THE PRECIPITOUSNESS OF TAIL CLUB GUESSING IDEALS

TETSUYA ISHIU

Abstract. From a measurable cardinal, we build a model in which the non-stationary ideal on \( \omega_1 \) is not precipitous, but there is a precipitous tail club guessing ideal on \( \omega_1 \).

1. Introduction

Club guessing sequences were introduced by Shelah in 80’s, for example in [8]. Since then, they were proved to be effective tools to show the results under ZFC. The ideals associated with club guessing sequences are also used in various arguments. There are two types of club guessing sequences, tail club guessing sequences and fully club guessing sequences. In this paper, we shall concentrate on tail club guessing sequences.

When \( \overrightarrow{C} \) is a tail club guessing sequence on \( \kappa \), then we can define the filter \( \text{TCG}(\overrightarrow{C}) \) on \( \kappa \) associated with \( \overrightarrow{C} \), which is called the tail club guessing filter. The definition is essentially due to Shelah. The tail club guessing ideal simply refers to the dual ideal of the tail club guessing filter. When \( \Gamma \) is a property of ideals, we say that a filter \( F \) has \( \Gamma \) if and only if its dual ideal has \( \Gamma \). When \( F \) is a filter, \( \check{F} \) denotes the dual ideal of \( F \).

There are several results about the precipitousness of tail club guessing ideals. In [10], Woodin proved that it is consistent relative to the consistency of a Woodin cardinal that \( \text{NS}_{\omega_1} \) is \( \aleph_2 \)-saturated and there exists a tail club guessing ideal \( \overrightarrow{C} \) on \( \omega_1 \) such that \( \text{NS}_{\omega_1} = \text{TCG}(\overrightarrow{C}) \), in particular \( \text{TCG}(\overrightarrow{C}) \) is precipitous. In [5], the author showed that if we collapse a Woodin cardinal to \( \omega_2 \) by the Levy collapse, then \( \text{NS}_{\omega_1} \) is precipitous and so is every tail club guessing ideal on \( \omega_1 \). In the same paper, it was thus asked if it is consistent that \( \text{NS}_{\omega_1} \) is not precipitous but there is a precipitous tail club guessing ideal on \( \omega_1 \). This is the question we shall answer in this paper. In addition, the model is built from a measurable cardinal. Hence, it also shows that the existence of a precipitous tail club guessing ideal is equiconsistent with the existence of a measurable cardinal.

We follow the standard notations in set theory. \( \text{Lim} \) stands for the class of limit ordinals. When \( X \) and \( Y \) are sets of ordinals, we say that \( X \) is almost contained in \( Y \) if and only if there exists a \( \zeta < \sup(X) \) such that \( X \setminus \zeta \subseteq Y \). When \( X \) is a set of ordinals, we define \( \text{nacc}(X) = X \setminus \text{lim}(X) \) where \( \text{lim}(X) \) denote the set of limit points of \( X \). An ordinal \( \alpha \) is indecomposable if and only if for every \( \beta < \alpha \), \( \beta + \alpha = \alpha \). When \( F \) is a filter on \( \kappa \), we say that a subset \( X \) of \( \kappa \) is \( F \)-positive

Date: April 23, 2008.
This material is based upon work supported by the National Science Foundation under Grant No. 0700983.
if and only if $\kappa \setminus X \notin F$. $F^+$ denotes the set of all $F$-positive subsets of $\kappa$. We automatically assume that $\dot{x}$ is a name for $x$.

2. Tail club guessing ideals

The following notions were introduced by Shelah in [8] though he used different terminology.

**Definition 2.1.** Let $\kappa$ be an uncountable regular cardinal. We say that a sequence $\overrightarrow{C} = \langle C_\delta : \delta \in \kappa \cap \text{Lim} \rangle$ is a tail club guessing sequence on $\kappa$ if and only if

(i) for every $\delta \in \kappa \cap \text{Lim}$, $C_\delta$ is an unbounded subset of $\delta$, and

(ii) for every club subset $D$ of $\kappa$, there exists a $\delta \in \kappa \cap \text{Lim}$ such that $C_\delta \subseteq^* D$.

We say that $\overrightarrow{C}$ has order type $\varepsilon$ if and only if for every $\delta \in (\kappa \setminus \varepsilon) \cap \text{Lim}$, $\text{otp}(C_\delta) = \varepsilon$.

We define the tail club guessing filter $\text{TCG}(\overrightarrow{C})$ associated with $\overrightarrow{C}$ as the filter on $\kappa$ generated by the sets of the form $\{\delta \in \kappa \cap \text{Lim} : C_\delta \subseteq^* D \}$ for some club subset $D$ of $\kappa$. A tail club guessing ideal is the dual ideal of a tail club guessing filter.

In [8], Shelah showed that $\text{TCG}(\overrightarrow{C})$ is $\kappa$-complete and normal for every tail club guessing sequence $\overrightarrow{C}$ on $\kappa$. Note that not all club guessing sequences have order types. However, for every tail club guessing sequence $\overrightarrow{C} = \langle C_\delta : \delta \in \kappa \cap \text{Lim} \rangle$, either there exists an $X \in \text{TCG}(\overrightarrow{C})$ such that for every $\delta \in X$, $\text{otp}(C_\delta) = \delta$ or there exists an $\varepsilon < \kappa$ such that $\{\delta \in \kappa \cap \text{Lim} : \text{otp}(C_\delta) = \varepsilon\}$ is $\text{TCG}(\overrightarrow{C})$-positive.

The following forcing notion was used in [9] by Shelah.

**Definition 2.2.** Let $\overrightarrow{C} = \langle C_\delta : \delta \in \kappa \cap \text{Lim} \rangle$ be a tail club guessing sequence on an uncountable regular cardinal $\kappa$. For every $X \in \text{TCG}(\overrightarrow{C})^+$, we define the standard forcing $P(\overrightarrow{C}, X)$ to shoot a $\text{TCG}(\overrightarrow{C})$-measure one set through $X$ as follows: $p \in P(\overrightarrow{C}, X)$ if and only if $p$ is a closed bounded subset of $\kappa$ such that for every $\delta \in p \cap \text{Lim}$, if $C_\delta \subseteq^* p$, then $\delta \in X$. $P(\overrightarrow{C}, X)$ is ordered by end-extension.

It is easy to see that $P(\overrightarrow{C}, X)$ forces that $\overrightarrow{C}$ is a tail club guessing sequence on $\kappa$ and $X \in \text{TCG}(\overrightarrow{C})$. In [6], to investigate this forcing notion, the author defined the following properties of club guessing sequences.

**Definition 2.3.** Let $\kappa$ be an uncountable regular cardinal and $\tau : \kappa \rightarrow [\kappa]^{<\kappa}$ We say that a subset $X$ of $\kappa$ is $\tau$-weakly tight if and only if for every $\gamma \in \text{nacc}(X)$, $X \cap \gamma \in \tau^\gamma$.

**Definition 2.4.** Let $\overrightarrow{C} = \langle C_\delta : \delta \in \kappa \cap \text{Lim} \rangle$ be a tail club guessing sequence on an uncountable regular cardinal $\kappa$.

(i) We say that $\overrightarrow{C}$ is weakly tight if and only if there exists a function $\tau : \kappa \rightarrow [\kappa]^{<\kappa}$ such that for every $\delta \in \kappa \cap \text{Lim}$, $C_\delta$ is $\tau$-weakly tight.

(ii) We say that $\overrightarrow{C}$ is simple if and only if for every $\delta \in \kappa \cap \text{Lim}$ and $\gamma \in C_\delta \cap \text{Lim}$, $C_\gamma \setminus C_\delta$ is unbounded in $\gamma$.

For example, if $\overrightarrow{C}$ has order type $\varepsilon$, then $\overrightarrow{C}$ is simple. If $\overrightarrow{C}$ has order type $\omega$, then $\overrightarrow{C}$ is also weakly tight. Standard constructions of club guessing sequences often yields simple weakly tight sequences.

We shall review several properties of forcing notions.
Definition 2.5. A sequence of $\langle N_\alpha : \alpha < \eta \rangle$ is called a tower if and only if

(i) for every limit $\alpha < \eta$, $N_\alpha = \bigcup_{\beta < \alpha} N_\beta$, and
(ii) for every $\alpha < \eta$, $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$.

Typically, each $N_\alpha$ is a countable elementary submodel of $H(\theta)$ for some large regular cardinal $\theta$.

Definition 2.6. Let $\varepsilon < \omega_1$. A forcing notion $P$ is called $\varepsilon$-proper if and only if whenever $\langle N_\alpha : \alpha < \varepsilon \rangle$ is a tower of countable elementary submodels of $H(\theta)$ for some sufficiently large regular cardinal $\theta$ with $P \in N_0$, for every $p \in P \cap N_0$, there exists a $q \leq p$ that is $(N_\alpha, P)$-generic for every $\alpha < \varepsilon$.

We say that $P$ is $<\varepsilon$-proper if and only if $P$ is $\varepsilon'$-proper for every $\varepsilon' < \varepsilon$.

Lemma 2.7 (Shelah [9]). Let $\varepsilon < \omega_1$. Let $\langle P_\alpha, Q_\beta : \beta < \alpha \leq \eta \rangle$ be a countable-support iteration such that for each $\alpha < \eta$, $P_\alpha$ forces that $Q_\alpha$ is $\varepsilon$-proper. Then $P_\eta$ is $\varepsilon$-proper.

Definition 2.8. A forcing notion $P$ is called totally proper if and only if $P$ is proper and adds no new reals.

Let $N$ be a countable elementary submodel of $H(\theta)$ and some sufficiently large regular cardinal $\theta$ with $P \in N$. We say that $p \in P$ is $(N, P)$-generic if and only if $p$ is $(N, P)$-generic and decides all dense subsets of $P$ lying in $N$.

Unlike $\varepsilon$-properness, total properness is not preserved by countable support iteration. It was discussed by Shelah in [9].

The following lemma was proved by the author in [6].

Lemma 2.9. Let $\vec{C} = \langle C_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$ be a simple weakly tight tail club guessing sequence on $\omega_1$ and $X \in \text{TCG}(\vec{C})^+$. Then $P(\vec{C}, X)$ is totally proper.

If $\vec{C}$ has order type $\varepsilon$ for some indecomposable ordinal $\varepsilon$, then $P(\vec{C}, X)$ is $<\varepsilon$-proper.

The following notion is exactly the same as what was called an outside club guessing sequence by Džamonja and Shelah in [2].

Definition 2.10. Let $W$ be an inner model of $V$ and $\kappa$ an uncountable regular cardinal in $W$. Then, we say that a subset $C$ of $\kappa$ is a fast club subset of $\kappa$ over $W$ if and only if for every club subset $D$ of $\kappa$ lying in $W$, $C \subseteq \ast D$.

We also use the iteration in the sense of Donder and Fuchs in [1]. They use the system of projections instead of regular embeddings and only consider a sequence of complete Boolean algebras, but our definition is essentially the same in this situation.

Definition 2.11. Let $\langle P_\alpha : \alpha < \eta \rangle$ be a sequence of forcing notions. We say that $\langle P_\alpha : \alpha < \eta \rangle$ is an iteration if there exists a system of functions $\langle \sigma_{\beta, \alpha} : \beta < \alpha < \eta \rangle$ such that for every $\beta < \alpha < \eta$, $\sigma_{\beta, \alpha} : P_\beta \to P_\alpha$ is a regular embedding and for every $\gamma < \beta < \alpha < \eta$, $\sigma_{\gamma, \alpha} = \sigma_{\beta, \alpha} \circ \sigma_{\gamma, \beta}$.

By arranging the representation of $P_\alpha$'s, we may assume that $\sigma_{\beta, \alpha}$ is an inclusion map for every $\beta < \alpha < \eta$. 

Definition 2.12. Let $\langle P_\alpha : \alpha < \eta \rangle$ be an iteration witnessed by $\langle \sigma_{\beta, \alpha} : \beta < \alpha < \eta \rangle$. The direct limit of $\langle P_\alpha : \alpha < \eta \rangle$ is the forcing notion $P = \bigcup_{\alpha < \eta} P_\alpha$. When $\beta < \alpha$, $p \in P_\beta$ and $q \in P_\alpha$, we define $p \leq_P q$ if and only if $\sigma_{\beta, \alpha}(p) \leq_{P_\alpha} q$.

Then, if $G \subseteq P$ is generic, then for every $\alpha < \eta$, $G \cap P_\alpha$ is generic. Notice that if $\langle P_\alpha, \bar{Q}_\beta : \beta < \alpha < \eta \rangle$ is an iteration in Shelah’s sense, then $\langle P_\alpha \pi : \alpha < \eta \rangle$ is an iteration in this sense, and the direct limit in both senses coincide.

By the standard argument to find an inner model of a measurable cardinal from a precipitous ideal, we can prove the following lemma.

Lemma 2.13. Let $\kappa$ be a measurable cardinal, $U$ a normal measure on $\kappa$, and $j : V \rightarrow M$ the induced elementary embedding. Suppose $V = L[U]$. Let $P$ be a forcing notion and $G \subseteq P$ generic. Suppose that in $V[G]$, $I$ is a normal precipitous ideal on $\kappa$. Let $U_I \subseteq P(\kappa)/I$ be generic over $V[G]$ and $j_I : V[G] \rightarrow N$. Then, $U_I \cap V = U$ and $j_I \upharpoonright V = j$.

3. A precipitous tail club guessing ideal from a measurable cardinal

This section is devoted to the proof of the following theorem.

Theorem 3.1. Let $\kappa$ be a measurable cardinal and $\varepsilon < \kappa$ an indecomposable ordinal. Then, there is a forcing extension in which

(i) there exists a tail club guessing sequence $\overrightarrow{C}$ of order type $\varepsilon$ such that $\text{TG}^*(\overrightarrow{C})$ is precipitous,

(ii) no restriction of $\text{NS}_{\omega_1}$ to any stationary subset of $\omega_1$ is precipitous, and

(iii) for every tail club guessing sequence $\overrightarrow{C'}$ of order type $\varepsilon'$, $\text{TG}^*(\overrightarrow{C'})$ is not precipitous.

First let $\kappa$ be a measurable cardinal and $\varepsilon < \kappa$ an indecomposable ordinal. We shall construct a forcing extension in which there exists a precipitous tail club guessing ideal on $\omega_1$. The construction is eventually used to witness the theorem.

Let $U$ be a normal measure on $\kappa$, and $j : V \rightarrow M$ the elementary embedding induced by $U$. Let $P = \text{Coll}(\omega_1, <\kappa)$. Let $G \subseteq P$ be generic over $V$ and $G \subseteq j(P)$ generic over $M$ extending $G$. It is well known that $j$ can be extended to $j_0 : V[G] \rightarrow M[G]$. Work in $V[G]$. Define an ideal $I_0$ on $\omega_1$ by: $X \in I_0$ if and only if $1_{j_0(P)/G} \Vdash \langle \kappa \notin j_0(X) \rangle$. It is also well known that if we define $\pi_0 : P(\kappa)/I_0 \rightarrow B(j(P)/G)$ by $\pi_0(X) = \{ s \in j_0(X) \}$, then $\pi_0$ is a dense embedding and hence $j_0(P)/G \cong P(\omega_1)/I_0$. Note $(M[G])^{\aleph_1} \cap V[G] \subseteq M[G]$.

In $V[G]$, we shall define a countable support iteration $\langle Q_\alpha, \bar{R}_\beta : \beta < \alpha \leq \omega_2 \rangle$ by induction so that for every $\alpha < \omega_2$, $Q_\alpha$ forces that $|\bar{R}_\alpha| = \aleph_1$ and $\bar{R}_\alpha$ is $\varepsilon$-proper and totally proper. By a standard argument, we can show that $Q_\alpha$ has a dense subset of size $\aleph_1$. By Lemma 2.7, it follows that for every $\beta < \alpha < \omega_2$ and generic filter $H_\beta \subseteq Q_\beta$ over $V[G]$, $Q_\alpha/H_\beta$ is $\varepsilon$-proper. In addition, we will show that $Q_\alpha$ is totally proper. We shall also define a $Q_\alpha$-name $\bar{I}_\alpha$ for an ideal on $\omega_1$. Let $\langle X_\alpha : \alpha < \omega_2 \rangle$ be a bookkeeping of all subsets of $\omega_1$ in the extension of $V[G]$ by $Q_{\alpha+}$ so that every subset appears unboundedly many times. Since CH holds in $V[G]$, we can pick a bijection $\tau : \omega_1 \rightarrow [\omega_1]^{<\aleph_0}$. Since $Q_\alpha$ is totally proper for every $\alpha < \omega_2$, $\tau$ remains a bijection from $\omega_1$ onto $[\omega_1]^{<\aleph_0}$ in the extension of $V[G]$ by $Q_\alpha$.

At the zero-th stage, let $\bar{R}_0$ be the set of all functions $r$ such that $\text{dom}(r) = \delta \cap \text{Lim}$ for some ordinal $\delta < \omega_1$ and for every $\gamma \in \delta \cap \text{Lim}$, $r(\gamma)$ is an unbounded subset.
of γ, if γ ≥ ε, then \(\text{otp}(r(γ)) = ε\), and for every \(ξ \in \text{nacc}(r(γ))\), \(r(γ) \cap ξ \in τ'ξ\). \(R_0\) is ordered by extension. If \(H_0 \subseteq R_0\) is generic over \(V[G]\), for every \(δ \in ω_1 \cap \text{Lim}\), let \(C_δ = r(δ)\) for some (all) \(r \in H_0\) with \(δ \in \text{dom}(r)\). Define \(\tilde{C}\) \(=\) \(C_δ : δ \in ω_1 \cap \text{Lim}\). It is easy to see that \(\tilde{C}\) is a simple weakly tight tail club guessing sequence on \(ω_1\) in \(V[G][H_0]\). It automatically defines \(Q_1\).

Suppose that we have defined \(Q_α\) for some \(α \in [1, κ^+]\). Let \(H_α \subseteq Q_α\) be generic over \(V[G]\). Since \(Q_α\) has a dense subset of size \(n_1\) and \((M[G])^{ω_1} \cap V[G] \subseteq M[G]\), every \(Q_α\)-name for a subset of \(n_1\) has an equivalent name lying in \(M[G]\).

Work in \(V[G][H_α]\). Define \(I_α\) by: whenever \(\tilde{G} \subseteq j(P)\) is generic over \(M_α\) extending \(G\) with \(H_α \subseteq M[G]\) and \(\tilde{H}_α\) is generic over \(M[\tilde{G}]\) so that \(j_0^{-1} \tilde{H}_α = H_α\) and \(j_0(C)_κ\) is a fast club subset of \(κ\) over \(V[G][H_α]\), we have \(κ \not\in j_α(X)\). Here, \(j_α : V[G][H_α] → M[\tilde{G}][H_α]\) is the elementary embedding defined by: for every \(x \in V[G][H_α]\), if \(\tilde{x}\) is a \(Q_α\)-name for \(x\) in \(V[G]\), then \(j_α(x) = j_0(\tilde{x})^{H_α}\). This is a highly meta-mathematical definition, but we can modify it into the expression inside \(V[G][H_α]\). For simplicity, we let \(C_κ\) denote \(j_0(C)_κ\). If \(X_α \not\in I_α\), then let \(R_α\) be the trivial forcing notion. If \(X_α \in I_α\), then let \(R_α\) be the standard forcing to shoot a \(\text{TCG}(\tilde{C})\)-measure one set through \(κ \setminus X_α\). This completes the definition of \(⟨Q_α, R_β : β < α \leq ω_2⟩\) and \(⟨I_α : α < ω_2⟩\).

Let \(Q = Q_{ω_2}\). We shall show that in \(V[G]\), \(Q\) forces that \(\text{TCG}(\tilde{C})\) is precipitous.

**Lemma 3.1.** Let \(α < κ^+\). Let \(\tilde{G} \subseteq j(P)\) be generic over \(M_α\) extending \(G\) such that there exists a generic filter \(H_α \subseteq Q_α\) over \(V[G]\) lying in \(M[\tilde{G}]\). Then, there exists an unbounded subset \(C \subseteq M[\tilde{G}]\) of \(κ\) such that \(\text{otp}(C) = ε\), \(C\) is \(τ\)-weakly tight, and \(C\) is a fast club subset of \(κ\) over \(V[G]\).

**Proof.** Note \(|\mathcal{P}(κ)^{V[G][H_α]}| = n_0\) in \(M[\tilde{G}]\). So, we can enumerate all club subsets of \(κ\) lying in \(V[G][H_α]\) as \(⟨E_n : n < ω⟩\). Now, it is easy to build a \(C\) witnessing the claim. □

**Lemma 3.2.** Let \(β < α < ω_2\). Let \(H_β \subseteq Q_β\) be generic. Suppose that \(\tilde{G} \subseteq j(P)\) is generic over \(M_β\) extending \(G\) with \(H_β \subseteq M[G]\) and \(C \subseteq M[\tilde{G}]\) is a \(τ\)-weakly tight fast club subset of \(κ\) over \(V[G][H_β]\) of order type \(ε\). Then, for every \(q \in Q_α\) with \(q \restriction β \subseteq H_β\), there exists a generic filter \(H_α \subseteq Q_α\) over \(V[G]\) such that \(H_α \subseteq M[G]\), \(H_β = H_α \cap Q_β\), \(q \in H_α\), and \(C\) is a fast club subset of \(κ\) over \(V[G][H_α]\).

**Proof.** In \(M[\tilde{G}]\), \(|\mathcal{P}(Q_α)^{V[G]}| = n_0\). So, we can pick an enumeration \(⟨D_n : n < ω⟩ \subseteq M[\tilde{G}]\) of all dense open subset of \(Q_α/H_β\) lying in \(V[G][H_β]\).

Work in \(M[\tilde{G}]\). Let \(θ\) be a sufficiently large regular cardinal. We shall construct a sequence \(⟨N_{n, ξ} : n < ω \text{ and } ξ < ω_1⟩\) so that for every \(n < ω\), \(⟨N_{n, ξ} : ξ < ω_1⟩ \subseteq M[G][H_β]\). In \(M[G][H_β]\), pick a tower \(⟨N_{0, ξ} : ξ < ω_1⟩\) of countable elementary submodels of \(H(θ)^M[G][H_β]\) with \(Q_α/H_β, q, D_0 \in N_{0,0}\). Suppose that we have defined \(⟨N_{n, ξ} : ξ < ω_1⟩\) in \(M[G][H_β]\), pick a tower \(⟨N_{n+1, ξ} : ξ < ω_1⟩\) of countable elementary submodels of \(H(θ)^M[G][H_β]\) such that \(D_{n+1} \subseteq N_{n+1,0}\) and for every \(ξ < ω_1\), \(N_{α, ξ} \subseteq N_{n+1, ξ}\). For each \(n < ω\), define \(E_n = {ξ : ω_1 \cap N_n = ξ}\). Then, \(E_n\) is a club subset of \(ω_1\) lying in \(M[G][H_β]\). By assumption, \(C \subseteq E_n\). Let \(ζ_n \in E_n\) be a successor ordinal so that \(C \setminus ζ_n \subseteq E_n\). Without loss of generality, we may assume that \(⟨ζ_n : n < ω⟩\) is increasing.

Keep working in \(M[\tilde{G}]\). We shall build a decreasing sequence \(⟨q_n : n < ω⟩\) in \(Q_α/H_β\) as follows. Let \(q_0 = q\). Suppose that we have defined \(q_n\) so that \(q_n \in N_{n, ζ_n}\).
Since $\text{otp}(C) = \varepsilon$, $C \cap [\zeta_n, \zeta_{n+1})$ has order type $< \varepsilon$. Since $Q_\alpha/H_\beta$ is $< \varepsilon$-proper, there exists a $q_{n+1} \leq q_n$ such that $q_{n+1} \in D_n \cap N_{n, \zeta_{n+1}}$ and for every $\gamma \in C \cap [\zeta_n, \zeta_{n+1})$, $q_{n+1}$ is $(N_{n, \gamma}, Q_\alpha/H_\beta)$-generic. Notice that $q_{n+1} \in N_{n, \zeta_{n+1}} \subseteq N_{n+1, \zeta_{n+1}}$. Define $H_{\beta, \alpha}$ be the filter generated by $\{q_n : n < \omega\}$. Let $H_\alpha$ be defined by $q' \in H_\alpha$ if and only if $q' \upharpoonright \beta \in H_\beta$ and $q' \upharpoonright [\beta, \alpha) \models M[G][H]_{\beta, 0} \in H_{\beta, \alpha}$. Then, it is easy to see that $H_\alpha$ satisfies the desired conditions. \hfill \Box

For each $\alpha < \kappa^+$, we say that $q \in Q_\alpha$ is a flat condition of height $\zeta$ if and only if for every $\beta \in [1, \alpha)$, $q \upharpoonright \beta$ decides $q(\beta)$ and either $q(\beta) = \emptyset$ or $\max(q(\beta)) = \zeta$. For each $\alpha < \kappa^+$ and $\zeta < \kappa$, let $D_{\alpha, \zeta}$ be the set of all flat conditions in $Q_\alpha$ of height $\geq \zeta$.

Lemma 3.3. The following hold for every $\alpha < \omega_2$.

(i) $Q_\alpha$ is totally proper.

(ii) For every $\zeta < \omega_1$, $D_{\alpha, \zeta}$ is dense in $Q_\alpha$.

(iii) Suppose that $G \subseteq j(P)$ is generic over $M$ extending $G$ and $H_\alpha \subseteq Q_\alpha$ is generic over $V[G]$. Let $C \in M[\hat{G}]$ be a fast club subset of $\kappa$ over $V[G][H_\alpha]$ of order type $\varepsilon$. In $M[\hat{G}]$, define $m'_\alpha \in j_0(Q_\alpha)$ by for every $\beta < j(\alpha)$, $$m'_\alpha(\beta) = \begin{cases} \overline{C} \cup \{(\kappa, C)\} & \text{if } \beta = 0 \\ \emptyset & \text{if } \beta \notin j'\alpha \\ \emptyset & \text{if } \beta = j(\beta) \text{ and } X_\beta \not\in I_\beta \\ D_\beta \cup \{\kappa\} & \text{if } \beta = j(\beta) \text{ and } X_\beta \in I_\beta \end{cases}$$ Then, for every $H_\alpha \subseteq j_0(Q_\alpha)$ that is generic over $M[\hat{G}]$ with $M[\hat{G}][H]_{\beta, 0} = C_\kappa = C'$, $j_0^{-1}H_\alpha = H_\alpha$ if and only if $m'_\alpha \in H_\alpha$.

(iv) $Q_\alpha$ forces that $\check{I}_\alpha$ is non-trivial.

Proof. Go by induction on $\alpha$. Suppose that (i), (ii), (iii) and (iv) hold for every $\beta < \alpha$.

We already know that $Q_\alpha$ is proper. First, we shall show (i) and (ii). Let $q \in Q_\alpha$, $\zeta < \omega_1$, and $\check{x}$ be a $Q_\alpha$-name for a subset of $\omega$. Without loss of generality, we may assume that $\check{x} \in M$. Let $G \subseteq j(P)$ be generic over $M$ extending $G$. Work in $M[\hat{G}]$. Since $|P(Q_\alpha)|^{V[\hat{G}]} = \aleph_0$, we can find an enumeration $\langle E_n : n < \omega \rangle$ of all open dense subsets of $Q_\alpha$ lying in $V[\hat{G}]$. Also pick an increasing cofinal sequence $\langle \alpha_n : n < \omega \rangle$ in $\alpha$ and an increasing cofinal sequence $\langle \kappa_n : n < \omega \rangle$ in $\kappa$. It is easy to build a decreasing sequence $\langle q_n : n < \omega \rangle$ so that $q_0 = q$, $q_{n+1} \in E_n$, $q_{n+1} \upharpoonright \alpha_n \in D_{\alpha_n, \kappa_n}$, and $q_{n+1}$ decides $n \in \check{x}$.

Let $H_\alpha \subseteq Q_\alpha$ be the filter generated by $\{q_n : n < \omega\}$. Then, clearly $H_\alpha$ is generic over $V[G]$. Let $\check{C} = (C_\delta : \delta \in \omega_1 \cap \text{Lim})$ be the sequence added at the zero-th stage and $D_\beta$ the club subset of $\omega_1$ added at the $\beta$-th stage. Let $\check{x} = \{n < \omega : q_{n+1} \upharpoonright \check{n} \in \check{x}\}$. Let $C \in M[\hat{G}]$ be a $\tau$-weakly tight fast club subset of $\kappa$ over $V[G][H_\alpha]$ of order type $\varepsilon$. Define $m'_\alpha$ as in (iii).

Claim 3.3.1. $m'_\alpha \in j_0(Q_\alpha)$.

Proof. We have $\text{supp}(m'_\alpha) \subseteq j''\alpha$, which is countable in $M[\hat{G}]$. Thus, it suffices to show that $m'_\alpha \upharpoonright \beta \models \langle m'_\alpha(\beta) \in j_0(\hat{R}) \rangle$ for every $\beta < j(\alpha)$. If $\beta \not\in j''\alpha$, then this is trivial. Suppose $\beta < \alpha$ and show that $m'_\alpha \upharpoonright j(\beta) \models \langle m'_\alpha(j(\beta)) \in j_0(\hat{R}) \rangle$'. If either $\beta = 0$ or $X_\beta \not\in I_\beta$, this is again trivial. Assume that $\beta > 0$ and $X_\beta \in I_\beta$. Let
Lemma 3.5. \( \hat{H}_\beta \subseteq j_0(Q_\beta) \) be generic over \( M[\hat{G}] \) with \( m'_\alpha \upharpoonright j(\beta) \in \hat{H}_\beta \). Then by (iii) applied to \( \beta \), we have \( j_0^{-1}\hat{H}_\beta = H_\alpha \cap Q_\beta \). By the definition of \( I_\beta \), we have \( \kappa \not\in j_3(X_\beta) \). We also have \( j_3(X_\beta) \cap \kappa = X_\beta \). Therefore, \( m'_\alpha(j(\beta)) = D_\beta \cup \{ \kappa \} \in j_3(R_\beta) \). Since this holds for arbitrary \( \hat{H}_\beta, m'_\alpha \upharpoonright j(\beta) \models 'm'_\alpha(j(\beta)) \in j_0(R_\beta)' \).

Claim 3.3.2. For every \( n < \omega, m'_\alpha \leq j_0(q_n) \). In particular, \( m'_\alpha \leq j_0(q) \)

**Proof.** Since \( \text{supp}(m'_\alpha) \subseteq j^n \alpha \), it suffices to show that for all \( k \in (n, \omega), m'_\alpha \upharpoonright j(\alpha_k) \leq j_0(q_n \upharpoonright \alpha_k) \). By definition, \( q_{k+1} \upharpoonright \alpha_k \in D_{\alpha_k, \kappa, \xi} \) in particular \( q_{k+1} \upharpoonright \alpha_k \) is a flat condition of height \( \xi' \) for some \( \xi' < \kappa \). Thus, \( j_0(q_{k+1} \upharpoonright \alpha_k) \) is a flat condition of height \( \xi' \) with \( \text{supp}(j_0(q_{k+1} \upharpoonright \alpha_k)) \subseteq j^n \alpha \). Now it is easy to see \( m'_\alpha \upharpoonright j(\alpha_k) \leq j_0(q_{k+1} \upharpoonright \alpha_k) \leq j_0(q_n \upharpoonright \alpha_k) \).

Claim 3.3.3. \( m'_\alpha \models 'j_0(x) = \bar{x}' \).

**Proof.** Let \( n < \omega \). Recall that \( q_{n+1} \) decides \( n \in \bar{x} \). If \( n \in \bar{x} \), then we have \( q_{n+1} \models 'n \in \bar{x}' \) and hence \( j_0(q_{n+1}) \models 'n \in j_0(\bar{x})' \). Hence, \( m'_\alpha \models 'n \in j_0(\bar{x})' \). By the same argument, if \( n \not\in \bar{x} \), we have \( m'_\alpha \models 'n \not\in j_0(\bar{x})' \).

Therefore, in \( M[\hat{G}] \), there exists an \( m'_\alpha \leq j_0(q) \) such that \( m'_\alpha \in j_0(D_{\alpha, \kappa, \xi}) \) and \( m'_\alpha \models 'j_0(x) = \bar{x}' \) for some \( x \in M[\hat{G}] \). Since \( j_0 : V[G] \rightarrow M[\hat{G}] \) is an elementary embedding, it shows that in \( V[G] \), there exists a \( q' \leq q \) such that \( q' \in D_{\alpha, \kappa, \xi} \) and \( q' \models 'x = \bar{x}' \) for some \( x \in V[G] \). Therefore, \( Q_\alpha \) adds no new reals and \( D_{\alpha, \kappa, \xi} \) is dense in \( Q_\alpha \).

Since the set of flat conditions is dense in \( Q_\alpha \), (iii) can be easily seen.

To see (iv), suppose that for some \( q \in Q_\alpha, q \models ' \kappa \in I_\alpha' \). Let \( \hat{G} \subseteq j(P) \) be generic over \( M \) extending \( G \). In \( M[\hat{G}] \), we can find an \( H_\alpha \subseteq Q_\alpha \) generic over \( V[G] \) with \( q \not\in H_\alpha \). Let \( C \in M[\hat{G}] \) be a \( \tau \)-weakly tight fast club subset of \( \kappa \) over \( V[G][H_\alpha] \) of order type \( \varepsilon \). Let \( C'_\alpha \subseteq j_0(Q_\alpha) \) be generic over \( M[\hat{G}] \) with \( m'_\alpha \in \hat{H}_\alpha \). Since \( q \models ' \kappa \in I_\alpha' \), we have \( \kappa \not\in j_0(\kappa) \). This is a contradiction.

**Lemma 3.4.** \( \alpha < \omega_2 \) and \( H_\alpha \subseteq Q_\alpha \) be generic over \( V[G] \). Then, in \( V[G][H_\alpha], \text{TCG}(\bar{C}) \subseteq I_\alpha \), and hence \( \bar{C} \) is a tail club guessing sequence on \( \omega_1 \).

**Proof.** Work in \( V[G][H_\alpha] \). Suppose that there exists a club subset \( D \) of \( \kappa \) such that \( X := \{ \delta \in \omega_1 \cap \text{Lim} : C_\delta \not\subseteq D \} \not\in I_\alpha \). By the definition of \( I_\alpha \), there exist a generic filter \( \hat{G} \subseteq j(P) \) over \( M \) extending \( G \) with \( H_\alpha \subseteq \hat{G} \) and a generic filter \( H_\alpha \subseteq j(Q_\alpha) \) over \( M[\hat{G}] \) such that \( j_0^{-1}H_\alpha = H_\alpha, C_\alpha \) is a fast club over \( V[G][H_\alpha] \), and \( \kappa \in j_0(\alpha, X) \). Since \( \kappa \in j_0(\alpha, X) \), we have \( C_\alpha \not\subseteq j_0(D, \alpha) \), which implies \( C_\alpha \not\subseteq j_0(D) \). Since \( C_\alpha \) is a fast club over \( V[G][H_\alpha] \), this is a contradiction.

Work in \( M[\hat{G}] \) where \( \hat{G} \subseteq j(P) \) is generic over \( M \) extending \( G \). For every \( \alpha < \omega_2 \), let \( m_\alpha \in j_0(Q_\alpha) \) be the truth value of \( j_0^{-1}H_\alpha \) is generic over \( V[G] \) and \( \hat{C}_\alpha \) is a fast club over \( V[G][H_\alpha]' \). Let \( \hat{m}_\alpha \) be a \( j(P)/G \)-name for \( m_\alpha \).

Suppose that \( \hat{H}_\alpha \subseteq j_0(Q_\alpha) \) is generic over \( M[\hat{G}] \) with \( m_\alpha \in \hat{H}_\alpha \). Let \( H_\alpha = j_0^{-1}H_\alpha \). Since \( m_\alpha \in H_\alpha, H_\alpha \) is generic over \( V[G] \). So, we can define \( j_0(\alpha) \) as above.

**Lemma 3.5.** In \( M[G], \) for every \( \beta < \alpha < \kappa^+ \), \( j(P)/G \) forces that \( m_\beta \models m_\alpha \upharpoonright j(\beta) \).

**Proof.** Let \( \hat{G} \subseteq j(P) \) be generic over \( M \) extending \( G \) and work in \( M[\hat{G}] \).

First we shall show that \( m_\alpha \upharpoonright \beta \leq m_\beta \). Let \( \hat{H}_\alpha \subseteq j_0(Q_\alpha) \) be generic over \( M[\hat{G}] \) with \( m_\alpha \in \hat{H}_\alpha \). Define \( H_\alpha = j_0^{-1}H_\alpha \). By the definition of \( m_\alpha, H_\alpha \) is generic over
\[ V[G], \hat{H}_\beta = \hat{H}_\alpha \cap j_0(Q_\beta) \] is a generic filter of \( j_0(Q_\beta) \) over \( M[G] \). Let \( H_\beta = j_0^{-1}\hat{H}_\beta \). Then, it is easy to see \( H_\beta = H_\alpha \cap Q_\beta \). So, \( H_\beta \subseteq Q_\beta \) is generic over \( V[G] \). Since \( C_\kappa \) is a fast club over \( V[G][H_\alpha] \), it is a fast club over \( V[G][H_\beta] \), too. By definition, we have \( m_\beta \in \hat{H}_\beta \). Since this holds for arbitrary generic \( \hat{H}_\alpha \subseteq j_0(Q_\alpha) \) over \( M[G] \) with \( m_\alpha \in \hat{H}_\alpha \), we have \( m_\alpha \upharpoonright \beta \leq m_\beta \).

Then we shall show that \( m_\beta \leq m_\alpha \upharpoonright \beta \). Let \( \hat{H}_\beta \subseteq j_0(Q_\beta) \) be generic over \( M[G] \) with \( m_\beta \in \hat{H}_\beta \) and define \( H_\beta = j_0^{-1}\hat{H}_\beta \). By Lemma 3.2, there exists a generic filter \( H_\alpha \subseteq Q_\alpha \) over \( V[G] \) such that \( H_\alpha \in M[G] \), \( H_\alpha = H_\alpha \cap Q_\alpha \), and \( C_\kappa \) is a fast club over \( V[G][H_\alpha] \). Let \( m'_\alpha \in j_0(Q_\alpha) \) be defined as in (ii) of Lemma 3.3 where \( C_\kappa \) is used for the fast club. Then, we have \( m'_\alpha \upharpoonright j(\beta) \in \hat{H}_\beta \). Let \( \hat{H}_\alpha \subseteq j_0(Q_\alpha) \) be generic over \( M[G] \) extending \( \hat{H}_\beta \) with \( m'_\alpha \in \hat{H}_\alpha \). Then, it is easy to see \( m_\alpha \in \hat{H}_\alpha \). Hence, we get \( m_\beta \leq m_\alpha \upharpoonright \beta \).

Suppose that \( \hat{G} \subseteq j(P) \) is generic over \( M \) extending \( G \) and \( \hat{H}_\alpha \subseteq j_0(Q_\alpha) \) is generic over \( M[G] \) with \( m_\alpha \in \hat{H}_\alpha \). For every \( \beta < \alpha \), define \( H_\beta = H_\alpha \cap j_0(Q_\beta) \) and \( H_\beta = j_0^{-1}\hat{H}_\beta \). Since \( m_\beta = m_\alpha \upharpoonright j(\beta) \in \hat{H}_\beta \), we can define \( j_\beta : V[G][H_\beta] \to M[G][H_\beta] \) as for \( j_\alpha \). It is easy to see that \( j_\beta \) is an elementary embedding and \( j_\beta = j_\alpha \upharpoonright V[G][H_\beta] \).

The following lemma is a consequence of the Duality Theorem proved by Foreman in [3]. See [4] for information about the Duality Theorem.

**Lemma 3.6.** In \( V[G] \), for every \( \alpha < \kappa^+ \), \( Q_\alpha \models (P(\omega_1)/I_\alpha) \cong (j(P)/G) \upharpoonright (j_0(Q_\alpha)/\hat{m}_\alpha) \).

**Proof.** We shall define a function \( \pi_\alpha : Q_\alpha \models (P(\omega_1)/I_\alpha) \to B((j(P)/G) \upharpoonright (j_0(Q_\alpha)/\hat{m}_\alpha)) \).

For every \( (q, X) \in Q_\alpha \models (P(\omega_1)/I_\alpha) \), let \( \pi_\alpha(q, X) \) be the truth value of the following statement taken in \( M[G] \): there exists an \( H_\alpha \subseteq Q_\alpha \) such that \( H_\alpha \in M[\hat{G}] \), \( H_\alpha \) is generic over \( V[G] \), \( q \in H_\alpha \), \( \hat{C}_\kappa \) is a fast club over \( V[G][H_\alpha] \), \( \hat{j}_0^{-1}\hat{H}_\alpha = H_\alpha \), and \( \kappa \in j_\alpha(X^{H_\alpha}) \).

We shall show that \( \pi_\alpha \) is a dense embedding. It is easy to see that \( \pi_\alpha \) is well-defined and order-preserving. To see that \( \pi_\alpha \) preserves incomparability, let \( (q, X) \) and \( (q', X') \) be incomparable elements in \( Q_\alpha \models (P(\omega_1)/I_\alpha) \). It follows that whenever \( q'' \) is a common extension of \( q \) and \( q' \), \( q'' \upharpoonright X \cap X' \in I_\alpha \). Suppose that \( \pi_\alpha((q, X)) \) and \( \pi_\alpha((q', X')) \) are compatible in \( j(P \uplus Q_\alpha) \). Let \( \hat{G} \uplus H_\alpha \) be a generic filter of \( j(P \uplus Q_\alpha) \) with \( \pi_\alpha((q, X)) \) and \( \pi_\alpha((q', X')) \) in \( G \uplus H_\alpha \). Let \( H_\alpha = j_0^{-1}H_\alpha \). Then, both \( q \) and \( q' \) belong to \( H_\alpha \), and hence we have \( X \cap X' \in I_\alpha \). But we also have \( \kappa \in j_\alpha(X \cap X') = j_\alpha(X \cap X') \). This contradicts \( X \cap X' \in I_\alpha \).

We shall show that the image of \( \pi_\alpha \) is dense in \( B((j(P)/G) \upharpoonright (j_0(Q_\alpha)/\hat{m}_\alpha)) \).

Let \( (p, \bar{r}) \in (j(P)/G) \upharpoonright (j_0(Q_\alpha)/\hat{m}_\alpha) \). Since \( P(\omega_1)/I_0 \) is densely embedded into \( B(j(P)/G) \) by the mapping \( X \mapsto [\kappa \in j_0(X)] \), we can find an \( X \in P(\omega_1)/I_0 \) such that \( [\kappa \in j_0(X)] \leq p \). Notice that \( \bar{r} \) is represented by a function \( f_\beta \) with domain \( \omega_1 \) such that for every \( \xi < \omega_1 \), \( f_\beta(\xi) \) is a \( \mathcal{P}_1 \)-name for an element of \( Q_\alpha \). Define a function \( f_{\beta'} : \omega_1 \to Q_{\beta'} \) by \( f_{\beta'}(\xi) = (f_\beta(\xi))^{G} \). Define a \( Q_{\beta'} \)-name \( Y \) for a subset of \( \kappa \) so that for every \( \xi \in X \), \( [\xi \in Y] = f_{\beta'}(\xi) \) and for every \( \xi \in \kappa \setminus X \), \( 1_{Q_{\beta'}} \models [\xi \notin Y] \).

We claim that \( [\bar{Y} \notin I_\alpha] \neq 0 \). Let \( \hat{G} \subseteq j(P) \) be generic over \( M \) extending \( G \) with \( M[\hat{G}] \models \forall \kappa \in j_0(X) \) and \( H_\alpha \subseteq j_0(Q_\alpha) \) generic over \( M[\hat{G}] \) with \( m_\alpha, r \in \hat{H}_\alpha \).

Let \( \hat{H}_\alpha = j_0^{-1}\hat{H}_\alpha \) and \( \hat{H}_\alpha \subseteq j_0(Q_\alpha) \) generic over \( M[\hat{G}] \) with \( m_\alpha, r \in \hat{H}_\alpha \). By the definition of \( m_\alpha \), \( H_\alpha \subseteq Q_\alpha \) is generic over \( V[G] \) and \( C_\kappa \) is a fast club over \( V[G][H_\alpha] \). Since \( V[G] \models \forall \xi \in X(f_\beta(\xi) \models [\xi \notin Y]) \), we have
Lemma 3.7. Let $\beta < \alpha < \kappa^+$. Suppose that $\mathcal{H}_\alpha \subseteq \mathcal{Q}_\alpha$ is generic over $V[\mathcal{G}]$. Then, in $V[\mathcal{G}][\mathcal{H}_\alpha]$, $I_\beta = I_\alpha \cap V[\mathcal{G}][\mathcal{H}_\alpha \cap \mathcal{Q}_\beta]$.

Proof. Let $X \in I_\beta$. To show that $X \in I_\alpha$, let $\mathcal{G} \subseteq j(P)$ be generic over $M$ extending $\mathcal{G}$ with $\mathcal{H}_\alpha \subseteq V[\mathcal{G}]$ and $\mathcal{H}_\alpha \subseteq j_0(\mathcal{Q}_\beta)$ generic over $V[\mathcal{G}]$ such that $j_0^{-1}\mathcal{H}_\beta = \mathcal{H}_\alpha$ and $\mathcal{C}_\kappa$ is a fast club over $V[\mathcal{G}][\mathcal{H}_\alpha]$. Since $X \in I_\beta$, we have $\kappa \notin j_0(X)$. But we have $j_3 = j_0 \upharpoonright V[\mathcal{G}][\mathcal{H}_\alpha \cap \mathcal{Q}_\beta]$. So, $\kappa \notin j_0(X)$ and hence $X \in I_\alpha$.

To see the converse, suppose that for some $q \in \mathcal{Q}_\alpha$ and $\mathcal{Q}_\beta$-name $\dot{X}$ for a subset of $\omega_1$, $q \Vdash \dot{X} \in I_\alpha \setminus I_\beta^\prime$. Let $\mathcal{H}_\beta \subseteq \mathcal{Q}_\beta$ be generic with $q \Vdash \beta \in \mathcal{H}_\beta$. Then, we have $X \notin I_\beta$. By definition, there exist a generic filter $\mathcal{G} \subseteq j(P)$ over $M$ extending $\mathcal{G}$ with $\mathcal{H}_\beta \subseteq V[\mathcal{G}]$ and a generic filter $\dot{\mathcal{H}}_\beta \subseteq j_0(\mathcal{Q}_\beta)$ over $V[\mathcal{G}]$ such that $j_0^{-1}\dot{\mathcal{H}}_\beta = \mathcal{H}_\alpha$ and $\mathcal{C}_\kappa$ is a fast club over $V[\mathcal{G}][\mathcal{H}_\alpha]$, and $\kappa \in j_0(X)$. By Lemma 3.2, there exists a generic filter $\mathcal{H}_\alpha \subseteq \mathcal{Q}_\alpha$ over $V[\mathcal{G}]$ such that $\mathcal{H}_\alpha \supseteq \mathcal{H}_\beta \cap \mathcal{Q}_\beta$, $q \in \mathcal{H}_\alpha$, and $\mathcal{C}_\kappa$ is a fast club over $V[\mathcal{G}][\mathcal{H}_\alpha]$. Let $\dot{m}_\alpha \in j_0(\mathcal{Q}_\alpha)$ be defined as in (iii) of Lemma 3.3 where $C = \mathcal{C}_\kappa$. Then, $\dot{m}_\alpha \upharpoonright j(\beta) \in \mathcal{H}_\beta$. Let $\dot{\mathcal{H}}_\alpha \subseteq j_0(\mathcal{Q}_\alpha)$ be generic over $M[\mathcal{G}]$ extending $\dot{\mathcal{H}}_\beta$ with $\dot{m}_\alpha \in \dot{\mathcal{H}}_\alpha$. Then, $j_0^{-1}\mathcal{H}_\alpha = \mathcal{H}_\alpha$. Since $q \in \mathcal{H}_\alpha$, we have $X \in I_\alpha$. Therefore, $\kappa \notin j_0(X)$. This is a contradiction since $j_3 = j_0 \upharpoonright V[\mathcal{G}][\mathcal{H}_\beta]$.

Lemma 3.8. In $V[\mathcal{G}]$, for every $\beta < \alpha < \omega_2$, $\mathcal{Q}_\beta \ast (\mathcal{P}(\omega_1)/I_\beta)$ is regularly embedded into $\mathcal{Q}_\alpha \ast (\mathcal{P}(\omega_1)/I_\alpha)$ by the identity mapping.

Proof. This holds since the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{Q}_\beta \ast (\mathcal{P}(\omega_1)/I_\beta) & \overset{\pi_\beta}{\longrightarrow} & (j(P)/G) \ast (j_0(\mathcal{Q}_\beta)/\dot{m}_\beta) \\
\downarrow \text{id} & & \downarrow \text{id} \\
\mathcal{Q}_\alpha \ast (\mathcal{P}(\omega_1)/I_\alpha) & \overset{\pi_\alpha}{\longrightarrow} & (j(P)/G) \ast (j_0(\mathcal{Q}_\alpha)/\dot{m}_\alpha)
\end{array}
\]

Let $\dot{I}$ be a $\mathcal{Q}$-name for $\bigcup_{\beta < \alpha^+} I_\beta$. The following lemma is an easy consequence of Lemma 3.7.

Lemma 3.9. In $V[\mathcal{G}]$, $\mathcal{Q} \ast (\mathcal{P}(\omega_1)/\dot{I})$ is the direct limit of $(\mathcal{Q}_\alpha \ast (\mathcal{P}(\omega_1)/I_\alpha) : \alpha < \omega_2)$.

Lemma 3.10. In $V[\mathcal{G}]$, for every $\alpha < \kappa^+$, let $\mathcal{H}_\alpha \subseteq \mathcal{Q}_\alpha$ be generic over $V[\mathcal{G}]$. Then, in $V[\mathcal{G}][\mathcal{H}_\alpha]$, $I_\alpha$ is precipitous. Moreover, suppose that $\mathcal{U}_\alpha \subseteq \mathcal{P}(\omega_1)/I_\alpha$ is generic over $V[\mathcal{G}][\mathcal{H}_\alpha]$ and $k_\alpha : V[\mathcal{G}][\mathcal{H}_\alpha] \rightarrow N \subseteq V[\mathcal{G}][\mathcal{H}_\alpha][\mathcal{U}_\alpha]$ is the induced
generic elementary embedding. Let $\hat{G} \ast \hat{H}_\alpha = \pi_\alpha (H_\alpha \ast U_\alpha)$. Let $j_\alpha = \hat{g}^{\hat{G} \ast \hat{H}_\alpha}$. Then, $N = M[\hat{G}][\hat{H}_\alpha]$ and $k_\alpha = j_\alpha$.

Proof. First, we claim that for every $X \in U_\alpha$, we have $\kappa \in j_\alpha(X)$. Let $X$ be a $Q_\alpha$-name for $X$. Then, there exists a $q \in H_\alpha$ such that $q \Vdash \exists q \bar{X} \notin \hat{I}_\alpha$. Then, we have $\langle q, \bar{X} \rangle \in H_\alpha \ast U_\alpha$. So, $\pi_\alpha (q, \bar{X}) \in \hat{G} \ast \hat{H}_\alpha$. By the definition of $\pi_\alpha$, it follows that $\kappa \in j_\alpha(X)$.

Let $U_0 = U_\alpha \cap V[G]$. By a standard argument, we can show that $(V[G])^\kappa / U_0 \simeq M[\hat{G}]$ and $j_\alpha$ coincides with the generic elementary embedding induced by $U_0$.

Define a function $\sigma_\alpha : (V[G][H_\alpha])^\kappa / U_0 \to M[\hat{G}][\hat{H}_\alpha]$ as follows. Let $f : \kappa \to V[G][H_\alpha]$ be a function lying in $V[G][H_\alpha]$. Let $\sigma_\alpha (f|U_0) = j_\alpha (f) (\kappa)$. First, we shall show that $\sigma_\alpha$ is well defined. Suppose that $[f]_{U_0} = [g]_{U_0}$. Let $X = \{ \xi : f(\xi) = g(\xi) \}$. Then, since $X \in U_\alpha$, we have $\kappa \in j_\alpha (X)$. So, $j_\alpha (f)(\kappa) = j_\alpha (g)(\kappa)$ and hence $\sigma_\alpha (f|U_0) = \sigma_\alpha (g|U_0)$. By a similar argument, we can show that $[f]_{U_0} \in j_\alpha ([g]_{U_0})$ if and only if $\sigma_\alpha (f|U_0) \in \sigma_\alpha ([g]_{U_0})$.

To see that $\sigma_\alpha$ is onto, let $x \in M[\hat{G}][\hat{H}_\alpha]$. Then, there exists a $j(Q_\alpha)$-name $\dot{x} \in M[\hat{G}]$ such that $x = \dot{x}^{\hat{H}_\alpha}$. Since $(V[G])^\kappa / U_0 \simeq M[\hat{G}]$, there exists a function $f : \kappa \to V[G]$ lying in $V[G]$ such that for every $\xi < \omega_1$, $f(\xi)$ is a $Q_\alpha$-name and $j_\alpha (f)(\kappa) = \dot{x}$. Define a function $g : \kappa \to V[G][H_\alpha]$ by $g(\xi) = f(\xi)^{\hat{H}_\alpha}$. Then, $\sigma_\alpha (g|U_0) = j_\alpha (g)(\kappa) = j_\alpha (f)(\kappa)^{\hat{H}_\alpha} = \dot{x}^{\hat{H}_\alpha} = x$.

Therefore, $\sigma_\alpha : (V[G][H_\alpha])^\kappa / U_0 \to M[\hat{G}][\hat{H}_\alpha]$ is an isomorphism. Since $M[\hat{G}][\hat{H}_\alpha]$ is well-founded, so is $(V[G][H_\alpha])^\kappa / U_0$. The proof also shows that the generic elementary embedding induced by $U_\alpha$ coincides with $j_\alpha$.

Let $H \subseteq Q$ be generic over $V[G]$. Work in $V[G][H]$. It is easy to see that $I = TCG(C)$ is precipitous. Suppose not. Then for some generic filter $\bar{U} \subseteq \mathcal{P}(\omega_1)/I$ over $V[G][H]$, there exists a sequence $\langle f_n : n < \omega \rangle$ in $V[G][H][\bar{U}]$ such that for every $n < \omega$, $f_n$ is a function from $\alpha$ into ordinals lying in $V[G][H]$ and $\{ \xi : f_{n+1}(\xi) < f_n(\xi) \} \in \bar{U}$.

For each $\alpha < \kappa^+$, let $H_\alpha = H \cap Q_\alpha$ and $U_\alpha = \bar{U} \cap V[G][H_\alpha]$. Since $Q_\alpha \ast (\mathcal{P}(\omega_1)/I_\alpha)$ is regularly embedded into $Q \ast (\mathcal{P}(\omega_1)/I)$ by the identity mapping, $H_\alpha \ast U_\alpha$ is a generic filter of $Q_\alpha \ast (\mathcal{P}(\kappa)/I_\alpha)$ over $V[G]$. By Lemma 3.10, in $V[G][H_\alpha]$, $I_\alpha$ is precipitous. Moreover, if we let $\hat{G} \ast \hat{H}_\alpha = \pi_{\alpha''}(H_\alpha \ast U_\alpha)$, then $(V[G][H_\alpha])^\kappa / U_\alpha \simeq M[\hat{G}][\hat{H}_\alpha]$, and the induced elementary embedding is equal to $j_\alpha$.

By a standard argument, for every $n < \omega$, there exists an $\alpha_n < \kappa^+$ such that $f_n \in V[G][H_{\alpha_n}]$. Let $\gamma_n = j_{\alpha_n}(f_n)(\kappa)$. We claim that $\gamma_{n+1} < \gamma_n$, which is a contradiction. Since $\{ \xi : f_{n+1}(\xi) < f_n(\xi) \} \in \bar{U} \cap V[G][H_{\alpha_n+1}] = U_{\alpha_n+1}$, we have $\gamma_{n+1} = j_{\alpha_n+1}(f_{n+1})(\kappa) < j_{\alpha_n}(f_n)(\kappa)$. But since $j_{\alpha_n} = j_{\alpha_n+1} \downarrow V[G][H_{\alpha_n}]$, we have $\gamma_n = j_{\alpha_n+1}(f_n)(\kappa)$. Therefore, $\gamma_{n+1} < \gamma_n$. Thus, we have proved the following theorem.

**Theorem 3.2.** Let $\kappa$ be a measurable cardinal and $\varepsilon$ an indecomposable ordinal. Then, there exists a forcing extension in which $\kappa = \aleph_1$ and there exists a tail club guessing sequence $\hat{C}$ on $\omega_1$ of order type $\varepsilon$ such that $TCG(C)$ is precipitous.

From now on, we assume $V = L[U]$ in addition. We shall show that any restriction of $NS_{\omega_1}$ or $TCG(C')$ for any tail club guessing sequence $C'$ of order type $< \varepsilon$ is not precipitous.
Work in $M$. For every $\alpha < \kappa^+$, define $\bar{m}_\alpha \in j(P \ast Q_\alpha)$ to be the truth value that $j^{-1}(\hat{G} \ast \hat{H}_\alpha)$ is a generic filter of $P \ast Q_\alpha$ over $M$. As in Lemma 3.5, we can show that for every $\beta < \alpha < \kappa^+$, $\bar{m}_\beta = \bar{m}_\alpha \cap j(P \ast Q_\beta)$.

For every $\alpha < \kappa^+$, define a function $\bar{\pi}_\alpha$ as follows. The domain of $\bar{\pi}_\alpha$ is the set of all triples $\langle p, \bar{q}, S \rangle$ such that $p \in P$, $\bar{q}$ is a $P$-name for an element of $Q_\alpha$, and $\hat{S}$ is a $P \ast Q_\alpha$-name for a subset of $\kappa$. Let $\langle p, \bar{q}, \hat{S} \rangle$ be in the domain of $\bar{\pi}_\alpha$. Define $\bar{\pi}_\alpha(p, \bar{q}, \hat{S}) = \langle (\langle p, \bar{q} \rangle) \wedge [\kappa \in j(\hat{S})] \in \mathcal{B}(j(P \ast Q_\alpha)) \rangle$. Note that $j(\hat{S})$ is a $j(P \ast Q_\alpha)$-name for a subset of $j(\kappa)$, so this is a valid definition.

For every $\alpha < \kappa^+$, define a $P \ast \hat{Q}_\alpha$-name $\hat{I}_\alpha$ for a subset of $\mathcal{P}(\kappa)$ so that for every $P \ast Q_\alpha$-name for a subset of $\kappa$, $\langle p, \bar{q} \rangle \Vdash 'S \in \hat{I}_\alpha'$ if and only if $\bar{\pi}_\alpha(p, \bar{q}, \hat{S})$ is incompatible with $\bar{m}_\alpha$ in $j(P \ast Q_\alpha)$. It is easy to verify that this indeed defines a $P \ast Q_\alpha$-name.

We can easily check that for every $\beta < \alpha < \kappa^+$, if $G \ast H_\alpha \subseteq P \ast Q_\alpha$ is generic over $V$ and $H_\beta = H_\alpha \cap Q_\beta$, then $I_\beta = \mathcal{P}(\kappa)^{V[G][H_\alpha]} \cap \hat{I}_\alpha$. Let $\hat{I}$ be a $P \ast \hat{Q}$-name so that $1_{P \ast \hat{Q}} \Vdash '\hat{I} = \bigcup_{\alpha < \kappa^+} \hat{I}_\alpha'$.

**Lemma 3.11.** Let $G \ast H \subseteq P \ast \hat{Q}$ be generic over $V$. Let $J$ be a precipitous ideal on $\kappa$ in $V[G][H]$. Then, $\hat{I} \subseteq J$.

**Proof.** We shall show that for every $S \in J^+$, we have $S \in \hat{I}^+$. In $V[G][H]$, let $U_J \subseteq \mathcal{P}(\kappa)/J$ be generic over $V[G][H]$ with $S \in U_J$ and $j_J : V[G][H] \rightarrow N_J$ the induced generic elementary embedding. By Lemma 2.13, we have $j_J \upharpoonright V = j$. Define $\hat{G} = j_J(G)$ and $\hat{H} = j_J(H)$. Then, $\hat{G}$ is a generic filter of $j(P)$ over $M$ and $\hat{H}$ is a generic filter of $j_j(Q)$ over $M[\hat{G}]$. For every $x \in V[G][H]$, we have $j_j(x) = j(x)^{\hat{G} \ast \hat{H}}$ where $\hat{x}$ is a $P \ast \hat{Q}$-name for $x$.

Let $\hat{S}$ be a $P \ast \hat{Q}$-name for $S$. There exists an $\alpha < \kappa^+$ such that $\hat{S}$ is a $P \ast \hat{Q}_\alpha$-name and for some $\langle p, \bar{q} \rangle \in G \ast (H \cap Q_\alpha)$, $\langle p, \bar{q} \rangle \Vdash '\hat{S} \notin \hat{J}'$. Let $H_\alpha = H \cap Q_\alpha$ and $\hat{H}_\alpha = j_J(H_\alpha)$. Note that $\kappa \in j_J(S)$. Since $j_J(S) = j_j(S)^{\hat{G} \ast \hat{H}_\alpha}$, there exists a $\langle p', \bar{q}' \rangle \in \hat{G} \ast \hat{H}_\alpha$ such that $\langle p', \bar{q}' \rangle \Vdash '\kappa \in j_j(S)'$. So, $\bar{\pi}_\alpha(p, \bar{q}, \hat{S}) \in \hat{G} \ast \hat{H}_\alpha$. Notice that $j_j^{-1}(\hat{G} \ast \hat{H}_\alpha) = G \ast H_\alpha$ is generic over $V$. Therefore, we have $\bar{m}_\alpha \in \hat{G} \ast \hat{H}_\alpha$. Hence, $\bar{\pi}_\alpha(p, \bar{q}, \hat{S})$ and $\bar{m}_\alpha$ are compatible. It means that $\langle p, \bar{q} \rangle \Vdash '\hat{S} \notin \hat{I}_\alpha'$ since this holds for every sufficiently large $\alpha < \kappa^+$ and every $\langle p, \bar{q} \rangle$ with $\langle p, \bar{q} \rangle \Vdash '\hat{S} \notin \hat{J}'$, we have $S \notin I$.

**Lemma 3.12.** Suppose $G \ast H \subseteq P \ast \hat{Q}$ is generic. For every stationary subset $S$ of $\omega_1$, $NS_{\omega_1} \upharpoonright S$ is not precipitous in $V[G][H]$.

**Proof.** We shall show that for every stationary subset $S$ of $\omega_1$, there exists a stationary subset $S'$ of $S$ such that $S' \subseteq I$. It suffices by Lemma 3.11.

Suppose that $\alpha < \kappa^+$, $\langle p, \bar{q} \rangle \in P \ast \hat{Q}_\alpha$, $\hat{S}$ is a $P \ast \hat{Q}_\alpha$-name such that $\langle p, \bar{q} \rangle \Vdash '\hat{S}$ is stationary'. Without loss of generality, we may assume that $\langle p, \bar{q} \rangle \Vdash '\bar{X}_\alpha \in \hat{I}_\alpha'$. If $\langle p, \bar{q} \rangle \Vdash '\hat{S} \notin \hat{I}_\alpha'$, then there is nothing to prove. Suppose not. It means that $\bar{\pi}_\alpha(p, \bar{q}, \hat{S})$ is compatible with $\bar{m}_\alpha$. Let $\hat{G} \ast \hat{H}_\alpha$ be a generic filter of $j(P \ast Q_\alpha)$ over $M$ with $\bar{\pi}_\alpha(p, \bar{q}, \hat{S}), \bar{m}_\alpha \in \hat{G} \ast \hat{H}_\alpha$. Define $G \ast H_\alpha = j^{-1}(\hat{G} \ast \hat{H}_\alpha)$. Since $P$ forces that $[\hat{Q}_\alpha] = \pi_\alpha$, we have $G \ast H_\alpha \in M[\hat{G}][\hat{H}_\alpha]$. Since $\bar{m}_\alpha \in \hat{G} \ast \hat{H}_\alpha$, $G \ast H_\alpha$ is a generic filter of $P \ast \hat{Q}_\alpha$ over $V$. So, we can define $j_\alpha : V[G][H_\alpha] \rightarrow M[\hat{G}][\hat{H}_\alpha]$ by letting $j_\alpha(\bar{z}^{\hat{G} \ast \hat{H}_\alpha}) = j(\bar{z})^{G \ast H_\alpha}$. Since $\bar{\pi}_\alpha(p, \bar{q}, \hat{S}) \in \hat{G} \ast \hat{H}_\alpha$, we have $\kappa \in j_\alpha(S)$.
Since \( \langle p, q \rangle \Vdash 'X_\alpha \in \dot{I}_\alpha ' \) and \( \langle p, q \rangle \in G * H_\alpha \), we have \( X_\alpha \in I_\alpha \) in \( V[G][H_\alpha] \). So, in \( V[G][H_\alpha] \), \( R_\alpha \) is the forcing to shoot a \( \text{TCG}(\mathcal{C}) \)-measure one set through \( \kappa \setminus X_\alpha \). Thus, \( j_\alpha(R_\alpha) \) is the forcing to shoot a \( \text{TCG}(\mathcal{C}) \)-measure one set through \( j_\alpha(\kappa \setminus X_\alpha) \). Define \( r' \) in \( j_\alpha(R_\alpha) \) to be the truth value of \( 'G_{R_\alpha} \cap \kappa ' \) is not generic over \( V[G][H_\alpha] \). There exists a function \( f_{r'} : \kappa \rightarrow R_\alpha \) such that \( j_\alpha(f_{r'})(\kappa) = r' \). Let \( S' \) be the \( R_\alpha \)-name such that for every \( \xi \in S \), \( [\xi \in S'] = f_{r'}(\xi) \) and for every \( \xi \in \omega_1 \setminus S \), \( 1_{R_\alpha} \Vdash '\xi \notin S' \).

Claim 3.12.1. \( 1_{R_\alpha} \Vdash 'S' \) is stationary'.

Proof. Let \( r \in R_\alpha \) and \( \dot{D} \) be an \( R_\alpha \)-name for a club subset of \( \omega_1 \). Note \( j_\alpha(r) = r \). Let \( \langle N_\alpha : \gamma < \omega_1 \rangle \in M[G][H_\alpha] \) be a tower of countable elementary submodel of \( H(\theta)^{M[G][H_\alpha]} \) for some sufficiently large regular cardinal \( \theta \) with \( r, \dot{D} \in N_0 \). For every \( \gamma < \omega_1 \), let \( \delta_\gamma = N_\gamma \cap \omega_1 \). In \( M[G] \), pick an increasing cofinal sequence \( \langle \kappa_n : n < \omega \rangle \) in \( \kappa \). We shall define a decreasing sequence \( \langle r_n : n < \omega \rangle \) in \( R_\alpha \) and an increasing sequence \( \langle \mu_n : n < \omega \rangle \) in \( \kappa \) so that \( \kappa \setminus N_{\mu_\alpha+1} \). Let \( \mu_0 = \kappa_0 \). Suppose that we have defined \( r_n \) so that \( r_n \in N_{\mu_n+1} \). Then \( \mu_n+1 < \kappa \) be so large that \( \mu_n+1 \geq \kappa_n+1 \) and \( \mu_{n+1} = \kappa_n+1 \). Let \( r_{n+1} = r_n \uplus \{ \mu_{n+1} \} \). Then we have \( r_{n+1} \in N_{\mu_{n+1}+1} \cap R_\alpha \). Hence, there exists an \( r_{n+1} \leq r_n \) such that \( r_{n+1} \in \dot{N}_{\mu_{n+1}+1} \) and \( r_{n+1} \uplus \{ \delta_{n+1} = \delta_n+1 \} \neq \emptyset \).

Define \( \tilde{r} = \bigcup_{n \in \omega} r_n \cup \{ \kappa \} \). It is easy to see that \( j_\alpha(C) \kappa \nsubseteq \tilde{r} \). So, we have \( \tilde{r} \cap j(\dot{R}_\alpha) \). Since \( \tilde{r} \cap \{ \delta_\gamma : \gamma < \omega_1 \} = \emptyset \), \( \tilde{r} \cap \kappa \) cannot be generic over \( M[G][H_\alpha] \). It follows that \( \tilde{r} \leq r' \). Since \( \kappa \in j(S') \), we have \( \tilde{r} \Vdash 'j(\dot{S}'), r' \). Since \( j_\alpha(\dot{D}) \) is a \( j_\alpha(R_\alpha) \)-name for a club subset and \( \tilde{r} \Vdash '\dot{D} \) is unbounded in \( \kappa \), we have \( \tilde{r} \Vdash '\kappa \in j(\dot{D}) \). Therefore, \( \tilde{r} \Vdash 'j(S') \cap j(\dot{D}) \neq \emptyset ' \). By elementarity, in \( V[G][H_\alpha] \), there exists a \( \tilde{r} \leq r \) such that \( \tilde{r} \Vdash 'S' \cap D \neq \emptyset ' \). Hence \( 1_{R_\alpha} \Vdash 'S' \) is stationary'.

However, it is easy to see \( 1_{R_\alpha} \Vdash 'S' \in \dot{I}_\alpha ' \). If we identify \( \dot{S}' \) with a \( P \uplus \dot{Q}_{\alpha+1} \)-name, then we have \( \langle p, q \rangle \Vdash 'S' \in \dot{I}_\alpha ' \). \qed

The same argument yields the following lemma. Note that \( Q \) is \( <\varepsilon \)-propper, so it is possible to build a decreasing sequence \( \langle r_n : n < \omega \rangle \) so that \( \tilde{r} \Vdash 'j(\dot{C}') \subseteq ^* j(\dot{D}) \).

**Lemma 3.13.** Let \( G * H \subseteq P * \dot{Q} \) be generic over \( V \). For every indecomposable ordinal \( \varepsilon' < \varepsilon \), there is no tail club guessing sequence \( \overline{\mathcal{C}}\) of order type \( \varepsilon' \) such that \( \text{TCG}(\mathcal{C}) \) is precipitous in \( V[G][H] \).

This finishes the proof of Theorem 3.1.

4. Open problems

While Theorem 3.1 answers some questions asked in [5], it leaves the following question open.

**Question 1.** Is it consistent that \( \text{NS}_{\omega_1} \) is precipitous but there is a tail club guessing sequence \( \overline{\mathcal{C}} \) on \( \omega_1 \) such that \( \text{TCG}(\overline{\mathcal{C}}) \) is not precipitous?

In [7], Jech, Magidor, Mitchell, and Prikry constructed from a measurable cardinal that \( \text{NS}_{\omega_1} \) is precipitous. We can show that there is a tail club guessing sequence in this model. If the tail club guessing ideal associated with it is not precipitous, then the previous questions is solved negatively. Otherwise, it solves the following question, which is also interesting.
Question 2. What is the consistency strength that \( \text{NS}_{\omega_1} \) is precipitous and there is a precipitous tail club guessing ideal on \( \omega_1 \)?

Note that the author constructed from a Woodin cardinal a model in which \( \text{NS}_{\omega_1} \) and all tail club guessing ideals on \( \omega_1 \) are precipitous. Thus, the existence of one Woodin cardinal is an upper bound of the consistency strength. However, it is expected to be much lower.

The following question is not solved in this paper either.

Question 3. Is it consistent that there are two tail club guessing sequences \( \overrightarrow{C} \) and \( \overrightarrow{C}' \) such that \( \text{TCG}(\overrightarrow{C}) \) is precipitous, \( \text{TCG}(\overrightarrow{C}') \) is not precipitous, and \( \overrightarrow{C} \) has smaller order type than \( \overrightarrow{C}' \)?

The technique used in the proof of Theorem 3.1 seems to be applicable to many other “natural ideals”, particularly those associated with guessing principles.

Question 4. Which natural ideals can be precipitous? Are there any relationships between the precipitousness of natural ideals?

Acknowledgement. I am grateful to Paul Larson, who gave me a hint about this argument. This research was partially supported by the CFR Summer Research Grant from Miami University.

References


