NONCOMMUTATIVE ALGEBRAIC GEOMETRY

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Abstract. After a brief review of commutative algebraic geometry, I introduce the new field of noncommutative projective algebraic geometry. Examples are given and my research is briefly described.

1. Algebraic geometry

One of the pervasive themes in mathematics over the last decade has been the concept of quantization. This term is borrowed from physics and has come to mean taking a commutative structure and generalizing its important properties to a noncommutative structure, just as classical Newtonian mechanics has been generalized and corrected by quantum mechanics. Arguably, at the core of mathematics lies the study of algebraic geometry. It has a power and elegance unrivaled in modern mathematics. Here I will give a broad introduction to the new and growing field of "noncommutative algebraic geometry," that is, a quantization of classical algebraic geometry.

First, we must examine the commutative case. This paper will give a very sparse treatment of the material, as it is really the subject of a year-long course. For the algebraic background, I recommend [AM]. For the algebraic geometry, one should read [H].

Algebraic geometry is concerned with the study of curves, surfaces, and higher dimensional objects (called *varieties*) which are solutions of polynomial equations. When one graphs a curve such as $y^2 = x^3$, one is revealing a small part of this connection. Over the past 70 years, mathematicians have made this connection strong, beautiful, and exact.

There is a strong correlation (category equivalence) between certain commutative rings and geometric objects known as *affine* algebraic varieties. More specifically, affine algebraic varieties correspond to rings finitely generated over a field. For simplicity, I will take that field to be $\mathbb{C}$. As an example, consider the curves $y = x^2$ and $y = 0$. Then

$$\begin{aligned} y = x^2 &\quad \text{corresponds to} \quad \mathbb{C}[x,y]/(y - x^2) \text{ and} \\ y = 0 &\quad \text{corresponds to} \quad \mathbb{C}[x,y]/(y). \end{aligned}$$

However, these two rings are isomorphic since

$$\begin{aligned} \mathbb{C}[x,y]/(y - x^2) &\cong \mathbb{C}[x,x^2] = \mathbb{C}[x] \\ \mathbb{C}[x,y]/(y) &\cong \mathbb{C}[x]. \end{aligned}$$

So in terms of algebraic geometry, $y = x^2$ and $y = 0$ are the same.

On the other hand

$$\mathbb{C}[x,y]/(y^3 - x^2) \ncong \mathbb{C}[x]$$

so $y^3 = x^2$ is not the same (geometrically) as $y = x^2$ or $y = 0$. One can describe these differences more precisely through the language of commutative algebra than would be possible with graphs.

2. Projective algebraic geometry

Affine varieties are the building blocks of general varieties. By "glueing" together affine varieties, one can build more complicated structures with useful properties. *Projective* varieties are arguably the most important class of varieties. For instance, when working over $\mathbb{C}$, they are varieties which are compact sets in the Euclidean topology. (One must be careful when speaking of the topology of an algebraic variety; the usual topology is the "Zariski topology," which is quite different from the Euclidean.)

The simplest example of a projective variety is $\mathbb{P}^n$ – projective n-space. This has $n+1$ coordinates $(a_0 : a_1 : \cdots : a_n)$, $a_i \in \mathbb{C}$, with some $a_i \neq 0$. Further

$$(a_0 : a_1 : \cdots : a_n) = (\lambda a_0 : \lambda a_1 : \cdots : \lambda a_n)$$

if $\lambda \neq 0$. So in particular, $\mathbb{P}^1$ has coordinates $(a_0 : a_1)$.

Setting $a_0 = 1$, we get an affine space $\mathbb{A}^n$. That is, we have the set of all $(a_1, a_2, \ldots, a_n)$. Similarly, setting any $a_i = 1$ yields an affine space $\mathbb{A}^n$. Thus $\mathbb{P}^n$ is $n+1$ copies of $\mathbb{A}^n$ glued together.

In general, projective varieties are closed subsets of projective n-space. They can be defined by *homogeneous* equations, that is, polynomials where each monomial has the same total degree. For example $xy - zw = 0$ in $\mathbb{P}^3$ is a projective variety, known as a quadric surface.

There are many methods for studying projective varieties. If the variety has relatively few singularities (for example normal), one of the most useful tactics is studying a variety's $n-1$-dimensional closed subvarieties, called *prime divisors.* More generally, one studies sums and differences of prime divisors, which are called *divisors. Ample* divisors are especially important. Some multiple mD of an ample divisor is a hyperplane slice through X. (Actually, there are two kinds of divisors: Weil divisors and Cartier divisors. It is beyond the scope of this paper to handle the different definitions. See [H, Ch. II.6] for more details.)

It is also useful to study *sheaves* on a variety X. A sheaf is a collection of abelian groups associated to the open subsets of X, along with extra axioms. If $U \subset X$ is open and $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}(U)$ is the associated abelian group. $\mathcal{F}(X)$ are the *global sections* of $\mathcal{F}$, sometimes denoted $\Gamma(X, \mathcal{F})$ or $\Gamma(\mathcal{F})$.

If X is nice enough (for example smooth) then one can associate a sheaf $\mathcal{L}$ to each divisor D. Further one can tensor these sheaves, so

$$\mathcal{L}^n = \mathcal{L} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}.$$

One can define a graded ring with the degree n piece equal to $\Gamma(\mathcal{L}^n)$, where $\mathcal{L}$ is associated to an ample divisor. This is known as a *homogeneous coordinate ring.* For example, there is an ample sheaf $\mathcal{L}$ on $\mathbb{P}^1$ with

$$\Gamma(\mathcal{L}^n) = \{\text{degree } n \text{ homogenous polynomials in } x, y\}.$$

The associated ring is $\mathbb{C}[x, y]$. For the quadric surface, a homogenous coordinate ring is $\mathbb{C}[x, y, z, w]/(xy - zw)$.

A homogeneous coordinate ring had several nice properties:

(1) Finitely generated over a field,

(2) Noetherian,
(3) The graded pieces grow "like a polynomial."

3. Noncommutative projective algebraic geometry

We have seen that a commutative homogeneous coordinate ring can be associated to a projective variety and these rings have nice properties since they come from geometric data. We would like to associate a noncommutative ring to a projective variety and hopefully get similar nice properties. One solution is *twisted homogeneous coordinate rings*. These were introduced in [ATV] and their basic properties were examined in [AV]. (Noncommutative algebraic geometry has now grown to include more than just twisted homogeneous coordinate rings. However, unanswered questions still remain about twisted homogeneous coordinate rings and this paper will only deal with them. Consult [AZ] for other developments.)

To build such a twisted ring, let X be a projective variety, σ be an automorphism of X, and $\mathcal{L}$ be a sheaf associated to a divisor. Define a ring with nth graded piece

$$B_n = \Gamma(\mathcal{L} \otimes \sigma^*\mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L}).$$

If $x \in B_n$ and $y \in B_m$, then $x * y = x\sigma^n(y)$.

Example 1. $\mathbb{P}^1$ has an automorphism $\sigma : (a_0 : a_1) \mapsto (qa_0 : a_1)$ with $q \in \mathbb{C}^*$. We let $\mathcal{L}$ be the *Serre twisting sheaf* $\mathcal{O}(1)$. $B_1 = \Gamma(\mathcal{O}(1))$ has basis x, y and the induced action is given by $\sigma(x) = qx, \sigma(y) = y$.

So the new multiplication is given by

$$\begin{aligned} x * y &= x\sigma(y) = xy \\ y * x &= y\sigma(x) = yqx = qxy = q(x * y). \end{aligned}$$

Or more succinctly, $y * x = qx * y$. The associated twisted ring is then

$$\mathbb{C}\{x, y\}/(y * x - qx * y).$$

Example 2. $\mathbb{P}^1$ has another automorphism, $\sigma : (a_0 : a_1) \to (a_0 : a_1 - a_0)$. With the same $\mathcal{L}$, we again have an induced action on $x, y \in B_1$, which leads to the multiplication

$$\begin{aligned} x * y &= x\sigma(y) = x(y - x) \\ y * x &= y\sigma(x) = yx = xy = x * y + x^2 \end{aligned}$$

So $y * x = x * y + x^2$ and the associated ring is

$$\mathbb{C}\{x, y\}/(y * x - x * y - x^2).$$

We would like these twisted homogeneous coordinate rings to have the same nice properties as the "usual" homogeneous coordinate rings. When do they?

4. σ-ample divisors

In the classical case, ample divisors gave good coordinate rings, so it might be useful to generalize the definition of ampleness. It turns out that there is such a useful generalization, σ-ample divisors, which correspond to σ-ample sheaves.

Since we are dealing with noncommutative objects, a priori, we may need to define σ-ampleness on the right and on the left. The definition uses sheaf cohomology, a subject best explained in [H, Ch. III]. We call $\mathcal{L}$ a right σ-ample sheaf if for all coherent sheaves $\mathcal{F}$, there exists an m_0 such that

$$H^i(\mathcal{F} \otimes \mathcal{L} \otimes \cdots \otimes (\sigma^{m-1})^*\mathcal{L}) = 0,$$

for $i > 0$ and $m \geq m_0$. The ring is then finitely generated and right Noetherian. A sheaf $\mathcal{L}$ is left σ-ample if for all coherent sheaves $\mathcal{F}$, there exists an m_0 such that

$$H^i(\mathcal{L} \otimes \cdots \otimes (\sigma^{m-1})^*\mathcal{L} \otimes (\sigma^m)^*\mathcal{F}) = 0,$$

for $i > 0$ and $m \geq m_0$. In this case the ring is finitely generated and left Noetherian.

This definition is hard to check. So when is the sheaf corresponding to a divisor σ-ample? When do they even exist? Is there a connection between right and left σ-ampleness? My research answers those questions.

I have found a simple criterion for σ-ampleness for any variety, which was already known for curves and smooth surfaces. (At the conference I mentioned that I was still working on the characteristic p case; since then, I have completed the necessary proofs. Details of my research will be forthcoming in [K].) The criterion is

Theorem 1. *Let X be a projective variety with automorphism σ and P the action of σ on numerical equivalence classes of Cartier divisors. Let D be a Cartier divisor. D is σ-ample if and only if P is quasi-unipotent (i.e., all eigenvalues are roots of unity) and*

$$D + \sigma D + \cdots + \sigma^{m-1} D$$

is ample for some $m > 0$.

The major consequence of this criteria is the equivalence of right and left σ-ampleness. Thus the twisted homogeneous coordinate ring corresponding to a σ-ample sheaf is both right and left Noetherian. Also, a σ-ample sheaf exists if and only if P is quasi-unipotent, since in this case any ample sheaf is σ-ample.

I have also shown σ-ampleness causes the twisted homogeneous coordinate ring to grow "like a polynomial." More precisely, the ring has finite Gel'fand-Kirillov dimension in the sense of [KL].

Noncommutative projective algebraic geometry is a very new and exciting field of study. I am glad to have contributed a bit to its growth and look forward to seeing how the subject develops in the years to come.

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