{-.06t}\right]_0^{15} \\ &= -50000(e^{-.9} - 1) \approx \$29671.52. \\ FV &= \int_0^{15} 3000e^{.06(15-t)}\, dt = 3000\left[-\frac{1}{.06}e^{.06(15-t)}\right]_0^{15} \approx \$72980.16. \end{aligned}$$