Making Geometric Connections to Pre-Calculus and Calculus Topics Using The Geometer’s Sketchpad

Ohio Council of Teachers of Mathematics 53rd Annual Conference ~ Cleveland, Ohio October 2-4, 2003

Suzanne R. Harper  
Department of Mathematics & Statistics  
Miami University  
HarperSR@muohio.edu

Shannon S. Driskell  
Department of Mathematics  
University of Dayton  
Shannon.Driskell@notes.udayton.edu
Table of Contents

Exploring Geometric Constructions of Parabolas .....................................3

Exploring Infinite Series through Baravelle Spirals...............................9

Exploring Trigonometric Functions .......................................................15

Investigating the Concept of Derivative ..............................................23

Investigating the Area under a Curve ...............................................25

Additional Resources .......................................................................27
Exploring Geometric Constructions of Parabolas

Activity Description

This activity is an introduction to geometric constructions of parabolas. We will investigate their properties and characteristics. This activity has been adapted from the following article: Olmstead, E.A. (1998). Exploring the locus definition of the conic sections. *Mathematics Teacher, 91*(5), 428-434.

**Mathematics:** The students will develop geometric constructions of parabolas as loci of lines, and as loci of points. They will use the distance formula to algebraically derive the standard form of a quadratic function. Students will conjecture and prove the following theorem:

- The distance from the parabola to its focus and the distance from the parabola to its directrix are equal.

In this activity, students will use problem-solving skills to simulate the geometric construction of a parabola using *The Geometer’s Sketchpad*. Students will also be asked to conjecture and prove the aforementioned theorem.

**Technology:** This activity uses several Sketchpad commands such as: *Line, Segment, Midpoint, Perpendicular Line*, and *Locus*.

The following sketch illustrates a geometric construction of a parabola using *The Geometer’s Sketchpad*.

A complete copy of this activity can be found on the web: [http://www.teacherlink.org/content/math/activities/skpv4-parabola/home.html](http://www.teacherlink.org/content/math/activities/skpv4-parabola/home.html)
Exploring Geometric Constructions of Parabolas

Activity Description

In 350 B.C. Menaechmus, a pupil of Plato and Eudoxus, studied the curves formed when a plane intersects the surface of a cone. In how many different ways can you slice a cone to result in a uniquely shaped cross-section? Compare your results with the diagram below.

![Diagram of conic sections: parabola, ellipse, hyperbola, and circle.](http://mathworld.wolfram.com/ConicSection.html)

We give credit to Apollonius for naming the conic sections (parabola, hyperbola, and ellipse) around 220 B.C. Only a few short years later in 212 B.C., Archimedes studied the properties of the conic sections. In this activity, we are going to explore the characteristics of one specific conic section, the parabola.

We will investigate the locus of points in the plane that are equidistant from a point and a line by using a paper-folding technique and by using *The Geometer's Sketchpad*. Below you can find the definitions for three geometric terms that we'll be using frequently in this activity.

- **directrix** - a fixed line that is the same distance as the focus from each point on the parabola.
- **focus** - a fixed point that is the same distance as the directrix from each point on the parabola.
- **locus** - the set of all points that satisfy a given condition or a set of given conditions.

Activity Guide

Part 1: Parabola - paper folding activity

Each student will need a 6-inch square piece of wax paper (or patty paper) and a straight edge.
• With a straight edge, construct a line on your piece of wax paper. This line will serve as the directrix.

• Construct a point anywhere on the paper, except on the line, and label it F. This point will serve as a focus. Construct another point anywhere on the line, and label it G.

• Using your straightedge, construct the line segment FG. Fold your paper so that the two points are concurrent, and deliberately crease it so that you can easily see the fold mark when the paper has been flattened out. Conjecture any relationships you might see between the line segment FG and the folded line.

• Fold your wax paper so that point F falls anywhere along the directrix. Again, deliberately crease your wax paper so that you can easily see the fold. Continue this process approximately ten more times so that each time point F falls on a different location of the directrix.

We are interested in the pattern of the creases that are formed when point F is folded along the directrix. Flatten out your paper and look for any geometric patterns. Describe the boundary of the shape of the area containing the focus that is bounded by all of the fold lines.

• Make a conjecture about the relationship of the distance from the focus to the boundary, and the distance from the boundary to the directrix. How could you test your conjecture?

Part 2: Parabola as loci of lines

To see the pattern described in Part 1 distinctly, we would have to fold the paper dozens of times. Instead, we are going to simulate the activity using The Geometer's Sketchpad.

• With your neighbor, plan a geometric construction we could use to simulate the folding process as described in Part 1. Write down the key steps of the construction and share your ideas with the class. Carry out your construction using The Geometer's Sketchpad.

There are several ways to simulate the activity in Part 1, using different features of The Geometer's Sketchpad. Specific steps have been recorded below for three different methods. (Instructor Note: A Sketchpad file illustrating the simulation using the animate feature has been saved as parabola-animate.gsp.)

All three constructions start in a similar manner.

• Construct a line (d) that will serve as the directrix.
• Construct two points, one as the focus (F) and one on the directrix (G).
• Construct a segment from the focus to the point on the directrix (segment FG).
• Construct the perpendicular bisector segment FG.

Simulation using the Trace feature:

• Select the perpendicular bisector, and choose the Trace Perpendicular Line command under the Display menu.
• Drag the point G along the directrix.
• To erase the traces, choose the Erase Traces command under the Display menu.

Simulation using the Animate feature:
Simulation using the *Animate* feature:

- Select the perpendicular bisector, and choose the **Trace Perpendicular Line** command under the **Display** menu.
- Select point G, and choose **Animate Point** from the **Display** menu.
- To cease animation, you may click on the solid square button (■) on the **Motion Controller** window, or choose **Stop Animation** under the **Display** menu.

Simulation using the *Locus* feature:

- Select point G (the driver point) and the perpendicular bisector. Select **Locus** from the **Construct** menu.

Note: There is an advantage to the construction using the **Locus** feature over the others. Once you have constructed the sketch, you can easily manipulate the position of the focus and directrix and investigate the connection between them. (Instructor Note: A Sketchpad file illustrating the simulation using the locus of lines has been saved as parabola-locus.gsp.)

- What is the general shape formed by the loci of the perpendicular bisector? Drag the focus point to manipulate the loci. What do you observe about the shape formed?

- Conjecture what happens when the focus is below the directrix. Test your conjecture.

- What is the relationship between the location of the focus and directrix and the general shape formed by the loci of lines?

- What is the relationship of each of the perpendicular bisectors to the parabola?
Part 3: Parabola as a loci of points

Why does the construction tracing perpendicular bisectors of segment from the focus to the directrix produce a parabola? To be able to answer this question we will need to slightly modify our construction. In Part 2, we constructed a parabola by loci of lines. Since each of the lines were tangent to the parabola, we could not locate any specific points on the parabola. We would like to be able to construct only those points on the parabola.

- Re-simulate the activity in Part 2 to construct a parabola as a locus of points. (Instructor Note: A Sketchpad file illustrating the simulation using the locus of points has been saved as parabola-locuspts.gsp.)

<table>
<thead>
<tr>
<th>To construct a parabola as a locus of points:</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Construct a dashed line through point G that is perpendicular to the directrix.</td>
</tr>
<tr>
<td>- Construct the point of intersection of the dashed line and the perpendicular bisector of segment FG. Call this point of intersection P.</td>
</tr>
<tr>
<td>- Construct the locus of point P as G moves along the directrix.</td>
</tr>
</tbody>
</table>

- What is the general shape formed by the loci of point P? Drag the focus point to manipulate the loci. What do you observe about the shape formed?

- Confirm that point P satisfies the definition of a parabola, that is, measure the distance from P to G and from P to F. Formally prove that these distances are equal.

- Algebraically, derive the standard form of a quadratic function. (Hints: Use the labeled sketch below to equate the distances from P to G and from P to F. Let the distance (FV) = p.)

- Manipulate the locations of the directrix and the focus to classify the movement of the graph.
• How do the constant h, k, and p affect the graph of the parabola? You may want to use a more appropriate technology to investigate. (Try the following interactive website: http://www.exploremath.com/activities/Activity_page.cfm?ActivityID=14)

• Another common form of a quadratic function is \( y = ax^2 + bx + c \). Further explore how the constant a, b, and c affect the graph of the parabola. Compare and contrast these two algebraic forms of quadratic functions.

Extensions:

• In Part 2 of this activity, instead of using the perpendicular bisector of FG, conjecture what would happen when an arbitrary perpendicular line to FG is used.

• Construct a line perpendicular to FG through any point other than the midpoint. Trace this perpendicular line. Is the shape formed a parabola? Does it meet the definition of a parabola? If so, where are its focus and directrix? If not, how would you classify its shape?

• Further manipulate the sketch to find when point F is the focus? When is line \( d \) the directrix? What is special about the relationship of the perpendicular bisector of segment FG to the parabola?

• Refer back to Part 3 of this activity. Since segment GP = segment FP, a circle centered at P will pass through both F and G and be tangent to the directrix at G. On the basis of these relationships, make up an alternative definition of a parabola.
Exploring Infinite Series through Baravelle Spirals

This activity is an introduction to the concept of convergent infinite series using an iterative geometric construction. This activity has been adapted from the following article: Choppin, J. M. (1994). Spiral through recursion. *Mathematics Teacher*, 87(7), 504-508.

**Mathematics Topics:** The students will explore the concept of a convergent *infinite series* using *partial sums*. This activity connects a number of *algebra* and *geometry* topics, as well as to involve students in formal *calculations*, *visualizations*, *pattern discovery* and *iterations*.

**Mathematical Thinking:** Students will be asked to investigate *numerically* and *visually* the sums of infinite series. Students will further *conjecture* and *prove* their ideas behind the sums of specific infinite series.

**Technology:** This activity uses several Sketchpad commands such as: *New Sketch*, *Midpoint*, *Segment*, *Iterate*, *Polygon Interior* and *Area*.

The following sketches illustrate geometric representations of three different convergent infinite series summing to 1/3, 1, and 1/4 respectively.

![Geometric representations of infinite series](image)

\[
\frac{\text{Area of Spiral}}{\text{Area of Original Triangle}} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots = \frac{1}{3}
\]

\[
\frac{\text{Area of Spiral}}{\text{Area of Original Square}} = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = \frac{1}{4}
\]

\[
\frac{\text{Outer Edge of Spiral}}{\text{Outer Edge of Original Triangle}} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1
\]

A complete copy of this activity can be found on the web: http://www.teacherlink.org/content/math/activities/skpv4-baravelle/home.html
Exploring Infinite Series through Baravelle Spirals

Part 1:

- With your partner, discuss and draft a definition for a spiral. Share your definition with the class, and formulate a class definition.

A spiral is a curve traced by a point that moves around a fixed point, called the pole, from which the point continually moves towards or away. There are many different types of spirals. We are going to create a specific type of spirals called Baravelle spirals. They are created by constructed nested regular polygons and shaded triangles in a clockwise manner.

- The first Baravelle spiral we will construct is formed by a group of nested equilateral triangles. Let's begin by constructing a large equilateral triangle, \( T_0 \), in your sketch window. Construct the midpoints of each of the three sides of the triangle and connect them by segments. The figure you have constructed separates your equilateral triangle into smaller triangles. How many smaller triangles are there? What kind of triangles are they? What other conjectures can you make about the four triangles? (Instructor Note: A Sketchpad file illustrating the construction of an equilateral triangle has been saved as equil.gsp.)

- Construct the polygon interior of one of the corner triangles and label it \( T_1 \). Formally prove that \( T_1 \) is congruent to the three other inner triangles. The area of \( T_1 \) is what fraction of \( T_0 \)? Verify your conjecture by a formal proof.

To label a figure in a sketch window:

- Select the Text Tool in the tool box.
- Double-click anywhere in your sketch window.
- When a flashing cursor appears, you may type in the box.
- To edit your text box once you’ve clicked outside of it, select the Text Tool and click inside the text box. A flashing cursor should appear, allowing you to edit.

- We would like to continue the construction of the Baravelle spiral by finding and connecting the midpoints of each of the sides of the center triangle. The figure you have constructed separates the center equilateral triangle into smaller triangles. How many smaller triangles are there? What kind of triangles are they? What else can you say about the four smaller triangles?

- Moving in a clockwise manner, construct the polygon interior of the corner inner triangle, which is adjacent to \( T_1 \), and label it \( T_2 \). Be sure to keep the color of your polygon interiors consistent. The area of \( T_2 \) is what fraction of \( T_0 \)?
We have created two levels of this construction. Create two or three additional shaded triangles using the same process as above. If this construction was continued indefinitely, the resulting shaded region is called a Baravelle spiral.

Part 2:

- Looking at your constructed Baravelle spiral, visually estimate the spiral’s area as a fraction of the area of $T_0$. Justify your answer. How could we find the actual area of the spiral?

- In your own words, what is a series? How does a series differ from a sequence? Can an infinite series sum to a finite number? If so, how can we find that number?

- Record the ratios of the areas of the shaded triangles, $T_N$, in comparison to the area of the original equilateral triangle, $T_0$. Fill in the chart below, computing the partial sums of the areas of each of the levels of triangles.

<table>
<thead>
<tr>
<th>Level of Triangle</th>
<th>Ratio of Area of $T_N$: ( \frac{\text{The area of } T_N}{\text{The area of } T_0} )</th>
<th>Computed Partial Sums of the areas $T_1$ through $T_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0$</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>$T_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_6$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- What did you notice about consecutive ratios of the areas of the triangles? Furthermore, what did you notice about the column of computed partial sums? As $N$ increased, what number did the column of partial sums approach? Describe your observations in your own words. Use your knowledge of infinite series to verify this sum. See the spreadsheet activity Exploring and Analyzing Sequences.

- Express the area of the Baravelle spiral as an infinite series. The spiral's area is what fraction of the area of the original triangle?

- Shade (each in a different color) two other Baravelle spirals using your original equilateral triangle. (Instructor Note: A Sketchpad file illustrating the construction of a Baravelle spiral generated by an equilateral triangle has been saved as eqtri-spiral.gsp.)

Part 3:

- Another infinite series is similarly found by considering the length of the outer edge of the spiral. The outer edge of a spiral is the sum of the lengths of one side of each of the shaded triangles. Record the length of each side of the triangle, $T_N$, in comparison to the length of each side of the original equilateral triangle, $T_0$. Fill in the chart below, computing the partial sums of the length of the outer edge for each of the levels of triangles.
What did you notice about consecutive ratios of the lengths of a side of the triangles? Furthermore, what did you notice about the column of computed partial sums? As N increased, what number did the column of partial sums approach? Describe your observations in your own words. Use your knowledge of infinite series to verify this sum.

Express the outer edge of the spiral as an infinite series.

Part 4:

"Mathematically, *iteration* refers to the process of repeatedly applying some mathematical construction, calculation, or other operation to the previous result of that same operation. The operation must define an output in terms of some input, and the iteration uses the output of one step as the input for the next step" (Key Press, 2001).

As you can see from the construction in Part 1, the process to construct a Baravelle spiral is a *iterative* process. In this context, discuss with your neighbor what it means to be a iterative process? Share your ideas with the class.

To further explore infinite series and iterations, we will direct focus on a Baravelle spiral generated by a square. To begin, construct a large square in your sketch window. (Instructor Note: A Sketchpad file illustrating the construction of a square has been saved as *square.gsp*.)

The basic construction for a Baravelle spiral is to find and connect the midpoints of each of the sides of the center square *multiple* times. Sketchpad allows us to create an iterated construction of one or more objects in terms of points and parameters. Follow the steps in the gray box below to create an iterative construction of a Baravelle spiral generated by a square.

### To construct a Baravelle spiral generated by a square iteratively:

- Construct the midpoints of each side of the square and connect consecutive midpoints by segments.
- Construct the polygon interior of the corner triangles, and shade each in a different color.
- Select the two independent vertices of your square and choose the *Iterate* command under the *Transform* menu.
- To specify this repetition rule (find and connect the midpoints of each of the sides of the center square), you must define a map that identifies the image of...
each independent point. Click on the image midpoints of A and B, respectively, in your sketch window. You will see the image of the iterative construction outlined in your square, and the number of iterations displayed in your Iterate window.

- You may increase and decrease the number of iterations displayed under the Display menu in the Iterate window. You may also slightly change the iterative construction under the Structure menu in the Iterate window.
- To finish the construction, click on the Iterate button.

The figure you have constructed separates the original square into a smaller squares and triangles. Within the first level of the construction, how many smaller triangles are there? What kind of triangles are they? What other conjectures can you state about the four corner triangles?

Record the ratios of the areas of the shaded triangles, $T_N$, in comparison to the area of the original square. Fill in the chart below, computing the partial sums of the areas of each of the levels of triangles.

<table>
<thead>
<tr>
<th>Level of Triangle</th>
<th>Ratio of the area of $T_N$: the area of the square</th>
<th>Computed Partial Sums of the areas $T_1$ through $T_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_0$</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>$T_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_6$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- What did you notice about consecutive ratios of the areas of the triangles? Furthermore, what did you notice about the column of computed partial sums? As $N$ increased, what did the column of partial sums approach?

- Express the area of the spiral as an infinite series. The spiral’s area is what fraction of the area of the original square?

- Construct the three other Baravelle spirals using your original square, each in a different color.
Extensions:

- Explore a similar investigation using a hexagon as the original starting figure. By constructing the midpoints of each of the six sides of the hexagon and connecting them by segments, continue a similar procedure as the triangle and square. The shaded spiral's area is approximately what fraction of the original hexagon? Use your knowledge about infinite series to verify this sum?

- In each of the Baravelle spirals constructed we constructed the midpoints of each of the sides of our polygon and connected them by segments. A different type of spiral could be constructed by choosing points other than the midpoints of each of the sides of the polygon. Iteratively construct a spiral that does not use the midpoints of the sides of the polygon. Write a report describing your construction and showing examples of your spirals.
Exploring Trigonometric Functions

This activity has students geometrically construct and investigate the graphs of sine and cosine based upon the lengths of the sides of a reference triangle.

Mathematics: Students will learn a little history of trigonometry while defining the sine and cosine functions using the unit circle. Students will qualitatively and quantitatively compare the numerical and graphical characteristics of the sine and cosine functions.

Mathematical Thinking: Students will be asked to predict the basic shape of the functions and how the graphs will differ once the radius of the circle is changed.

Technology: This activity uses several new Sketchpad commands such as: Define Coordinate System, Circumference, Animation button, Movement button, Trace Point, and Hide/Show button.

The following sketch illustrates the trigonometric functions, sine and cosine.

A complete copy of this activity can be found on the web:
http://www.teacherlink.org/content/math/activities/skpv4-trig/home.html
The study of trigonometry, which translates verbatim as "triangle measurement", began more than 2000 years ago, partially as a means to solving land surveying problems. The trigonometric functions we use today are not the same as those used 2000 years ago. Early trigonometry related the length of a chord of a circle as the value of a trigonometric function.

In a circle of a fixed radius, the problem was to find the length of the chord subtended by a given angle. For a unit circle, the length of the chord subtended by the angle $x$ is, $y = \frac{2 \sin \frac{x}{2}}{2}$.

Although the first known tables of chords have not survived, it is claimed that twelve books containing these tables were written by the Greek mathematician Hipparchus, around 140 B.C. Due to this, it is claimed that he is the founder of trigonometry.

In the sixteenth century, right triangles were used to define the trigonometric functions that we are more familiar with today. We will use a modified right triangle approach to define the trigonometric functions by placing one of the acute angles of a right triangle on a coordinate plane. We will explore its measurements using The Geometer’s Sketchpad.

Part 1:

- Open a new sketch and define a coordinate system in your sketch window.

To define a coordinate system:

- Select Define Coordinate System, under the Graph menu.
- You may drag the point (1,0) along the x-axis to define the size of the scaling unit.
• Construct a unit circle centered at the origin; that is, circle with a radius of one inch centered at the origin. Be sure to first change your Distance Unit to inches in your Preferences window.

• Label or relabel the point (1,0) as point D.

To relabel an already labeled figure:

- Select the Text Tool from the toolbox.
- Position the hand pointer on any label.
- Double click and type the new label in the Label dialog box.

We would like to investigate a group of right triangles, where (1) the length of the hypotenuse is equal to the radius of the circle; (2) the vertex of the right angle is confined to the x-axis; and (3) a vertex is at the origin.

• Using the three parameters above, sketch a representative right triangle on paper. Describe and compare your triangle with your neighbor's.

• Construct an arbitrary point on the unit circle and drag this point around the circle until it lies in the first quadrant. Re-label it as point B. Construct the radius of the circle, by constructing a segment from the origin to point B. This construction satisfies the first parameter, the length of the hypotenuse is equal to the radius of the circle.

• Construct a perpendicular line to the x-axis through point B. Construct the intersection point of the line and the x-axis. Re-label it as point C. Construct a segment between points B and C. Change the display of the segment BC to thick and red. Hide the perpendicular line. This construction satisfies the second parameter, the vertex of the right angle is confined to the x-axis.

• Construct a segment from the origin to point C. Change the segment's color to green. This construction satisfies the third parameter, a vertex is at the origin.

• Observe the constructed triangle on the inside of the circle. Label the sides of the triangle in the conventional manner. Compare your sketch with the sketch below. (Instructor Note: A Sketchpad file illustrating the unit circle has been saved as unitcircle.gsp.)
• What kind of triangle have you constructed? How can you be sure?

• As you drag point B around the circle, focus your attention on the lengths of sides a and b. Qualitatively describe the lengths of a and b as point B is dragged around the circle. Write down your observations.

• It is difficult to describe specifics about lengths a and b without being able to pinpoint "locations" around the circle. We shall use the measurement of the angle DAB to refer to the location of point B on circle. Change your Angle Unit to directed degrees in your Preferences window.

• Measure angle DAB. Observe the angle measurement as point B is dragged around the circle. Describe the angle’s measurement. Can you have an angle that has the measure of 240°? Why or why not? What does this tell you about The Geometer’s Sketchpad measurement techniques? Reconcile the differences between the angle measurement techniques of The Geometer’s Sketchpad and the conventional unit circle.

• Measure the lengths of sides a and b. At what point on the circle are lengths a and b congruent? Drag point B around the circle to find where lengths a and b reach their maximum and minimum lengths.

Part 2:

In your sketch, you notice the lengths of the sides of the triangle changing as point B is dragged around the circle. We would like to graph the length of side a as point B is dragged around the circle and observe the patterns displayed. To graph, we will be using both the animate and trace features of The Geometer’s Sketchpad.

• Once again, drag point B around the circle. On a sheet of paper, sketch a graph of the length of side a as a function of the measure of < DAB.

• Using The Geometer’s Sketchpad, we can graph the length of side a versus the measure of angle DAB. We would like to trace the length of side a as the measure of angle DAB changes. Estimate the distance, in inches, point B will travel around the unit circle once. Use The Geometer’s Sketchpad to measure this distance.

To measure a circle’s circumference:

• Select the circle.
• Select Circumference from the Measure menu.

• Mentally compare your estimate to the computed circumference of the circle. Was your estimate reasonable? Why or why not? To serve as the x-axis of the graph, construct a segment the same length as the circle’s circumference starting at the origin, along the positive x-axis. Construct a point on this new segment, and color it blue. Drag the point along the segment to be sure it is confined to the segment.

• This segment defines the x-coordinates of our graph. In order to define the y-coordinates, we would like to construct a point whose distance from the x-axis is the same as the length of side a. Construct a dashed line parallel to the x-axis through the point B. Construct another dashed line, parallel to the y-axis through the blue point. Construct the point of intersection of the two dashed lines. Change its color to green. Drag the green point. Does its distance from the x-axis equal the length of segment a?
• Describe what happens to the green point as point B is dragged around the circle. Describe what happens to the green point as the blue point is dragged along its segment.

We would like to trace the green point while point B travels around the circle AND while the blue point travels along its segment. However we cannot drag both at the same time with our mouse....but we can have Sketchpad animate both points together.

• Create an animation button to trace the green point as point B travels counter-clockwise around the circle once slowly and while the blue point travels forward along its segment once slowly.

To identify which point to trace:

- Select the point.
- Choose Trace Point under the Display menu.

To create an animation button for two objects:

- Select the objects you wish to animate.
- Choose Action Button under the Edit menu, and then Animation.
- In the Properties of Action Button window, make travel selections for your object (i.e., its direction and speed).
- Click on the Label tab to rename your action button, and click OK.
- An Animate button will appear in your sketch window.

• Before you activate your button, predict the shape of the graph by a brief sketch on your paper. Activate your button by clicking on the Animate button. To cease the animation, re-click on the Animate button.

• Describe the path that the green point traced. Compare your predicted graph with that one in your Sketchpad window. In what ways is it accurate? In what ways is it inaccurate? What is the length of segment a when the measure of angle DAB is 0°?

• We need to make sure we have an appropriate starting point to trace the length of side a. What would be an appropriate starting point? To start our tracing properly we need to set the measure of angle DAB to 0°. How could we change the angle measurement of angle DAB?
• One way is to have Sketchpad move a point to another point’s location in your sketch window. This can be done with a Movement button. Create a movement button that will move point B to point D and the blue point to the origin.
To create a movement button:

- Select the first point you would like to move; then select the point where you would like it to move to.
- Select the second point you would like to move; and the point you would like it to move to.
- Choose Action Button under the Edit menu, and then Movement.
- Select the movement speed you desire and click OK.
- A Movement button will appear in your sketch window.

- Click on the Movement button in your sketch window. Describe what happens. Let’s give this button a more descriptive name than "Move."

To re-name a button:

- Using the Text Tool double click on the button.
- Type in the window a more descriptive name.
- Then click OK.

- Click on the animate button in your sketch window. How has the graph changed? At what angle measure(s) does the graph cross the x-axis? (Instructor Note: A Sketchpad file illustrating the unit circle and the sine function has been saved as sine.gsp.)

- If you change the radius of your circle, predict how the graph will change. Change the radius of the circle, reset your graph and double click on the animate button and assess your predictions.

The function you have graphed is called the sine function. On a historical note,

the Hindu word jya for the sine was adopted by the Arabs who called the sine jiba, a meaningless word with the same sound as jya. Now jiba became jaib in later Arab writings and this word does have a meaning, namely a ‘fold’. When European authors translated the Arabic mathematical works into Latin they translated jaib into the word sinus meaning fold in Latin. In particular Fibonacci's use of the term sinus rectus arcus soon encouraged the universal use of sine (http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Trigonometric_functions.html, retrieved April 15, 1999).

Formally, the definitions of cos \( \theta \) and sin \( \theta \) can be generalized as follows. If \( \theta \) is an angle in standard position and if (x,y) is any point (other than the origin) on the terminal side of \( \theta \), then

\[
\sin \theta = \frac{y}{r} \quad \text{and} \quad \cos \theta = \frac{x}{r} \quad \text{where} \quad r = \sqrt{x^2 + y^2}
\]

- Using the calculator and the lengths of the sides of the triangle, compute the sine and cosine ratios. Compare those values with the calculator Functions sine/cosine for the angle DAB.
• Drag point B around the circle. How do the ratios and the function values compare? Are they always equivalent?

Part 3:

In your sketch, we graphed the length of side a as the measurement of angle DAB increases. We would now like to do a similar investigation by graphing the length of side b as the angle DAB changes.

• Drag point B around the circle. On a sheet of paper, sketch a graph of the length of side b as the measurement of angle DAB increases.

• Using a slightly modified process as above, create another Animate button in your sketch window that will trace the length of side b as the measurement of angle DAB increases. You may want to change the color of the point being traced to distinguish it from the sine function.

To trace the length of side b as the measure of <DAB changes:

• Drag point B to the first quadrant.
• Drag the dashed vertical line to the right of the origin.
• Similar to side a, we would like to have a vertical segment the same length as side b. To do this, rotate side b 90° about the origin. Select the origin and Mark Center from the Transform menu.
• Select both side b and its endpoints. Rotate the segment 90° by choosing Rotate from the Transform menu.
• Construct a dashed perpendicular line to the rotated segment through its endpoint.
• Construct the point of intersection between this dashed line and the vertical dashed line. Change its color to red. Hide both the rotated segment and its endpoint.
• Select the red point and Trace Point under the Display menu.
• Now we are ready to animate. Select point B and blue point along the x-axis.
• Choose Action Button under the Edit menu, and then Animation.
• In the Properties of Action Button window, make travel selections for your objects as follows: point B travels counter-clockwise around the circle once slowly and while the blue point travels forward along its segment once slowly. Click OK.
• An Animate button will appear in your sketch window. Rename the button to distinguish it from the other Animate button in your sketch window.
• Click on the movement button to reset the points; and then click on your new Animate button.

• Describe your observations when the button was activated.

• Qualitatively and quantitatively compare and contrast the sine and cosine functions.

• For what angle measurements are the sine function positive? Negative? For what angle measurements are the cosine function positive? Negative? What are the roots of each of the functions? How could you find the roots of the functions mathematically?
• For what angle measurement does the sine function equal the cosine function?

Part 4:

• Both the red and green points are traced when you click on either of your Animate buttons resulting in both functions being graphed simultaneously. What was the difference between the Animate buttons? How could you have only one of the functions to be graphed?

• *The Geometer’s Sketchpad* has a type of *Animation* button which allows you to show and hide objects in your sketch window. Create *Hide/Show* buttons to hide the traces for each of the trigonometric functions. (*Instructor Note:* A *Sketchpad* file illustrating both the sine and cosine functions has been saved as [trig.gsp](#)).

  **To create a Hide/Show button:**
  
  • Select the point you wish to hide.
  • Choose *Action Button* under the *Edit* menu, and then *Hide/Show*.
  • A button will appear in your sketch window. Rename the button with a more descriptive name.

• Click on the *Hide* button you have created for the sine function. Click on the Animate button. What changed in your sketch window? Experiment with the other *Hide/Show* button.

• What are the advantages and disadvantages to the Hide/Show buttons?

Extensions:

• Why was the *unit* circle chosen to construct the sine and cosine curves?

• Using your unit circle, construct the graphs of other trigonometric functions.

• Conjecture what the graph would look like if point B traveled clockwise.

• Sketch a graph, by hand, of the area of the triangle verses the angle measurement.

Resources:

• Trigonometric Functions article from the MacTutor History of Mathematics Archive -- an integrated collection of over 1000 biographies and historical articles of a mathematical nature, alongside interactive birthplace maps and the famous curve applet. [http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Trigonometric_functions.html](http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/Trigonometric_functions.html)

Investigating the Concept of Derivative

This activity was designed to investigate the rate of change of a function at a specific point, or the instantaneous rate of change, which is a fundamental concept in calculus.

Part 1:

- Open the sketch Derivatives.gsp in the Derivatives folder. This sketch is original, however, the tangent line has been constructed using a built-in custom tool in Sketchpad, which is located in the following folder: C:\ProgramFiles\Sketchpad\Samples\CustomTools\AdvancedTools.gsp

In this sketch, you will find the graph of a general cubic function, \( f(x) = ax^3 + bx^2 + cx + d \), with sliders to adjust the coefficients \( a, b, c, \) and \( d \).

- Change the value of the \( a \) slider, or the coefficient for \( x^3 \), and explain how changing this value effects the graph. Repeat this process for each slider.

- Translate point \( P \) along the graph. When does point \( P \) coincide with the y-axis? How does this value relate to the function \( f(x) \)?

- What are the coordinates for the relative maximum and relative minimum of the function? Make a note of these values.

Part 2:

- Click the Show Tangent Line button. Describe the behavior of the tangent line as you transverse point \( P \) along the graph of the function.
  - When is the slope of the tangent line positive? Record the range of \( x \)-values for when the slope of the tangent line is positive. Describe the behavior of the function \( f(x) \) when the tangent line is positive.
  - When is the slope of the tangent line negative? Describe the behavior of the function \( f(x) \) when the tangent line is negative.
  - When is the slope of the tangent line zero? Describe the behavior of the function \( f(x) \) when the tangent line is zero.

- As you continue to transverse point \( P \) along the graph of the function, describe when the slope of the tangent line is increasing and decreasing. Record the range of these \( x \)-values.

- Measure the slope of the tangent line and confirm when the slope of the tangent line is increasing and decreasing.

- Describe where the slope of the tangent line changes from increasing to decreasing or vice versa.

- Describe the relationship of the tangent line to the graph of the function as the slope of the tangent line is increasing and decreasing.

- Analyze the values you recorded in the previous tasks and predict the equation and draw a graph of \( f'(x) \).
Part 3:

- Calculate and graph the derivative of the function, \( f(x) \).

  To calculate and graph the derivative of a function:
  - Select the equation for the function (i.e., \( f(x) = ax^3 + bx^2 + cx + d \)).
  - From the Menu, select Graph: Derivative.
  - With the function of the derivative highlighted, select Graph: Plot Function.

- Construct point \( D \) on the derivative and measure its coordinates.

- Translate point \( D \) along \( f'(x) \). When does point \( D \) coincide with the \( x \)-axis? How do these coordinates compare to the corresponding coordinates of the function \( f(x) \)?

- Assess your prediction of the equation and graph of \( f'(x) \) from Part 2. Reconcile and differences.

Extension:

- Complete similar tasks as described above to investigate the second derivative.

References:

Investigating the Area under a Curve

Finding the area between the graph of a function and the x-axis has a number of applications. For example, finding the surface area of non-rectangular shapes, so you can find out how much grass seed you need for a curved lawn area. Likewise, the area under a velocity curve is the object’s change in position, or the net distance traveled over the given time interval (Clements, Pantoazzi, & Steketee, 2002). In this activity students will find the approximate area between the graph of a function and the x-axis using three methods.

Part 1:

- Open the sketch **Riemann Sums.gsp** in the **Integrals** folder. This file is a slightly modified version of the built-in Riemann Sums file in GSP (Jackiw, 2001). You may access the original file through the following directories: `C:\Program Files\Sketchpad\Samples\Sketches\Advanced\Riemann Sums.gsp`.

In this sketch, you will find the graph of a cubic function, \( f(x) = ax^3 + bx^2 + cx + d \), with sliders to adjust the coefficients \( a, b, c, \) and \( d \). The upper and lower bounds of the domain may be adjusted by moving points A and B along the x-axis. Also notice that point P is confined to the graph of \( f(x) \).

- Experiment with the sliders associated with the coefficients of the cubic function, \( f(x) \). Briefly describe the effect each slider has on the graph.

- Click the **Show Upper Sum** button in the top right-hand corner of your sketch window. Describe how the rectangle shown in your sketch window was formed. How was the rectangle’s height determined? What is its area? Do you think this is a good estimate of the area under the curve along the interval \([a, b]\)? Why or why not?

- Click the **Animate Number of Rectangles** button. How does the Upper Sum change as the number of rectangles increases? [Note: You may cease animation at any time by simply clicking on the **Animate Number of Rectangles** button.]

- Using the **Upper Sum**, what is the best approximation for the area under the curve? Justify your response.

- Click the **Hide Upper Sum** button before you start the next part of the activity.

Part 2:

- Click the **Show Lower Sum** button in the top right-hand corner of your sketch window. Describe how the rectangle shown in your sketch window was formed. How was the rectangle’s height determined? What is its area? Do you think this is a good estimate of the area under the curve along the interval \([a, b]\)? Why or why not?
• Click the Animate Number of Rectangles button. How does the Lower Sum change as the number of rectangles increases?

• Using the Lower Sum, what is the best approximation for the area under the curve? Justify your response.

• Why do you think there is a large difference between the Lower Sum and Upper Sum values that you found? What could you do to decrease this difference?

• Click the Hide Lower Sum button before you start the next part of the activity.

Part 3:

• Click the Show Midpoint Sum button in the top right-hand corner of your sketch window. Describe how the rectangle shown in your sketch window was formed. How was the rectangle’s height determined? What is its area? Do you think this is a good estimate of the area under the curve along the interval [a, b]? Why or why not?

• Click the Animate Number of Rectangles button. How does the Midpoint Sum change as the number of rectangles increases?

• Using the Midpoint Sum, what is the best approximation for the area under the curve? Justify your response. How does this value compare to the best approximations you found in Parts 1 and 2?

Part 4:

• Review your best approximations of the area between f(x) and the x-axis using the Upper Sum, Lower Sum and Midpoint Sum. Which is the best approximation of area between f(x) and the x-axis? Justify your response.

• You have seen that increasing the number of rectangles leads to a more accurate estimate of the area under a curve. Can you think of another way to improve the accuracy for your area estimate? (Hint: What type of polygon could you use to calculate a better estimate for the area between f(x) and the x-axis? Why would this improve your area approximation?)

The notation for the exact area is \( \int_a^b f(x) \, dx \), read “the integral from x = a to x = b of the function f of x.” So the integral \( \int_a^b f(x) \, dx \) is defined as the limit of the sum of the rectangles as their width h approaches 0 on the interval [a, b], or \( \int_a^b f(x) \, dx = \lim_{h \to 0} \sum f(x_i) \cdot h_i \).

References:


Additional Resources

Center for Technology and Teacher Education
http://www.teacherlink.org/content/math/activities/index.html

The Center's math group has developed activities that prepare teachers to use technology to enhance and extend students' learning of mathematics. Their focus has been to develop activities using graphing calculators, The Geometer's Sketchpad, Microsoft Excel, the ExploreMath.com website, Global Positioning Systems, and MicroWorlds logo for pre-service secondary mathematics students. Many activities address content spanning several years of secondary content; thus, *some modifications may be necessary for classroom implementation.*

The Geometer’s Sketchpad Resource Center
http://www.keypress.com/sketchpad/

Download handouts and sketches from the 2003 Exploring Calculus Summer Institute.