Compact topological cliques in sparse graphs

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Abstract
Let $\epsilon$ be a real number such that $0 < \epsilon < 1$ and $t$ a positive integer. Let $n$ be a sufficiently large positive integer as a function of $t$ and $\epsilon$. We show that every $n$-vertex graph with at least $2^{7t^2}n^{1+\epsilon}$ edges contains a subdivision of $K_t$ in which each edge of $K_t$ is subdivided at most $\max\{2, \frac{10}{\epsilon}\ln\frac{1}{\epsilon}\} - 1$ times. This improves the main result in [12]. Some related problems are proposed.

1 Introduction

We consider only simple graphs in this paper unless otherwise specified. Generally speaking, in extremal graph theory we study how global parameters of a graph such as its edge density or chromatic number affect the existence of certain substructures. One of the most fundamental substructures is a subgraph. Given a graph $H$, how many edges do we need in an $n$-vertex graph $G$ to force $H$ to appear as a subgraph of $G$? Turán was one of the first to explore this question. The classic Turán number $ex(n, H)$ of a graph $H$ is defined as the maximum number of edges in an $n$-vertex graph not containing $H$ as a subgraph. Turán determined the exact value of $ex(n, K_r)$. The celebrated Erdős-Simonovits-Stone Theorem determined $ex(n, H)$ asymptotically for all non-bipartite $H$, showing that $ex(n, H) \sim (1 - \frac{1}{p-1})\binom{n}{2}$, where $p = \chi(H)$ is the chromatic number of $p$.

Another fundamental substructure of interest is the so-called minor. Given a graph $H$ and another graph $L$, $L$ is called a $H$-minor if $H$ can be obtained from $L$ by a series of edges contractions. If a graph $G$ contains a subgraph that is an $H$-minor, then we can also say that $G$ contains $H$ as a minor. (In other words, $G$ contains a subgraph that can be contracted into $H$.) A variant of the notion of a minor is that of a topological minor of subdivision. Subdividing an edge $uv$ in a graph, as usual, means replacing the edge $uv$ by path $uwv$ through a new vertex $w$. Given a graph $H$ and another graph $L$, we say that $L$ is an $H$-subdivision (a subdivision

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of \( H \)) if \( L \) can be obtained from \( H \) by series of edge subdivisions. For example, any cycle \( C \) can be viewed as a \( C_3 \)-subdivision, as it can be obtained from \( C_3 \) via edge-subdivisions. If a graph \( G \) contains a subdivision of \( H \), we also say that \( G \) contains \( H \) as a topological minor. A famous example of use of the notion of subdivisions is the Kuratowski’s Theorem which says a graph is planar if and only if it contains no subdivisions of \( K_5 \) or \( K_{3,3} \).

Mader was one of the first to explore Turán type questions for (topological) minors. We have seen from the Erdős-Simonovits-Stone Theorem that in order to force any given non-bipartite \( H \) to appear as a subgraph in an \( n \)-vertex host graph, we need \( \Omega(n^2) \) edges. Indeed, \( K_{n/2,n/2} \) has \( n^2/4 \) edges and avoids any non-bipartite \( H \) as a subgraph. So, loosely speaking it takes a very dense host graph to force a particular \( H \) to appear as a subgraph. Contrasting this phenomenon for subgraphs, Mader showed that it only takes \( O(n) \) edges in an \( n \)-vertex host graph to force a subdivision of \( H \), as implied by the following theorem of his.

**Theorem 1.1 (Mader [13])** Given a positive integer \( t \), there exists a positive constant \( c_t \) depending only on \( t \) such that every \( n \)-vertex graph \( G \) with at least \( c_t n \) edges contains a subdivision of \( K_t \).

While Mader’s theorem was proved in 1967, it was not until mid til late 1990s that Bollobás and Thomason [3] and independently Komlós and Szemerédi [10] determined the growth rate of the optimal constant \( c_t \) as a function of \( t \), showing that Mader’s Theorem holds for some \( c_t \leq ct^2 \) for some absolute constant \( c \).

We see between Turán’s theorem and Mader’s theorem that there is a big gap between required edge-densities for forcing a graph \( H \) directly as a subgraph versus as a topological minor (subdivision), with the former much higher. What seems to make it easier to force a subdivision of \( H \) is that we allow as many subdivisions on an edge of \( H \) as needed and this gives us a lot of leeway. This is reminiscent of forcing cycles: it is much easier to force a cycle of length at least a given length \( k \) than to force a cycle of length at most \( k \) (for the former \( O(n) \) edges suffice while for the latter \( \Omega(n^{1+1/(k+1)}) \) edges are required).

This brings up a natural question: what if we require that each edge in the original \( H \) is only allowed to be subdivided some bounded number \( p \) times? What edge density is needed in a host graph \( G \) to force such a more restricted subdivision of \( H \)? Formally, we introduce the following definition.

**Definition 1.2** Let \( p \) be a positive integer. A subdivision \( L \) of a graph \( H \) is called a \( p \)-subdivision of \( H \) if in forming \( L \) each edge of \( H \) is subdivided at most \( p-1 \) times.

Note that a 1-subdivision of \( H \) is just \( H \) itself. We pose the following general question.

**Question 1.3** Fix a positive integer \( p \) and a graph \( H \), how many edges does an \( n \)-vertex graph \( G \) need to have to force a \( p \)-subdivision of \( H \)?

We may rephrase Question 1.3 in terms of Turán numbers as follows. Given a positive integer \( p \) and a graph \( H \), let \( H^{(\leq p)} \) denote the family of \( p \)-subdivisions of \( H \) (i.e. graphs obtainable from \( H \) by subdividing each edge of \( H \) at most \( p-1 \) times).

**Problem 1.4** Given positive integers \( p, n \) and a graph \( H \), determine \( ex(n, H^{(\leq p)}) \).
In this paper, we focus on the case where $H = K_t$, the complete graph, as it provides natural bounds for the general $H$. Another motivation for our work in this paper comes from the following work of Kostochka and Pyber [12]. It is well-known that any $n$-vertex planar graph with more than $3n - 6$ edges contains a non-planar subgraph. However, if we want to force a non-planar subgraph of a bounded (constant) order then it is easy to see that $cn$ edges are not sufficient no matter how large $c$ is. Erdős [5] asked the following: Is it true that every $n$-vertex graph with at least $n^{1+\epsilon}$ edges contains a non-planar subgraph of order at most $c(\epsilon)$, where $c(\epsilon)$ is a constant depending only on $t$? Kostochka and Pyber [12] answered the question in the affirmative, proving the following more general result.

**Theorem 1.5 (Kostocha and Pyber [12])** Let $\epsilon$ be a positive real such that $0 < \epsilon < 1$. Let $n, t$ be positive integers. Every $n$-vertex graph with at least $4t^2n^{1+\epsilon}$ edges contains a subdivision of $K_t$ on at most $7t^2\ln t/\epsilon$ vertices.

Since a subdivision of $K_5$ is non-planar, the special case of $k = 5$ of Theorem 1.5 answers Erdős’ question in the affirmative. In proving Theorem 1.5, Kostochka and Pyber in effect proved

**Theorem 1.6 [12]** Let $\epsilon$ be a positive real such that $0 < \epsilon < 1$. Let $n, t$ be positive integers. Every $n$-vertex graph with at least $4t^2n^{1+\epsilon}$ edges contains a $p$-subdivision of $K_t$ with $p \leq 2(1 + 2\epsilon(1 + 2\ln t)) \leq 14\ln t/\epsilon$.

Note that in Theorem 1.6, the bound on $p$ varies with $t$. If we use Theorem 1.6 to derive a bound on $ex(n, K_t^{(\leq p)})$, then we only get $ex(n, K_t^{(\leq p)}) \leq 4t^2n^{1+14\ln t/\epsilon}$, which is not very useful since as $t$ grows the exponent of $n$ quickly exceeds 2 and thus trivializing the bound.

In this paper, we eliminate this dependency of the bound of $p$ on $t$ (Theorem 1.10). First we note that when $\epsilon \geq \frac{1}{2}$ the following result of Alon, Krivelevich and Sudakov [1] readily implies the existence of a 2-subdivision of $K_t$ in any $n$-vertex graph with at least $t^2n^{1+\epsilon}$ edges.

**Proposition 1.7 [1]** Let $H$ be a bipartite graph with maximum degree $r$ on one side. Then exists a constant $c_H > 0$, depending on $H$, such that $ex(n, H) \leq c_Hn^{2-\frac{1}{r}}$.

From the proof of Theorem 2.2 in [1], one can easily check that $c_H \leq n(H)$ when $r = 2$. Thus, we have

**Corollary 1.8** Let $H$ be a bipartite graph with $h$ vertices such that vertices in one partite set all have degree at most 2, then $ex(n, H) \leq \frac{h}{2}n^{3/2}$.

Let $K_t^{(2)}$ denote the graph obtained from $K_t$ by subdividing each edge exactly once. Observe that $K_t^{(2)}$ is a bipartite graph on $t + \frac{t}{2} < t^2$ vertices in which the vertices used to subdivide the edges form one part $X$ and the branching vertices form the other part $Y$. Further, each vertex in $X$ has degree two. By Corollary 1.7, we have

**Proposition 1.9** For each positive integer $t$, we have $ex(n, K_t^{(\leq 2)}) \leq ex(n, K_t^{(2)}) \leq \frac{t^2}{2} \cdot n^{3/2}$.
Proposition 1.9 gives the correct order of magnitude for $ex(n, K_t^{(\leq 2)})$ and $ex(n, K_t^{(2)})$ because of the well-known fact that there are $n$-vertex graphs with $\Omega(n^{3/2})$ edges and no $C_3$ or $C_4$. Proposition 1.9 implies that for $\epsilon \geq \frac{1}{2}$ every $n$-vertex graph with at least $\frac{t^2}{2}n^{1+\epsilon}$ edges contains a 2-subdivision of $K_t$. Thus, for the rest of the paper, we restrict our attention to $\epsilon < 1/2$. Our main result is as follows.

**Theorem 1.10** Let $t$ be a positive integer. Let $0 < \epsilon < 1/2$ be a real. Let $G$ be an $n$-vertex graph with at least $2^{t^2} \cdot n^{1+\epsilon}$ edges, where $n$ is sufficiently large as a function of $t$ and $\epsilon$. Then $G$ contains a $\frac{10}{\epsilon} \ln \frac{1}{\epsilon}$-subdivision of $K_t$.

It is well-known that there are $n$-vertex graphs with $\Omega(n^{1+1/g})$ edges and girth at least $g$. Note that a $p$-subdivision of $K_t$, where $t \geq 3$, necessarily contains a cycle of length at most $3p$. Hence, there are $n$-vertex graphs with $\Omega(n^{1+1/(3p+1)})$ edges and no $p$-subdivision of $K_t$. This implies

**Proposition 1.11** For each real $0 < \epsilon < 1$ and positive integer $t \geq 3$, there are $n$-vertex graphs with $\Omega(n^{1+\epsilon})$ edges and no $\left\lfloor \frac{1}{3}\left(\frac{1}{\epsilon} - 1\right) \right\rfloor$-subdivision of $K_t$.

Proposition 1.11 shows that Theorem 1.10 is not too far from being optimal. However, it is natural to ask whether one can get rid of the $\ln \frac{1}{\epsilon}$ factor in Theorem 1.10. Most importantly, compared to Theorem 1.5, the new bound on the number of times each edge of $K_t$ is subdivided now depends only on $\epsilon$ and does not vary with $t$. This, together with Proposition 1.9, allows us to obtain the following bound on $ex(n, K_t^{(\leq p)})$. For natural reasons, we may assume $p \geq 2$ and $t \geq 3$.

**Corollary 1.12** Let $p, t$ be fixed positive integers where $p \geq 2$ and $t \geq 3$. As a function of $n$, we have $ex(n, K_t^{(\leq p)}) = O(n^{1+\min\left\{\min\left\{\frac{1}{\epsilon}, \frac{1}{3}\right\}, \frac{1}{2}\right\}})$ and $ex(n, K_t^{(\leq p)}) = \Omega(n^{1+1/(3p+1)})$.

If one can somehow remove the $\ln \frac{1}{\epsilon}$ factor in Theorem 1.10, then one would improve the upper bound on $ex(n, K_t^{(\leq p)})$ to $O(n^{1+\frac{c}{p}})$ for some constant $c$. For the rest of the paper, we prove Theorem 1.10. We will combine ideas from [12], [8], as well as some new ideas. In our arguments, we will drop floors and ceilings whenever appropriate. For terms and notation not defined here, the reader is referred to [16].

## 2 Forcing compact subdivisions in graphs with no dense subgraphs of small radius

To prove Theorem 1.10, we first prove in this section that if a graph is reasonably dense itself but contains no dense subgraph of small radius, then we can find a compact subdivision of $K_t$. The following notion will be used frequently in our proofs.

**Definition 2.1** Let $c, \epsilon$ be positive reals where $0 < \epsilon < 1$. A graph $H$ is called $(c, \epsilon)$-dense if $e(H) \geq c|n(H)|^{1+\epsilon}$.
In our proofs, we will often use the well-known fact that a graph $G$ satisfying $e(G) \geq d \cdot n(G)$ contains an induced subgraph of minimum degree at least $d$. The original statement of next lemma is slightly weaker and the proof is probabilistic. The version we present here is suggested by Kostochka [11].

**Lemma 2.2** Let $a, m, q$ be positive integers. Let $A_1, \ldots, A_m$ be a collection of sets of size $a$. Suppose each elements of $A = \bigcup_i A_i$ lies in at most $q$ different $A_i$’s. Then for each $i \in [m]$, there exists $B_i \subseteq A_i$ of size $\lfloor a/q \rfloor$ such that $B_1, \ldots, B_m$ are pairwise disjoint.

**Proof.** Let $p = \lfloor a/q \rfloor$. Create a bipartite graph $H$ with a bipartition $(X, A)$ where $|X| = mp$ as follows. Label the vertices of $X$ by $x_1^1, \ldots, x_1^p, x_2^1, \ldots, x_2^p, \ldots, x_m^1, \ldots, x_m^p$. For each $i \in [m]$ and $y \in A_i$, if $y \in A_i$ then we add edges between $y$ and $x_i^1, \ldots, x_i^p$. By our construction, each vertex in $X$ has degree $a$. Also, since each $y \in A$ lies in at most $t$ different $A_i$’s, each vertex in $A$ has degree at most $pq \leq a$ in $H$. By Hall’s Theorem, it is easy to see that $H$ has a matching $M$ saturating all of $X$. For each $i \in [m]$, the elements of $A$ that $x_i^1, \ldots, x_i^p$ are matched to by $M$ are elements of $A_i$. Hence, we obtain disjoint $B_1, \ldots, B_m$ each of size $p = \lfloor a/q \rfloor$, with $B_i \subseteq A_i$ for each $i \in [m]$. ■

In the following lemma and theorem, in order to simplify the presentation while not affecting the final main proof, we assume $c$ to be a constant real that is at least 1 and $G$ to be a bipartite graph. Similar statements still hold when $c$ is any positive real and when $G$ is any graph. The next technical lemma provides the most crucial ingredient of our proof of Theorem 1.10.

**Lemma 2.3** Let $t$ be a positive integer and $c, \epsilon$ positive reals, where $0 < \epsilon < 1/2$ and $c \geq 1$. Let $n$ be a sufficiently large positive integer (depending on $t$ and $c, \epsilon$). Let $G$ be an $n$-vertex bipartite graph with minimum degree at least $cn^{1/t}$ Let $l(\epsilon) = \lfloor \frac{c}{2} \ln \frac{1}{\epsilon} \rfloor$. Suppose that $G$ has no $(\frac{c}{2t}, \epsilon)$-dense subgraph of radius at most $l(\epsilon)$. Then for each vertex $u$ in $G$, there are at least $n^{1-2/(t+1)}$ vertices $x$ each of which is joined to $u$ by at least $l(\epsilon) \cdot t^2$ internally disjoint paths of length at most $l(\epsilon)$.

**Proof.** First we show that $G$ has good expansion properties.

**Claim 1.** Let $H$ be a subgraph of $G$ of radius at most $l(\epsilon) - 1$, where $n(H) \leq 3n^a$ and $a < 1$. Let $W$ be a set of vertices outside $H$ having neighbors in $H$. Suppose $G$ contains at least $\frac{\epsilon}{4}n^{a+\epsilon}$ edges between $V(H)$ and $W$. Then $|W| \geq 2n^{a+\frac{\epsilon}{t+1}(1-a)}$.

**Proof of Claim 1.** Let $F$ denote the subgraph of $G$ induced by $V(H) \cup W$. Since $H$ has radius at most $l(\epsilon) - 1$, $F$ has radius at most $l(\epsilon)$. By our assumption of $G$, $F$ is not $(\frac{c}{2t}, \epsilon)$-dense. Suppose $n(F) = n^b$. We have

$$\frac{c}{4}n^{a+\epsilon} \leq e(F) \leq \frac{c}{2^\epsilon}(n^b)^{1+\epsilon} \quad (1)$$

From this, we get

$$n^b \geq 16^{1/(t+1)} n^{2\epsilon/(t+1)} \geq 5n^{a+\frac{\epsilon}{t+1}(1-a)}.$$

Hence, $|W| = n(F) - n(H) \geq 5n^{a+\frac{\epsilon}{t+1}(1-a)} - 3n^a \geq 2n^{a+\frac{\epsilon}{t+1}(1-a)}$. ■
Let \( d = \lceil cn^\epsilon \rceil \). We know that each vertex of \( G \) has degree at least \( d \). Let \( u \) be any vertex in \( G \). We iteratively define a sequence of disjoint sets \( L_1, L_2, \ldots, L_{l(e)-1} \). During our procedure, we will maintain the following conditions:

(1) Each \( L_i \) is called a level and will be designated as strong or weak. Set \( L_1 \) will be strong.

(2) Each \( L_i \) is partitioned into some \( d_i \) subsets \( L^j_i \) called sectors of equal size. If \( L_i \) is a strong level then each sector consists only of a single vertex. If \( L_i \) is a weak level, then \( \frac{1}{2}d_{i-1} \leq d_i \leq d_{i-1} \).

(3) Each vertex in a strong level \( L_i \) beyond \( L_1 \) has neighbors in at least \( l(e) \cdot l^2 \) sectors of \( L_{l(e)} \).

(4) If \( L_i \) is a weak level, then there exists an injection \( f \) from the collection of sectors of \( L_i \) into the collection of sectors of \( L_{i-1} \) such that if \( L^j_i \) is a sector of \( L_i \) then each vertex in \( L^j_i \) has at least one neighbor in \( f(L^j_i) \). We call \( f(L^j_i) \) the parent sector of \( L^j_i \) in \( L_{i-1} \).

(5) For each \( i \), suppose \( |L_i| = n^{a_i} \). If \( L_{i+1} \) is a strong level, then \( |L_{i+1}| \geq 2|L_i| \cdot n^{\frac{\epsilon}{10}}(1-a_i) \).

Thus, \( a_{i+1} \geq a_i + \frac{\epsilon}{10} (1-a_i) \). If \( L_{i+1} \) is a weak level, \( |L_{i+1}| \geq 2|L_i| \cdot n^{\frac{\epsilon}{10}}(1-a_i + 0.9\epsilon) \).

Recall that \( G \) has minimum degree at least \( cn^\epsilon \geq n^\epsilon \). To start, let \( L_1 = \{x_1, \ldots, x_{n^{\epsilon}}\} \) be a set of \( d_1 = \lceil n^{\epsilon} \rceil \) neighbors of \( u \). We designate \( L_1 \) as a strong level. For each \( j \in [d_1] \), let \( L^j_1 = \{x_j\} \). Suppose \( L_1, \ldots, L_i \) have been defined so that (1)-(5) hold for \( L_j, j \leq i \). Let \( U_{i-1} = \{u\} \cup L_1 \cup L_2 \cup \ldots \cup L_{i-1} \). By (5), for each \( j \leq i \), \( |L_j| \geq 2|L_{j-1}| \cdot n^{\frac{\epsilon}{10}}(1-a_{j-1}) \geq 2|L_{i-1}| \).

So, \( |U_{i-1}| \leq |L_{i-1}|(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots) \leq 2|L_{i-1}| \leq |L_i| \). It is clear from conditions (3) and (4) that there is a path from each vertex in \( L_i \) back to \( u \) through \( U_{i-1} \). Thus there is a subgraph \( H_i \) of \( G \) containing \( L_i \) of radius at most \( i \) and order at most \( |U_{i-1}| + |L_i| \leq 2|L_i| \).

Let \( d \) denote the minimum degree of \( G \). We have \( d \geq cn^\epsilon \). Let \( L^s_i \) denote the set of vertices in \( L_i \) having at least \( \frac{d}{2} \) neighbors in \( U_{i-1} \). Since \( L_i \) is itself an independent set, each vertex in \( L_i - L^s_i \) has at least \( \frac{d}{2} \) neighbors outside \( U_i = U_{i-1} \cup L_i \). Let \( H^s \) be the subgraph of \( G \) induced by \( U_{i-1} \cup L^s_i \). Then \( H^s \) has radius at most \( l(e) \). If \( |L^s_i| > \frac{1}{4}|U_{i-1}| \), then \( |L^s_i| \geq \frac{n^{\epsilon}}{5} \) and \( \epsilon(H^s) \geq |L^s_i| \cdot \frac{\epsilon}{5} \geq |L^s_i| \cdot \frac{\epsilon}{5} n^{\epsilon} \geq \frac{\epsilon}{10}[n(H^s)]^{1+\epsilon} \), contradicting \( G \) containing no \((\frac{d}{2}, \epsilon)\)-dense subgraph of radius at most \( l(e) \). Hence \( |L^s_i| \leq \frac{1}{4}|U_{i-1}| \leq \frac{1}{4}|L_i| \).

Consider the \( d_i \) sectors \( L^j_i \) of \( L_i \). If \( |L^j_i| \cap L^s_i > \frac{1}{2}|L^j_i| = \frac{1}{2} \frac{|L_i|}{d_0} \), we say \( L^j_i \) is bad. Otherwise we say it is good. If more than \( \frac{d}{2} \) of the sectors are bad, then \( |L^s_i| > \frac{d}{2} \cdot |L^s_i| = \frac{1}{4}|L_i| \), a contradiction. So at most \( \frac{d}{2} \) of the sectors are bad and at least \( \frac{d}{2} \) of the sectors are good.

Now, let \( J = \{j : L^j_i \) is a good sector of \( L_i\} \). For each \( j \in J \), let \( W^j_i \) denote the set of neighbors of \( L^j_i \) outside \( U_i \) and suppose \( |L^j_i| = n^{a_i,j} \).

**Claim 2.** For each \( j \in J \), we have \( |W^j_i| \geq 2n^{a_i,j+\epsilon}(1-a_{i,j}) = 2|L^j_i| \cdot n^{\frac{\epsilon}{10}}(1-a_{i,j}) \).

**Proof of Claim 2.** Let \( j \in J \). By our assumption, at least \( \frac{1}{2}|L^j_i| \) vertices in \( L^j_i \) have at least \( \frac{d}{2} \) neighbors outside \( U_i \). If \( L_i \) is a strong level, then \( L^j_i \) is a single vertex \( v \) and \( v \) has at least \( \frac{d}{2} \) neighbors outside \( U_i \). So \( |W^j_i| \geq \frac{d}{2} \geq \frac{\epsilon}{5} n^{\epsilon} \geq 2|L^j_i| \cdot n^{\frac{\epsilon}{10}}(1-a_{i,j}) \), for large \( n \), noting that \( |L^j_i| = 1 \).
Suppose now that $L_i$ is a weak level. Then $L_i^t$ has a parent sector $L_i^{t-1}$ in $L_{i-1}$ of size $|L_{i-1}|/d_{i-1}$. Since $d_i \leq d_{i-1}$, we have $|L_i^t| / |L_i^{t-1}| \geq |L_i| / |L_{i-1}| \geq 2n \frac{\epsilon}{1-a_i} \geq 2$. By backtracking, we can find a tree $H_i^t$ containing $L_i^t$ that is rooted at $u$ and has depth $i \leq l(\epsilon) - 1$ and order at most $|L_i^t| \left( 1 + \frac{1}{2} + \frac{1}{3^2} + \cdots + \frac{1}{(l(\epsilon) - 1)} \right) \leq 2|L_i^t| + l(\epsilon)$. (Indeed, a sector in a weak level has a parent sector in the previous level that is much smaller, while any sector in a strong level is linked to $u$ by a path of length at most $l(\epsilon)$.) If $a_{i-1} \geq \frac{3n}{2}$, then $|L_i^t| / |L_i^{t-1}| \geq 2n^{\frac{\epsilon}{1-a_i}} \frac{1}{2}$, which implies $|L_i^t| \geq l(\epsilon)$ for large $n$. If $a_{i-1} \geq \frac{3n}{2}$ instead, then $|L_i^t| \geq 2|L_i^{t-1}| = 2|L_{i-1}| / d_{i-1} \geq 2n^{\frac{\epsilon}{1-a_i}} / n^* \geq l(\epsilon)$, for large $n$. Hence, we always have $|L_i^t| \geq l(\epsilon)$. So $H_i^t$ has order at most $2|L_i^t| + l(\epsilon) \leq 3|L_i^t|$.

By our assumption, at least $\frac{1}{2}|L_i^t|$ vertices in $L_i^t$ have at least $\frac{1}{2}$ neighbors outside $U_i$. So, there are at least $\frac{1}{2}|L_i^t| = \frac{\epsilon}{4} n^{a_i+\epsilon}$ edges between $H_i^t$ and $W_i^t$. Since $n(H_i^t) \leq 3|L_i^t| = 3n^{a_i-j}$ and $H_i^t$ has radius at most $l(\epsilon) - 1$, applying Claim 1 with $H = H_i^t$ and $W = W_i^t$, we have $|W_i^t| \geq 2n^{a_i+\epsilon} \frac{1}{1-a_i-j} = 2|L_i^t| \cdot n^{\frac{\epsilon}{1-a_i-j}}$.

Let $W_i = \bigcup_{j \in J} W_i^j$. Let $q = t^2 \cdot l(\epsilon)$. We say a vertex $y$ in $W_i$ is heavy if it belongs to at least $q$ different $W_i^j$'s. Otherwise we say it is light. Let $W_i^+$ denote the set of heavy vertices in $W_i$ and $W_i^-$ the set of light vertices in $W_i$. We consider two cases. Recall that there is a subgraph $H_i$ containing $L_i$ that has radius at most $i \leq l(\epsilon) - 1$ and order at most $2|L_i^t|$. 

**Case 1.** At least half of the edges between $L_i$ and $W_i$ are incident to $W_i^+$

In this case, we let $L_{i+1} = W_i^+$ and designate it as a strong level. By our earlier discussion, at most $\frac{1}{4}|L_i^t|$ vertices in $L_i^t$ are bad. So, there are at least $\frac{3}{4}|L_i^t| > \frac{\epsilon}{8} n^{a_i+\epsilon}$ edges between $L_i$ and $L_{i+1}$. Recall that $|L_i| = n^{a_i}$, $H_i$ contains $L_i$, $n(H_i) \leq 2|L_i| = 2n^{a_i}$, and $H_i$ has radius at most $l(\epsilon) - 1$. Applying Claim 1 with $H = H_i$, $a = a_i$, $W = |L_{i+1}|$, we have $|L_{i+1}| \geq 2n^{a_i+\epsilon} \frac{1}{1-a_i-j} = 2|L_i^t| \cdot n^{\frac{\epsilon}{1-a_i-j}}$. Let $d_{i+1} = |L_{i+1}|$. We partition $L_{i+1}$ into $d_{i+1}$ many subsets of equal size. We define them to be the sectors of $L_{i+1}$ and rename them $L_{i+1}^1, \ldots, L_{i+1}^{d_{i+1}}$, respectively. We will designate $L_{i+1}$ as a weak level. It remains to verify that (2), (4) and (5) hold for $L_{i+1}$. We have $d_{i+1} = |J| \geq \frac{1}{2} d_i$. So (2) holds. For (4), if $L_{i+1}^t = A_j^t \subseteq W_i^t$, we let $f(L_{i+1}^t) = L_j^t$. It is readily seen that such $f$ satisfies (4).

Since (1), (2) and (5) hold for $L_i, \ldots, L_{i+1}$, it is easy to see that $d_i \geq d_{i+1} > n^* / 2^i$. Hence $|L_i| / |L_i^t| = d_i \geq n^* / 2^i \geq n^{0.95}$, for large $n$. This implies $a_i - a_{i+1} \geq 0.95 \epsilon$, or $a_{i+1} \leq a_i - 0.95 \epsilon$. So, $|Z_i^t| \geq 2|L_i^t| \cdot n^{\frac{\epsilon}{1-a_i+0.95}} \geq 4q \cdot |L_i^t| \cdot n^{\frac{\epsilon}{1-a_i+0.95}}$, for large $n$. Hence, $|A_j^t| = |Z_j^t| / q \geq 4|L_i^t| \cdot n^{\frac{\epsilon}{1-a_i+0.95}}$. Since $|J| \geq \frac{d_i}{2}$, we have $|L_{i+1}| = \sum_{j \in J} |A_j^t| \geq \frac{d_i}{2} \cdot 4|L_i^t| \cdot n^{\frac{\epsilon}{1-a_i+0.95}} = 2|L_i^t| \cdot n^{\frac{\epsilon}{1-a_i+0.95}}$. So (5) holds for $L_{i+1}$.

We have now constructed the sequence $L_1, L_2, \ldots$, that satisfy conditions (1)–(5).
Claim 3. Each vertex $x$ in a strong level $L_i$ is joined to $u$ by at least $q$ internally disjoint paths of length $i$.

Proof of Claim 3. By (3), $x$ has neighbors in at least $q$ sectors of $L_{i-1}$. Let $y^1_{i-1}, \ldots, y^q_{i-1}$ be neighbors of $x$ in $L_{i-1}$ all from different sectors. If $L_{i-1}$ is a weak level, then by condition (4), the $q$ sectors involving $y^1_{i-1}, \ldots, y^q_{i-1}$ all have different parent sectors in $L_{i-2}$. So, for each $j \in [q]$, we can find a neighbor $y^j_{i-2}$ of $y^j_{i-1}$ in $L_{i-2}$ so that $y^1_{i-2}, \ldots, y^q_{i-2}$ are from different sectors of $L_{i-2}$. If $L_{i-1}$ is a strong level instead and $i > 1$, then by (3) each $y^j_{i-1}$, $j \in [q]$, has neighbors in at least $q$ different sectors of $L_{i-2}$. In this case, it is easy to see that we can still find $y^1_{i-2}, \ldots, y^q_{i-2}$, all from different sectors of $L_{i-2}$, such that $y^j_{i-2}$ is a neighbor of $y^j_{i-1}$ for each $j \in [q]$. We can continue this process and build $q$ internally disjoint paths of length $i$ from $x$ back to $u$.

By (5), for each $i \leq l(\epsilon) - 1$ we have $a_{i+1} \geq a_i + \frac{\epsilon}{1+\epsilon}(1-a_i)$. Solving the recurrence, with $a_i \geq \epsilon$, we get $a_i \geq 1 - \frac{1}{1+\epsilon} \cdot \frac{1}{(1-\epsilon)^{l(\epsilon)-1}}$. Let $m = \min\{i : a_i \geq 1 - \frac{\epsilon^2}{2(1+\epsilon)}\}$. To have $a_i \geq 1 - \frac{\epsilon^2}{2(1+\epsilon)}$, it suffices that $1 - \frac{1}{(1+\epsilon)^{l(\epsilon)-1}} \geq 1 - \frac{\epsilon^2}{2(1+\epsilon)}$ which holds if $(i-2)\ln(1+\epsilon) \geq \ln \frac{2}{\epsilon^2}$. Since we assume $0 < \epsilon < 0.5$, one can check that $\ln(1+\epsilon) \geq 0.8\epsilon$. In order for (*) to hold, it suffices to have $(i-2)(0.8\epsilon) \geq \ln \frac{1}{\epsilon}$, from which we get $i \geq 2 + \frac{3}{0.8\epsilon} \ln \frac{1}{\epsilon}$. It follows that $m \leq 2 + \frac{3}{0.8\epsilon} \ln \frac{1}{\epsilon} \leq 2 + \frac{3}{\epsilon} \ln \frac{1}{\epsilon} \leq l(\epsilon) - 2$.

Claim 4. $L_{m+1}$ is a strong level.

Proof of Claim 4. Suppose instead that $L_{m+1}$ is a weak level. By condition (5), we have $|L_{m+1}| \geq 2|L_m| \cdot n^{1/\epsilon - \frac{a_m + 0.9\epsilon}{1+\epsilon}} \geq 2|L_m| \cdot n^{0.9\epsilon^2/\epsilon^2} \geq 2n^{1 - \frac{\epsilon^2}{\epsilon^2} + \frac{0.9\epsilon^2}{\epsilon^2}} \gg n$, a contraction. So, $L_{m+1}$ must be a strong level.

Now, by our choice of $m$ and (5), $|L_{m+1}| \geq 2n^{1 - \frac{\epsilon^2}{\epsilon^2}} \geq n^{1 - \frac{\epsilon^2}{\epsilon^2}}$. By Claim 3, each vertex $x$ in $L_{m+1}$ is joined to $u$ by at least $q = l^2(\epsilon)$ internally disjoint paths of length $m + 1 \leq l(\epsilon)$.

Since $u$ is arbitrary, this proves the lemma.

Theorem 2.4 Let $t$ be a positive integer and $c, \epsilon$ fixed reals, where $0 < \epsilon < \frac{1}{2}$ and $c \geq 1$. Let $n$ be a sufficiently large positive integer (depending on $t$ and $c, \epsilon$). Let $G$ be an $n$-vertex bipartite graph with minimum degree at least $cn'$. Let $l(\epsilon) = \lceil \frac{3}{\epsilon} \ln \frac{1}{\epsilon} \rceil$. Suppose $G$ has no $(c\frac{\epsilon}{2}, \epsilon)$-dense subgraph of radius at most $l(\epsilon)$. Then $G$ contains a $2l(\epsilon)$-subdivision of $K_t$.

Proof. By Lemma 2.3, for each vertex $x$ in $G$, there are at least $n^{1 - \frac{\epsilon^2}{\epsilon^2}}$ vertices $y$ each of which is joined to $x$ by at least $l(\epsilon) \cdot t^2$ internally disjoint paths of length at most $l(\epsilon)$.

Let’s define a new graph $H$ with $V(H) = V(G)$ such that $xy \in E(H)$ if and only if $x$ and $y$ are joined by at least $l(\epsilon) \cdot t^2$ internally disjoint paths of length at most $l(\epsilon)$ in $G$. By our earlier discussion, each vertex in $H$ has degree at least $n^{1 - \frac{\epsilon^2}{\epsilon^2}} \gg t^2 \cdot n^{1/2}$, for large $n$. Thus, $e(H) \geq \frac{c}{2}n^{3/2}$. By Proposition 1.9, $H$ contains a 2-subdivision $F$ of $K_t$. From $F$ we obtain an $l(\epsilon)$-subdivision $M$ of $K_t$ in $G$ as follows. Let $p = n(F) = t + \left( \frac{1}{2} \right) < t^2$. Suppose $V(F) = \{x_1, \ldots, x_p\}$ and $E(F) = \{e_1, e_2, \ldots, e_q\}$. Initially, let $V(M) = V(F)$ and $E(M) = \emptyset$. Suppose without loss of generality that $e_1$ joins $x_1$ and $x_2$ in $F \subseteq H$. By our definition of $H$, there exist at least $t^2 \cdot l(\epsilon)$ internally disjoint paths in $G$ between $x_1$ and $x_2$. 

8
Since \( t^2 \cdot l(\epsilon) > n(F) \), one of these paths \( P_1 \) avoids \( V(F) - \{x_1, x_2\} \). In other words, \( P_1 \) is a path of length at most \( l(\epsilon) \) in \( G \) between \( x_1 \) and \( x_2 \) that intersects \( V(F) \) only at \( x_1 \) and \( x_2 \).

Now let \( V(M) = V(M) \cup V(P_1) \) and \( E(M) = E(M) \cup E(P_1) \).

In general, suppose we have processed \( e_1, e_2, \ldots, e_i \) and for each \( j = 1, \ldots, i \), we have added to \( M \) some path \( P_j \) in \( G \) of length at most \( l(\epsilon) \) that joins the two endpoints of \( e_j \) and intersects the previous \( M \) only at these two endpoints. Now, suppose \( e_{i+1} \) joins \( x_a \) and \( x_b \) in \( F \). By definition, there exist at least \( t^2 \cdot l(\epsilon) \) internally disjoint paths in \( G \) between \( x_a \) and \( x_b \). Note that at this point \( |V(M)| \leq t + \left( \frac{1}{2} \right) \cdot l(\epsilon) < t^2 \cdot l(\epsilon) \). Thus we can find one of these paths \( P_{i+1} \) that avoids \( V(M) - \{x_a, x_b\} \). That is, \( P_{i+1} \) intersects \( M \) only at \( x_a \) and \( x_b \). Let \( V(M) = V(M) \cup V(P_{i+1}) \) and \( E(M) = E(M) \cup E(P_{i+1}) \). We continue like this till all the edges of \( F \) is processed. It is easy to see that the final \( M \) is an \( l(\epsilon) \)-subdivision of \( F \) in \( G \), which also forms a \( 2l(\epsilon) \)-subdivision of \( K_t \) in \( G \).

\[ \blacksquare \]

**Remark 2.5** From the proof of Theorem 2.4, one can see that a weaker version of Lemma 2.3 will suffice. Indeed, we only need to ensure that for each vertex \( u \) there are say at least \( t^2 n^{1/2} \) vertices \( x \) each of which is joined to \( u \) by at least \( t^2 l(\epsilon) \) internally disjoint paths. However, in the proof of Lemma 2.3 we deliberately argued that there are at least \( n^{1 - \frac{1}{2} \epsilon^2} \) such \( x \) as a way to ensure that there are at least \( t^2 n^{1/2} \) such \( x \). In fact, this was what introduced the \( \ln \frac{1}{\epsilon} \) factor in Theorem 1.10. So, in order to improve Theorem 1.10, one will likely have to improve the arguments used in the proof of Lemma 2.3.

# 3 Proof of Theorem 1.10

In this section, we prove Theorem 1.10. Let us first recall some facts from [12]. For a vertex \( u \) in a graph \( G \), and a positive integer \( i \), let \( D^u_i(G) = \{x \in V(G) : d_G(u, x) = i\} \). In other words, \( D^u_i(G) \) is the set of vertices at distance \( i \) from \( u \) in \( G \). When the host graph \( G \) is clear, we write \( D^u_i \) for \( D^u_i(G) \). The following lemma was proved in [12]. For completeness, we include its short proof.

**Lemma 3.1** [12] Let \( c, \epsilon \) be positive reals, where \( 0 < \epsilon < 1 \). Let \( G \) be a \((c, \epsilon)\)-dense graph and \( u \) a vertex in \( G \). Then there exists an \( i \) such that \( G[D^u_i \cup D^u_{i+1}] \) is \((\frac{c}{2}, \epsilon)\)-dense.

**Proof.** Suppose there is no such \( i \). For each \( i \) let \( d_i = |D^u_i \cup D^u_{i+1}| \). We have \( e(G[D^u_i \cup D^u_{i+1}]) < (c/2)(d_i)^{1+\epsilon} < (c/2)d_i n^\epsilon \) for all \( i \), where \( n = n(G) \). This yields

\[ e(G) \leq \sum_i e(G[D^u_i \cup D^u_{i+1}]) < cn^\epsilon (1/2) \sum_i d_i \leq cn^{1+\epsilon}, \]

contradicting \( G \) being \((c, \epsilon)\)-dense.

\[ \blacksquare \]

The following simple observation somewhat surprisingly played a key role in [12].

**Observation 3.2** Let \( H \) be a bipartite graph with a bipartition \((X, Y)\). Suppose there is a path \( P \) of \( 2m \) vertices in \( H \), where the vertices on \( P \) are \( a_1, b_1, a_2, b_2, \ldots, a_m, b_m \) in order. Then either all the \( a_i \)'s or all the \( b_i \)'s are contained in \( X \).
Now, we are ready to prove our main result.

**Proof of Theorem 1.10:** Let \( l(\varepsilon) = \frac{\ln 2}{\varepsilon} \ln \frac{1}{\varepsilon} \). It is well-known that \( G \) contains a spanning bipartite subgraph \( G' \) with \( e(G') \geq \frac{1}{2} e(G) \geq 2^{n^2} \cdot n^{\varepsilon + \varepsilon} \). Let \( G_0 = G' \), we iteratively define a sequence of subgraphs of \( G' \) as follows. Note that these graphs are all bipartite since \( G' \) is bipartite. For convenience let \( \beta = 2^{n^2} \). Let \( G_1 \) be a \((\beta/2^6, \varepsilon)\)-dense subgraph of \( G_0 \) of radius at most \( l(\varepsilon) \) with center \( u_1 \), if it exists. By Lemma 3.1, for some two consecutive distance classes \( X_1, Y_1 \) from \( u_1 \) in \( G_1 \) the subgraph \( H_1 \) induced by them is \((\beta/2^7, \varepsilon)\)-dense, where \( X_1 \) denotes the distance class of the two that is closer to \( u_1 \). Note that since \( G_1 \) is bipartite, each distance class from \( u_1 \) is an independent set. Thus, \((X_1, Y_1)\) is a bipartition of \( H_1 \). If it exists, let \( G_2 \) be a subgraph of \( H_1 \) that is \((\beta/2^{13}, \varepsilon)\)-dense and has radius at most \( l(\varepsilon) \). Let \( u_2 \) be a vertex in the center of \( G_2 \). By Lemma 3.1, for some two consecutive distance classes \( X_2, Y_2 \) from \( u_2 \) in \( G_2 \) the subgraph \( H_2 \) induced by them is \((\beta/2^{24}, \varepsilon)\)-dense, where \( X_2 \) denotes the distance class of the two that is closer to \( u_2 \). As before, \((X_2, Y_2)\) forms a bipartition of \( H_2 \). We continue like this. Suppose we have defined \( G_1, H_1, G_2, H_2, \ldots, G_i, H_i \), where \( G_i \) is \((\beta/2^{7i-1}, \varepsilon)\)-dense and has radius at most \( l(\varepsilon) \) and \( u_i \) is a vertex in the center, and \( H_i \) is a subgraph of \( G_i \) induced by some two consecutive distance classes from \( u_i \) and is \((\beta/2^{7i}, \varepsilon)\)-dense. If it exists, let \( G_{i+1} \) be a subgraph of \( H_i \) that is \((\beta/2^{7i+6}, \varepsilon)\)-dense and has radius at most \( l(\varepsilon) \); let \( u_{i+1} \) be a vertex in its center. Then for some two consecutive distance classes \( X_{i+1}, Y_{i+1} \) of \( G_{i+1} \) from \( u_{i+1} \) the subgraph \( H_{i+1} \) induced by them is \((\beta/2^{7(i+1)}, \varepsilon)\)-dense, where \( X_{i+1} \) is closer to \( u_{i+1} \) than \( Y_{i+1} \) and \((X_{i+1}, Y_{i+1})\) forms a bipartition of \( H_{i+1} \).

Clearly, this process eventually terminates, suppose \( G_m, H_m \) are the last graphs in the sequence. We consider two main cases.

**Case 1.** \( m \geq t(t-1) \).

Let \( s = t(t-1) \). Since \( H_s \) is \((\beta/2^{7s}, \varepsilon)\)-dense, in particular \( e(H_s)/n(H_s) \geq \beta n^s/2^{7s} \geq 2^{t-1}2^t \). Thus, \( H_s \) contains a subgraph \( F \) with \( \delta(F) \geq 2t \). \( F \) contains a path \( P \) on \( 2t \) vertices. Since \( H_1 \supseteq H_2 \supseteq \cdots \supseteq H_s \), \( P \) lies in all of \( H_1, \ldots, H_s \). By our construction for each \( i \), \( H_i \) is a bipartite subgraph of \( G_i \) induced by some two consecutive distance classes \( X_i, Y_i \) from the center \( u_i \) where \((X_i, Y_i)\) forms a bipartition of \( H_i \) and \( X_i \) is the distance class of the two that is closer to \( u_i \).

Let \( a_1, b_1, \ldots, a_t, b_t \) be the vertices on \( P \) in order. By Observation 3.2, for each \( i \in [s] \) we have either \( a_1, \ldots, a_t \subseteq X_i \) or \( b_1, \ldots, b_t \subseteq X_i \). Let \( I \) denote the set of indices \( i \) for which the first scenario occurs and \( I' \) the set of indices for which the second scenario occurs. Without loss of generality, we may assume that \( |I| \geq s/2 \).

For each \( i \in I \), we can connect any two \( a_x, a_y \in \{a_1, \ldots, a_t\} \) by a path \( Q_{x,y} \) of length at most \( 2\text{rad}(G_i) \leq 2l(\varepsilon) \) through the center \( u_i \) such that \( V(Q_{x,y}) \cap V(G_{i+1}) = \{a_x, a_y\} \). Using \( G_i \), \( i \in I \), we can find such \( Q_{x,y} \) for all \( x, y \in \{1, \ldots, t\}, x \neq y \). These paths have no common internal vertices and their union is a subdivision of \( K_t \) in which each edge of \( K_t \) is replaced by a path of length at most \( 2l(\varepsilon) = \frac{10}{\varepsilon} \ln \frac{1}{\varepsilon} \).

**Case 2.** \( m < t(t-1) \).

By our assumption, \( H_m \) is \((\beta/2^{7m}, \varepsilon)\)-dense but contains no \((\beta/2^{7m+6}, \varepsilon)\)-dense subgraph of radius at most \( l(\varepsilon) \), since otherwise \( G_{m+1} \) would have been defined. Since \( e(H_m) \geq \frac{\beta}{2^{7m}} n(H_m)^{\varepsilon} \cdot n(H_m) \geq \frac{\beta}{2^{7m}} [n(F)]^{\varepsilon} \). Also, since \( F \subseteq H_m \), \( F \) contains no \((\beta/2^{7m+6}, \varepsilon)\)-dense subgraph of radius at most
l(ε). By Corollary 2.4, with \( c = \frac{2}{7}m \geq \frac{2n^2 - 1}{2m - 1} \geq 1 \), \( F \) contains a \( 2l(\epsilon) \)-subdivision of \( K_t \). Hence, \( G \) contains a \( \frac{10}{\epsilon} \ln \frac{1}{\epsilon} \)-subdivision of \( K_t \).

4 Concluding Remarks

It would be interesting to see if the \( \ln \frac{1}{\epsilon} \) factor in Theorem 1.10 can be eliminated. As mentioned in Remark 2.5, this likely requires an improvement of Lemma 2.3. It would be interesting to study \( ex(n, H(\leq p)) \) for general graphs \( H \). One could get general bounds using our approach for \( ex(n, K_t^{(\leq p)}) \), but it is conceivable that for special classes of graphs such as graphs with bounded maximum degree sharper bounds can be obtained.

One could similarly define \( H^{(p)} \) to be the graph obtained from \( H \) by subdividing each edge of \( H \) exactly \( p - 1 \) times and study \( ex(n, H^{(p)}) \). One earlier result of this type was obtained by Faudree and Simonovits [8], described as follows. Let \( C_{m,k} \) denote the graph obtained from joining two vertices \( x \) and \( y \) by \( m \) internally disjoint paths of length \( k \). Faudree and Simonovits [8] showed for fixed \( m \) and \( k \) that \( ex(n, C_{m,k}) = O(n^{1+1/k}) \), extending the classic result of Bondy and Simonovits [4] that \( ex(n, C_{2k}) = O(n^{1+1/k}) \). We may view \( C_{m,k} \) as obtained from the graph consisting of two vertices joined by \( m \) multiple edges by subdividing each of those edges \( k - 1 \) times.

In general, there are very few results on graphs whose Turán number is \( O(n^{1+\epsilon}) \) for \( \epsilon \) close to 0. It is easy to see by the usual Turán lower bound in terms of density that such a graph \( H \) must have average degree close to 2. Since degree 1 vertices also don’t affect the Turán number substantially such a graph must then have most of its vertices having degree 2. This means \( H \) is some sort of “long” subdivision of another graph. This observation somewhat indicates that studying \( p \)-subdivisions is a meaningful step towards the study of graphs whose Turán number is \( O(n^{1+\epsilon}) \) for small \( \epsilon \).

References


