Using Dynamic Geometry Software to Explore Topics in Mathematics History

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Complete handout has been posted at:
I. Hippocrates' Quadrature of the Lune (ca. 440 B.C.)

Heath's translation: “After proving this, he proceeded to show in what way it was possible to square a lune the outer circumference of which is that of a semicircle. This he affected by circumscribing a semicircle about an isosceles right-angled triangle and a segment of a circle similar to those cut off by the sides. Then, since the segment about the base is equal to the sum of those about the sides, it follows that, when the part of the triangle above the segment about the base is added to both alike, the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared.”

From an article written by J.J. O'Connor and E.F. Robertson located at: http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Hippocrates.html

II. Quadratrix of Hippias (ca. 430 B.C.)

ABCD is a square and BED is part of a circle, centre A radius AB. As the radius AB rotates about A to move to the position AD then the line BC moves at the same rate parallel to itself to end at AD. Then the locus of the point of intersection F of the rotating radius AB and the moving line BC is the quadratrix (from Book IV of Pappus' Synagoge circa 340).

III. The Pythagorean Theorem (ca. 300 B.C.)

Proposition I-47. In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle (from Book I of Euclid’s Elements circa 300 B.C.).

IV. Archimedes' Process for Finding Circular Area (ca. 225 B.C.)

Archimedes used the method of exhaustion by inscribing and circumscribing 2n-gons to calculate the area of a circle. His proof was a double reductio ad absurdum or a double proof by contradiction.

V. The Study of the Conics Sections (ca. 210 B.C., restudied in 1579)

We give credit to Apollonius for naming the conic sections (parabola, hyperbola, and ellipse) around 220 B.C. Only a few short years later in 212 B.C., Archimedes studied the properties of the conic sections. Much later, conic sections were defined as loci in the plane. Guidobaldo del Monte, for example, in 1579 defined the ellipse as the locus of points the sum of whose distances from the foci is a constant.

VI. Witch of Agnesi (ca. 1665)

The name of the function, "Witch of Agnesi", is a complete misnomer. Many people who have heard of the function are not aware that Agnesi was a woman (Maria Gaetana Agnesi, 1718-1799) and ironically, was not the discoverer of the function. The French mathematician Pierre de Fermat is said to have written about the function in 1665, almost a hundred years before Ms. Agnesi ever did.
Hippocrates' Quadrature of the Lune

In his attempt to square a circle or to construct a square with the same area as a circle, Hippocrates (ca. 470 B.C. – 410 B.C.) calculated the area of a lune, which is a crescent-shaped figure (see Figure 1). He demonstrated how to calculate three types of lunes: lunes with an outer circumference greater than, less than, and equal to a semicircle. His process for finding the area of the later type of lune is discussed in this activity.

- Have the students read and interpret Hippocrates’ method for finding the area of a lune.

Below is Heath’s translation of Hippocrates’ method for finding the area of a lune:

“[Hippocrates] proceeded to show in what way it was possible to square a lune the outer circumference of which is that of a semicircle. This he affected by circumscribing a semicircle about an isosceles right-angled triangle and a segment of a circle similar to those cut off by the sides. Then, since the segment about the base is equal to the sum of those about the sides, it follows that, when the part of the triangle above the segment about the base is added to both alike, the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared” (O’Connor & Robertson, 1999).

- Since Hippocrates initially circumscribed “a semicircle about an isosceles right-angled triangle” (O’Connor & Robertson, 1999), ask the students how to construct an isosceles right-angled triangle using The Geometer’s Sketchpad. You may want to first ask the students how to construct a right triangle using properties of a circle (i.e., all angles inscribed in a semi-circle are right angles). Follow-up by asking how to constrain this right triangle to be an isosceles right-angled triangle.

To construct an isosceles right-angled triangle, construct a diameter of circle H. Label this diameter \( \overline{AC} \). Construct a perpendicular line to the diameter \( \overline{AC} \) and center point H. Construct the intersections of this newly created line with circle H. Labels these points B and D. Construct segment \( \overline{BD} \). Hide the newly created line. Construct segments \( \overline{AB} \) and \( \overline{BC} \).

- Verify that the triangle is an isosceles triangle by measuring \( \angle ABC \), \( \overline{AB} \), and \( \overline{BC} \), and by dragging each vertex of \( \triangle ABC \).

- Knowing that Hippocrates formed the lune by the intersection of two circles where the second circle has “a segment of a circle similar to those cut off by the sides” of \( \triangle ABC \), construct a circle with these properties (i.e., construct a circle with a radius equal to the measure of the legs of \( \triangle ABC \) and that intersects points A and C).

- Construct radii \( \overline{DC} \) and \( \overline{DA} \). Analyze the properties of triangles \( \triangle ABH \) and \( \triangle ACD \). What can we conclude about these triangles? (They are similar triangles since they are both isosceles-right triangles.)
• Let \( m(BH) \) and \( m(AH) \) equal 1 unit. Determine the ratio: \( \frac{m(AC)^2}{m(AB)^2} \).

• What can we conclude about the area magnification ratio of circle H and circle D? (Hint: The diameter of circle D is \( 2(m(DC)) = 2(\sqrt{2}) \). Thus, \( \frac{2(m(DC)^2}{m(AC)^2} = \frac{(2\sqrt{2})^2}{2^2} = \frac{2}{1} \).

• Since parts of similar circles are in the same ratio as the magnification ratio, what can we conclude about the arc segment AEC and arc segment AFB? With this in mind, generalize the relationship between arc segment AEC and arc segments AFB and BGC. (Hint: \( m(AB) = m(BC) \))

• Again, since parts of similar circles are in the same ratio as the magnification ratio, what can we conclude about the area of semicircle CAI and semicircle ABC? Using this information, what can we conclude about half the area of semicircle CAI?

• After analyzing the relationship of the quarter circle CAI and semicircle ABC, calculate the area of the lune ABC.

• Discuss how this process relates to Hippocrates’ method for calculating the area of the lune.

Resources:


Quadratrix of Hippias

From ancient Greek times, mathematicians have considered three famous geometrical construction problems. These problems are: (1) the duplication of the cube – construct the edge of a cube whose volume is twice that of a given cube; (2) angle trisection – construct an angle that equals one-third that of a given angle; and (3) the squaring of a circle – construct a square whose area equals the area of a given circle. Given the only tools at the time a compass and an unmarked ruler, these problems challenged mathematicians for centuries.

History

Hippias of Elis (born around 460 B.C.) was a statesman and philosopher who traveled from place to place being paid for his lectures on poetry, grammar, history, politics, archaeology, mathematics and astronomy. Plato described him as a vain man being both arrogant and boastful, having a wide but superficial knowledge (Burton, 2003). Hippias’ contribution to mathematics was small, but significant. In his attempt to trisect an angle, he created a new transcendental curve which unfortunately could not be constructed with only a compass and unmarked ruler; but the curve can be used to divide an angle into not only three, but any number of congruent angles.

Construction

Hippias’ quadratrix was described as follows: “ABCD is a square and BED is part of a circle, centre A radius AB. As the radius AB rotates about A to move to the position AD then the line BC moves at the same rate parallel to itself to end at AD. Then the locus of the point of intersection F of the rotating radius AB and the moving line BC is the quadratrix” (from Book IV of Pappus’ Synagoge circa 340).

To construct the quadratrix using Geometer’s Sketchpad, you will first need to construct a square in your sketch window. Select any point on CD and call it C’. Construct a parallel line to BC, through the point C’. Name the point of intersection of AB and the parallel line point B’. Shorten the parallel line to CB. Construct the arc BED formed by a circle with center A and radius AB. Construct any radius of the arc BED, namely AE. Find the intersection of BEC’ and AE, and name the point of intersection F. The quadratrix is formed by point F as point C’ moves along CD, and point E moves along arc BED. Construct an animation button such that point C’ takes exactly the same amount of time to traverse CD, as point E takes to traverse arc BED. Under the Display menu, select point F to be traced. You will also want to construct a movement button to reset the sketch by moving point E to point B and point C’ to point C. Reset the sketch using your movement button, and activate the animation button. The locus of point F is called the “Quadratrix of Hippias”.

How to use the quadratrix for trisecting an angle

Once the quadratrix has been formed, move point E anywhere along arc BED. Using Hippias’ method, we will find an angle one-third the measure of <DAE. Move point C’ such that point F is the intersection of the quadratrix and AE. Construct a segment perpendicular line to AD through the point F. Shorten the perpendicular line to FH. The trisection of an angle problem has now been reduced to trisecting the line segment FH. Formally construct a point (K) along FH such that HK is one-third the length of FH. Construct a parallel line to AD through the point K. Identify the intersection of the parallel line with the quadratrix, and name this point J. Construct AJ. The measure of <DAJ is one-third the measure of <DAE.

Resource:
The Pythagorean Theorem

Activity Description  Activity Guide  Resources

Students create both a visual and formal proof of the Pythagorean theorem, as well as view four additional geometric demonstrations of the theorem. These demonstrations are based on proofs by Bhaskara, Leonardo da Vinci, and Euclid. Students will value this activity more if they have already had some experience with the Pythagorean Theorem.

**Mathematics:** The students will construct a square and conjecture the following theorem:

- *The sum of the areas of the squares on the legs of a right triangle is equal to the area of the square on the hypotenuse.*

The extension activity includes investigating the converse of the Pythagorean theorem, and generalizing the theorem to shapes other than squares.

**Mathematical Thinking:** Students will be asked to *conjecture, prove* and *generalize* the aforementioned theorem.

**Technology:** The students will use several Sketchpad features such as: *Custom Tools, Text Tool, Area, Polygon Interior,* and *Sample Sketches.*

The following sketch illustrates a visual demonstration of the Pythagorean Theorem.
The Pythagorean Theorem

Part 1:

- State and illustrate the Pythagorean theorem. Discuss and interpret the meaning of the algebraic equation, \( a^2 + b^2 = c^2 \). What do \( a, b \) and \( c \) represent? What do \( a^2, b^2 \) and \( c^2 \) represent?

- Create a custom tool for the construction of a right triangle if you have not already done so. Then use your custom tool to construct a right triangle in your sketch window. (Instructor Note: A Sketchpad file illustrating the construction of a right triangle has been saved as righttri.gsp.)

- Does the Pythagorean theorem hold true for your triangle? It will probably be easier for you to relabel your triangle’s sides and vertices to follow the usual conventions (i.e., vertices A, B, C where the sides opposite these vertices are \( a, b, c \); and \( c \) is the hypotenuse). Using the Sketchpad measurement tools and calculator, numerically verify the Pythagorean theorem.

  To relabel an already labeled figure:

  - Select the Text Tool \( \text{A} \) from the toolbox.
  - Position the hand pointer on any label.
  - Double click and type the new label in the Label dialog box.

- Drag a vertex (or side) to further verify the theorem.

- Measure the area of your right triangle.

  To construct the polygon interior of a closed figure:

  - Select the vertices of the closed figure, in order.
  - Choose from the Construct menu, the Polygon Interior command.
  - Click anywhere inside the sketch window to de-select the polygon interior.

  To measure the area of a polygon:

  - Select the polygon interior.
  - From the Measure menu, select the Area command.
the lengths of sides and area of the triangle. Use these measurements to verify the formula for the area of a right triangle.

Part 2:

- Discuss how can you think about the Pythagorean theorem in terms of areas of squares.

- To construct squares on each side of your right triangle, it would be helpful to have a custom tool for the construction of a square. In a New Sketch window, create a custom tool for the construction of a square. **Hint:** Combine your techniques from constructing an isosceles triangle and a right triangle. (Instructor Note: A Sketchpad file illustrating the construction of a square has been saved as square.gsp.)

- Manipulate your square. How does each of the vertices move when you drag them around your sketch window? Which vertices are the most restricted and which are the least restricted? As you manipulate your figure, does it always stay a square? How does your construction guarantee that your square will remain a square?

- Use your custom tool to create several squares in your sketch window. Pay close attention to how the custom tool creates a square.

- Now we are ready to construct squares on the sides of a right triangle. In a New Sketch window and using your custom tool, construct a right triangle. Further construct squares (that do not overlap your right triangle) on each side of your right triangle. If a square overlaps the triangle, don’t worry. You have selected the two vertices in the wrong order. Under the Edit menu, select **Undo Square.** Try using your custom tool again, keeping in mind that the order of your vertices is important.

- Measure the area of each of the three squares. What do you notice about the areas? Manipulate your triangle. What conclusions can you draw from your observations? (Instructor Note: A Sketchpad file illustrating the Pythagorean Theorem has been saved as pythag.gsp.)
To construct the polygon interior of a quadrilateral:

- Select the vertices of the quadrilateral, in order.
- Choose from the Construct menu, the Quadrilateral Interior command.
- Click anywhere inside the sketch window to de-select the quadrilateral interior.

To measure the area of a polygon:

- Select the polygon interior.
- From the Measure menu, select the Area command.

- Formally state and prove the Pythagorean theorem.

Part 3:

- Under the File menu choose Open... In the new window, look in the C:. Double-click on Program Files » Sketchpad » Samples » Sketches » Geometry » Pythagoras.gsp.

- Click on the Behold Pythagoras! button in your sketch window. The twelfth century Hindu scholar, Bhaskara demonstrated the Pythagorean theorem with a similar figure to the one found in your sketch window. The only text accompanying the figure was the word “Behold!” . Drag point D and observe what happens to the right triangles in the square on the left. Do the interior figures change shape? How is the hypotenuse of one of the triangles related to the side of the original square? Can you describe the area of this square in terms of the hypotenuse? Verify algebraically that the area of the five figures sum to the area of the larger square. Hint: To start, write an expression for (1) the area of the whole square in terms of c; (2) the areas of the four right triangles in terms of a and b; and (3) the area of the interior square. You should be able to write an equation involving a, b, and c.

- Click on the Contents button and then the Leonardo da Pythagoras button in your sketch window. Leonardo da Vinci (1452-1519) was a great Italian painter, engineer, and inventor during the Renaissance. He was also an amateur mathematician credited with the following proof of the Pythagorean theorem. Before you click on any of the buttons, describe what is displayed in the sketch window. What does each of the regions of the figure represent? Double click on the “reflect top” button. What happened? Did the area of the regions change? Double click on the “turn Δ’s” button. Again, describe what happened. Further explore this sketch and interpret in your own words Leonardo’s proof of the Pythagorean theorem.

- Click on the Contents button and then the Puzzled Pythagoras button in your sketch window. Follow the directions within the sketch window. Can you explain how this demonstrates that the area of the square on the hypotenuse is equal to the sum of the areas on the squares on the other two legs of the triangle? This dissection demonstration convincingly illustrates the truth of the Pythagorean theorem, but does not provide a formal proof of the theorem. Using the “Puzzled Pythagoras” sketch and an argument of congruent triangles, formally prove the Pythagorean theorem.

- Click on the Contents button and then the Shear Pythagoras! button in your sketch window. Shearing is a transformation which translates every point in a figure in a direction parallel to a given line by a distance proportional to a point’s distance from the line. In your sketch window, drag point P to shear
the parallelogram back and forth. Note that the parallelogram’s area doesn’t change as you change the figure’s shape. Finish the shear by dragging P to lie on the red line. Explain why shearing does not affect the area of the parallelogram. Shear the other parallelogram on the other leg of the triangle. Drag until point Q is on the line. Measure the area of each of the three squares, and then follow the directions given. Comment on your observations. Prove that the two shaded figures are congruent.

Extensions:

- State the converse of the Pythagorean theorem. How could you investigate the converse of the Pythagorean theorem? One way is by constructing squares on the sides of an arbitrary triangle. Measure the areas of the three squares and calculate the sum of two of them. Then drag a vertex until the sum equals the area of the third square. What kind of triangle do you have? Try to formally prove the converse of the Pythagorean theorem.

- Construct a right triangle and place a non-overlapping equilateral triangle on each of its sides. Conjecture a relationship between the areas of the equilateral triangles. Measure the areas to assess your conjecture. Formally prove your conjecture. Try placing other figures (e.g., pentagons, hexagons, semicircles, etc.) on the sides of a right triangle. Conjecture a relationship between the areas outside of the right triangle. Then measure the areas to assess your conjecture.

Resources:

- This website contains several dynamic proofs of the Pythagorean Theorem. http://www.ies.co.jp/math/java/geo/pythagoras.html

- One of Dudeny’s proofs of the Pythagorean Theorem is located on this website. http://www.geocities.com/CapeCanaveral/Launchpad/3740/dudeny.html

- Stephanie J. Morris a Mathematics Education graduate student at the University of Georgia has presented many visual and algebraic proofs of the Pythagorean Theorem on this website. She claims that "Students who are taught the Pythagorean Theorem using [multiple] methods will see the connections, and thus, benefit greatly". http://jwilson.coe.uga.edu/EMT669/Student.Folders/Morris.Stephanie/EMT.669/Essay.1/Pythagorean.html

Related Articles:

Archimedes' Process for Finding Circular Area

In his treatise *The Measurement of the Circle*, Archimedes’ approximated the area of a circle by proving that the measure for a circumference of a circle is between the measure for the perimeters of inscribed and circumscribed regular 2n-gons. Archimedes began with the following:

Proposition: The area of any circle is equal to a right-angled triangle with the base equal to the radius of the circle and the other leg equal to the circumference of the circle (see Figure 1).

![Figure 1. Circle O and ΔOPQ with base = r and the other leg = circumference of circle O.](image)

Letting the area of circle $O = A$ and the area of $ΔOPQ = T$, Archimedes' proved that $A = T$ by a double *reductio ad absurdum* proof. Thus, he considered two cases:

**Case 1:** $A > T$
**Case 2:** $A < T$

**Investigating Case 1: $A > T$:**

- Using The Geometer’s Sketchpad, construct a circle and inscribe a square. Calculate the area of the square and compare this area to the area of the circle.

- Inscribe an octagon and calculate the area. Compare the area of the octagon to the area of the circle and square.

- Continue this process until the difference of the area of the circle and the area of the inscribed regular 2n-gon is less than $A - T$. That is: $A - $ Area of inscribed 2n-gon < $A - T$

- It is important to understand how Archimedes calculated the area of a polygon before interpreting the next step in Archimedes’ proof. Given an octagon, Archimedes calculated the area of a triangle with base $b$ and height $h$. Knowing there are eight congruent triangles in an octagon, he calculated the area of the octagon as: $\frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh + \frac{1}{2}bh = \frac{1}{2}h(b + b + b + b + b + b + b + b) = \frac{1}{2}hQ$.

- Using the inequality $A - $ Area of inscribed 2n-gon < $A - T$, add the quantity “Area of inscribed 2n-gon + $T - A$” to both sides. Thus, $T < $ Area of inscribed 2n-gon. However, Area of inscribed 2n-gon = $\frac{1}{2}hQ < \frac{1}{2}(rC) = T$. This is obviously a contradiction.
Investigating Case 2: $A < T$:

- Circumscribe a square about circle $O$. Calculate the area of the square and compare this area to the area of the circle.

- Circumscribe an octagon and calculate the area. Compare the area of the octagon to the area of the circle and square.

- Continue this process until the difference of the area of the circumscribed regular $2n$-gon and the area of the circle is less than $T - A$. That is: Area of circumscribed $2n$-gon - $A < T - A$. Add $A$ to both sides of the inequality. Thus, Area of circumscribed $2n$-gon < $T$. However, as discussed above, Area of circumscribed $2n$-gon = $\frac{1}{2} hQ > \frac{1}{2} (rC) T$. This is obviously a contradiction. Thus, $A = T$, which is described above in Proposition 1.

Resource:

Exploring Geometric Constructions of Parabolas

Activity Description

This activity is an introduction to geometric constructions of parabolas. We will investigate their properties and characteristics. This activity has been adapted from the following article: Olmstead, E.A. (1998). Exploring the locus definition of the conic sections. *Mathematics Teacher, 91*(5), 428-434.

**Mathematics:** The students will develop geometric constructions of parabolas as loci of lines, and as loci of points. They will use the distance formula to algebraically derive the standard form of a quadratic function. Students will conjecture and prove the following theorem:

- The distance from the parabola to its focus and the distance from the parabola to its directrix are equal.

In this activity, students will use problem-solving skills to simulate the geometric construction of a parabola using *The Geometer’s Sketchpad*. Students will also be asked to conjecture and prove the aforementioned theorem.

**Technology:** This activity uses several Sketchpad commands such as: Line, Segment, Midpoint, Perpendicular Line, and Locus.

The following sketch illustrates a geometric construction of a parabola using *The Geometer’s Sketchpad*. 
Exploring Geometric Constructions of Parabolas

Activity Description

Activity Guide

In 350 B.C. Menaechmus, a pupil of Plato and Eudoxus, studied the curves formed when a plane intersects the surface of a cone. In how many different ways can you slice a cone to result in a uniquely shaped cross-section? Compare your results with the diagram below.

We give credit to Apollonius for naming the conic sections (parabola, hyperbola, and ellipse) around 220 B.C. Only a few short years later in 212 B.C., Archimedes studied the properties of the conic sections. In this activity, we are going to explore the characteristics of one specific conic section, the parabola.

We will investigate the locus of points in the plane that are equidistant from a point and a line by using a paper-folding technique and by using The Geometer's Sketchpad. Below you can find the definitions for three geometric terms that we'll be using frequently in this activity.

**directrix** - a fixed line that is the same distance as the focus from each point on the parabola.

**focus** - a fixed point that is the same distance as the directrix from each point on the parabola.

**locus** - the set of all points that satisfy a given condition or a set of given conditions.
Part 1: Parabola - paper folding activity

Each student will need a 6-inch square piece of wax paper (or patty paper) and a straight edge.

- With a straight edge, construct a line on your piece of wax paper. This line will serve as the directrix.
- Construct a point anywhere on the paper, except on the line, and label it F. This point will serve as a focus. Construct another point anywhere on the line, and label it G.
- Using your straightedge, construct the line segment FG. Fold your paper so that the two points are concurrent, and deliberately crease it so that you can easily see the fold mark when the paper has been flattened out. Conjecture any relationships you might see between the line segment FG and the folded line.
- Fold your wax paper so that point F falls anywhere along the directrix. Again, deliberately crease your wax paper so that you can easily see the fold. Continue this process approximately ten more times so that each time point F falls on a different location of the directrix.

We are interested in the pattern of the creases that are formed when point F is folded along the directrix. Flatten out your paper and look for any geometric patterns. Describe the boundary of the shape of the area containing the focus that is bounded by all of the fold lines.

- Make a conjecture about the relationship of the distance from the focus to the boundary, and the distance from the boundary to the directrix. How could you test your conjecture?

Part 2: Parabola as loci of lines

To see the pattern described in Part 1 distinctly, we would have to fold the paper dozens of times. Instead, we are going to simulate the activity using The Geometer's Sketchpad.

- With your neighbor, plan a geometric construction we could use to simulate the folding process as described in Part 1. Write down the key steps of the construction and share your ideas with the class. Carry out your construction using The Geometer's Sketchpad.

There are several ways to simulate the activity in Part 1, using different features of The Geometer's Sketchpad. Specific steps have been recorded below for three different methods. (Instructor Note: A Sketchpad file illustrating the simulation using the animate feature has been saved as parabola-animate.gsp.)

**All three constructions start in a similar manner.**

- Construct a line (d) that will serve as the directrix.
- Construct two points, one as the focus (F) and one on the directrix (G).
- Construct a segment from the focus to the point on the directrix (segment FG).
- Construct the perpendicular bisector segment FG.
Simulation using the *Trace* feature:

- Select the perpendicular bisector, and choose the **Trace Perpendicular Line** command under the **Display** menu.
- Drag the point G along the directrix.
- To erase the traces, choose the **Erase Traces** command under the **Display** menu.

Simulation using the *Animate* feature:

- Select the perpendicular bisector, and choose the **Trace Perpendicular Line** command under the **Display** menu.
- Select point G, and choose **Animate Point** from the **Display** menu.
- To cease animation, you may click on the solid square button ( ■ ) on the **Motion Controller** window, or choose **Stop Animation** under the **Display** menu.

Simulation using the *Locus* feature:

- Select point G (the driver point) and the perpendicular bisector. Select **Locus** from the **Construct** menu.

**Note:** There is an advantage to the construction using the **Locus** feature over the others. Once you have constructed the sketch, you can easily manipulate the position of the focus and directrix and investigate the connection between them. (Instructor Note: A *Sketchpad* file illustrating the simulation using the locus of lines has been saved as *parabola-locus.gsp*.)

- What is the general shape formed by the loci of the perpendicular bisector? Drag the focus point to manipulate the loci. What do you observe about the shape formed?
- Conjecture what happens when the focus is below the directrix. Test your conjecture.
• What is the relationship between the location of the focus and directrix and the general shape formed by the loci of lines?

• What is the relationship of each of the perpendicular bisectors to the parabola?

Part 3: Parabola as a loci of points

Why does the construction tracing perpendicular bisectors of segment from the focus to the directrix produce a parabola? To be able to answer this question we will need to slightly modify our construction. In Part 2, we constructed a parabola by loci of lines. Since each of the lines were tangent to the parabola, we could not locate any specific points on the parabola. We would like to be able to construct only those points on the parabola.

• Re-simulate the activity in Part 2 to construct a parabola as a locus of points. (Instructor Note: A Sketchpad file illustrating the simulation using the locus of points has been saved as parabola-locuspts.gsp.)

To construct a parabola as a locus of points:

• Construct a dashed line through point G that is perpendicular to the directrix.
• Construct the point of intersection of the dashed line and the perpendicular bisector of segment FG. Call this point of intersection P.
• Construct the locus of point P as G moves along the directrix.

• What is the general shape formed by the loci of point P? Drag the focus point to manipulate the loci. What do you observe about the shape formed?

• Confirm that point P satisfies the definition of a parabola, that is, measure the distance from P to G and from P to F. Formally prove that these distances are equal.

• Algebraically, derive the standard form of a quadratic function. (Hints: Use the labeled sketch below to equate the distances from P to G and from P to F. Let the distance \( FV = p \).)
Algebraic derivation of the standard form of a quadratic function:

\[
\sqrt{(x-z)^2 + ((k - p) - y)^2} = \sqrt{(x-h)^2 + (y-(k+p))^2}
\]

\[
(x-x)^2 + (k-p-y)^2 = (x-h)^2 + (y-k-p)^2
\]

\[
k^2 - kp - ky + k^2 + p^2 + py + py + y^2 = (x-h)^2 + y^2 - ky - ky + k^2 + kp - py + kp - py + p^2
\]

\[
k^2 + p^2 + y^2 - 2kp - 2ky + 2py = (x-h)^2 + k^2 + p^2 + y^2 + 2kp - 2ky - 2kp + 2py = (x-h)^2 + 2kp - 2py
\]

\[
4py = (x-h)^2 + 4kp
\]

\[
y = 1/4p(x-h)^2 + k
\]

- Manipulate the locations of the directrix and the focus to classify the movement of the graph.

- How do the constant \(h\), \(k\), and \(p\) affect the graph of the parabola? You may want to use a more appropriate technology to investigate. (Try the following interactive website: [http://www.exploremath.com/activities/Activity_page.cfm?ActivityID=14](http://www.exploremath.com/activities/Activity_page.cfm?ActivityID=14))

- Another common form of a quadratic function is \(y = ax^2 + bx + c\). Further explore how the constant \(a\), \(b\), and \(c\) affect the graph of the parabola. Compare and contrast these two algebraic forms of quadratic functions.

Extensions:

- In Part 2 of this activity, instead of using the perpendicular bisector of \(FG\), conjecture what would happen when an arbitrary perpendicular line to \(FG\) is used.

- Construct a line perpendicular to \(FG\) through any point other than the midpoint. Trace this perpendicular line. Is the shape formed a parabola? Does it meet the definition of a parabola? If so, where are its focus and directrix? If not, how would you classify its shape?

- Further manipulate the sketch to find when point \(F\) is the focus? When is line \(d\) the directrix? What is special about the relationship of the perpendicular bisector of segment \(FG\) to the parabola?

- Refer back to Part 3 of this activity. Since segment \(GP = segment FP\), a circle centered at \(P\) will pass through both \(F\) and \(G\) and be tangent to the directrix at \(G\). On the basis of these relationships, make up an alternative definition of a parabola.
Exploring the Witch of Agnesi

This activity has students construct the graph of the Witch of Agnesi, and investigate both its asymptotes and inflection points. Fermat studied this function in the seventeenth century.

Mathematics: Students will construct the graph of the Witch of Agnesi and conjecture the asymptotes and inflection points of the function. Furthermore, students will algebraically derive the equation of the function using similar triangles and then formally compute the asymptotes and inflection points of the function.

Mathematical Thinking: Students will be asked to *conjecture* the basic shape of the function, and *derive* the graph’s equation.

Technology: This activity uses several Sketchpad commands such as: Define Coordinate System, Circle by Center+Point, Plot Points, Ray, Color, Trace Point, and Action Button -> Animation.

The following sketch illustrates the Witch of Agnesi function.

![Sketch of the Witch of Agnesi](image)

The equation of "The Witch" is \( y = \frac{64}{x^2 + 16} \).
Exploring the Witch of Agnesi

The name of the function, "Witch of Agnesi", is a complete misnomer. Many people who have heard of the function are not aware that Agnesi was a woman (Maria Gaetana Agnesi, 1718-1799) and ironically, was not the discoverer of the function. The French mathematician Pierre de Fermat is said to have written about the function in 1665, almost a hundred years before Ms. Agnesi ever did. Furthermore, the word witch in its title is the result of an inaccurate English translation. It has been recorded in 1703 that another mathematician, Guido Grandi, named the function Versaria, meaning "turning in every direction". In the course of time the word versaria took on another meaning. The Latin words adversaria, by aphaeresis, and versaria, signify a female that is contrary to God. Gradually the function versaria came to be known in English as "the witch".

Part 1:

"The Witch" is a function, so we would like to graph it on a coordinate system. Define a Coordinate System in your sketch window.

To define a coordinate system:

- Select Define Coordinate System, under the Graph menu.
- You may drag the point (1,0) along the x-axis to define the size of the scaling unit.

• Construct a circle centered at the point (0,2), with a radius of 2 units. Remember, to construct a circle you can use the Circle by Center+Point command under the Construct menu.

To plot a point on the coordinate system:

- Select Plot Points, under the Graph menu.
- Enter the x and y coordinates of a point you would like to be plotted, and click the Plot button.
- Click the Done button.

• Construct a line tangent to the circle, above and parallel to the x-axis.

• Construct an arbitrary point on the circle, and drag this point around the circle until it lies in the first quadrant. Change its color to green.
To change the color of an object in your sketch window:

- Select the object.
- Select **Color** under the **Display** menu.
- A palette of colors will appear. Click the mouse on a color of your liking.

- Construct a Ray from the origin through the green point.
- Find the point of intersection of the tangent line and the ray. Change its color to blue.
- Using the Line Style command under the Display menu, construct a dashed perpendicular line through the blue point to the x-axis; and a dashed perpendicular line through the green point to the y-axis.
- Construct the point of intersection between the two dashed lines, and color it red.

The function formed by tracing the red point as the green point travels around the circle is called the Witch of Agnesi.

- Drag the green point around the circle, paying particular attention to the movement of the red point. Make a conjecture about the general shape of the function. On paper, draw a quick sketch illustrating your ideas.

**Part 2:**

- We would like to trace the red point as the green point travels around the circle. Create a button to animate the green point traveling one way around the circle slowly. (Instructor Note: A Sketchpad file illustrating the construction of the Witch of Agnesi has been saved as *witch.gsp*.)

**To identify which point to trace:**

- Select the point.
- Choose **Trace Point** under the **Display** menu.

**To create an animation button:**

- Select the object you wish to animate.
- Choose **Action Button** under the **Edit** menu, and then **Animation**.
- In the **Properties of Action Button** window, make travel selections for your object (i.e., its direction and speed).
- Click on the **Label** tab to rename your action button, and click **OK**.
- An **Animate** button will appear in your sketch window.

- Your action button toggles on and off by clicking on it. Activate your animation button, and observe the traced intersection point.
- How does the resulting function compare with your previous conjecture? Reconcile any differences.
• While the red point is being traced, there is a difference in the density of the points being plotted. Explain why this occurs.

Part 3:

• Describe the characteristics of an asymptote. Does this function have any asymptotes? Play around with the scale of your graph. How does changing the graph’s scale influence your perception of the location of the function’s asymptotes?

• Describe the characteristics of an inflection point. Does the function have any inflection points? Explain and give the approximate coordinates of each.

• Derive the equation of the function such that the red point will be all (x,y) pairs which satisfy the equation. Hint: Look for a pair of similar triangles and equate ratios of their sides.

The derivation of the equation of The Witch of Agnesi:

Go back to the construction on the first page of this activity and equate the ratios of the sides of the similar triangles.

\[
\frac{EA}{IA} = \frac{BG}{IF}
\]

Since the radius of the circle is 2, \( F = (x_c, y) \) and \( H = (x, y) \), then

\( EA = 4; \quad IA = y \) (That is, the y-coordinate of both F and H);

\( EG = x \) (That is, the x-coordinate of H); and \( IF = x_c \) (That is, the x-coordinate of F)

Substituting these values in the above ratio equation, we have

\[
\frac{4}{y} = \frac{x}{x_c}
\]

From the equation of our circle \( x_c^2 + (y-2)^2 = 2^2 \) we know that \( x_c = \sqrt{2^2 - (y-2)^2} \). We use \( (y-2) \) instead of \( y \) here to account for the displacement of the circle in our figure, whose center is not at the origin.

Substituting this value of \( x_c \) in the previous equation we get

\[
\frac{4}{y} = \frac{x}{\sqrt{2^2 - (y-2)^2}}
\]

Squaring both sides of the preceding equation and doing some other simplification, we get

\[
\frac{16}{y^2} = \frac{x^2}{4 - (y-2)^2} = \frac{x}{4 - y^2 + 4y - 4} = \frac{x^4}{y(4 - y)}
\]

and finally \( x^2y = 16(4-y) \) or \( x^2 + 16 \).
Once the equation has been derived, mathematically verify the existence of both your conjectured asymptote(s) and inflection point(s).

Extensions:

- Derive the equation for the Witch of Agnesi using a polar coordinate system.
- Think of other investigations/activities that could be explored with this function. Write a summary of your ideas, and be prepared to present them to the class.
- Research the life of Maria Gaetana Agnesi (or another female mathematician) and prepare a presentation about her life and mathematics.
- Create another construction that uses the animation button in a useful manner. Write up a description about the construction and demonstrate your construction to the class.

Resources:

- There are multiple sites that provide a detailed biography of Maria Gaetana Agnesi. Explore some of the ones listed below:
  "Biographies of Women Mathematicians" web site from Agnes Scott College
  http://www.AgnesScott.edu/lriddle/women/agnesi.htm
  "4000 Years of Women in Science"
  http://www.astr.ua.edu/4000WS/AGNESI.html
  "The MacTutor History of Mathematics Archive"
  http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Agnesi.html
  "The Witch of Agnesi Java Sketchpad" web site allows students to manipulate a Geometer’s Sketchpad sketch on the web to produce the function. The site also has a brief history and description of the function.
  http://www.keypress.com/sketchpad/java_gsp/witch.html
  "4000 Years of Women in Science" web site has an animated GIF that illustrates the Witch of Agnesi function.
  http://www.astr.ua.edu/4000WS/witch-of-agnesi.html
  "Famous Curves Index" at St. Andrew’s Web site can be JAVA enabled to allow students to manipulate a sketch of the function on the web. The site also has a brief history and description of the function.
  http://www-groups.dcs.st-andrews.ac.uk/~history/Curves/Witch.html