Noncommutative ample divisors

by

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To my parents, who always encouraged me to learn

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CHAPTER 1

Introduction

The interplay between algebra and geometry has been one of the most important mathematical ideas of the last century. Through the use of "ample invertible sheaves," it is well known that the class of projective schemes over a field k is nearly equivalent to the class of commutative, finitely generated, graded k-algebras. Many theorems for projective schemes can be translated into theorems for commutative graded rings, and vice versa. In the past ten years a study of "noncommutative projective geometry" has flourished. By using and generalizing techniques of commutative projective geometry, one can study certain noncommutative rings and obtain results for which no purely algebraic proof is known.

The most basic building block of the theory is the twisted homogeneous coordinate ring. Let X be a projective scheme over an algebraically closed field k with σ a scheme automorphism and let $\mathcal{L}$ be an invertible sheaf on X. In [ATV] a twisted version of the homogeneous coordinate ring $B = B(X, \sigma, \mathcal{L})$ of X was invented with the grading $B = \oplus B_m$ for

$$B_m = H^0(X, \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}})$$

where $\mathcal{L}^{\sigma} = \sigma^* \mathcal{L}$ is the pullback of $\mathcal{L}$. Multiplication on sections is defined by $a \cdot b = \varphi(a \otimes b^{\sigma^m})$ where $a \in B_m$, $b \in B_n$ and φ is the natural map $B_m \otimes (\sigma^m)^* B_n \to B_{m+n}$.

Soon after their seminal paper, Artin and Van den Bergh formalized much of the theory of these twisted homogeneous coordinate rings in [AV]. In the commutative case, the most useful homogeneous coordinate rings are associated with an ample invertible sheaf. A generalization of ampleness was therefore needed and defined as follows.

An invertible sheaf $\mathcal{L}$ is called right σ-*ample* if for any coherent sheaf $\mathcal{F}$,

$$H^q(X, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}) = 0$$

for $q > 0$ and $m \gg 0$. Similarly, $\mathcal{L}$ is called left σ-*ample* if for any coherent sheaf $\mathcal{F}$,

$$H^q(X, \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}} \otimes \mathcal{F}^{\sigma^m}) = 0$$

for $q > 0$ and $m \gg 0$. A divisor D is called right (respectively left) σ-ample if $\mathcal{O}_X(D)$ is right (respectively left) σ-ample. If σ is the identity automorphism, then these conditions are the same as saying $\mathcal{L}$ is ample. Artin and Van den Bergh proved that if $\mathcal{L}$ is right (respectively left) σ-ample, then B is a finitely generated right (respectively left) noetherian k-algebra [AV].

Twisted homogeneous coordinate rings have been instrumental in the classification of rings, such as the 3-dimensional Artin-Schelter regular algebras [ATV, St1, St2] and the 4-dimensional Sklyanin algebras [SS]. Artin and Stafford showed that any connected (i.e. $B_0 = k$) graded domain of GK-dimension 2 generated by B_1 is the twisted homogeneous coordinate ring (up to a finite dimensional vector space) of some projective curve X, with automorphism σ and (left and right) σ-ample $\mathcal{L}$ [AS]. Therefore any such ring is automatically noetherian!

While the concept of noncommutative schemes has grown to encompass more than just twisted homogeneous coordinate rings (cf. [AZ]), they remain a guide for how such a scheme ought to behave. However, fundamental open questions about these coordinate rings and σ-ample divisors have persisted for the past decade. In [AV], the authors derived a simple criterion for a divisor to be σ-ample in the case X is a curve, a smooth surface, or certain other special cases. With this criterion, they showed that B must have finite GK-dimension. In other words, they showed that B has polynomial growth. They ask

Questions 1.0.1. [AV, Question 5.19]

1. What is the extension of our simple criterion to higher dimensions?

2. Does the existence of a σ-ample divisor imply that B has polynomial growth?

The second question was asked again after [AS, Theorem 4.1].

One would also like to know if the conditions of right and left σ-ampleness are related and if B could be right noetherian, but not left noetherian. One might ask for which (commutative) schemes and automorphisms a σ-ample divisor even exists and if one can be easily found.

In this thesis, all these questions will be settled very satisfactorily. The main results are given in Chapter 3 where we prove:

Theorem 1.0.2 (See §3.5, 3.6). *The following are true for any projective scheme X over an algebraically closed field.*

1. *Right and left σ-ampleness are equivalent. Thus every associated B is (right and left) noetherian.*

2. *A projective scheme X has a σ-ample divisor if and only if the action of σ on*

numerical equivalence classes of divisors is quasi-unipotent (cf. §3.3 for definitions). In this case, every ample divisor is σ-ample.

3. GKdim B *is an integer if* $B = B(X, \sigma, \mathcal{L})$ *and* $\mathcal{L}$ *is* σ*-ample. Here* GKdim B *is the Gel'fand-Kirillov dimension of* B *in the sense of* [KL].

These facts are all consequences of

Theorem 1.0.3 (See Remark 3.5.2). *Let X be a projective scheme with automorphism σ. Let D be a Cartier divisor. D is (right) σ-ample if and only if σ is quasi-unipotent and*

$$D + \sigma D + \cdots + \sigma^{m-1} D$$

is ample for some $m > 0$.

This is the "simple criterion" which was already known if X is a smooth surface [AV, Theorem 1.7]. We obtain the result mainly by use of Kleiman's numerical theory of ampleness [K].

Besides the results above, we derive other corollaries in §3.5 and find bounds for the GK-dimension in §3.6 via Riemann-Roch theorems. We also examine what happens in the non-quasi-unipotent case and obtain

Theorem 1.0.4 (See Theorem 3.6.3). *Let X be a projective scheme with automorphism σ. Then the following are equivalent:*

1. *The automorphism σ is quasi-unipotent.*

2. *For all ample divisors D, $B(X, \sigma, \mathcal{O}_X(D))$ has finite GK-dimension.*

3. *For all ample divisors D, $B(X, \sigma, \mathcal{O}_X(D))$ is noetherian.*

In Chapter 4 we will then generalize our results to the case of multi-homogeneous coordinate rings in the sense of [C]. These rings will be defined more fully in that chapter. In part, we obtain

Theorem 1.0.5 (See Corollaries 4.2.4, 4.2.5). *Let X be a projective scheme with automorphism σ and σ-ample invertible sheaf $\mathcal{L}$. Let Y be a projective scheme with automorphism τ and τ-ample invertible sheaf $\mathcal{M}$. Set $B = B(X, \sigma, \mathcal{L})$ and $B' = B(Y, \tau, \mathcal{M})$. Then*

1. *If B is generated in degree one, then the Rees ring $B[It] = \oplus I^r t^r$ is noetherian where $I = B_{>0}$.*

2. *The ring $B \otimes B'$ is noetherian.*

Chapter 2 contains previously known results which are relevant to this thesis. In particular, it covers the definition of twisted homogeneous coordinate rings in detail and connects these rings to the more recent category-theoretic work of [AZ]. We also review well-known results from classical algebraic geometry, mainly related to intersection theory.

Chapter 3 is then the heart of the thesis, proving new results for twisted homogeneous coordinate rings. In particular it covers Theorems 1.0.2–1.0.4. Most of the material in that chapter has appeared in [Ke1]. The new results of Chapter 4 pertain to multi-homogeneous coordinate rings and will appear in [Ke2].

CHAPTER 2

Background material

2.1 Introduction

This chapter will cover previously known results pertaining to this thesis. Section 2.2 will define twisted homogeneous coordinate rings of a projective scheme. We save most of the analysis of these rings for §2.5.

Sections 2.3 and 2.4 describe a more general construction of "coordinate rings" which does not require the presence of a projective scheme, but rather just a nicely behaved abelian category. This construction greatly simplifies proofs and reveals more of what is "really going on." Then in §2.5, we relate twisted homogeneous coordinate rings to these categorical coordinate rings to deduce noetherian properties for twisted homogeneous coordinate rings.

Finally, in §2.6 we recall well-known results of classical algebraic geometry which will be of use, particularly intersection theory and the numerical theory of ampleness.

2.2 Definitions and an example

In this section, we will introduce the twisted homogeneous coordinate ring more formally and give a simple example. The idea of a twisted homogeneous coordinate ring relies heavily on commutative geometry, so we must assume familiarity with the ideas in [H2]. However, we will attempt to point the reader to the appropriate material in that lucid book.

First, we must fix our terminology and notation for graded rings. A *graded ring* is a ring R with an abelian group decomposition

$$R = \bigoplus_{i=0}^{\infty} R_i$$

such that $R_i R_j \subseteq R_{i+j}$. One can also grade over $\mathbb{Z}$ or even other monoids, but in this thesis, all rings will be $\mathbb{N}$-graded. Usually in noncommutative geometry, these rings are k-algebras, where k is an algebraically closed field. In this case, each R_i is a vector space over k and $k \subseteq R_0$. If $\dim_k R_i < \infty$ for all i, then R is called *finitely graded.* Note that this is automatic if R is finitely generated as a k-algebra and $\dim_k R_0 < \infty$. *All graded rings in this thesis will be finitely graded.* If $R_0 = k$ and R is finitely graded, then R is defined to be *connected.* A *graded right R-module* M is a right R-module with a decomposition

$$M = \bigoplus_{i=-\infty}^{\infty} M_i$$

such that $M_i R_j \subseteq M_{i+j}$. Note that M is $\mathbb{Z}$-graded.

The submodule $M_{\geq s}$ is the module

$$M_{\geq s} = \bigoplus_{i=s}^{\infty} M_i.$$

We define $M_{>s}$ similarly. The module $M[n]$ is the *shifted module* with graded pieces $M[n]_i = M_{n+i}$. We call this functor the *degree shift.*

Now we need to recall some algebraic geometry. Let X be a proper scheme over an algebraically closed field k and let $\mathcal{F}$ be a coherent sheaf on X. The global sections $H^0(X, \mathcal{F})$ form a finite dimensional vector space over k [H2, p. 252, Remark 8.8.1]. If $\mathcal{L}$ is an invertible sheaf on X, one can form a finitely graded ring $B = B(X, \mathcal{L})$, known as a *homogeneous coordinate ring.* The graded pieces are

$$B_m = H^0(X, \mathcal{L}^m),$$

where $\mathcal{L}^m = \mathcal{L}^{\otimes m}$. The isomorphisms $\mathcal{L}^m \otimes \mathcal{L}^n \tilde{\to} \mathcal{L}^{m+n}$ induce a natural multiplication $B_m \otimes B_n \to B_{m+n}$.

Now we will look at a twisted homogeneous coordinate ring. Again, let X be a proper scheme and let $\mathcal{L}$ be an invertible sheaf. In addition, let σ be an automorphism of X and denote the pullback $\sigma^* \mathcal{F}$ by $\mathcal{F}^\sigma$. For notational convenience we set

(2.2.1) $$\mathcal{L}_m = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}.$$

Let $B = B(X, \sigma, \mathcal{L})$ be the ring with graded pieces

$$B_m = H^0(X, \mathcal{L}_m).$$

We will first briefly sketch the usual presentation of multiplication in B. Then we will give a more detailed presentation that will show the connection with a category-theoretic notion of homogeneous coordinate rings defined in §2.3. Recall that for any coherent $\mathcal{F}$, one has $H^0(X, \mathcal{F}) \cong \operatorname{Hom}(\mathcal{O}_X, \mathcal{F})$ [H2, p. 234, Proposition 6.3(c)].

For a coherent sheaf $\mathcal{F}$ and integer n, we may define a k-vector space isomorphism $\operatorname{Hom}(\mathcal{O},\mathcal{F})\tilde{\to}\operatorname{Hom}(\mathcal{O},\mathcal{F}^{\sigma^n})$, which we also denote as σ^n, as follows.

Let $f \in \operatorname{Hom}(\mathcal{O},\mathcal{F})$. So f is a collection of maps $f|_U : \mathcal{O}(U) \to \mathcal{F}(U)$, where $f|_U$ is an $\mathcal{O}(U)$-module map. Now the ring $\mathcal{O}(U)$ has an $\mathcal{O}(\sigma^n U)$-module structure, defined via the isomorphism $\varphi^n : \mathcal{O}(U)\tilde{\to}\mathcal{O}(\sigma^n U)$, with φ^n induced by the isomorphism $\mathcal{O}\tilde{\to}\sigma^n_*\mathcal{O}$. The map $\sigma^n(f)|_U$ should be a map from $\mathcal{O}(U)$ to $\mathcal{F}(\sigma^n U) \otimes_{\mathcal{O}(\sigma^n U)} \mathcal{O}(U)$. So we may define

$$\sigma^n(f)|_U = (f|_{\sigma^n U} \circ \varphi^n) \otimes 1.$$

The vector space map σ^n is easily seen to be an isomorphism. We then have a natural multiplication

$$(2.2.2) \qquad H^0(X,\mathcal{L}_n) \otimes H^0(X,\mathcal{L}_m)\tilde{\to}H^0(X,\mathcal{L}_n) \otimes H^0(X,\mathcal{L}_m^{\sigma^n}) \to H^0(X,\mathcal{L}_{n+m}).$$

Now we will define our multiplication a second time, via composition, so that in §2.5 we may study B with the methods of §2.3. First, note that canonically $\mathcal{O}_X \cong \mathcal{O}_X^\tau$ for any automorphism τ. To see this, given any open U, the natural isomorphisms,

$$\mathcal{O}(U)\tilde{\to}\mathcal{O}(\tau U) \otimes_{\mathcal{O}(\tau U)} \mathcal{O}(U) = \mathcal{O}^\tau(U),$$

are given by $1 \mapsto 1 \otimes 1$.

Now we also have natural isomorphisms

$$\mathcal{L}_m\tilde{\to}\mathcal{L}_m \otimes \mathcal{O}^{\sigma^m}.$$

Thus, we may define B with

$$B_m = \operatorname{Hom}(\mathcal{O},\mathcal{L}_m \otimes \mathcal{O}^{\sigma^m}).$$

If $b \in B_m$, there is a unique corresponding map

$$b' \in \operatorname{Hom}(\mathcal{O}^{\sigma^n},\mathcal{L}_m^{\sigma^n} \otimes \mathcal{O}^{\sigma^{n+m}})$$

for any $n \in \mathbb{Z}$. We write this b' as $\sigma^n(b)$. If one considers σ^* as pulling back not only objects but also homomorphisms, then $\sigma^n(b) = (\sigma^n)^*(b)$. Also note that this $\sigma^n(b)$ is nearly the same as the $\sigma^n(b)$ in our first sketch of multiplication.

Then if $a \in B_n, b \in B_m$,

$$a \in \operatorname{Hom}(\mathcal{O},\mathcal{L}_n \otimes \mathcal{O}^{\sigma^n}),$$
$$b' \equiv 1 \otimes \sigma^n(b) \in \operatorname{Hom}(\mathcal{L}_n \otimes \mathcal{O}^{\sigma^n},\mathcal{L}_n \otimes \mathcal{L}_m^{\sigma^n} \otimes \mathcal{O}^{\sigma^{n+m}}).$$

So there is a composition map $(1 \otimes \sigma^n(b)) \circ a$. In order to be consistent with the earlier definition of the multiplication in B we let automorphisms act on the right, so that

$$(1 \otimes \sigma^n(b)) \circ a = a\sigma^n(b) \in \operatorname{Hom}(\mathcal{O},\mathcal{L}_{n+m} \otimes \mathcal{O}^{\sigma^{n+m}}) = B_{n+m}.$$

Finally, we note that using the sheaf isomorphisms $\mathcal{O}\tilde{\to}\sigma^n_*\mathcal{O}$, one can show that the multiplication (2.2.2) is naturally isomorphic to this new multiplication.

Definition 2.2.3. Let $B = \oplus B_m$ where

$$B_m = H^0(X, \mathcal{L}_m \otimes \mathcal{O}^{\sigma^m})$$

and define multiplication by

$$a \cdot b = a\sigma^n(b) \in B_{m+n} \tag{2.2.4}$$

for $a \in B_n, b \in B_m$, where the product $a\sigma^n(b)$ is the one described above. We denote B as $B(X, \sigma, \mathcal{L})$ and call it a *twisted homogeneous coordinate ring.*

Lemma 2.2.5. *Let X be a proper scheme with automorphism σ and invertible sheaf $\mathcal{L}$. Then $B(X, \sigma^{-1}, \mathcal{L}) \cong B(X, \sigma, \mathcal{L})^{\mathrm{op}}$.*

Proof. Set $B = B(X, \sigma^{-1}, \mathcal{L})$ and $B' = B(X, \sigma, \mathcal{L})^{\mathrm{op}}$. Let $\cdot$ be multiplication in B and $*$ be multiplication in B'. There is a natural map $\tau \colon B \to B'$ given by $\tau(a) = \sigma^{n-1}(a)$ for $a \in B_n$, where $\sigma^{n-1}(a)$ is the pullback of a via σ^{n-1} as above. Extend τ linearly so it is a vector space map. It is obviously a vector space isomorphism. Finally, for $a \in B_n, b \in B_m$,

$$\tau(a \cdot b) = \tau(a\sigma^{-n}(b)) = \sigma^{n+m-1}(a)\sigma^{m-1}(b),$$
$$\tau(a) * \tau(b) = \sigma^{n-1}(a) * \sigma^{m-1}(b) = \sigma^{m-1}(b)\sigma^{n+m-1}(a).$$

Thus $\tau(a \cdot b) = \tau(a) * \tau(b)$, as required. □

Example 2.2.6. Let σ be an automorphism of $X = \mathbb{P}^1$. Form the twisted homogeneous coordinate ring

$$B = B(\mathbb{P}^1, \sigma, \mathcal{O}_X(1)) = \bigoplus_{m=0}^{\infty} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^1}(1)^{\sigma^{m-1}}).$$

Note that since $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$, we have $\sigma^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$, so actually $B_m \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$.

For $\mathbb{P}^1$, any automorphism σ is induced by an automorphism (also written σ) of the fraction field $k(u)$ of $\mathbb{P}^1$ [H2, p. 46, Exercise 6.6]. Here $u = y/x$, thinking of $\mathbb{P}^1$ as $\operatorname{Spec} k[x/y] \cup \operatorname{Spec} k[y/x]$. The sheaves $\mathcal{O}_{\mathbb{P}^1}(m)$ can be embedded in the sheaf of rational functions $\mathcal{K}$ so that $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$ is generated by $\{1, u, \dots, u^m\}$ as a k-vector space. So if $x = 1 \in B_1$ and $y = u \in B_1$, then in the commutative multiplication,

$$x^2 = 1 \otimes 1 \mapsto 1,$$
$$xy = 1 \otimes u \mapsto u,$$
$$y^2 = u \otimes u \mapsto u^2 \in B_2.$$

If $q \in k^*$, then $\sigma\colon u \mapsto qu$ is an automorphism of $k(u)$. We have the multiplication rules

$$x \cdot y = 1 \otimes \sigma(u) = 1 \otimes qu \mapsto qu,$$
$$y \cdot x = u \otimes \sigma(1) = u \otimes 1 \mapsto u \in B_2$$

where $\cdot$ is the new multiplication. So we have the multiplication rule $x \cdot y = qy \cdot x$. Thus there is a homomorphism

$$U_q \overset{\text{def}}{=} k\{x, y\}/\langle x \cdot y - qy \cdot x\rangle \xrightarrow{\varphi} B(\mathbb{P}^1, \sigma, \mathcal{O}_{\mathbb{P}^1}(1)). \tag{2.2.7}$$

It is easy to see that φ is surjective by taking a geometric point of view. Because $\mathcal{O}_{\mathbb{P}^1}(1)$ is generated by global sections, there is an exact sequence

$$0 \to \operatorname{Ker} f \to H^0(\mathbb{P}^1, \mathcal{O}_X(1)) \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{f} \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

Tensoring with $\mathcal{O}_{\mathbb{P}^1}(n)$ and taking global sections, we get part of a long exact sequence

$$0 \to H^0(\mathbb{P}^1, \operatorname{Ker} f \otimes \mathcal{O}_{\mathbb{P}^1}(n)) \to B_1 \otimes B_n \to B_{1+n} \to H^1(\mathbb{P}^1, \operatorname{Ker} f \otimes \mathcal{O}_{\mathbb{P}^1}(n)) \to 0.$$

The sheaves involved are the same whether we are in the commutative or noncommutative case since $\sigma^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1)$. So we have in either case $H^1(\mathbb{P}^1, \operatorname{Ker} f \otimes \mathcal{O}_{\mathbb{P}^1}(n)) = 0$ (since we know that this particular H^1 must be 0 for the commutative multiplication maps to be surjective). Now φ respects the grading, each graded piece is finite dimensional, and the dimensions match, so φ must be an isomorphism.

One can repeat the argument with the automorphism $u \mapsto u + 1$. In this case, one constructs the Jordan quantum plane

$$U_J \overset{\text{def}}{=} k\{x, y\}/\langle x \cdot y - y \cdot x - x^2\rangle. \tag{2.2.8}$$

These are the only twisted homogeneous coordinate rings of $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ generated in degree 1, up to isomorphism. We know this because any automorphism acts on x, y as an element of $\mathrm{PGL}(1, k)$ [H2, p. 151, Example 7.1.1], and hence is conjugate to $x \mapsto x, y \mapsto qy$ or $x \mapsto x, y \mapsto x + y$.

Recall that as a set, $X = \operatorname{Proj} R$ is all homogeneous prime ideals of R which do not contain $R_{>0}$, where R is a commutative finitely generated, finitely graded k-algebra. If $B = B(X, \mathcal{L})$ is a commutative homogeneous coordinate ring of $X = \operatorname{Proj} R$, generated in degree one with $\mathcal{L}$ ample, then $\operatorname{Proj} B \cong \operatorname{Proj} R$. We would like to make such a statement in the noncommutative case also. However, noncommutative rings do not have many prime ideals in general. For instance consider

$$U_J/\langle y - \alpha x\rangle = k\{x, y\}/\langle x \cdot y - y \cdot x - x^2, y - \alpha x\rangle \cong k[x]/\langle x^2\rangle,$$

for any $\alpha \in k$. Thus $\langle y - \alpha x\rangle$ is not a prime ideal. Indeed, if $\operatorname{char} k = 0$, then the only homogeneous prime ideals are $\langle 0\rangle$, $\langle x\rangle$, and $\langle x, y\rangle$. By following the usual definition of Proj as a set, one has $\operatorname{Proj} U_J = \{\langle 0\rangle, \langle x\rangle\}$. Another tactic is needed. We will use categories to make a new definition for $\operatorname{Proj} R$ in the next section. We will then return to twisted homogeneous coordinate rings B in §2.5 and discuss how they are related to this new definition of Proj.

2.3 Category-theoretic proj

As stated above, the goal of this section will be to give a new definition of Proj which works well for noncommutative rings. Our treatment is only partial; for the full story, see [AZ, §2–4].

Let R be a right noetherian finitely graded k-algebra. Then $\operatorname{gr} R$ will denote the category of noetherian graded right R-modules. Maps in $\operatorname{gr} R$ are *graded homomorphisms* of degree 0. This means that if $f\colon M \to N$, then $f(M_i) \subseteq N_i$ for all i. Note that the functor $M \mapsto M[n]$ is an autoequivalence of $\operatorname{gr} R$ for any $n \in \mathbb{Z}$.

A module M is *left bounded* if $M_i = 0$ for all $i \ll 0$. Similarly, if $M_i = 0$ for all $i \gg 0$, then M is *right bounded*. We say that M is *bounded* if it is both left and right bounded. Note that any finitely generated graded R-module must be left bounded since R is $\mathbb{N}$-graded. An element $m \in M$ is *torsion* if there exists an s such that $mR_{\geq s} = 0$. Let $\tau(M)$ be the set of all torsion elements of M. The set $\tau(M)$ is easily seen to be a submodule. If $\tau(M) = M$, then M is called *torsion* and if $\tau(M) = 0$, then M is *torsion-free*.

Lemma 2.3.1. *A finitely generated graded module M over a graded ring R is torsion if and only if it is bounded.*

Proof. Suppose that M is torsion and let $m_1, \dots, m_t$ be generators of M. Let $j = \max_i \deg(m_i)$. Then $\sum m_i R_{\geq s - \deg(m_i)} \subseteq M_{\geq s}$ for $s \geq j$. Since M is finitely generated, we have equality for $s \gg 0$. But since M is torsion, the left hand side is 0 for $s \gg 0$. So M is right bounded and since it is finitely generated, it is bounded. The converse is easily seen to be true even if M is not finitely generated. □

Let $\operatorname{tors} R$ be the full subcategory of all torsion modules. This is a *dense subcategory* in the sense that for a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

one has $M \in \operatorname{tors} R$ if and only if $M', M'' \in \operatorname{tors} R$ [P, p. 165]. Thus we can form the *quotient category* $\operatorname{qgr} R = \operatorname{gr} R / \operatorname{tors} R$. The details of this standard construction are in [P, p. 165–173, §4.3]. This category has the same objects as $\operatorname{gr} R$. Let πM denote the image of M in $\operatorname{qgr} R$. Then we define [AZ, Equation 2.2.1]

$$\operatorname{Hom}_{\operatorname{qgr} R}(\pi M, \pi N) = \lim_{s\to\infty} \operatorname{Hom}_{\operatorname{gr} R}(M_{\geq s}, N). \tag{2.3.2}$$

Specifically, given a map $f \in \operatorname{gr} R$, the corresponding map $f \in \operatorname{qgr} R$ is an isomorphism if and only if the map $f \in \operatorname{gr} R$ has bounded kernel and cokernel. Thus, if there exists s such that $M_{\geq s} \cong N_{\geq s}$, then $M \cong N$ in $\operatorname{qgr} R$. The converse holds since M and N are assumed to be noetherian. Because of this, $\operatorname{qgr} R$ is sometimes called $\operatorname{tails} R$. We define $\operatorname{proj} R$ to be the pair $(\operatorname{qgr} R, \pi R)$.

Now if R is commutative and finitely generated by R_1, then $\operatorname{qgr} R \cong \operatorname{coh}(\operatorname{Proj} R)$, where $\operatorname{coh}(X)$ is the category of coherent sheaves on X [H2, p. 125, Exercise 5.9]. This is the idea which can be generalized to noncommutative rings. First, we need a few more definitions. A category $\mathcal{C}$ will be called *k-linear* if it is an abelian category and its objects and Hom groups are k-vector spaces. For more on abelian categories, one may consult [Mac, Chapter VIII] or [P]. For our purposes, one can think of an abelian category as a category of modules over a ring. Of course the module categories we work with are k-linear. Further, we will assume $\mathcal{C}$ is a *noetherian category*. That is, for each object $\mathcal{M}$ in $\mathcal{C}$, any set of subobjects of $\mathcal{M}$ has a maximal member.

We wish to construct a ring from $\mathcal{C}$, mimicking the construction of a homogeneous coordinate ring when $\mathcal{C} = \operatorname{coh}(X)$ for a proper scheme X. As we saw in the last section, given X and an invertible sheaf $\mathcal{L}$, the coordinate ring $B(X, \mathcal{L})$ has the form

$$\bigoplus_{i=0}^{\infty} \operatorname{Hom}(\mathcal{O}_X, \mathcal{L}^i).$$

Also, the functor $\mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{L}$ is an autoequivalence of $\operatorname{coh}(X)$, so we will include an autoequivalence in our construction.

In general, for a noetherian k-linear category $\mathcal{C}$, distinguished object $\mathcal{O}$, and autoequivalence s, we define the *homogeneous coordinate ring* of $(\mathcal{C}, \mathcal{O}, s)$ to be

$$B = \Gamma(\mathcal{C}, \mathcal{O}, s)_{\geq 0} = \Gamma(\mathcal{O})_{\geq 0} = \bigoplus_{i=0}^{\infty} \operatorname{Hom}(\mathcal{O}, s^i\mathcal{O}).$$

Given an object $\mathcal{M} \in \mathcal{C}$, there is a corresponding graded right B-module of the form

$$M = \Gamma(\mathcal{M}) = \bigoplus_{i=-\infty}^{\infty} \operatorname{Hom}(\mathcal{O}, s^i\mathcal{M}).$$

The multiplication on B is given by composition of maps. More specifically, if $a \in B_m = \operatorname{Hom}(\mathcal{O}, s^m\mathcal{O})$ and $b \in B_n = \operatorname{Hom}(\mathcal{O}, s^n\mathcal{O})$, then

$$a \cdot b = s^n(a) \circ b \in \operatorname{Hom}(\mathcal{O}, s^{m+n}\mathcal{O}) = B_{m+n}. \tag{2.3.3}$$

The module structure of M is defined similarly. Notice that in $\operatorname{gr} B$, one has a natural autoequivalence $[+1]$, the degree shift, and

$$\Gamma(\mathcal{M})[+1] = \Gamma(s\mathcal{M}). \tag{2.3.4}$$

We are interested in situations where this induces an equivalence of categories $\Gamma : \mathcal{C} \xrightarrow{\sim} \operatorname{qgr} B$, such that $\mathcal{O} \mapsto \pi B$. Since (2.3.4) automatically holds, we say the triples $(\mathcal{C}, \mathcal{O}, s)$ and $(\operatorname{qgr} B, \pi B, [+1])$ are equivalent. More generally, two triples $(\mathcal{C}, \mathcal{O}, s)$ and $(\mathcal{C}', \mathcal{O}', s')$ are equivalent if there exists an equivalence of categories $\theta \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$ such that $\theta(\mathcal{O}) = \mathcal{O}'$ and for any $\mathcal{M} \in \mathcal{C}$, we have $\theta(s\mathcal{M}) = s'\theta(\mathcal{M})$.

In the case of $\mathcal{C} = \operatorname{coh}(X)$, tensoring with an ample invertible sheaf $\mathcal{L}$ plays a special role, since then one has $\operatorname{qgr} B \cong \operatorname{coh}(X)$, as mentioned above. Thus we need a definition for an ample autoequivalence s or more specifically a pair $(\mathcal{O}, s)$. In order to make the analogy clearer, consider the standard definition of ampleness in the classical case:

Definition 2.3.5. [H2, p. 145, Theorem 7.6, p. 229, Proposition 5.3] An invertible sheaf $\mathcal{L}$ on a proper scheme X is *ample* if one of the following three equivalent conditions hold:

1. For all coherent sheaves $\mathcal{F}$, there exists m_0 such that for all $m \geq m_0$ and $q > 0$,
$$H^q(X, \mathcal{F} \otimes \mathcal{L}^m) = 0.$$
2. For all coherent sheaves $\mathcal{F}$, there exists m_0 such that for all $m \geq m_0$, $\mathcal{F} \otimes \mathcal{L}^m$ is generated by global sections.
3. There exists m such that $\mathcal{L}^m$ is very ample, i.e., $\mathcal{L}^m \cong \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ for some n and some closed immersion φ.

We now make a definition for ample autoequivalences of $\mathcal{C}$, similar to the one above. For an autoequivalence s, there is a quasi-inverse s^{-1}. That is, s^{-1} is an autoequivalence that is a left and right adjoint of s. Given $\mathcal{M} \in \mathcal{C}$, we will denote $s^d\mathcal{M}$ as $\mathcal{M}[d]$ for $d \in \mathbb{Z}$.

Definition 2.3.6. Let $\mathcal{C}$ be a noetherian, k-linear category with a distinguished object $\mathcal{O}$. Let s be an autoequivalence of $\mathcal{C}$. The pair $(\mathcal{O}, s)$ is *ample* if both of the following two conditions hold:

1. For all epimorphisms $\mathcal{M} \twoheadrightarrow \mathcal{N} \in \mathcal{C}$, there exists m_0 such that for all $m \geq m_0$, the natural maps
$$\operatorname{Hom}(\mathcal{O}, \mathcal{M}[m]) \to \operatorname{Hom}(\mathcal{O}, \mathcal{N}[m])$$
are surjective.

2. For all $\mathcal{M} \in \mathcal{C}$, there exist integers $l_0, \ldots, l_t \geq 1$ such that there exists an epimorphism

$$\bigoplus_{i=0}^{t} \mathcal{O}[-l_i] \twoheadrightarrow \mathcal{M}.$$

Let us emphasize again that ampleness depends both on $\mathcal{O}$ and s.

Note that the first condition for ampleness is a weakening of the cohomological definition of ampleness of an invertible sheaf $\mathcal{L}$ (or more formally of $(\mathcal{O}_X, - \otimes \mathcal{L})$). The second condition is a weakening of the global sections definition.

Before stating the main theorem of [AZ], we need one more definition. It will be convenient to have the notation

$$\underline{\mathrm{Hom}}(N, M) = \bigoplus_{i=-\infty}^{\infty} \mathrm{Hom}_{\mathrm{gr}\, R}(N, M[i])$$

and similarly for $\underline{\mathrm{Ext}}^q$. When N and M are finitely generated, we have $\underline{\mathrm{Hom}}(N, M) \cong \mathrm{Hom}_R(N, M)$, the group of all R-module homomorphisms, since any homomorphism can be decomposed into a sum of homomorphisms which preserve the grading, but may shift degrees. In our arguments, we will use whichever representation is more convenient.

Definition 2.3.7. Let R be a finitely graded right noetherian k-algebra. The ring R is said to satisfy χ_j if for all finitely generated modules M and all $l \leq j$,

$$\dim_k \underline{\mathrm{Ext}}^l(R/R_{>0}, M) < \infty.$$

If R satisfies χ_j for all $j \geq 0$, we say R satisfies χ.

The authors of [AZ] discuss many rings which satisfy χ_1. Perhaps most importantly, any commutative noetherian graded k-algebra satisfies χ_1 and, in fact, χ [AZ, Proposition 3.11(c)].

We now state a specialized version of the main theorem of [AZ].

Theorem 2.3.8. ([AZ, Theorem 4.5]) *Let $\mathcal{C}$ be a noetherian k-linear category with distinguished object $\mathcal{O}$ and autoequivalence s. Suppose that*

1. $\dim_k \mathrm{Hom}(\mathcal{O}, \mathcal{M}) < \infty$ *for all* $\mathcal{M} \in \mathcal{C}$.

2. *The pair* $(\mathcal{O}, s)$ *is ample.*

Then $B = \Gamma(\mathcal{O})_{\geq 0}$, the homogeneous coordinate ring of $(\mathcal{C}, \mathcal{O}, s)$, is a right noetherian, finitely graded k-algebra satisfying χ_1. There is an equivalence of triples

$$(\mathcal{C}, \mathcal{O}, s) \cong (\mathrm{qgr}\, B, \pi B, [+1]).$$

Conversely, suppose that R is a right noetherian, finitely graded k-algebra satisfying χ_1. Then $(\operatorname{qgr} R, \pi R, [+1])$ is ample. Also, R and $\Gamma(\operatorname{qgr} B, \pi B, [+1])_{\geq 0}$ differ only by a finite dimensional vector space.

We will only prove part of this theorem, namely that $\Gamma(\mathcal{C}, \mathcal{O}, s)_{\geq 0}$ is right noetherian. We first need a standard lemma.

Lemma 2.3.9. *Let R be a graded ring. The ring R is right noetherian if and only if every homogeneous right ideal of R is finitely generated.*

Proof. The proof is standard and left as an exercise for the reader. □

Proposition 2.3.10. *Let $(\mathcal{C}, \mathcal{O}, s)$ be as in the first paragraph of Theorem 2.3.8. Then $B = \Gamma(\mathcal{C}, \mathcal{O}, s)_{\geq 0}$ is right noetherian.*

Proof. Let N be a homogeneous right ideal of B. Any homogeneous element $x \in N_r \subset B_r$ is an element of $\operatorname{Hom}(\mathcal{O}, \mathcal{O}[r])$. Thus x induces a map $f_x \in \operatorname{Hom}(\mathcal{O}[-r], \mathcal{O})$. Given any finite set X of homogeneous elements of N, let

$$\mathcal{P}_X = \bigoplus_{x \in X} \mathcal{O}[-r_x],$$

where $r_x = \deg(x)$. There is a map $f_X \colon \mathcal{P}_X \to \mathcal{O}$ given by the direct sum of the f_x for $x \in X$.

Set $\mathcal{N}_X = \operatorname{Im}(f_X) \subset \mathcal{O}$. Since $\mathcal{O}$ is noetherian, there is a (unique) maximal $\mathcal{N}_X$. Fix a set X corresponding to the maximal $\mathcal{N}_X$, writing $\mathcal{N} = \mathcal{N}_X$ and $\mathcal{P} = \mathcal{P}_X$. Further, set

$$N'' = \Gamma(\mathcal{N})_{\geq 0}, \quad P'' = \bigoplus_{x \in X} B[-r_x], \quad N' = \sum_{x \in X} xB.$$

Note that $N' = \operatorname{Im}(P'' \to B)$ where the map is given on each component $B[-r_x]$ by left multiplication by x. Also, $N' \subseteq N$.

Let $y \in N_r$ for some r. Since $\mathcal{N}$ is maximal, $\operatorname{Im}(f_y) \subseteq \mathcal{N}$. Thus

$$y \in \operatorname{Hom}(\mathcal{O}, \operatorname{Im}(f_y)[r]) \subseteq \operatorname{Hom}(\mathcal{O}, \mathcal{N}[r]) = \Gamma(\mathcal{N})_r = N''_r$$

and so $N \subseteq N''$.

By definition of $\mathcal{N}$, there is an epimorphism $\mathcal{P} \twoheadrightarrow \mathcal{N}$. Since the pair $(\mathcal{O}, s)$ is ample, the maps $\operatorname{Hom}(\mathcal{O}, \mathcal{P}[n]) \to \operatorname{Hom}(\mathcal{O}, \mathcal{N}[n])$ are onto for $n \gg 0$. Thus $\Gamma(\mathcal{N}) / \operatorname{Im} \Gamma(\mathcal{P})$ is right bounded and

$$N''_{\geq n} = \Gamma(\mathcal{N})_{\geq n} = \operatorname{Im} \Gamma(\mathcal{P})_{\geq n}$$

for $n \gg 0$.

Now consider typical summands $\mathcal{O}[-r]$ and $B[-r]$ of $\mathcal{P}$ and P'' respectively. We have $\Gamma(\mathcal{O}[-r])_n = \mathrm{Hom}(\mathcal{O}, \mathcal{O}[-r+n]) = B_{-r+n}$ *if* $n \geq r$. And so for $n \geq \max_x r_x$, we have $\Gamma(\mathcal{P})_n = P''_n$. Thus $\Gamma(\mathcal{P})/P''$ is right bounded.

Thus we have a chain

$$N'' \supseteq \mathrm{Im}\,\Gamma(\mathcal{P})_{\geq 0} \supseteq \mathrm{Im}\,P'' = N'$$

with each factor bounded. So N''/N' is bounded, hence a finite dimensional k-vector space, hence a right noetherian B_0-module. Thus the submodule N/N' is a right noetherian B_0-module. But it is then also a finitely generated right B-module. By construction, N' is finitely generated. So N is finitely generated, which is what we wished to prove. □

We have chosen to work with the category of noetherian modules since it is more convenient for our purposes. However, the main ideas of this section can be made in terms of $\mathrm{Gr}\,R$, the category of all graded R-modules. A class of objects $\mathcal{C}'$ in a category $\mathcal{C}$ generates $\mathcal{C}$ if for any $\mathcal{M}, \mathcal{N} \in \mathcal{C}$ and $f, g \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ with $f \neq g$, there exists $\mathcal{P} \in \mathcal{C}'$ and $h \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{P}, \mathcal{M})$ such that $fh \neq gh$ [P, p. 4]. The subcategory $\mathrm{gr}\,R$ generates $\mathrm{Gr}\,R$. To see this, suppose that $f, g\colon M \to N$ with $f \neq g$. Choose $m \in M$ with $f(m) \neq g(m)$. The module $P = mR$ is obviously finitely generated and if $h\colon P \to M$ is the natural injection, we have $fh \neq gh$.

If $\mathrm{Tors}\,R$ is the full subcategory of torsion modules in $\mathrm{Gr}\,R$, then $\mathrm{tors}\,R$ generates $\mathrm{Tors}\,R$. (We emphasize here that R is assumed to be right noetherian; otherwise, $\mathrm{Tors}\,R$ must be defined as the category of submodules $\tau(M) \subset M \in \mathrm{Gr}\,R$ such that $\tau(M)$ is the smallest submodule with $M/\tau(M)$ torsion-free [AZ, p. 233].) One can then form the quotient category $\mathrm{QGr}\,R = \mathrm{Gr}\,R/\,\mathrm{Tors}\,R$ and $\mathrm{qgr}\,R$ determines $\mathrm{QGr}\,R$ up to equivalence [AZ, Proposition 2.3]. Thus Theorem 2.3.8 can be expressed without essential change in terms of a *locally noetherian* category $\mathcal{C}$ (meaning it has a noetherian generating subcategory) and $\mathrm{QGr}\,R$. So $\mathrm{Proj}\,R$ is defined to be $(\mathrm{QGr}\,R, \pi R)$.

2.4 Comments on χ

Let us say a few words about the importance of the condition χ_1. We will present a special case where it is clear that the condition χ_1 is necessary to achieve the results of Theorem 2.3.8. Consider the exact sequence

$$0 \to R_{\geq 1} \to R \to R/R_{\geq 1} \to 0.$$

Choose a torsion-free noetherian right R-module M. Applying $\underline{\mathrm{Hom}}(-, M)$, there is an exact sequence

$$0 \to \underline{\mathrm{Hom}}(R/R_{\geq 1}, M) \to M \to \underline{\mathrm{Hom}}(R_{\geq 1}, M) \to \underline{\mathrm{Ext}}^1(R/R_{\geq 1}, M) \to 0.$$

Since M is torsion-free, $\underline{\mathrm{Hom}}(R/R_{\geq 1}, M) = 0$. For any $n > 0$, there is a natural map $\varphi\colon \underline{\mathrm{Hom}}(R_{\geq 1}, M) \to \underline{\mathrm{Hom}}(R_{\geq n}, M)$. Suppose that $f \in \underline{\mathrm{Hom}}(R_{\geq 1}, M)$ goes to the zero map, i.e., $f(R_{\geq n}) = 0$. If $r \in R_i$, $1 \leq i < n$, then $f(r)R_{\geq n-i} = 0$. Since M is torsion-free, we must have $f(r) = 0$, i.e., f is the zero map. Thus φ is an injection. So by (2.3.2), $\underline{\mathrm{Hom}}(R_{\geq 1}, M)$ injects into $\Gamma(\pi M)$.

Now if χ_1 does not hold for M, then $\underline{\mathrm{Ext}}^1(R/R_{\geq 1}, M)$ is not bounded. But then the cokernel of the map $M \to \Gamma(\pi M)$ cannot be bounded, so we do not get the desired isomorphism in $\mathrm{qgr}\, R$. In particular, if R is torsion-free and χ_1 fails for $M = R$, then R and $\Gamma(\pi R)_{\geq 0}$ differ by an infinite dimensional vector space.

Example 2.4.1. Unfortunately, not all right noetherian graded k-algebras satisfy χ_1. Let k be a field of characteristic 0 and $U = U_J = k\{x, y\}/\langle xy - yx - x^2\rangle$, as in Example 2.2.6, and let $R = k + Uy$. In [StaZ, §2] it is shown that R does not satisfy χ_1. In addition, R is (left and right) noetherian. However, one still has that the degree shift $[+1]$ is ample and $(\mathrm{qgr}\, R, \pi R, [+1]) \cong (\mathrm{coh}(\mathbb{P}^1), \mathcal{O}_{\mathbb{P}^1}, s)$ for some s. Thus the coordinate ring $\Gamma(\mathrm{qgr}\, R, \pi R, [+1])_{\geq 0}$ has χ_1 and in fact this ring is U_J [StaZ, Proposition 2.7].

It is not the case that the degree shift is ample for any ring. If one takes $R = k+Uy$ as above and $T = R[z]$ with z a central indeterminate, then T is noetherian by the Hilbert Basis Theorem. Also, $\Gamma(\mathrm{qgr}\, T, \pi T, [+1])_{\geq 0} = T$ [StaZ, Corollary 2.11]. However, T does not satisfy χ_1 and thus $(\pi T, [+1])$ cannot be ample.

One may wonder what effect the stronger condition χ has on a ring. The authors of [AZ] show

Theorem 2.4.2. ([AZ, Corollary 7.5]) *Let $(\mathcal{C}, \mathcal{O}, s)$ be as in the first paragraph of Theorem 2.3.8. Let $H^q(\mathcal{M}) = \mathrm{Ext}^q(\mathcal{O}, \mathcal{M})$ for $q \geq 0$ and $\mathcal{M} \in \mathcal{C}$. Assume that*

1. *$\dim_k H^q(\mathcal{M}) < \infty$ for all $q \geq 0$ and all $\mathcal{M}$.*
2. *For each $\mathcal{M}$, there exists m_0 such that $H^q(\mathcal{M}[m]) = 0$ for all $m \geq m_0$ and $q > 0$.*

Then $\Gamma(\mathcal{C}, \mathcal{O}, s)_{\geq 0}$ satisfies χ.

Conversely, if R is a right noetherian, finitely graded k-algebra which satisfies χ, then $(\mathrm{qgr}\, R, \pi R, [+1])$ satisfies the conditions (1) and (2) of the first paragraph. □

We will see that the twisted homogeneous coordinate rings $B(X, \sigma, \mathcal{L})$, the main objects of study in this thesis, do satisfy χ. However, there are right noetherian rings R which satisfy χ_1 but not χ_2 [StaZ, Corollary 4.4].

2.5 Autoequivalences of coh(X)

We now wish to show the connection between the twisted homogeneous coordinate rings of §2.2 and the category-theoretic coordinate rings of §2.3. To do so, let us discuss the possible autoequivalences of the category coh(X) where X is a proper scheme over k, not necessarily projective. Such an X is also called a complete scheme. Since X is proper, we know $\dim_k H^q(X, \mathcal{F}) < \infty$ for all $q \geq 0$ and any coherent $\mathcal{F}$ [H2, p. 252, Remark 8.8.1]. In §3.2 we will see that non-projective schemes are not interesting for our purposes, so the reader may wish to think of X as being projective.

The autoequivalences of coh(X) can be given in terms of bimodules in a category, which for us will mean the following special objects:

Definition 2.5.1. Let X be a proper scheme with automorphism σ and invertible sheaf $\mathcal{L}$. An *invertible bimodule* is the symbol $\mathcal{L}_\sigma$ with the following actions on a coherent sheaf $\mathcal{F}$:

1. $\mathcal{L}_\sigma^m \otimes \mathcal{F} = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}} \otimes \mathcal{F}^{\sigma^m}$,
2. $\mathcal{F} \otimes \mathcal{L}_\sigma^m = \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$.

The symbol $\mathcal{L}_\sigma$ is called an invertible bimodule because on an open set $U \subseteq X$, $\mathcal{L}_\sigma$ acts like an $\mathcal{O}_X(U)$-module on the right, but as as $\mathcal{O}_X(\sigma U)$-module on the left. There is a more general theory of $\mathcal{O}_X$-bimodules, however to go into this theory would take us too far afield; see [AV, §2] for the general definition.

Consider the following functor $s = \mathcal{L}_\sigma \otimes -$ from coh(X) to itself. So for $\mathcal{F} \in$ coh(X), we set $s(\mathcal{F}) = \mathcal{L} \otimes \mathcal{F}^\sigma$. If $f : \mathcal{F} \to \mathcal{G}$, then $s(f)$ is the natural map $1 \otimes \sigma(f)$. One has

$$s^m \mathcal{F} = \mathcal{L}_m \otimes \mathcal{F}^{\sigma^m},$$

where $\mathcal{L}_m$ is given by the formula (2.2.1). Recall that canonically, $\mathcal{O}_X^\sigma \cong \mathcal{O}_X$. So, if $t = ((\mathcal{L}^{\sigma^{-1}})^{-1})_{\sigma^{-1}} \otimes -$, then $st \cong ts \cong 1_{\mathrm{coh}(X)}$ canonically. Therefore s is an autoequivalence.

We now examine the coordinate ring $R = \Gamma(\mathcal{C}, \mathcal{O}, s)_{\geq 0}$. One has a multiplication as in (2.3.3), setting $R_n = \mathrm{Hom}(\mathcal{O}_X, s^n \mathcal{O}_X)$. For $a \in R_n, b \in R_m$,

$$a \cdot b = s^m(a) \circ b.$$

Now

$$s^m(a) = 1 \otimes \sigma^m(a) \in \mathrm{Hom}(\mathcal{L}_m \otimes \mathcal{O}^{\sigma^m}, \mathcal{L}_m \otimes \mathcal{L}_n^{\sigma^m} \otimes \mathcal{O}^{\sigma^{n+m}}),$$
$$b \in \mathrm{Hom}(\mathcal{O}, \mathcal{L}_m \otimes \mathcal{O}^{\sigma^m}).$$

Thus

(2.5.2) $$a \cdot b = s^m(a) \circ b \in \mathrm{Hom}(\mathcal{O}, \mathcal{L}_{m+n} \otimes \mathcal{O}^{\sigma^{m+n}}) = R_{m+n}.$$

In the notation of (2.2.4), we write this as $\sigma^m(a)b$.

This multiplication is closely related to the one seen in (2.2.4) for twisted homogeneous coordinate rings. First, $R = B(X, \sigma, \mathcal{L})$ as vector spaces over k. Comparing (2.5.2) and (2.2.4) one sees that

Proposition 2.5.3. ([AZ, p. 262]) *Let X be a proper scheme with automorphism σ and invertible sheaf $\mathcal{L}$. Let $B = B(X, \sigma, \mathcal{L})$ and $R = \Gamma(\mathrm{coh}(X), \mathcal{O}_X, \mathcal{L}_\sigma \otimes -)_{\geq 0}$. Then $R \cong B^{\mathrm{op}}$.* □

The reason for our giving more than one presentation for the multiplication is historical; in [AV], the multiplication is defined as in (2.2.2), whereas the multiplication above is more natural in terms of §2.3, which follows [AZ].

For the most part, we simply need that the left action of $\mathcal{L}_\sigma$ is an autoequivalence, as explained above, so that we may use Theorem 2.3.8. However, to make some category-theoretic deductions in §3.7, we will use

Proposition 2.5.4. ([AZ, Corollary 6.9], [AV, Proposition 2.15]) *Let X be a proper scheme. Then any autoequivalence s of* $\mathrm{coh}(X)$ *is naturally isomorphic to $\mathcal{L}_\sigma \otimes -$ for some automorphism σ and invertible sheaf $\mathcal{L}$.* □

Thus, by studying the twisted homogeneous coordinate rings, we are studying all possible coordinate rings $\Gamma(\mathrm{coh}(X), \mathcal{O}_X, s)_{\geq 0}$.

We now define a concept of ampleness for $\mathcal{L}$ and σ. This is the definition given in [AV]. We will then explain how this implies ampleness of the pair $(\mathcal{O}_X, \mathcal{L}_\sigma \otimes -)$ in the sense of Definition 2.3.6.

Definition 2.5.5. Let X be a proper scheme with automorphism σ and invertible sheaf $\mathcal{L}$.

1. We say $\mathcal{L}$ is *right σ-ample* if for all coherent sheaves $\mathcal{F}$, there exists m_0 such that
$$H^q(X, \mathcal{F} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}) = 0,$$
for $m \geq m_0$ and $q > 0$.

2. We say $\mathcal{L}$ is *left σ-ample* if for all coherent sheaves $\mathcal{F}$, there exists m_0 such that
$$H^q(X, \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}} \otimes \mathcal{F}^{\sigma^m}) = 0,$$
for $m \geq m_0$ and $q > 0$.

These two conditions are not unrelated. In fact, one of our main goals in this thesis is to show they are equivalent. But for now, we have

Lemma 2.5.6. ([St3, p. 31]) *An invertible sheaf $\mathcal{L}$ is right σ^{-1}-ample if and only if $\mathcal{L}$ is left σ-ample.*

Proof. Let $\mathcal{L}$ be right σ^{-1}-ample. Then for any coherent sheaf $\mathcal{F}$, there exists an m_0 such that

$$H^q(X, \mathcal{F}^{\sigma} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{-m+1}}) = 0$$

for $q > 0$ and $m \geq m_0$. Since cohomology is preserved under automorphisms, pulling back by σ^{m-1}, we have

$$H^q(X, \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}} \otimes \mathcal{F}^{\sigma^m}) = 0$$

for $q > 0$ and $m \geq m_0$. So $\mathcal{L}$ is left σ-ample. Clearly, the argument can be reversed. □

As promised, these concepts of σ-ampleness imply the categorical ampleness. Suppose that $s = \mathcal{L}_\sigma \otimes -$ and $\mathcal{L}$ is left σ-ample. Let $\mathcal{F} \overset{\varphi}{\twoheadrightarrow} \mathcal{G}$ be an epimorphism of coherent sheaves. Then $H^1(X, s^m(\operatorname{Ker}\varphi)) = 0$ for $m \gg 0$, so $H^0(X, s^m\mathcal{F}) \to H^0(X, s^m\mathcal{G})$ is an epimorphism for $m \gg 0$. Thus $(\mathcal{O}_X, s)$ satisfies condition (1) of Definition 2.3.6. For condition (2) we need

Proposition 2.5.7. ([AV, Proposition 3.2]) *Let X be a projective scheme with automorphism σ and invertible sheaf $\mathcal{L}$. Suppose that $\mathcal{L}$ is left σ-ample. Then for any coherent sheaf $\mathcal{F}$, there exists an integer m_0 such that*

$$\mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}} \otimes \mathcal{F}^{\sigma^m}$$

is generated by global sections for $m \geq m_0$. □

Note that a similar statement holds for right σ-ample via Lemma 2.5.6, since the proposition will hold for $\mathcal{L}$ left σ^{-1}-ample.

Now given this proposition, we have that $s^m\mathcal{F}$ is generated by global sections for some large m. Then there is an epimorphism $\oplus\mathcal{O}_X \twoheadrightarrow s^m\mathcal{F}$ and hence an epimorphism $\oplus s^{-m}\mathcal{O}_X \twoheadrightarrow \mathcal{F}$. So we obtain

Proposition 2.5.8. *Let X be a projective scheme with automorphism σ and invertible sheaf $\mathcal{L}$. If $\mathcal{L}$ is left σ-ample, then $(\mathcal{O}_X, \mathcal{L}_\sigma \otimes -)$ is ample in the sense of (2.3.6).* □

Theorem 2.5.9. ([AV, Theorem 1.4]) *Let X be a projective scheme with automorphism σ and invertible sheaf $\mathcal{L}$. Let $B = B(X, \sigma, \mathcal{L})$. If $\mathcal{L}$ is right σ-ample, then B is a right noetherian, finitely generated k-algebra. If $\mathcal{L}$ is left σ-ample, then B is a left noetherian, finitely generated k-algebra.*

Proof. As noted in Proposition 2.5.3, we have $B^{\mathrm{op}} = \Gamma(\mathrm{coh}(X), \mathcal{O}_X, s)_{\geq 0}$ with $s = \mathcal{L}_\sigma \otimes -$. If $\mathcal{L}$ is left σ-ample, then $(\mathcal{O}_X, s)$ is ample. Thus by Theorem 2.3.8, B is left noetherian.

If $\mathcal{L}$ is right σ-ample, then $\mathcal{L}$ is left σ^{-1}-ample by (2.5.6). So $B(X, \sigma^{-1}, \mathcal{L}) \cong B^{\mathrm{op}}$ is left noetherian and hence B is right noetherian. □

Remark 2.5.10. The main results of this thesis will be proved in terms of Cartier divisors rather than line bundles. However, the reader should note that, unravelling the definitions, one has $\mathcal{O}_X(\sigma D) \cong \mathcal{O}_X(D)^{\sigma^{-1}}$. It is therefore notationally more convenient to work with a right σ^{-1}-ample line bundle $\mathcal{L} = \mathcal{O}_X(D)$, since then D is right σ^{-1}-ample if and only if

$$H^q(X, \mathcal{F} \otimes \mathcal{O}_X(D + \sigma D + \cdots + \sigma^{m-1} D)) = 0$$

for all $q > 0$ and $m \gg 0$. Obviously, this will have no effect on the final theorems. Throughout this thesis, we will use the notation $\Delta_m = D + \sigma D + \cdots + \sigma^{m-1} D$.

Before ending this section, let us say a few more words about notation. In Chapter 3, we will deal with only one σ-ample invertible sheaf $\mathcal{L}$. It will be important to consider the actual invertible sheaves $\mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$, thus we will not use the invertible bimodule notation in that chapter. In Chapter 4, however, we must deal with several pairs of invertible sheaves $\mathcal{L}$ and automorphisms σ. Thus it will be more convenient to use the compact notation $\mathcal{L}_\sigma$. In that chapter, when we wish to speak of the invertible sheaf $\mathcal{O}_X \otimes \mathcal{L}_\sigma^m$ we will use the following notation.

Notation 2.5.11. Given an invertible bimodule $\mathcal{L}_\sigma$, the notation $|\mathcal{L}_\sigma|$ will mean the underlying invertible sheaf $\mathcal{O}_X \otimes \mathcal{L}_\sigma$. For example, given $\mathcal{L}_\sigma$, one has

$$|\mathcal{L}_\sigma^m| = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}.$$

It is often useful to replace $\mathcal{L}$ with $|\mathcal{L}_\sigma^m|$ and σ with σ^m to assume $\mathcal{L}$ and σ have a desired property. Using standard techniques, one can also show

Lemma 2.5.12. ([AV, Lemma 4.1]) *Let $\mathcal{L}$ be an invertible sheaf on X. Given a positive integer m, $\mathcal{L}$ is right σ-ample if and only if $|\mathcal{L}_\sigma^m|$ is right σ^m-ample.*

Proof. We follow the method of [H2, p. 154, Proposition 7.5]. If $\mathcal{L}$ is right σ-ample, it is clear that $|\mathcal{L}_\sigma^m|$ is right σ^m-ample.

Conversely, suppose that $|\mathcal{L}_\sigma^m|$ is right σ^m-ample. Then by definition, given a coherent sheaf $\mathcal{F}$, there exists n_0 such that the higher cohomology of $\mathcal{F} \otimes (\mathcal{L}_\sigma^m)^n$ vanishes for $n \geq n_0$. Similarly, for $j = 1, \ldots, m-1$, there exists n_j such that the higher cohomology of $\mathcal{F} \otimes \mathcal{L}_\sigma^j \otimes (\mathcal{L}_\sigma^m)^n$ vanishes for $n \geq n_j$. If we take $N = m \cdot \max\{n_j | j = 0, \ldots, m-1\}$, then the higher cohomology of $\mathcal{F} \otimes \mathcal{L}_\sigma^n$ vanishes for $n \geq N$. Thus $\mathcal{L}$ is right σ-ample. □

2.6 Intersection theory and ampleness

In this section, we will review commutative algebraic geometry results which are pertinent to this thesis. Let us briefly summarize for the benefit of the reader already well-versed in the theory.

Recall that throughout this thesis, we will work in the case of a proper scheme X over an algebraically closed base field k of arbitrary characteristic. Usually, X will be projective. A variety will mean a reduced, irreducible scheme. All divisors will be *Cartier divisors* unless otherwise stated. For a projective scheme, the group of Cartier divisors, modulo linear equivalence, is naturally isomorphic to the Picard group of invertible sheaves [H2, p. 144, Remark 6.14.1]. Since much of our work will entail intersection theory, we will often work from the divisor point of view. Several times we use the facts that the ample divisors form a cone, that ampleness depends only on the numerical equivalence class of a divisor, and that ampleness is preserved under an automorphism. Hence the cone of ample divisors and its closure, the cone of numerically effective divisors, are invariant under an automorphism. As a reference for these and related facts we suggest [K]. A short review also appears in [D].

Let us now give more details. Since this material is more standard than that on twisted homogeneous coordinate rings, we will only prove a few selected results in order to present the flavor of the theory.

We recall some facts about invertible sheaves on a proper scheme X.

Proposition 2.6.1. ([H2, p. 169, Exercise 7.5]) *Let X be a proper scheme with invertible sheaves $\mathcal{L}$ and $\mathcal{M}$.*

1. *If $\mathcal{L}$ is ample and $\mathcal{M}$ is generated by global sections, then $\mathcal{L} \otimes \mathcal{M}$ is ample.*
2. *If $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then $\mathcal{M} \otimes \mathcal{L}^n$ is very ample for $n \gg 0$.*
3. *If $\mathcal{L}$ and $\mathcal{M}$ are both ample, then so is $\mathcal{L} \otimes \mathcal{M}$.*
4. *If $\mathcal{L}$ is very ample and $\mathcal{M}$ is generated by global sections, then $\mathcal{L} \otimes \mathcal{M}$ is very ample.* □

The following proposition is invaluable for induction arguments on projective schemes. We use the shorthand $\mathcal{F}(-H)$ for $\mathcal{F} \otimes \mathcal{O}_X(-H)$.

Proposition 2.6.2. *Let X be a projective scheme with coherent sheaf $\mathcal{F}$ and invertible sheaf $\mathcal{L}$. Suppose that $\mathcal{L}$ is ample and generated by global sections. Then there exists $s \in H^0(X, \mathcal{L})$ such that there is an exact sequence*

$$0 \to \mathcal{F} \otimes \mathcal{L}^{-1} \xrightarrow{-\otimes s} \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_H \to 0,$$

where H is the Cartier divisor defined by $s = 0$. Further, $\dim(\operatorname{Supp}\mathcal{F} \otimes \mathcal{O}_H) < \dim(\operatorname{Supp}\mathcal{F})$.

Proof. One possible proof appears in [Mu, Theorem 2]. However, we wish to demonstrate the utility of the category equivalence between $\mathrm{coh}(\operatorname{Proj} S)$ and $\mathrm{qgr}\, S$.

For any $s \in H^0(X, \mathcal{L})$ we have an exact sequence

$$\mathcal{F}(-H) \stackrel{-\otimes s}{\to} \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_H \to 0,$$

so we need to show that the map given by $- \otimes s$ is injective for sufficiently general s.

Let $B = B(X, \mathcal{L})$. Under the category equivalence explained in Theorem 2.3.8 or [H2, p. 125, Exercise 5.9], we may choose a torsion-free B-module M with $M = \oplus H^0(X, \mathcal{F} \otimes \mathcal{L}^m)$ corresponding to $\mathcal{F}$ and $M[-1]$ corresponding to $\mathcal{F} \otimes \mathcal{L}^{-1}$. Tensoring with $s \in H^0(X, \mathcal{L})$ then corresponds to multiplication by a homogeneous degree one element x_s of B.

Now since $\mathcal{L}$ is generated by global sections, there is an exact sequence

$$0 \to \operatorname{Ker} f \to \mathcal{O}_X \otimes B_1 \to \mathcal{L} \to 0.$$

Tensoring with $\mathcal{L}^m$ and taking global sections, we see the map $B_n \otimes B_1 \to B_{n+1}$ is surjective for $n \gg 0$. So suppose that for all $x \in B_1$ the multiplication $m \mapsto mx$, with $m \in M[-1]$, is not injective. Then multiplication by any $x \in B_n$ is not injective for $n \gg 0$.

And so there exists n_0 such that for all $x \in B_{\geq n_0}$, multiplication by x is not injective. But then there exists $m \in M[-1]$ with $m \neq 0$ such that $mB_{\geq n_0} = 0$ [B, p. 266, Proposition 8]. This contradicts the fact that M is torsion-free. So there must be $x_s \in B_1$ with $m \mapsto mx_s$ injective.

To see that $\dim(\operatorname{Supp}\mathcal{F} \otimes \mathcal{O}_H) < \dim(\operatorname{Supp}\mathcal{F})$, consider each irreducible component X_i of $\operatorname{Supp}\mathcal{F}$. The local ring S_i at the generic point of X_i is a local ring of dimension zero. Thus any element in the maximal ideal must be nilpotent. So tensoring with s must correspond to multiplication by an element outside of the maximal ideal, hence an invertible element. So at the generic points of each X_i, tensoring with s is an isomorphism. Hence the cokernel is 0 in an open subset of each X_i and so $\operatorname{Supp}(\mathcal{F} \otimes \mathcal{O}_H) \cap X_i \subsetneq X_i$. Thus $\operatorname{Supp}(\mathcal{F} \otimes \mathcal{O}_H)$ has smaller dimension than $\operatorname{Supp}(\mathcal{F})$ on each X_i and hence on all of X □

The following proposition can be thought of as a weak version of the Riemann-Roch Theorem. It is necessary to define the intersection numbers used in this thesis. We recall that the *Euler characteristic* $\chi(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ is defined as

$$\chi(\mathcal{F}) = \sum_{q=0}^{\infty} (-1)^q \dim H^q(X, \mathcal{F}).$$

This formula makes sense when X is proper (so each H^q is finite dimensional [H2, p. 252, Remark 8.8.1]). Further, $H^q(X, \mathcal{F}) = 0$ for $q > \dim(\operatorname{Supp} \mathcal{F})$ [H2, p. 208, Theorem 2.7]. We also recall that a *numerical polynomial* is a polynomial with rational coefficients which is integer valued on integers.

Proposition 2.6.3. ([K, p. 295, Theorem (Snapper)]) *Let X be a proper scheme. Let $\mathcal{F}$ be a coherent sheaf with $\dim(\operatorname{Supp} \mathcal{F}) = s$. Let $\mathcal{L}_1, \ldots, \mathcal{L}_t$ be invertible sheaves on X. Then*

$$\chi(\mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t})$$

is a numerical polynomial in the n_i of total degree $\leq s$.

Proof. For simplicity, we will only prove this when X is projective and use the method of [D, Proposition 4.1]. In this case, any invertible sheaf $\mathcal{L} = \mathcal{M} \otimes \mathcal{N}^{-1}$ for some very ample invertible sheaves $\mathcal{M}, \mathcal{N}$ by Proposition 2.6.1(2). We may therefore replace each $\mathcal{L}_i^{n_i}$ by $\mathcal{M}_i^{n_i} \otimes \mathcal{N}_i^{-n_i}$ and assume that each $\mathcal{L}_i$ is very ample.

We proceed by induction on $\dim(\operatorname{Supp} \mathcal{F}) = s$. If $\mathcal{F} = 0$, then the result is obvious. For $s \geq 0$, by standard methods we may assume $\operatorname{Supp} \mathcal{F}$ is a variety. So if $s = 0$, we may assume that $\operatorname{Supp} \mathcal{F}$ is a point. Then the restriction of any invertible sheaf to $\operatorname{Supp} \mathcal{F}$ is the trivial invertible sheaf, i.e., $\mathcal{O}_{\operatorname{Supp} \mathcal{F}}$, so again the result is obvious.

For $s > 0$, write $\mathcal{L}_1 = \mathcal{O}_X(H)$ and $\mathcal{F}_H = \mathcal{F} \otimes \mathcal{O}_H$. By Proposition 2.6.2, there is an H so that we have an exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{L}_1^{-1} \to \mathcal{F} \to \mathcal{F}_H \to 0$$

with $\dim(\operatorname{Supp} \mathcal{F}_H) < s$. Tensoring with the $\mathcal{L}_i^{n_i}$ and calculating χ we have

$$\chi(\mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}) - \chi(\mathcal{F} \otimes \mathcal{L}_1^{n_1 - 1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}) = \chi(\mathcal{F}_H \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t})$$

The right hand side is a numerical polynomial of total degree $< s$. Now by induction on t, we may assume $\chi(\mathcal{F} \otimes \mathcal{L}^{n_2} \otimes \cdots \otimes \mathcal{L}^{n_t})$ is a polynomial of total degree $\leq s$. The result then follows by taking the telescoping sum over n_1. □

Definition 2.6.4. Let a be the coefficient of $n_1 \ldots n_t$ in the polynomial above. Then a is an integer due to general facts regarding numerical polynomials. The *intersection number* $(\mathcal{L}_1. \ldots .\mathcal{L}_t.\mathcal{F})$ is this number a. If $\mathcal{L}_i \cong \mathcal{O}(H_i)$, we also write this number as $(H_1. \ldots .H_t.\mathcal{F})$. If $\mathcal{F} = \mathcal{O}_V$ for some subscheme V, then we also write this number as $(\mathcal{L}_1. \ldots .\mathcal{L}_t.V) = (\mathcal{L}_1. \ldots .\mathcal{L}_t)_V$, or similarly with the H_i replacing $\mathcal{L}_i$. If $V = X$, then we write $(\mathcal{L}_1. \ldots .\mathcal{L}_t)_V = (\mathcal{L}_1. \ldots .\mathcal{L}_t)$.

The following properties of the intersection numbers come from appropriate short exact sequences. The proofs are in [K, p. 296–301].

Proposition 2.6.5. *The number* $(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}) = 0$ *if* $\dim(\operatorname{Supp}\mathcal{F}) < t$. *Also,* $(\mathcal{F}) = \dim H^0(X, \mathcal{F})$ *if* $\dim(\operatorname{Supp}\mathcal{F}) = t = 0$. □

Proposition 2.6.6. *The number* $(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}) = 0$ *is a symmetric* t*-linear form in the* $\mathcal{L}_i$. □

Proposition 2.6.7. *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is exact, then*

$$(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}) = (\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}') + (\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}''). \quad \square$$

Proposition 2.6.8. *Assume that* H *is an effective divisor such that* $\mathcal{L}_1 \cong \mathcal{O}(H)$ *and that there exists an exact sequence*

$$0 \to \mathcal{F}(-H) \to \mathcal{F} \to \mathcal{F}_H \to 0,$$

where $\mathcal{F}_H = \mathcal{F} \otimes \mathcal{O}_H$. *Then* $(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}) = (\mathcal{L}_2.\dots.\mathcal{L}_t.\mathcal{F}_H)$. *In particular, if* $\dim X \leq t$, *then* $(\mathcal{L}_1.\dots.\mathcal{L}_t) = (\mathcal{L}_2.\dots.\mathcal{L}_t.H)$. □

If $i\colon W \hookrightarrow X$ is the inclusion of a closed subscheme, we use the notation $\mathcal{L}|_W = i^*\mathcal{L}$.

Proposition 2.6.9. *If* W *is a closed subscheme with* $\operatorname{Supp}\mathcal{F} \subseteq W \subseteq X$, *then* $(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F})_X = (\mathcal{L}_1|_W.\dots.\mathcal{L}_t|_W.\mathcal{F})_W$. *Thus in particular,* $(\mathcal{L}_1.\dots.\mathcal{L}_t.W)_X = (\mathcal{L}_1|_W.\dots.\mathcal{L}_t|_W)_W$. □

Corollary 2.6.10. *Let* $V = \operatorname{Supp}\mathcal{F}$, *with irreducible components* $V_1, \dots, V_j$. *Let* $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{O}_{V_i}$. *Then*

$$(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}) = \sum_i (\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}_i). \quad \square$$

Corollary 2.6.11. *Let* X *be irreducible with* $\dim X \leq t$. *Let* $x \in X$ *be the generic point. Set* $l = \operatorname{length}_{\mathcal{O}_{X,x}} \mathcal{F}_x$. *Then* $(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}) = l(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{O}_{X_{red}})$. □

Proposition 2.6.12. *If* $f\colon X' \to X$ *is a morphism with* $t \geq \max\{\dim X', \dim X\}$, *then*

$$(f^*\mathcal{L}_1.\dots.f^*\mathcal{L}_t)_{X'} = \deg f(\mathcal{L}_1.\dots.\mathcal{L}_t)_X. \quad \square$$

Proposition 2.6.13. *The intersection form* $(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F})$ *is uniquely defined by the results of Propositions 2.6.5–2.6.12.* □

We will now look at how these intersection numbers relate to the ampleness of invertible sheaves. These are the main ideas we use in this thesis. We begin with the well-known Nakai criterion for ampleness.

Theorem 2.6.14. ([H1, p. 30, Theorem 5.1]) *Let X be a proper scheme with invertible sheaf $\mathcal{L}$. The sheaf $\mathcal{L}$ is ample if and only if $(\mathcal{L}^s.V) > 0$ for all s and all closed subvarieties $V \subseteq X$ with $\dim V = s$.* □

The following proposition regarding ample invertible sheaves are now easy to see, given the Nakai criterion. The reader should be warned, however, that the proof of the Nakai criterion depends on some of these propositions, so our reasoning is circular. To actually prove this proposition, one must use cohomological methods [H2, p. 232, Exercise 5.7] or [H1, pp. 23–26].

Proposition 2.6.15. *Let X be a proper scheme with invertible sheaf $\mathcal{L}$.*

1. *If $\mathcal{L}$ is ample on X, then $\mathcal{L}|_V$ is ample on V for any closed subscheme $V \subset X$.*
2. *$\mathcal{L}$ is ample on X if and only if $\mathcal{L}|_{X_{red}}$ is ample on X_{red}.*
3. *Let X be reduced. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L}|_{X_i}$ is ample on X_i for each irreducible component X_i.* □

Also using the Nakai criterion and Proposition 2.6.12, one can show

Proposition 2.6.16. ([H2, p. 232, Exercise 5.7]) *Let $f\colon Y \to X$ be a finite morphism of proper schemes. If $\mathcal{L}$ is ample on X, then $f^*\mathcal{L}$ is ample on Y. If f is finite and surjective, and if $f^*\mathcal{L}$ is ample on Y, then $\mathcal{L}$ is ample on X. In particular, ampleness is preserved under automorphisms.* □

One should be warned that in the Nakai criterion, it is not sufficient that $(\mathcal{L}.C) > 0$ for all integral curves $C \subset X$ [K, p. 326, Example 2]. However, we do have

Proposition 2.6.17. ([H1, p. 27, Proposition 4.6]) *Let X be a proper scheme with invertible sheaf $\mathcal{L}$. If $\mathcal{L}$ is generated by global sections and $\mathcal{L}|_C$ is ample for all integral curves C, then $\mathcal{L}$ is ample. In other words, if $\mathcal{L}$ is generated by global sections and $(\mathcal{L}.C) > 0$ for all curves C, then $\mathcal{L}$ is ample.* □

The property of having non-negative intersection with every curve C is of central importance to our work. So we define

Definition 2.6.18. An invertible sheaf $\mathcal{L}$ (respectively divisor D) is *numerically effective* if $(\mathcal{L}.C) \geq 0$ (respectively $(D.C) \geq 0$) for all integral curves $C \subset X$.

The concept of numerical effectiveness behaves better than ampleness. It is clear that the statements of Proposition 2.6.15 hold for numerically effective divisors, since the concept only depends on the integral curves in X. We have an even stronger version of Proposition 2.6.16.

Proposition 2.6.19. ([K, p. 303, Proposition 1]) *Let $f\colon Y \to X$ be any morphism of proper schemes. If $\mathcal{L}$ is numerically effective on X, then $f^*\mathcal{L}$ is numerically effective on Y. If f is surjective, and if $f^*\mathcal{L}$ is numerically effective on Y, then $\mathcal{L}$ is numerically effective on X. In particular, numerical effectiveness is preserved under automorphisms.* □

So we see that any invertible sheaf $\mathcal{L}$ generated by global sections is numerically effective since $\mathcal{L} = \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ for some morphism $\varphi\colon X \to \mathbb{P}^n$ [H2, p. 150, Theorem 7.1].

One particularly nice fact about numerically effective invertible sheaves, which make them "better" than ample invertible sheaves, is that having non-negative intersection with curves is enough to force non-negative intersection with all closed subvarieties.

Proposition 2.6.20. ([K, p. 320, Theorem 1]) *Let X be a proper scheme. Let $\mathcal{F}$ be coherent with $\dim(\operatorname{Supp}\mathcal{F}) = t$. If $\mathcal{L}_1, \dots, \mathcal{L}_t$ are numerically effective invertible sheaves, then $(\mathcal{L}_1.\dots.\mathcal{L}_t.\mathcal{F}) \geq 0$. So in particular, $\mathcal{L}$ is numerically effective if and only if $(\mathcal{L}^s.V) \geq 0$ for all s and all subvarieties $V \subset X$ with $\dim V = s$.* □

Recall that $\operatorname{Pic}(X)$ is the *Picard group* of X, the abelian group of isomorphism classes of invertible sheaves on X, with addition given by the tensor product. Intersecting with curves also gives an extremely useful equivalence relation on $\operatorname{Pic}(X)$. This will allow us to reduce many questions to facts about finite dimensional $\mathbb{R}$-vector spaces and cones in $\mathbb{R}^n$.

Definition 2.6.21. Two invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$ are called *numerically equivalent* if $(\mathcal{L}_1.C) = (\mathcal{L}_2.C)$ for all integral curves $C \subset X$. Numerical equivalence for divisors is defined similarly. We denote this equivalence relation by $\equiv$ and set $A^1_{\text{Num}}(X) = \operatorname{Pic}(X)/\equiv$. We let $A^1_{\mathbb{R}} = A^1_{\mathbb{R}}(X) = A^1_{\text{Num}}(X) \otimes \mathbb{R}$. When speaking of an invertible sheaf $\mathcal{L} \in A^1_{\mathbb{R}}$, formally we mean the equivalence class of $\mathcal{L}$. We sometimes denote this equivalence class as $[\mathcal{L}]$ for emphasis.

Remark 2.6.22. Clearly, the property of being numerically effective only depends on the numerical equivalence class. We will see in Theorem 2.6.29 that this is true for ampleness as well. So when speaking of these properties, there is no danger in writing $\mathcal{L} = [\mathcal{L}]$. However, very ampleness is not preserved by numerical equivalence [H2, p. 368, Exercise 1.12].

Theorem 2.6.23. ([K, p. 305, Remark 3]) *The group $A^1_{\text{Num}}(X)$ is a finitely generated free abelian group.* □

This theorem is known as the Theorem of the Base or the Neron-Severi Theorem. For a proof when X is a nonsingular projective complex surface, see [H2, p. 367, Exercise 1.8]. If X is a nonsingular projective complex variety, another method is explained in [H2, p. 446–447, Appendix B, §5].

The rank of $A^1_{\mathrm{Num}}(X)$ is known as the *Picard number* of X and is denoted $\rho(X)$. So $A^1_{\mathbb{R}}$ is a $\rho(X)$-dimensional real vector space.

Definition 2.6.24. Let V be an n-dimensional vector space over $\mathbb{R}$. A subset $\kappa \subset V$ is a *cone* if

1. $\kappa + \kappa \subset \kappa$,
2. $a\kappa \subset \kappa$ for all $a \in \mathbb{R}_{>0}$.

The cone κ is *pointed* if $\kappa \cap -\kappa = \{0\}$. The cone κ is closed or open if it is closed or open as a topological subspace of $V \cong \mathbb{R}^n$. The cone κ is *solid* if the interior $\mathrm{Int}(\kappa) \neq \emptyset$.

We recall two well-known facts from cone theory.

Lemma 2.6.25. ([V, p. 1209]) *Let κ be a closed pointed cone in $\mathbb{R}^n$. Then $v \in \mathrm{Int}(\kappa)$ if and only if, for all $u \in \mathbb{R}^n$, there exists m_0 such that $mv - u \in \kappa$ for all $m \geq m_0$.* □

Lemma 2.6.26. ([K, p. 324, Lemma 1 (Caratheodory)]) *Let κ be a cone in $\mathbb{R}^n$ generated by possibly infinitely many elements $\{v_i\}$. Then any $N \in \kappa$ may be written as*

$$N = \sum_{j=1}^{n} a_j v_{i_j},$$

with $a_j \geq 0$. □

Proposition 2.6.27. ([K, p. 324–325]) *Let X be a proper scheme. Let $\kappa \subset A^1_{\mathbb{R}}$ be the set of all elements N such that $(N.C) \geq 0$ for all curves C. Then κ is a closed pointed cone in $A^1_{\mathbb{R}}$. The lattice points in κ correspond to numerically effective invertible sheaves and in fact κ is the closure of the cone generated by numerically effective invertible sheaves. The ample invertible sheaves generate an open cone κ° inside* $\mathrm{Int}(\kappa)$.

Proof. Clearly, κ is a cone. It is clear that κ° is a subcone of κ since every ample invertible sheaf is numerically effective and numerical effectiveness clearly only depends on numerical equivalence classes. Further, κ must be pointed since any element of $\kappa \cap -\kappa$ must have zero intersection with all curves.

To see that κ is closed, let N be in the closure of κ and let M_C be an equivalence class such that $(M_C.C) < 0$ for a given curve C. Since N is in the closure of κ, for all $n \gg 0$, $(N + (1/n)M_C.C) \geq 0$. Thus

$$(N.C) \geq \frac{1}{n}(-M_C.C) > 0.$$

So taking the limit, $(N.C) \geq 0$, so $N \in \kappa$. Since the numerically effective invertible sheaves determine the rational points of κ and $\mathbb{Q}$ is dense in $\mathbb{R}$, they uniquely determine κ.

To see that $\kappa^\circ \subset \operatorname{Int}(\kappa)$, we may assume X is projective, since otherwise $\kappa^\circ = \emptyset$. If $N \in \kappa^\circ$, then $N = \sum a_i[\mathcal{L}_i]$ for ample invertible sheaves $\mathcal{L}_i$ and $a_i > 0$. Given $u \in A^1_{\mathbb{R}}$, u is of the form $u = \sum_{i=1}^{\rho(X)} b_i[\mathcal{M}_i]$ for some invertible sheave $\mathcal{M}_i$. There exists m_0 such that for $m \geq m_0$, $\mathcal{L}_1^m \otimes \mathcal{M}_i^{-1}$ is ample for each i. Thus taking $m_1 = \rho(X)m_0/a_1$ we have $mN - u \in \kappa^\circ$ for all $m \geq m_1$. Thus $N \in \operatorname{Int}(\kappa)$ by Lemma 2.6.25 since $mN - u$ has positive intersection with any curve. Further, since $A^1_{\mathbb{R}}$ is finite dimensional, this argument shows that κ° contains sufficiently small balls around any $[\mathcal{L}_i] \in \kappa^\circ$. So κ° is open. □

Corollary 2.6.28. *Let X be projective. Given any endomorphism $f\colon X \to X$, the natural mapping $f^*\colon A^1_{\mathbb{R}} \to A^1_{\mathbb{R}}$ maps κ to κ. Thus f^* is represented by a matrix in $\mathrm{GL}_{\rho(X)}(\mathbb{Z})$ which preserves a closed pointed solid cone.*

Proof. We already know f^* preserves numerical effectiveness by (2.6.19). Because f^* preserves the lattice $A^1_{\mathrm{Num}}(X)$, its action is represented by $P \in \mathrm{GL}_\rho(\mathbb{Z})$. Since the map P is continuous, it preserves the closure of the cone generated by numerically effective invertible sheaves, namely κ. Since X is projective, κ is solid. □

We are now ready for the famous Kleiman criterion for ampleness. Briefly, it says $\operatorname{Int}(\kappa) = \kappa^\circ$ given the notation above. However, this is not true for every proper scheme X. It is only true if X is *quasi-divisorial*, meaning that for every integral closed subscheme Y which is not reduced to a point, there is an invertible sheaf $\mathcal{L}_Y$ on Y with $\mathcal{L}_Y \cong \mathcal{O}_Y(D)$ for some non-zero effective Cartier divisor D on Y. All projective schemes are quasi-divisorial, since one may take the D to be a very ample divisor. Also any locally factorial scheme X (hence any nonsingular scheme) is quasi-divisorial [K, p. 326, Example 1].

Theorem 2.6.29. ([K, p. 326, Theorem 2]) *Let X be a quasi-divisorial, proper scheme. Then $\kappa^\circ = \operatorname{Int}(\kappa)$.*

Proof. We have already seen that $\kappa^\circ \subseteq \operatorname{Int}(\kappa)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and κ° is open, it suffices to prove that the rational points of $\operatorname{Int}(\kappa)$ are in κ°. So let $N \in \operatorname{Int}(\kappa)$ be rational and we may assume N is the equivalence class of an invertible sheaf by multiplying by a sufficiently large integer. So we wish to show that N satisfies the Nakai criterion for ampleness (2.6.14).

We proceed by induction on the dimension of a subvariety $Y \subseteq X$. If $\dim Y = 0$, then $(N^0.Y) = (Y) = (\mathcal{O}_Y) = 1$ by (2.6.5). So suppose that $\dim Y = s > 0$ and that $(N^t.W) > 0$ for all subvarieties $W \subsetneq Y$ with $\dim W = t < s$. By (2.6.10) and

(2.6.11), N has positive intersection with any subscheme of Y of dimension less than s.

Since X is quasi-divisorial, find a non-zero effective Cartier divisor D on Y. Then $(N^{s-1}.D) = (N^{s-1}.D.Y) > 0$. Since $N \in \mathrm{Int}(\kappa)$, for $m \gg 0$, $mN - [D]$ is numerically effective by (2.6.25). And so by Proposition 2.6.20,

$$m(N^{s-1}.N.Y) - (N^{s-1}.D.Y) \geq 0.$$

And thus $(N^s.Y) > 0$. □

Corollary 2.6.30. *The property of being ample depends only on the numerical equivalence class of an invertible sheaf.* □

Corollary 2.6.31. *If $\mathcal{L}$ is ample and $\mathcal{M}$ is numerically effective, then $\mathcal{L} \otimes \mathcal{M}$ is ample.* □

CHAPTER 3

Twisted Homogeneous Coordinate Rings

3.1 Introduction

In this chapter, we will prove our main theorems regarding σ-ample invertible sheaves $\mathcal{L}$ on a projective scheme X and their associated twisted homogeneous coordinate rings $B = B(X, \sigma, \mathcal{L})$. For the convenience of the reader, we reprint some of the theorems from Chapter 1. We show

Theorem 3.1.1. *The following are true for any projective scheme X over an algebraically closed field.*

1. *Right and left σ-ampleness are equivalent. Thus every associated B is (right and left) noetherian.*

2. *A projective scheme X has a σ-ample divisor if and only if the action of σ on numerical equivalence classes of divisors is quasi-unipotent (cf. §3.3 for definitions). In this case, every ample divisor is σ-ample.*

3. GKdim B *is an integer if $B = B(X, \sigma, \mathcal{L})$ and $\mathcal{L}$ is σ-ample. Here* GKdim B *is the Gel'fand-Kirillov dimension of B in the sense of* [KL].

These facts are all consequences of

Theorem 3.1.2 (See Remark 3.5.2). *Let X be a projective scheme with automorphism σ. Let D be a Cartier divisor. D is (right) σ-ample if and only if σ is quasi-unipotent and*

$$D + \sigma D + \cdots + \sigma^{m-1} D$$

is ample for some $m > 0$.

In §3.2, we reduce the question of σ-ampleness to one about the (classical) ampleness of certain sheaves. This allows us to use cone theory to deduce in §3.3 that if a σ-ample invertible sheaf exists, then the action of σ on numerical equivalence classes must be quasi-unipotent. We then use cone theory again in §3.4 to prove the rest of Theorem 3.1.2.

We then examine numerous corollaries of Theorem 3.1.2 in §3.5. The most important corollary is that left and right σ-ampleness are equivalent and hence the twisted homogeneous coordinate rings associated to σ-ample divisors are noetherian. We then use Riemann-Roch type theorems in §3.6 to show that such rings have finite, integral GK-dimension.

Finally, in §3.7, we show how our results are related to the category-theoretic coordinate rings of §2.3. We end the chapter in §3.8 by examining the possibility of a σ-ample invertible sheaf on a proper, non-projective scheme.

3.2 Reductions

Since X is projective, the invertible sheaves on X and the Cartier divisors modulo linear equivalence are in one-to-one correspondence, as noted in the beginning of §2.6. So without loss of generality we may use divisors in our arguments, which we do since it is more customary for intersection theory.

Before deriving our main criterion for σ-ampleness, we must prove other equivalent criteria. We will need

Lemma 3.2.1. ([Fj, p. 520, Theorem 1]) *Let $\mathcal{F}$ be a coherent sheaf on a projective scheme X and let H be an ample divisor on X. Then there exists an integer c_0 such that for all $c \geq c_0$,*

$$H^q(X, \mathcal{F} \otimes \mathcal{O}_X(cH + N)) = 0$$

for $q > 0$ and any numerically effective divisor N. □

Proposition 3.2.2. *Let X be a projective scheme with σ an automorphism. Let D be a divisor on X and $\Delta_m = D + \sigma D + \cdots + \sigma^{m-1} D$, as in Remark 2.5.10. Then the following are equivalent:*

1. *For any coherent sheaf $\mathcal{F}$, there exists an m_0 such that*

   $$H^q(X, \mathcal{F} \otimes \mathcal{O}_X(\Delta_m)) = 0$$

   for $q > 0$ and $m \geq m_0$. That is, D is right σ^{-1}-ample.

2. *For any coherent sheaf $\mathcal{F}$, there exists an m_0 such that $\mathcal{F} \otimes \mathcal{O}_X(\Delta_m)$ is generated by global sections for $m \geq m_0$.*

3. *For any divisor H, there exists an m_0 such that $\Delta_m - H$ is very ample for $m \geq m_0$.*

4. *For any divisor H, there exists an m_0 such that $\Delta_m - H$ is ample for $m \geq m_0$.*

5. *For any divisor H, there exists an m_0 such that $\Delta_m - H$ is numerically effective for $m \geq m_0$.*

Proof. (1) $\Rightarrow$ (2) is Proposition 2.5.7.

(2) $\Rightarrow$ (3) Given any divisor H and a very ample divisor H', we may choose m_0 such that $\Delta_m - H - H'$ is generated by global sections for $m \geq m_0$. Then $\Delta_m - H - H' + H' = \Delta_m - H$ is very ample for $m \geq m_0$ by Proposition 2.6.1(4).

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are trivial.

(5) $\Rightarrow$ (1). For any ample divisor H and any $c \geq 0$, one can choose m_0 so that for $m \geq m_0$, we have $N = \Delta_m - cH$ is a numerically effective divisor. Then (1) follows immediately from Lemma 3.2.1. □

A similar proposition holds for left σ^{-1}-ample divisors, with $\mathcal{F}$ and H replaced by $\mathcal{F}^{\sigma^{-m}}$ and $\sigma^m H$. One deduces this easily from Lemma 2.5.6.

We note that if σ is the identity, then condition (5) says exactly that D is in the interior κ° of the cone of numerically effective divisors, as defined in §2.6. Thus this proposition is a generalization of Kleiman's criterion for ampleness, Theorem 2.6.29.

3.3 The non-quasi-unipotent case

Recall the definition of $A^1_{\text{Num}}(X)$ from (2.6.21). We let P denote the action of σ on $A^1_{\text{Num}}(X)$; hence $P \in \text{GL}_\ell(\mathbb{Z})$ for some ℓ by Theorem 2.6.23.

A matrix is called *quasi-unipotent* if all of its eigenvalues are roots of unity. We call an automorphism σ quasi-unipotent if P is. The main goal of this section is to show that a non-quasi-unipotent σ cannot give a σ-ample divisor.

First, we must review an useful fact about integer matrices.

Lemma 3.3.1. *Let $P \in \text{GL}_\ell(\mathbb{Z})$. Then P is quasi-unipotent if and only if all eigenvalues of P have absolute value* 1*. Thus if P is not quasi-unipotent, then P has an eigenvalue of absolute value greater than* 1*.*

Proof. The first claim is [AV, Lemma 5.3]. To see this, suppose all eigenvalues λ_i of P have absolute value 1. Each λ_i is an algebraic integer whose conjugate roots all have absolute value 1. The only such algebraic integers are the roots of unity [L, p. 353, VII, Exercise 5].

For the second claim, the property of P not being quasi-unipotent is reduced to saying P has an eigenvalue of absolute value not 1. Since P has determinant ± 1, P has an eigenvalue of absolute value greater than 1. □

Recall that the *spectral radius* of a matrix P is the nonnegative real number $r = \text{sr}(P) = \max\{|\lambda|\colon \lambda \text{ an eigenvalue of } P\}$. The following lemma shows a relationship between the spectral radius r and the intersection numbers $(\sigma^m D.C)$, where D is an ample divisor.

Lemma 3.3.2. *Let P be as described above with spectral radius $r = \operatorname{sr}(P)$. There exists an integral curve C with the following property: If D is an ample divisor, then there exists $c > 0$ such that*

$$(\sigma^m D.C) \geq cr^m \qquad \text{for all } m \geq 0.$$

Proof. Let κ be the cone generated by numerically effective divisors in $A^1_{\mathrm{Num}}(X) \otimes \mathbb{R}$. In the terminology of (2.6.24), κ is a solid cone since it has a non-empty interior by Proposition 2.6.27. Since P maps κ to κ, the spectral radius r is an eigenvalue of P and r has an eigenvector $v \in \kappa$ [V, Theorem 3.1].

Since $v \in \kappa \setminus \{0\}$, there exists a curve C with $(v.C) > 0$. Given an ample divisor D, there is a positive ℓ so that $\ell D - v$ is in the ample cone by Lemma 2.6.25. Thus

$$\ell(\sigma^m D.C) = \ell(P^m D.C) > (P^m v.C) = r^m(v.C).$$

Taking $c = (v.C)/\ell$, we have the lemma. □

Now a graded ring $B = \oplus_{i \geq 0} B_i$ is finitely graded if $\dim B_i < \infty$ for all i. Such a ring B has exponential growth (see [SteZ]) if

$$\limsup_{n\to\infty} \left(\sum_{i \leq n} \dim B_i \right)^{\frac{1}{n}} > 1. \tag{3.3.3}$$

Theorem 3.3.4. ([SteZ, Theorem 0.1]) *Let B be a finitely generated, finitely graded k-algebra. If B has exponential growth, it is neither right nor left noetherian.* □

This fact combined with the intersection numbers above allow us to prove

Theorem 3.3.5. *Let X be a projective scheme with automorphism σ. If X has a right σ^{-1}-ample divisor, then σ is quasi-unipotent.*

Proof. Suppose that D is a right σ^{-1}-ample divisor. Let $\Delta_m = D + \sigma D + \cdots + \sigma^{m-1} D$. By (3.2.2) and (2.5.12), we may replace D with Δ_m and σ with σ^m and assume that D is ample.

Let P be the action of σ on $A^1_{\mathrm{Num}}(X)$. Suppose that P is non-quasi-unipotent with spectral radius $r > 1$ and choose an integral curve C as in Lemma 3.3.2. Let $\mathcal{I}$ be the ideal sheaf defining C in X. Since D is right σ^{-1}-ample, the higher cohomologies of $\mathcal{I}(\Delta_m) = \mathcal{I} \otimes \mathcal{O}_X(\Delta_m)$ and $\mathcal{O}_C(\Delta_m)$ vanish for $m \gg 0$. So one has an exact sequence

$$0 \to H^0(X, \mathcal{I}(\Delta_m)) \to H^0(X, \mathcal{O}_X(\Delta_m)) \to H^0(C, \mathcal{O}_C(\Delta_m)) \to 0.$$

For $m \gg 0$, the Riemann-Roch formula for curves [Fl, p. 360, Example 18.3.4] gives

$$\dim H^0(C, \mathcal{O}_C(\Delta_m)) = (\Delta_m.C) + \text{a constant term}.$$

Thus using the exact sequence and the previous lemma, there exists $c > 0$ so that

$$\dim H^0(X, \mathcal{O}_X(\Delta_m)) > cr^m$$

for $m \gg 0$. Thus the associated twisted homogenous coordinate ring has exponential growth and hence is not (right or left) noetherian by (3.3.4). So D cannot be right σ^{-1}-ample, by Theorem 2.5.9. □

Remark 3.3.6. One can give a more elementary proof of Theorem 3.3.5. Indeed, examining the Jordan form of P gives an upper bound on $(\sigma^m D.C)$. Further, using the full strength of [V, Theorem 3.1] and asymptotic estimates, one can improve the lower bound of Lemma 3.3.2. We then have

$$c_1 m^k r^m > (\sigma^m D.C) > c m^k r^m \tag{3.3.7}$$

for $m > 0$, where $k+1$ is the size of the largest Jordan block associated to r. Then using estimates similar to those in the proof of [AV, Lemma 5.10], one can find an ample divisor H such that

$$(\Delta_m - H.\sigma^m C) < 0$$

for all $m \gg 0$. This contradicts the fourth equivalent condition for right σ^{-1}-ampleness in Proposition 3.2.2.

Even when an automorphism σ is not quasi-unipotent, one can form associated twisted homogeneous coordinate rings. As might be expected, some of these rings have exponential growth.

Proposition 3.3.8. *Let X be a projective scheme with non-quasi-unipotent automorphism σ. Let D be an ample divisor. Then there exists an integer $n_0 > 0$ such that for all $n \geq n_0$, the ring $B = B(X, \sigma, \mathcal{O}_X(nD))$ has exponential growth and is neither right nor left noetherian.*

Proof. Again choose a curve C as in Lemma 3.3.2 with ideal sheaf $\mathcal{I}$. By Lemma 3.2.1, there exists n_0 such that for all $n \geq n_0$ and $q > 0$,

$$H^q(X, \mathcal{I}(nD + N)) = H^q(C, \mathcal{O}_C(nD + N)) = 0$$

for any ample divisor N. In particular, the above cohomologies vanishes for $nD+N = nD + \sigma(nD) + \cdots + \sigma^{m-1}(nD)$ where $m > 1$. Then repeating the last paragraph of the proof of Theorem 3.3.5 shows that B has exponential growth. □

When X is a nonsingular surface, [AV, Corollary 5.17] shows that the above proposition is true for $n_0 = 1$. Their proof makes use of the relatively simple form of the Riemann-Roch formula and the vanishing of $H^2(X, \mathcal{O}_X(\Delta_m))$ when Δ_m is the sum of sufficiently many ample divisors. The proof easily generalizes to the singular surface case, but not to higher dimensions.

Question 3.3.9. Given a non-quasi-unipotent automorphism σ and ample divisor D on a scheme X, must $B(X, \sigma, \mathcal{O}_X(D))$ have exponential growth?

There do exist varieties with non-quasi-unipotent automorphisms. If the canonical divisor K is ample or minus ample, then any automorphism σ must be quasi-unipotent (cf. Proposition 3.5.7). So intuitively, one expects to find non-quasi-unipotent automorphisms far away from this case, i.e., when $K = 0$. Further, there are strong existence theorems for automorphisms of $K3$ surfaces (which do have $K = 0$). Indeed, a $K3$ surface with non-quasi-unipotent automorphism is studied in [W].

Example 3.3.10. *There exists a $K3$ surface with automorphism σ such that X has no σ-ample divisors.*

Proof. Wehler [W, Proposition 2.6, Theorem 2.9] constructs a family of $K3$ surfaces whose general member X has

$$\operatorname{Pic}(X) \cong A^1_{\mathrm{Num}}(X) \cong \mathbb{Z}^2, \qquad \operatorname{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

(That is, $\operatorname{Aut}(X)$ is the free product of two cyclic groups of order 2.) The ample generators H_1 and H_2 of $A^1_{\mathrm{Num}}(X)$ have intersection numbers

$$(H_1^2) = (H_2^2) = 2, \qquad (H_1.H_2) = 4.$$

$\operatorname{Aut}(X)$ has two generators σ_1, σ_2 whose actions on $A^1_{\mathrm{Num}}(X)$ can be represented as two quasi-unipotent matrices

$$\sigma_1 = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix}.$$

However, the action of $\sigma_1\sigma_2$ has eigenvalues $7 \pm 4\sqrt{3}$. So X *has no $\sigma_1\sigma_2$-ample divisor.* Note that by Corollary 3.5.5 below, any ample divisor is σ_1-ample and σ_2-ample. □

3.4 The quasi-unipotent case

Now let σ be a quasi-unipotent automorphism with P its action on $A^1_{\mathrm{Num}}(X)$. We will have several uses for a particular invariant of σ.

Definition 3.4.1. Let $k+1$ be the rank of the largest Jordan block of P. We define $J(\sigma) = k$.

Note that $J(\sigma) = J(\sigma^m)$ for all $m \in \mathbb{Z} \setminus \{0\}$. It may be that k is greater than 0, as seen in [AV, Example 5.18]. We will see in the next section that k must be even, but this is not used here.

To prove Theorem 3.1.2, it remains to show that (for σ quasi-unipotent) if D is a divisor such that Δ_m is ample for some m, then D is right σ^{-1}-ample. So fix such a D. We may again replace D with Δ_n and σ with σ^n via (2.5.12), so that D is ample and P is unipotent, that is $P = I + N$, where N is the nilpotent part of P. In this case, $k = J(\sigma)$ is the smallest natural number such that $N^{k+1} = 0$.

We let $\equiv$ denote numerical equivalence and reserve $=$ for linear equivalence. We then have, for all $m \geq 0$,

$$\sigma^m D \equiv P^m D = \sum_{i=0}^{k} \binom{m}{i} N^i D, \tag{3.4.2}$$

$$\Delta_m \equiv \sum_{i=0}^{k} \binom{m}{i+1} N^i D. \tag{3.4.3}$$

Once a basis for $A^1_{\mathrm{Num}}(X)$ is chosen, one can treat $N^i D$ as a divisor. Of course, this representation of $N^i D$ is not canonical. However, since ampleness and intersection numbers only depend on numerical equivalence classes, this is not a problem.

Lemma 3.4.4. *Let σ be a unipotent automorphism with $P = I + N$ and $k = J(\sigma)$. If D is an ample divisor, then $N^k D \not\equiv 0$ in $A^1_{\mathrm{Num}}(X)$.*

Proof. Since $N^k \neq 0$, there exists a divisor E and curve C such that $(N^k E.C) > 0$. Choose ℓ so that $\ell D - E$ is ample. By Equation 3.4.2, the intersection numbers $(\sigma^m(\ell D - E).C)$ are given by a polynomial in m with leading coefficient $(\ell N^k D - N^k E.C)/k!$. Since this polynomial must have positive values for all m, we must have $N^k D \not\equiv 0$. □

We now turn towards proving that for any divisor H, there exists m_0 such that $\Delta_{m_0} - H$ is ample, when σ is unipotent and D is ample. Then since D is ample, $\Delta_m - H$ is ample for $m \geq m_0$. For certain H, this is true even if σ is not quasi-unipotent.

Lemma 3.4.5. *Let X be a projective scheme with automorphism σ (not necessarily quasi-unipotent). Let D be an ample divisor and H a divisor whose numerical equivalence class is fixed by σ. Then there exists an m such that $\Delta_m - H$ is ample.*

Proof. Choose m such that $D' = mD - H$ is ample. Let

$$\Delta'_j = D' + \sigma D' + \cdots + \sigma^{j-1} D'.$$

Then $\Delta'_m \equiv m\Delta_m - mH$ is ample and thus $\Delta_m - H$ is ample. □

Proposition 3.4.6. *Let X be a projective scheme with unipotent automorphism σ. Let D be an ample divisor and H any divisor. Then there exists an m_0 such that $\Delta_{m_0} - H$ is ample. Hence $\Delta_m - H$ is ample for $m \geq m_0$.*

Proof. Let $W \subset A^1_{\mathrm{Num}}(X) \otimes \mathbb{R}$ be the span of $D, ND, \ldots, N^k D$. Then W is a $k+1$-dimensional vector space by Lemma 3.4.4. By Equation 3.4.2, it contains the real cone κ generated by $S = \{\sigma^i D | i \in \mathbb{N}\}$. Using Lemma 2.6.26, any element of κ can be written as a linear combination of $k+1$ elements of S with non-negative real coefficients. Thus for all $m \in \mathbb{N}$,

$$\Delta_m \equiv \sum_{i=0}^{k} f_i(m) \sigma^{g_i(m)} D$$

where $f_i \colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ and $g_i \colon \mathbb{N} \to \mathbb{N}$. Expanding the $\sigma^{g_i(m)} D$ above and comparing the coefficient of D with Equation 3.4.3, one finds that

$$\sum_{i=0}^{k} f_i(m) = m.$$

Since $f_i(m) \geq 0$, for each m, there must be some j such that $f_j(m) \geq m/(k+1)$.

Now choose l such that $lD - H$ is ample and m_0 such that, $m_0/(k+1) \geq l$. Then

$$f_j(m_0) \sigma^{g_j(m_0)} D - \sigma^{g_j(m_0)} H$$

is in the ample cone for the given j. Set $g = g_j(m_0)$. The other $f_i(m_0)$ are non-negative. Then $\Delta_{m_0} - \sigma^g H$ is in the ample cone as it is a sum of elements in the ample cone. Since it is a divisor, it is ample by Proposition 2.6.27.

We now prove the lemma by induction on q, the smallest positive integer such that $N^q H \equiv 0$. Since N is nilpotent, there is such a q for any H. The case $q = 1$ is handled by the previous lemma.

Now as $\sigma \equiv I + N$, we know $\sigma^{-m_0}(\sigma^g H - H)$ is killed by N^{q-1}. So there is an m_1 so that

$$Y = \Delta_{m_1} + \sigma^{-m_0}(\sigma^g H - H)$$

is ample. Then as σ fixes the ample cone

$$\Delta_{m_0} - \sigma^g H + \sigma^{m_0} Y = \Delta_{m_0+m_1} - H$$

is ample. □

We now immediately have by Propositions 3.2.2, 3.4.6, and Theorem 3.3.5:

Theorem 3.4.7. *Let X be a projective scheme with an automorphism σ. A divisor D is right σ^{-1}-ample if and only if σ is quasi-unipotent and $D + \sigma D + \cdots + \sigma^{m-1} D$ is ample for some m.* □

3.5 Corollaries

The characterization of (right) σ^{-1}-ampleness has many strong corollaries which are now easy to prove, but were only conjectured before.

Corollary 3.5.1. *Right σ-ample and left σ-ample are equivalent conditions. Further, σ-ampleness and σ^{-1}-ampleness are equivalent.*

Proof. Let D be right σ^{-1}-ample. By Theorem 3.4.7, σ is quasi-unipotent and Δ_m is ample for some m. Then σ^{-1} is quasi-unipotent and

$$\sigma^{-(m-1)}\Delta_m = D + \sigma^{-1}D + \cdots + \sigma^{-(m-1)}D$$

is ample. Applying the theorem again, we have that D is right σ-ample. Thus D is left σ^{-1}-ample by Lemma 2.5.6. The same lemma gives the second statement of the corollary. □

Remark 3.5.2. Combined with (3.4.7), this proves Theorem 3.1.2:

Theorem 3.5.3. *Let X be a projective scheme with automorphism σ. Let D be a Cartier divisor. D is (right) σ-ample if and only if σ is quasi-unipotent and*

$$D + \sigma D + \cdots + \sigma^{m-1} D$$

is ample for some $m > 0$. □

Thus we may refer to a divisor as being simply "σ-ample."

In [AV], left σ-ampleness was shown to imply the associated twisted homogeneous coordinate ring is left noetherian. However, as noted in the footnote of [AS, p. 258], the paper says, but does not prove, that B is noetherian. This actually is the case.

Corollary 3.5.4. *Let $B = B(X, \sigma, \mathcal{O}_X(D))$ be the twisted homogeneous coordinate ring associated to a σ-ample divisor D. Then B is a (left and right) noetherian ring, finitely generated over the base field.* □

Analysis of the GK-dimension of B will be saved for the next section.

From the definition of σ-ample, it is not obvious when σ-ample divisors even exist. Theorem 3.4.7 makes the question much easier.

Corollary 3.5.5. *A projective scheme X has a σ-ample divisor if and only if σ is quasi-unipotent. In particular, every ample divisor is a σ-ample divisor if σ is quasi-unipotent.* □

Thus, it is important to know when an automorphism σ is quasi-unipotent. From the bounds in Equation 3.3.7, we obtain

Proposition 3.5.6. *Let D be an ample divisor. Then σ is quasi-unipotent if and only if for all curves C, the intersection numbers $(\sigma^m D.C)$ are bounded by a polynomial for positive m.* □

Proposition 3.5.7. *Let X be a projective scheme such that*

1. *X has a canonical divisor K which is an ample or minus-ample divisor, or*
2. *the Picard number of X, i.e., the rank of $A^1_{\mathrm{Num}}(X)$, is 1.*

Then any automorphism σ of X is quasi-unipotent. Indeed, some power of σ is numerically equivalent to the identity.

Proof. In the first case, for K to be ample or minus-ample, it must be a Cartier divisor. Thus the intersection numbers $(\sigma^m K.C)$ are defined, where C is a curve. Since K must be fixed by σ, some power of σ must be numerically equivalent to the identity by Equation 3.3.7. In the second case, the action of σ itself must be numerically equivalent to the identity. □

Thus for many important projective varieties, such as curves, projective n-space, Grassmann varieties [Fl, p. 271], and Fano varieties [Ko, p. 240, Definition 1.1], one automatically has that any automorphism must be quasi-unipotent.

Returning to corollaries of Theorem 3.4.7, we see that building new σ-ample divisors from old ones is also possible.

Corollary 3.5.8. *Let D be a σ-ample divisor and let D' be a divisor with one of the following properties:*

1. *σ-ample,*
2. *generated by global sections, or*
3. *numerically effective.*

Then $D + D'$ is σ-ample.

Proof. Take m such that Δ_m is ample and $\Delta'_m = D' + \cdots + \sigma^{m-1}D'$ is respectively ample, generated by global sections, or numerically effective. Then $\Delta_m + \Delta'_m$ is ample by the results of §2.6 and we again apply the main theorem. □

The following could be shown directly from the definition, but also using a similar method to the proof above, one can see

Corollary 3.5.9. *Let σ and τ be automorphisms. Then D is σ-ample if and only if τD is $\tau\sigma\tau^{-1}$-ample.* □

Note that τ need not be quasi-unipotent.

Finally, as in the case of ampleness, σ-ampleness is a numerical condition.

Corollary 3.5.10. *Let D, D' be numerically equivalent divisors and σ, σ' be numerically equivalent automorphisms (i.e., their actions on $A^1_{\mathrm{Num}}(X)$ are equal). Then D is σ-ample if and only if D' is σ'-ample.*

Proof. As $\Delta_m \equiv D' + (\sigma')D' + \cdots + (\sigma')^{m-1}D'$ and ampleness depends only on the numerical equivalence class of a divisor, the corollary follows from our main theorem. □

3.6 GK-dimension of B

Recall the definition of GK-dimension for a finitely generated, finitely graded k-algebra,

Definition 3.6.1. ([KL, p. 62]) Let $R = \oplus R_i$ be a finitely generated finitely graded k-algebra. Then

$$\operatorname{GKdim} R = \limsup_{n\to\infty} \left\{ \log_n \sum_{i=0}^{n} \dim_k R_i \right\}.$$

If R is commutative, then the GK-dimension of R is equal to the classical Krull dimension of R. Hence the GK-dimension is an integer in this case. We generalize this to twisted homogeneous coordinate rings.

As mentioned, our main goal of this section is to prove

Theorem 3.6.2. *Let $B = B(X, \sigma, \mathcal{L})$ for some projective scheme X and σ-ample invertible sheaf $\mathcal{L}$.*

1. $\operatorname{GKdim} B$ *is an integer. Hence B is of polynomial growth. Also,* $\operatorname{GKdim} B$ *is independent of the σ-ample $\mathcal{L}$ chosen.*
2. *If $\sigma^m \equiv I$ for some m, then* $\operatorname{GKdim} B = \dim X + 1$.
3. *If X is an equidimensional scheme,*

$$k + \dim X + 1 \le \operatorname{GKdim} B \le k(\dim X - 1) + \dim X + 1$$

where $k = J(\sigma)$ (cf. Definition 3.4.1) is an even natural number depending only on σ.

We now have all the necessary pieces of Theorem 1.0.4, which we reprint for the reader.

Theorem 3.6.3. *Let X be a projective scheme with automorphism σ. Then the following are equivalent:*

1. *The automorphism σ is quasi-unipotent.*
2. *For all ample divisors D, $B(X, \sigma, \mathcal{O}_X(D))$ has finite GK-dimension.*
3. *For all ample divisors D, $B(X, \sigma, \mathcal{O}_X(D))$ is noetherian.*

Proof. (1) $\implies$ (2) is the theorem above. (1) $\implies$ (3) is from Corollaries 3.5.4 and 3.5.5. And finally (2) $\implies$ (1) and (3) $\implies$ (1) both follow from Proposition 3.3.8. □

Theorem 3.6.2 generalizes [AV, Proposition 1.5, Theorem 1.7]. The authors of [AV] further show that if X is a smooth surface, then $k = 0, 2$ and thus the only possible GK-dimensions are 3 and 5. The proof that $k \leq 2$ in the surface case uses the Hodge Index Theorem and thus far we have been unable to find a similar bound in higher dimensions. Note that if X is a curve or $X = \mathbb{P}^n$, then $\operatorname{rank} A^1_{\mathrm{Num}}(X) = 1$ and hence by Proposition 3.5.7, some power of σ is numerically equivalent to the identity (in fact, $P = I$). So the theorem implies that $\operatorname{GKdim} B = \dim X + 1$.

In studying the GK-dimension of $B = B(X, \sigma, \mathcal{O}_X(D))$ with D σ-ample, [AV, p. 263] proves that

$$\operatorname{GKdim} B(X, \sigma, \mathcal{O}_X(D)) = \operatorname{GKdim} B(X, \sigma^m, \mathcal{O}_X(\Delta_m)) \tag{3.6.4}$$

for any positive m. This comes from

Lemma 3.6.5. *Let R be a finitely generated, finitely graded k-algebra. Suppose that there exists an integer i_0 such that for all $i \geq i_0$ there exists j_i so that for $j \geq j_i$,*

$$R_j R_i = R_{i+j}.$$

Let $R^{(d)}$ be any Veronese subring $R^{(d)} = \oplus R_{di}$. Then $\operatorname{GKdim} R = \operatorname{GKdim} R^{(d)}$.

Proof. Since $R^{(d)}$ is a subring of R, certainly $\operatorname{GKdim} R^{(d)} \leq \operatorname{GKdim} R$. On the other hand, R is a $R^{(d)}$-module. We claim R is a finite $R^{(d)}$-module. It suffices to prove this for $d \gg 0$, since the finite generation of R over $R^{(\ell d)}$ for some ℓ implies finite generation over $R^{(d)}$.

Let d be such that $d \geq i_0$. Find $\ell d \geq j_d$ and let R' be the $R^{(d)}$-module generated by $R_1, R_2, \ldots, R_{(\ell+1)d-1}$. Let $N \in \mathbb{N}$ and assume by induction that $R_n \subset R'$ for all $n < N$. Now if $N - d < j_d$, then $N < (\ell+1)d$, so R_N is trivially generated by the set above. If $N - d \geq j_d$, then $R_d R_{N-d} = R_N$ and $R_{N-d} \subset R'$ by induction, so $R_N \subset R'$ and hence $R = R'$. Therefore, R is finitely generated over $R^{(d)}$ by a vector space basis of $R_1, \ldots, R_{(\ell+1)d-1}$. So the claim is proven and thus $\operatorname{GKdim} R^{(d)} \geq \operatorname{GKdim} R$ [KL, p. 52, Proposition 5.1(d)]. □

Lemma 3.6.6. (cf. [AV, Theorem 3.14]) *The ring $B = B(X, \sigma, \mathcal{L})$ satisfies the hypotheses of the previous lemma.*

Proof. Choose m_0 so that for $m \geq m_0$, the sheaf $\mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ is generated by global sections for $m \geq m_0$. Then we have an exact sequence

$$0 \to \operatorname{Ker} f_m \to \mathcal{O}_X \otimes B_m \xrightarrow{f_m} \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}} \to 0.$$

Now choose n_0 such that the higher cohomology of $\operatorname{Ker} f_m \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$ vanishes for $n \geq n_0$. Then from the above exact sequence we have the surjection $B_n \otimes B_m \to B_{m+n}$ as required. □

So by replacing D with Δ_m and σ with σ^m, in order to prove Theorem 3.6.2 we may again assume P is unipotent, D is ample, and $H^q(X, \mathcal{O}_X(\Delta_m)) = 0$ for $q > 0$ and *all* $m > 0$. Then

$$\dim B_m = \dim H^0(X, \mathcal{O}_X(\Delta_m)) = \chi(\mathcal{O}_X(\Delta_m))$$

where χ is the Euler characteristic on X. Recall that as in §2.6, a polynomial with rational coefficients, integer valued on integers, is called a *numerical polynomial*. We will soon see that $\chi(\mathcal{O}_X(\Delta_m))$ is a numerical polynomial in m with positive leading coefficient. Any such polynomial of degree d is of the form

$$a_d \binom{m}{d} + a_{d-1} \binom{m}{d-1} + \cdots + a_0$$

with $a_i \in \mathbb{Q}$. Then using standard combinatorial identities, we have that

$$\sum_{i=0}^{m-1} \dim B_i$$

is a numerical polynomial of degree $d+1$. By the definition of GK-dimension (3.6.1), we immediately have

$$\operatorname{GKdim} B = \deg(\dim B_m) + 1 = \deg(\chi(\mathcal{O}_X(\Delta_m))) + 1. \tag{3.6.7}$$

Thus far, we have only used the intersection numbers $(D.C)$, where D is a divisor and C is a curve. In studying the growth of Δ_m in terms of m, we will need to examine the intersection of divisors on higher dimensional subvarieties. More precisely, for an n-dimensional variety V, we use the symmetric n-linear form

$$(D_1. \dots .D_n)_V = (\mathcal{O}_X(D_1). \dots .\mathcal{O}_X(D_n).\mathcal{O}_V)$$

defined in §2.6.

We prove

Lemma 3.6.8. *Let X be a projective scheme with unipotent automorphism σ and ample divisors D and D' with $\Delta'_m = D' + \cdots + \sigma^{m-1}D'$. Further let V be a closed subvariety of X of dimension n. Then for $0 \leq i \leq n$,*

1. *$(D^i.\Delta_m^{n-i})_V$ is a numerical polynomial in m with positive leading coefficient.*
2. *$\deg(D^i.\Delta_m^{n-i})_V = \deg((D')^i.(\Delta'_m)^{n-i})_V$.*
3. *$\deg(D^{i-1}.\Delta_m^{n-i})_W \leq \deg(D^i.\Delta_m^{n-i})_V$ where $W \subset V$ is a closed subvariety with $\dim W = \dim V - 1$.*
4. *$\deg(D^i.\Delta_m^{n-i})_V < \deg(D^{i-1}.\Delta_m^{n-i+1})_V$.*
5. *$\deg(\Delta_m^j)_W < \deg(\Delta_m^n)_V$ where $W \subset V$ is a closed subvariety and $\dim W = j < n$.*

Proof. Since σ is unipotent and intersection numbers only depend on numerical equivalence classes, we may replace Δ_m by the divisor on the right hand side of Equation 3.4.3. As noted below that equation, it is not a problem to treat the N^iD as divisors. Since the intersection form is multilinear and integer valued on divisors, $(D^i.\Delta_m^{n-i})_V$ must be a numerical polynomial. By the Nakai criterion for ampleness (2.6.14) the function is positive for all positive m (since Δ_m is ample) and hence has a positive leading coefficient. Thus part (1) is proven.

Now for some fixed ℓ, we know that $\ell D' - D$ is ample. Hence

$$\ell(D'.D^{i-1}.\Delta_m^{n-i})_V - (D^i.\Delta_m^{n-i})_V = (\ell D' - D.D^{i-1}.\Delta_m^{n-i})_V > 0$$

for all $m > 0$. Thus

$$\deg(D'.D^{i-1}.\Delta_m^{n-i})_V \geq \deg(D^i.\Delta_m^{n-i})_V$$

and by symmetry the two degrees are equal. We can continue this argument, replacing each D with D', so $\deg(D^i.\Delta_m^{n-i}) = \deg((D')^i.\Delta_m^{n-i})$. By also noting that

$$\ell\Delta'_m - \Delta_m = (\ell D' - D) + \cdots + \sigma^{m-1}(\ell D' - D)$$

is ample, one can similarly replace each Δ_m with Δ'_m. Thus the second claim is proven.

Now let $W \subset V$ be a closed subvariety with $\dim W = \dim V - 1$. One has

$$(D^{i-1}.\Delta_m^{n-i})_W = (D^{i-1}.W.\Delta_m^{n-i})_V$$

by (2.6.9). We claim that for some fixed ℓ, the intersection number of $\ell D - W$ with any collection of $n-1$ ample divisors is positive. This is well-known if V is normal so W is a Weil divisor, so for some ℓ, the Weil divisor $\ell D - W$ is effective [R, p. 282]. The general case can be seen by pulling back to the normalization of V. Since normalization is a finite, birational morphism, ampleness (2.6.16) and intersection numbers (2.6.12) are both preserved under pull-back. Thus the claim is proven. An argument similar to the proof of part (2) proves the third claim of the lemma.

For part (4), Equation 3.4.3 shows that the leading coefficient of $(D^{i-1}.D'.\Delta_m^{n-i})_V$ is a sum of terms

$$a_\alpha(D^{i-1}.D'.N^{\alpha_1}D.\dots.N^{\alpha_{n-i}}D)_V$$

where $a_\alpha((k+1)!)^n$ is an integer. So any leading coefficient times $((k+1)!)^n$ is a positive integer. Thus given any set of ample divisors $\{D'\}$, there is a D' in that set such that $(D^{i-1}.D'.\Delta_m^{n-i})_V$ has the smallest leading coefficient.

Now let j be a natural number such that $(D^{i-1}.\sigma^j D.\Delta_m^{n-i})_V$ has the smallest leading coefficient of all $(D^{i-1}.\sigma^l D.\Delta_m^{n-i})_V$. Then for any $l \geq 0$,

$$\frac{(D^{i-1}.\sigma^l D.\Delta_m^{n-i})_V}{(D^{i-1}.\sigma^j D.\Delta_m^{n-i})_V}$$

is a rational function with limit, as $m \to \infty$, greater than or equal to 1. So given any natural number M,

$$\lim_{m\to\infty} \frac{(D^{i-1}.\Delta_m.\Delta_m^{n-i})_V}{(D^{i-1}.\sigma^j D.\Delta_m^{n-i})_V} \geq M.$$

Since this is true for any M, the limit must be $+\infty$. So

$$\deg(D^{i-1}.\Delta_m.\Delta_m^{n-i})_V > \deg(D^{i-1}.\sigma^j D.\Delta_m^{n-i})_V.$$

Examining the proof of part (2), we see the right hand side equals $\deg(D^i.\Delta_m^{n-i})_V$, proving part (4).

Finally, for part (5), find a chain of subvarieties $W = V_0 \subsetneq \cdots \subsetneq V_{n-j} = V$. Then part (3) combined with part (4) proves the claim for each part of the chain. □

By a version of the Riemann-Roch Theorem for an n-dimensional complete scheme X and coherent sheaf $\mathcal{F}$ [Fl, p. 361, Example 18.3.6]:

$$\chi(\mathcal{F}(\Delta_m)) = \sum_{j=0}^{n} \frac{1}{j!}\int_X (\Delta_m^j) \cap \tau_{X,j}(\mathcal{F}). \tag{3.6.9}$$

The $\tau_{X,j}(\mathcal{F})$ is a j-cycle, a linear combination of j-dimensional closed subvarieties of $\operatorname{Supp}\mathcal{F}$. In other words,

$$\tau_{X,j}(\mathcal{F}) = \sum_V a_V[V] \tag{3.6.10}$$

where V is a subvariety of X, $[\ ]$ denotes rational equivalence, and a_V is a rational number. The terms of (3.6.9), for $\mathcal{F} = \mathcal{O}_X$ can then be interpreted as

$$\int_X (\Delta_m^j) \cap \tau_{X,j}(\mathcal{O}_X) = \sum_V a_V (\Delta_m^j)_V.$$

If X_i is an irreducible component of X of dimension j, then $[X_i] = n[(X_i)_{\text{red}}]$ is a term in $\tau_{X,j}(\mathcal{O}_X)$, where n is the degree of the natural map $(X_i)_{\text{red}} \to X_i$. To see this, first note that $(\Delta_m^{\dim X_i})_{X_i}/(\dim X_i)!$ must be the $\dim X_i$ term of $\chi(\mathcal{O}_{X_i}(\Delta_m))$ [Fl, ibid.]. Also $a_{(X_i)_{\text{red}}} = n$ by (2.6.11). The short exact sequence

$$0 \to \mathcal{I}_i \to \mathcal{O}_X \to \mathcal{O}_{X_i} \to 0$$

gives $\chi(\mathcal{O}_X(\Delta_m)) = \chi(\mathcal{O}_{X_i}(\Delta_m)) + \chi(\mathcal{I}_i(\Delta_m))$. The support of $\mathcal{I}_i$ does not contain X_i and an irreducible component is rationally equivalent only to itself [Fl, p. 11, Example 1.3.2]. So there is no $[X_i]$ term in $\chi(\mathcal{I}_i(\Delta_m))$ which could cancel out the $[X_i]$ term in the first summand.

Lemma 3.6.11. *Let X be a projective scheme with unipotent automorphism σ and irreducible components X_i. Then*

$$\deg \chi(\mathcal{O}_X(\Delta_m)) = \max_{X_i} \deg(\Delta_m^{\dim X_i})_{X_i}.$$

Proof. If the left hand side is larger than the right hand side, then by the discussion before the lemma, there is a subvariety V with

$$\deg \chi(\mathcal{O}_X(\Delta_m)) = \deg(\Delta_m^{\dim V})_V > \deg(\Delta_m^{\dim X_j})_{X_j},$$

where X_j is an irreducible component *properly* containing V. This cannot happen by Lemma 3.6.8(5).

On the other hand, if the right hand side is larger, then there exists a subvariety V with $a_V < 0$ in the notation of Equation 3.6.10 and

$$\deg(\Delta_m^{\dim V})_V = \max_{X_i} \deg(\Delta_m^{\dim X_i})_{X_i}.$$

The earlier discussion shows that $a_{X_i} > 0$ for each i. Hence V is properly contained in some irreducible component. But again this cannot happen by Lemma 3.6.8(5). □

Lemma 3.6.12. *Let X be a projective scheme with unipotent automorphism σ. Let $V \subset X$ be a closed subscheme which does not contain (the reduction of) an irreducible component of X. Then* $\deg \chi(\mathcal{O}_V(\Delta_m)) < \deg \chi(\mathcal{O}_X(\Delta_m))$.

Proof. By Lemma 3.6.11 we may pick an irreducible component V_0 of V with

$$\deg \chi(\mathcal{O}_V(\Delta_m)) = \deg(\Delta_m^{\dim V_0})_{V_0}.$$

Then X has an irreducible component X_0 with $V_0 \subsetneq X_0$. The claim is then proven by combining Lemmata 3.6.8(5) and 3.6.11. □

Proposition 3.6.13. *Let X be a projective scheme with unipotent automorphism σ and ample divisor D. Then $\chi(\mathcal{O}_X(\Delta_m))$ is a numerical polynomial in m. The degree of this polynomial is independent of the ample divisor D chosen. Further, if σ is numerically equivalent to the identity, this polynomial has degree* $\dim X$.

Proof. The first claim is obvious since the intersection numbers in Equation 3.6.9 are numerical polynomials, as noted in Lemma 3.6.8. The independence of the degree comes from Lemma 3.6.8(2).

If σ is numerically equivalent to the identity, then $k = 0$. So $\chi(\mathcal{O}_X(\Delta_m)) = \chi(\mathcal{O}_X(mD))$ has degree $\dim X$. □

Combined with Equations 3.6.4 and 3.6.7, this proposition implies the first two parts of Theorem 3.6.2.

Considering Lemma 3.6.11 and Equation 3.6.7, we immediately have

Proposition 3.6.14. *Let X be a scheme with unipotent automorphism σ, ample divisor D, and irreducible components X_i. Let $B = B(X, \sigma, \mathcal{L})$. Then*

$$\operatorname{GKdim} B - 1 = \deg \chi(\mathcal{O}_X(\Delta_m)) = \max_{X_i} \deg(\Delta_m^{\dim X_i})_{X_i}.$$

In particular, if X is equidimensional, then

$$\operatorname{GKdim} B - 1 = \deg(\Delta_m^{\dim X})_X. \quad \square$$

Remark 3.6.15. Note that by replacing σ by a power, we may assume σ fixes each irreducible component. That is, σ is an automorphism of each component. Thus the soon to be proven bounds of Theorem 3.6.2 for equidimensional schemes can be used to find bounds in the general case.

Lemma 3.6.16. *Let σ be a unipotent automorphism with numerical action $P = I + N$, with $k = J(\sigma)$ (cf. Definition 3.4.1). Then k is even and* $\deg \chi(\mathcal{O}_X(\Delta_m)) \geq k + \dim X$.

Proof. Given an ample divisor D, one has $N^k D \neq 0$ by Lemma 3.4.4. So there exists a curve C such that $(N^k D.C) \neq 0$. Since $(\sigma^m D.C) > 0$ for all $m \in \mathbb{Z}$ and in particular for $m > 0$, $(N^k D.C) > 0$. However, if k is odd, (3.4.3) implies that the leading term of $(\sigma^{-m} D.C)$ is $-\binom{m}{k}(N^k D.C)$ where $m > 0$. Then $(\sigma^{-m} D.C) < 0$ for large m, which cannot occur.

For the lower bound, note $\deg \chi(\mathcal{O}_C(\Delta_m)) = \deg(\Delta_m.C) = k+1$. Constructing a chain of subvarieties between C and X, Lemma 3.6.12 shows that $\deg \chi(\mathcal{O}_X(\Delta_m)) \geq \dim X + k$. ☐

Lemma 3.6.17. *Let $n = \dim X$. Then $(\Delta_m^n)_X$ has degree at most $k(n-1)+n$.*

Proof. If $k = 0$ the lemma is trivial. So assume that $k > 0$. Let $P = I + N$.

Expanding (Δ_m^n) gives terms of the form

$$f(m)(N^{i_1}D.N^{i_2}D.\ldots.N^{i_n}D)$$

where $i_1 \leq i_2 \leq \cdots \leq i_n$ and $\deg_m f = n + \sum i_j$. We will show that if $\sum i_j > k(n-1)$ then $(N^{i_1}D.N^{i_2}D.\ldots.N^{i_n}D) = 0$.

Order $(i_1, \ldots, i_n)$ in the following way: $(i_1, \ldots, i_n) > (i_1', \ldots, i_n')$ if the right-most non-zero entry of $(i_1, \ldots, i_n) - (i_1', \ldots, i_n')$ is positive. We proceed by descending induction on this ordering.

The largest n-tuple in this ordering is $(k, k, \ldots, k)$. Since $k > 0$, $N^{k-1}D$ exists (taking $N^0 = I$) so

$$\begin{aligned}(N^{k-1}D.(N^k D)^{n-1}) &= (PN^{k-1}D.(PN^k D)^{n-1}) \\ &= (N^{k-1}D.(N^k D)^{n-1}) + ((N^k D)^n)\end{aligned}$$

and hence $((N^k D)^n) = 0$.

Now suppose that $(i_1, \ldots, i_n)$ is such that $\sum i_j > k(n-1)$ and we have proven our claim for all larger $(i_1', \ldots, i_n')$. Since $\sum i_j > k(n-1)$, we have $i_1 > 0$ so examine

$$(N^{i_1-1}D.N^{i_2}D.\ldots.N^{i_n}D) = (PN^{i_1-1}D.PN^{i_2}D.\ldots.PN^{i_n}D).$$

A typical term in the right-hand side is of the form

$$(N^{i_1-1+\delta_1}D.N^{i_2+\delta_2}D.\ldots.N^{i_n+\delta_n}D)$$

where $\delta_j = 0, 1$. The terms with $\delta_j = 1$ where $j > 1$ are all higher in the ordering than $(i_1, \ldots, i_n)$ and hence are zero. This only leaves

$$\begin{aligned}&(N^{i_1-1}D.N^{i_2}D.\ldots.N^{i_n}D) = \\ &\qquad (N^{i_1-1}D.N^{i_2}D.\ldots.N^{i_n}D) + (N^{i_1}D.N^{i_2}D.\ldots.N^{i_n}D)\end{aligned}$$

and so $(N^{i_1}D.N^{i_2}D.\ldots.N^{i_n}D) = 0$. ☐

Using Equation 3.6.4 and Proposition 3.6.14, these lemmata complete the proof of Theorem 3.6.2.

Even more restrictions can give sharper results.

Example 3.6.18. Let X be a smooth 3-fold and $k = J(\sigma) = 2$. Then for any σ-ample D, $\operatorname{GKdim} B = k + \dim X + 1 = 6$.

Assume again that $H^q(X, \Delta_m) = 0, q > 0, m > 0$, D is very ample and P is unipotent. Lemma 3.3.2 shows $\deg(\sigma^{-m}D.D^2) = \deg(D.(\sigma^m D)^2) \leq 2$. Expanding $(D.(\sigma^m D)^2)$ one writes

$$(D^3) + 2m(D^2.ND) + 2\binom{m}{2}(D^2.N^2D) + m^2(D.(ND)^2)$$
$$+2m\binom{m}{2}(D.ND.N^2D) + \binom{m}{2}^2(D.(N^2D)^2)$$

So the terms in the second line are both zero.

Now expand $(D^3) = ((\sigma^m D)^3) =$

$$(D^3) + 3m(D^2.ND) + 3\binom{m}{2}(D^2.N^2D) + 3m^2(D.(ND)^2)$$
$$+m^3((ND)^3) + 3m^2\binom{m}{2}((ND)^2.N^2D)$$
$$+6m\binom{m}{2}(D.ND.N^2D) + 3\binom{m}{2}^2(D.(N^2D)^2)$$
$$+3m\binom{m}{2}^2(ND.(N^2D)^2) + \binom{m}{2}^3((N^2D)^3)$$

Each term in the last line is zero by the proof of (3.6.17). The terms in the third line are zero by the comments above. Since (D^3) is constant, the coefficients of each m^l must also be zero. Since $\frac{3}{2}((ND)^2.N^2D)$ is the coefficient of m^4, it is zero. Then the coefficient of m^3 is $((ND)^3)$ which is also zero. So only the top line has non-zero terms.

So now expanding out (Δ_m^3) one sees the degree can be at most 5 and by (3.6.16) it must be $\dim X + k = 5$.

3.7 Categorical results

Now using the theorems of [AZ] as described in §2.3, we get the following result for rings which are close to commutative in the sense of the following theorem. Recall from §2.3 that $\operatorname{proj} R = (\operatorname{qgr} R, \pi R)$.

Theorem 3.7.1. *Let R be a finitely graded ring over an algebraically closed field k. Suppose that R is right noetherian and satisfies χ. Further suppose that* $\operatorname{proj} R \cong$

$(\mathrm{coh}(X), \mathcal{O}_X)$*, for some (classical) projective scheme* X*. Then* R *is left noetherian and has finite, integer GK-dimension.*

Proof. Since R is right noetherian and satisfies χ_1, the degree shift $(\pi R, [+1])$ is ample by Theorem 2.3.8. Thus $(\pi R, [+1])$ corresponds to some ample autoequivalence $(\mathcal{O}_X, s)$ of $\mathrm{coh}(X)$. Any autoequivalence of $\mathrm{coh}(X)$ has the form $\mathcal{L}_\sigma \otimes -$ by Proposition 2.5.4 for some automorphism σ and invertible sheaf $\mathcal{L}$. Thus $R \cong B = B(X, \sigma^{-1}, \mathcal{L})$ up to a finite dimensional vector space, by Theorem 2.3.8, Proposition 2.5.3, and Lemma 2.2.5.

Further, since R satisfies χ, for any coherent sheaf $\mathcal{F}$ we have the vanishing of the higher cohomology of $H^q(X, s^m\mathcal{F})$ and sufficiently large m by Theorem 2.4.2. Thus $\mathcal{L}$ is right σ^{-1}-ample and hence left σ^{-1}-ample. Then by Theorem 3.1.1, B is noetherian and has finite, integer GK-dimension. Thus R has these properties as well. □

One can also find bounds on the GK-dimension of such R which only depend on $\mathrm{qgr}\, R$, as D. Eisenbud pointed out to the author.

Corollary 3.7.2. *Let* R *satisfy the hypotheses of the above theorem, with* $\mathrm{proj}\, R \cong (\mathrm{coh}(X), \mathcal{O}_X)$ *for some projective scheme* X*. Let* $\rho = \rho(X)$ *be the Picard number of* X*. Then*

$$\dim X + 1 \leq \mathrm{GKdim}\, R \leq 2 \left\lfloor \frac{\rho - 1}{2} \right\rfloor (\dim X - 1) + \dim X + 1.$$

Proof. For any σ used in the previous theorem, the number $\ell = J(\sigma)$ is even by Lemma 3.6.16. Also $\ell \leq \rho - 1$ since any matrix acting on $A^1_{\mathrm{Num}}(X)$ has rank less than ρ. Then by Theorem 3.6.2(3) we have the corollary for equidimensional schemes. This bound can also be used in the non-equidimensional case by using the observation of Remark 3.6.15. □

3.8 A curious question

Let X be a proper, non-projective scheme X with automorphism X. We do not know if there could exist an invertible sheaf $\mathcal{L}$ such that $\mathcal{L}$ is σ-ample. This seems unlikely, since no such $\mathcal{L}$ can exist when σ is the identity by Definition 2.3.5(3).

We will now give further evidence that a non-projective proper scheme probably cannot have a σ-ample invertible sheaf.

Proposition 3.8.1. *Let* X *be a proper scheme with automorphism* σ *and invertible sheaf* $\mathcal{L}$*. Then the following two statements are equivalent:*

1. *For any coherent sheaf* $\mathcal{F}$*, there exists an* m_0 *such that* $\mathcal{F} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ *is generated by global sections for* $m \geq m_0$*.*

2. *For any invertible sheaf* $\mathcal{H}$, *there exists an* m_0 *such that* $\mathcal{H}^{-1} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ *is ample for* $m \geq m_0$.

Thus, if either condition holds, X *must be projective because it has an ample invertible sheaf.*

Proof. Suppose that (1) holds. Let C be any integral curve on X. Since C is a proper curve, it is projective [H2, p. 232, Exercise 5.8]. Thus there is an invertible sheaf $\mathcal{M}_C$ on C which has negative degree, and hence cannot be generated by global sections. But by the hypothesis (1), there exists n such that

$$\mathcal{M}_C \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$$

is generated by global sections. So in particular, there is an invertible sheaf on X which restricts to a sheaf of non-zero degree on C.

Let $A^1 = A^1_{\mathrm{Num}}(X)$. Then A^1 is a finitely generated free abelian group by Theorem 2.6.23. Let $\mathcal{H}_1, \ldots, \mathcal{H}_{\rho(X)}$ be a $\mathbb{Z}$-basis for A^1. Choose m_0 large enough so that for $i = 1, \ldots, \rho(X)$ and $m \geq m_0$, the invertible sheaves $\mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ and

$$\mathcal{H}_i^{\pm 1} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$$

are generated by global sections.

Since the $\mathcal{H}_i$ are a $\mathbb{Z}$-basis for A^1, there is some j with $(\mathcal{H}_j.C) \neq 0$. Without loss of generality, we may assume $\mathcal{H}_j$ has negative degree on C. But since $\mathcal{H}_j \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ is generated by global sections for $m \geq m_0$, that sheaf is numerically effective. Hence for $m \geq m_0$, the sheaf $\mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ has positive degree on C. Notice that m_0 does not depend on j and hence does not depend on C. So by Proposition 2.6.17, the sheaf $\mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ is ample for $m \geq m_0$. Hence X is projective.

Now let $\mathcal{H}$ be an arbitrary invertible sheaf and let $\mathcal{N}$ be an ample invertible sheaf. Choose m_1 so that

$$\mathcal{N}^{-1} \otimes \mathcal{H}^{-1} \otimes \mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$$

is generated by global sections for $m \geq m_1$. Tensoring with $\mathcal{N}$ and using Proposition 2.6.1(1), we have (2).

Now suppose that we have (2). Since an ample sheaf exists, the scheme X is projective by Definition 2.3.5(3). Then (2) $\implies$ (1) of the current proposition is the same as (4) $\implies$ (2) in Proposition 3.2.2. □

Unfortunately, the current proof of Proposition 2.5.7 requires X to be projective, so the previous proposition does not give that the existence of a σ-ample invertible sheaf implies that X is projective. The difficulty lies in showing that for some m, the sheaf $\mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ is generated by global sections; Proposition 2.5.7 uses a

Koszul resolution to do this. Assuming that $\mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ is generated by global sections for some m, one can proceed as in [H2, p. 229, Proposition 5.3] and show that Proposition 2.5.7 is true when X is proper. We then have

Proposition 3.8.2. *Let X be a proper, non-projective scheme with automorphism σ and invertible sheaf $\mathcal{L}$. Suppose that $\mathcal{L} \otimes \cdots \otimes \mathcal{L}^{\sigma^{m-1}}$ is generated by global sections for some m. Then $\mathcal{L}$ is not σ-ample.* $\square$

CHAPTER 4

Twisted Multi-homogeneous Coordinate Rings

4.1 A new generalization

Recently, Chan introduced twisted multi-homogeneous coordinate rings in [C]. Given a projective scheme X, one studies the "ampleness" of a finite collection of invertible sheaves and automorphisms $\{(\mathcal{L}_i, \sigma_i)\}$. Via these methods, Chan studies rings associated to twisted homogeneous coordinate rings, like the tensor product of two such rings. In this chapter, we will generalize our previous results to the multi-homogeneous case and thereby strengthen his findings.

Because of the notational difficulties associated with handling arbitrarily many pairs $(\mathcal{L}_i, \sigma_i)$, we remind the reader that we defined a concept of invertible bimodule $\mathcal{L}_\sigma$ in Definition 2.5.1. In this chapter it will be important to know how more than one invertible bimodule acts on a coherent sheaf. Let σ and τ be automorphisms and let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves. Then for a coherent sheaf $\mathcal{F}$,

$$\begin{aligned}\mathcal{L}_\sigma \otimes (\mathcal{M}_\tau \otimes \mathcal{F}) &= \mathcal{L}_\sigma \otimes (\mathcal{M} \otimes \tau^*\mathcal{F})\\ &= \mathcal{L} \otimes \sigma^*\mathcal{M} \otimes \sigma^*\tau^*\mathcal{F}\\ &= \mathcal{L} \otimes \sigma^*\mathcal{M} \otimes (\tau\sigma)^*\mathcal{F}\\ &= (\mathcal{L} \otimes \sigma^*\mathcal{M})_{\tau\sigma} \otimes \mathcal{F}.\end{aligned}$$

Given two invertible bimodules $\mathcal{L}_\sigma$ and $\mathcal{M}_\tau$, one then defines the tensor product to be

$$\mathcal{L}_\sigma \otimes \mathcal{M}_\tau = (\mathcal{L} \otimes \sigma^*\mathcal{M})_{\tau\sigma}, \tag{4.1.1}$$

where the second tensor product is the usual product on quasi-coherent sheaves. We will sometimes denote the product of invertible bimodules by juxtaposition if the meaning is clear.

We now sketch the construction of a twisted multi-homogeneous coordinate ring. Let $\{(\mathcal{L}_i)_{\sigma_i}\}$ be a collection of s invertible bimodules. For notational convenience, we will write $\mathcal{L}_{(i,\sigma_i)} = (\mathcal{L}_i)_{\sigma_i}$. Given these s invertible bimodules, one wishes to form an associated twisted multi-homogeneous coordinate ring $B = B(X; \{\mathcal{L}_{(i,\sigma_i)}\})$. For an s-tuple $\overline{n} = (n_1, \ldots, n_s)$ we define the multi-graded piece $B_{\overline{n}}$ as

$$B_{\overline{n}} = H^0(X, \mathcal{L}^{n_1}_{(1,\sigma_1)} \ldots \mathcal{L}^{n_s}_{(s,\sigma_s)}) \tag{4.1.2}$$

where the cohomology of an invertible bimodule is just cohomology of the underlying sheaf. Multiplication should be given by

$$a \cdot b = a\sigma^{\overline{m}}(b) \tag{4.1.3}$$

when $a \in B_{\overline{m}}$ and $b \in B_{\overline{n}}$. Here $\sigma^{\overline{m}}(b) = \sigma_1^{m_1}\sigma_2^{m_2} \ldots \sigma_s^{m_s}(b)$, as we defined the action of automorphisms on global sections in our discussion before Equation (2.2.2).

However, for this multiplication to be defined, the invertible bimodules must commute with each other. To see this, consider the bigraded case, with bimodules $\mathcal{L}_\sigma, \mathcal{M}_\tau$. Then

$$B_{(1,0)} = H^0(X, \mathcal{L}_\sigma), \quad B_{(0,1)} = H^0(X, \mathcal{L}_\sigma), \quad B_{(1,1)} = H^0(X, \mathcal{L}_\sigma\mathcal{M}_\tau).$$

Given the multiplication above, we have $B_{(1,0)}B_{(0,1)} \subset B_{(1,1)}$ and $B_{(0,1)}B_{(1,0)} \subset H^0(X, \mathcal{M}_\tau\mathcal{L}_\sigma)$. To guarantee that $B_{(1,1)} = H^0(X, \mathcal{M}_\tau\mathcal{L}_\sigma)$, so that we have a bigraded ring, we demand $\mathcal{L}_\sigma\mathcal{M}_\tau = \mathcal{M}_\tau\mathcal{L}_\sigma$.

Examining (4.1.1), we see two bimodules $\mathcal{L}_\sigma, \mathcal{M}_\tau$ commute when

$$\mathcal{L} \otimes \sigma^*\mathcal{M} \cong \mathcal{M} \otimes \tau^*\mathcal{L} \quad \text{and} \quad \sigma\tau = \tau\sigma. \tag{4.1.4}$$

Thus we need sheaf isomorphisms $\varphi_{ij} : \mathcal{L}_{(j,\sigma_j)}\mathcal{L}_{(i,\sigma_i)} \to \mathcal{L}_{(i,\sigma_i)}\mathcal{L}_{(j,\sigma_j)}$ for each $1 \leq i < j \leq s$. It is further noted in [C] that when there are three or more bimodules, these isomorphisms must be compatible on "overlaps" in the sense of Bergman's Diamond Lemma. In terms of the isomorphism φ_{ij} this means [C, p. 444]

$$\begin{aligned} (\varphi_{ij} \otimes 1_{\mathcal{L}_{(k,\sigma_k)}}) \circ (1_{\mathcal{L}_{(j,\sigma_j)}} \otimes \varphi_{ik}) \circ (\varphi_{jk} \otimes 1_{\mathcal{L}_{(i,\sigma_i)}}) \\ = (1_{\mathcal{L}_{(i,\sigma_i)}} \otimes \varphi_{jk}) \circ (\varphi_{ik} \otimes 1_{\mathcal{L}_{(j,\sigma_j)}}) \circ (1_{\mathcal{L}_{(k,\sigma_k)}} \otimes \varphi_{ij}) \end{aligned} \tag{4.1.5}$$

in $\operatorname{Hom}(\mathcal{L}_{(k,\sigma_k)}\mathcal{L}_{(j,\sigma_j)}\mathcal{L}_{(i,\sigma_i)}, \mathcal{L}_{(i,\sigma_i)}\mathcal{L}_{(j,\sigma_j)}\mathcal{L}_{(k,\sigma_k)})$. We will always assume that we have this compatibility when forming the ring B. Summarizing, we have

Proposition 4.1.6. *Let $\{\mathcal{L}_{(i,\sigma_i)}\}$ be a set of commuting invertible bimodules. Assume these bimodules have compatible pairwise commutation relations in the sense of* (4.1.5). *Then there is a multi-graded ring B with multi-graded pieces given by* (4.1.2) *and multiplication given by* (4.1.3). □

To study these rings, a multi-graded version of σ-ampleness is introduced. Since we will again be interested in both this version of ampleness and the usual one, we will call this (right) *NC-ampleness*, whereas [C] uses the terminology (right) ampleness. We define the ordering on s-tuples to be the standard one, i.e., $(n'_1, \dots, n'_s) \geq (n_1, \dots, n_s)$ if $n'_i \geq n_i$ for all i. For simplicity we write $\mathcal{L}_{\overline{\sigma}}^{\overline{m}} = \mathcal{L}_{(1,\sigma_1)}^{m_1} \dots \mathcal{L}_{(s,\sigma_s)}^{m_s}$.

Definition 4.1.7. Let X be a projective scheme with s commuting invertible bimodules $\{\mathcal{L}_{(i,\sigma_i)}\}$.

1. If for any coherent sheaf $\mathcal{F}$, there exists an $\overline{m_0}$ such that
$$H^q(X, \mathcal{F} \otimes \mathcal{L}_{\overline{\sigma}}^{\overline{m}}) = 0$$
for $q > 0$ and $\overline{m} \geq \overline{m_0}$, then the set $\{\mathcal{L}_{(i,\sigma_i)}\}$ is called *right NC-ample*.
2. If for any coherent sheaf $\mathcal{F}$, there exists an $\overline{m_0}$ such that
$$H^q(X, \mathcal{L}_{\overline{\sigma}}^{\overline{m}} \otimes \mathcal{F}) = 0$$
for $q > 0$ and $\overline{m} \geq \overline{m_0}$, then the set $\{\mathcal{L}_{(i,\sigma_i)}\}$ is called *left NC-ample*.

As in the case of one invertible bimodule, right and left NC-ampleness are related.

Lemma 4.1.8 (See Lemma 2.5.6). *Let X be a projective scheme with s commuting invertible bimodules $\{(\mathcal{L}_i)_{\sigma_i}\}$. Then the set $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$ commutes pairwise. Also, the set $\{(\mathcal{L}_i)_{\sigma_i}\}$ is right NC-ample if and only if the set $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$ is left NC-ample.*

Proof. Let $\mathcal{L}_\sigma, \mathcal{M}_\tau$ be two commuting invertible bimodules. Then (4.1.4) holds. Obviously $\sigma^{-1}\tau^{-1} = \tau^{-1}\sigma^{-1}$. Now since $\mathcal{L} \otimes \sigma^*\mathcal{M} \cong \mathcal{M} \otimes \tau^*\mathcal{L}$, pulling back by $(\sigma^{-1}\tau^{-1})$ we have

$$(\tau^{-1})^*(\sigma^{-1})^*\mathcal{L} \otimes (\tau^{-1})^*\mathcal{M} \cong (\sigma^{-1})^*(\tau^{-1})^*\mathcal{M} \otimes (\sigma^{-1})^*\mathcal{L}.$$

So $\mathcal{L}_{\sigma^{-1}}^{\sigma^{-1}} = ((\sigma^{-1})^*\mathcal{L})_{\sigma^{-1}}$ and $\mathcal{M}_{\tau^{-1}}^{\tau^{-1}} = ((\tau^{-1})^*\mathcal{M})_{\tau^{-1}}$ commute.

Now suppose that the set $\{(\mathcal{L}_i)_{\sigma_i}\}$ is right NC-ample. Then the higher cohomology of

$$Z = |\mathcal{F} \otimes (\mathcal{L}_1)_{\sigma_1}^{m_1} \dots (\mathcal{L}_s)_{\sigma_s}^{m_s}|$$

vanishes for all $(m_1, \dots, m_s)$ sufficiently large. We may write

$$Z = \mathcal{F} \otimes \mathcal{N}_1 \otimes \dots \otimes \mathcal{N}_s$$

where

$$\mathcal{N}_j = |((\mathcal{L}_j)_{\sigma_j}^{m_j})^{\sigma_1\sigma_2\dots\sigma_{j-1}}|.$$

Writing out the expression in the sense of (2.5.1), we see that each $\mathcal{N}_j$ is the tensor product

$$\prod_{l=0}^{m_j-1} \mathcal{L}_j^{\sigma_1^{m_1}\sigma_2^{m_2}\ldots\sigma_j^l}$$

of invertible sheaves. Now pulling back by $\tau = \sigma_1^{-m_1}\sigma_2^{-m_2}\ldots\sigma_s^{-m_s}$ we have that $\mathcal{N}_j^\tau$ is the tensor product

$$\prod_{l=0}^{m_j-1} \mathcal{L}_j^{\sigma_j^{l-m_j}\ldots\sigma_s^{-m_s}} = \prod_{l=0}^{m_j-1} (\mathcal{L}_j^{\sigma_j^{-1}})^{\sigma_j^{l-m_j+1}\ldots\sigma_s^{-m_s}}.$$

Thus we have

$$\mathcal{N}_1^\tau \otimes \cdots \otimes \mathcal{N}_s^\tau \otimes \mathcal{F}^\tau = |(\mathcal{L}_1^{\sigma_1^{-1}})^{m_1}_{\sigma_1^{-1}} \ldots (\mathcal{L}_s^{\sigma_s^{-1}})^{m_s}_{\sigma_s^{-1}} \otimes \mathcal{F}|.$$

These are the sheaves whose higher cohomology should vanish for $(m_1, \ldots, m_s)$ sufficient large so that $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$ is left NC-ample. Clearly the argument can be reversed. □

Notice that the commutation relation between the invertible bimodules forces the consideration of $\mathcal{L}_{\sigma^{-1}}^{\sigma^{-1}}$ instead of just $\mathcal{L}_{\sigma^{-1}}$ as in the "right vs. left" proofs of §2.2 and §2.5. This is similar to the situation in Lemma 2.5.6, where one pulled back by σ^{m-1} rather than σ^m.

We also have an analogue of Lemma 2.2.5.

Lemma 4.1.9. *Let X be a projective scheme with s commuting invertible bimodules $\{(\mathcal{L}_i)_{\sigma_i}\}$. Assume the commutation relations of $\{(\mathcal{L}_i)_{\sigma_i}\}$ and of $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$ are compatible in the sense of* (4.1.5)*. If $B' = B(X; \{(\mathcal{L}_i)_{\sigma_i}\})$ and $B = B(X; \{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\})$, then $B \cong (B')^{\text{op}}$.*

Proof. Proceed as in the proof of Lemma 2.2.5, with $\tau \colon B \to B'$ given by $\tau(a) = \sigma_1^{n_1}\ldots\sigma_s^{n_s}(a)$ for $a \in B_{(n_1,\ldots,n_s)}$. □

As in §3.2, we have simpler equivalent conditions for a set of bimodules to be right NC-ample.

Proposition 4.1.10. *Let X be a projective scheme with s commuting invertible bimodules $\{\mathcal{L}_{(i,\sigma_i)}\}$. Then the following are equivalent:*

1. *The set $\{\mathcal{L}_{(i,\sigma_i)}\}$ is right NC-ample.*
2. *For any coherent sheaf $\mathcal{F}$, there exists an $\overline{m_0}$ such that $\mathcal{F} \otimes \mathcal{L}_{\overline{\sigma}}^{\overline{m}}$ is generated by global sections for $\overline{m} \geq \overline{m_0}$.*

3. *For any invertible sheaf* $\mathcal{H}$*, there exists an* $\overline{m_0}$ *such that* $|\mathcal{H}^{-1} \otimes \mathcal{L}_{\overline{\sigma}}^{\overline{m}}|$ *is very ample for* $\overline{m} \geq \overline{m_0}$*.*

4. *For any invertible sheaf* $\mathcal{H}$*, there exists an* $\overline{m_0}$ *such that* $|\mathcal{H}^{-1} \otimes \mathcal{L}_{\overline{\sigma}}^{\overline{m}}|$ *is ample for* $\overline{m} \geq \overline{m_0}$*.*

A similar statement holds for left NC-ample.

Proof. Each step of the proof of Proposition 3.2.2 does not depend on the structure of the grading monoid $\mathbb{N}$, only that we have a partial order such that any finite set of elements of the monoid has an upper bound. There is a concept of $m \gg 0$ in any partial order. So the proof goes through exactly as in Proposition 3.2.2. □

We can now give a connection between right NC-ampleness and the concept of σ-ampleness for one invertible sheaf $\mathcal{L}$.

Lemma 4.1.11. *Let* X *be a projective scheme with* s *commuting invertible bimodules* $\{\mathcal{L}_{(i,\sigma_i)}\}$*. Suppose that* $\overline{n} = (n_1, \ldots, n_s) \in (\mathbb{N}^+)^s$ *and set* $\tau = \sigma_1^{n_1} \ldots \sigma_s^{n_s}$*. If the set of bimodules is right NC-ample, then* $|\mathcal{L}_{(1,\sigma_1)}^{n_1} \ldots \mathcal{L}_{(s,\sigma_s)}^{n_s}|$ *is* τ*-ample.*

Proof. Let $\mathcal{H}$ be an invertible sheaf and let $\overline{m_0}$ be such that for all $\overline{m} \geq \overline{m_0}$, the sheaf $|\mathcal{H}^{-1} \otimes \mathcal{L}_{\overline{\sigma}}^{\overline{m}}|$ is ample by Proposition 4.1.10(4).

Now there exists an integer l_0 such that for all $l \geq l_0$, we have $l\overline{n} \geq \overline{m_0}$. So $|\mathcal{H}^{-1} \otimes (\mathcal{L}_{\overline{\sigma}}^{\overline{n}})^l|$ is ample. Thus by Proposition 3.2.2(4), $|\mathcal{L}_{\overline{\sigma}}^{\overline{n}}|$ is τ-ample. □

We then have a new version of Theorem 3.1.2.

Theorem 4.1.12. *Let* X *be a projective scheme with* s *commuting invertible bimodules* $\{\mathcal{L}_{(i,\sigma_i)}\}$*. The set* $\{\mathcal{L}_{(i,\sigma_i)}\}$ *is (right) NC-ample if and only if every* σ_i *is quasi-unipotent and there exists* $\overline{m_0} \in \mathbb{N}^s$ *such that* $|\mathcal{L}_{\overline{\sigma}}^{\overline{m}}|$ *is ample for all* $\overline{m} \geq \overline{m_0}$*.*

Proof. Suppose that $\{\mathcal{L}_{(i,\sigma_i)}\}$ is right NC-ample. Then by Proposition 4.1.10(4), there exists $\overline{m_0} \in \mathbb{N}^s$ such that $|\mathcal{L}_{\overline{\sigma}}^{\overline{m}}|$ is ample for all $\overline{m} \geq \overline{m_0}$. Further, by the previous lemma, $\mathcal{L}_{(1,\sigma_1)}^{n_1} \ldots \mathcal{L}_{(s,\sigma_s)}^{n_s}$ is τ-ample when $\tau = \sigma_1^{n_1} \ldots \sigma_s^{n_s}$ and each $n_i > 0$. Now recall that all the automorphisms commute and hence their actions on $A^1_{\mathrm{Num}}(X)$ are commuting matrices. Thus the eigenvalues of the product $\sigma_1^{n_1} \ldots \sigma_s^{n_s}$ are products of eigenvalues from each σ_i. So if σ_1 were not quasi-unipotent, then either $\tau_1 = \sigma_1\sigma_2 \ldots \sigma_s$ or $\tau_2 = \sigma_1^2\sigma_2 \ldots \sigma_s$ would not be quasi-unipotent. But τ_1 and τ_2 must be quasi-unipotent by Theorem 3.1.2 since the corresponding sheaves $\mathcal{L}_{(1,\sigma_1)}^1 \ldots \mathcal{L}_{(s,\sigma_s)}^1$ and $\mathcal{L}_{(1,\sigma_1)}^2 \ldots \mathcal{L}_{(s,\sigma_s)}^1$ are τ_1-ample and τ_2-ample respectively. Thus each σ_i must be quasi-unipotent.

Now suppose that every σ_i is quasi-unipotent and there exists $\overline{m_0} \in \mathbb{N}^s$ such that $|\mathcal{L}_{\overline{\sigma}}^{\overline{m}}|$ is ample for all $\overline{m} \geq \overline{m_0}$. As the σ_i commute, $\tau = \sigma_1 \ldots \sigma_s$ is quasi-unipotent.

Then by Theorem 3.1.2, the invertible bimodule $\mathcal{L}_{(1,\sigma_1)} \dots \mathcal{L}_{(s,\sigma_s)}$ is τ-ample. So given any invertible sheaf $\mathcal{H}$, there exists $n_0 \in \mathbb{N}$ such that

$$|\mathcal{H}^{-1} \otimes (\mathcal{L}_{(1,\sigma_1)} \dots \mathcal{L}_{(s,\sigma_s)})^n| = |\mathcal{H}^{-1} \otimes \mathcal{L}^n_{(1,\sigma_1)} \dots \mathcal{L}^n_{(s,\sigma_s)}|$$

is ample for $n \geq n_0$ by Proposition 3.2.2(4). Then for all $\overline{m} \geq (n_0, n_0, \dots, n_0) + \overline{m_0}$ the invertible sheaf

$$|\mathcal{H}^{-1} \otimes \mathcal{L}^{\overline{m}}_{\overline{\sigma}}| = |\mathcal{H}^{-1} \otimes \mathcal{L}^{n_0}_{(1,\sigma_1)} \dots \mathcal{L}^{n_0}_{(s,\sigma_s)}| \otimes |\mathcal{L}^{m_1-n_0}_{(1,\sigma_1)} \dots \mathcal{L}^{m_s-n_0}_{(s,\sigma_s)}|$$

is the tensor product of two ample invertible sheaves. Hence it is ample and so the set of invertible bimodules is right NC-ample by Proposition 4.1.10(4). □

Corollary 4.1.13. *Let X be a projective scheme with s commuting invertible bimodules $\{\mathcal{L}_{(i,\sigma_i)}\}$. Then $\{\mathcal{L}_{(i,\sigma_i)}\}$ is right NC-ample if and only if it is left NC-ample.*

Proof. Suppose that $\{\mathcal{L}_{(i,\sigma_i)}\}$ is right NC-ample. Then each σ_i is quasi-unipotent and there exists $\overline{m_0}$ such that $|\mathcal{L}^{m_1}_{(1,\sigma_1)} \dots \mathcal{L}^{m_s}_{(s,\sigma_s)}|$ is ample for $(m_1, \dots, m_s) \geq \overline{m_0}$. Pulling back by $\sigma_1^{-m_1} \dots \sigma_s^{-m_s}$, we have that $|(\mathcal{L}_1^{\sigma_1^{-1}})^{m_1}_{\sigma_1^{-1}} \dots (\mathcal{L}_s^{\sigma_s^{-1}})^{m_s}_{\sigma_s^{-1}}|$ is ample. Thus by Theorem 4.1.12, the set $\{(\mathcal{L}_i^{\sigma_i^{-1}})_{\sigma_i^{-1}}\}$ is right NC-ample. So the original set $\{\mathcal{L}_{(i,\sigma_i)}\}$ is left NC-ample by Lemma 4.1.8. The argument is clearly reversible. □

Thus we may now refer to a set of bimodules as being simply NC-ample.

Note the difference between Theorems 3.1.2 and 4.1.12. The former requires only that $|\mathcal{L}^m_\sigma|$ is ample for one value of m, while the latter requires the product of bimodules to be "eventually" ample. To see this stronger requirement is necessary, let X be any projective scheme with $\mathcal{L}$ any ample invertible sheaf. We need to rule out the pair $\mathcal{L}, \mathcal{L}^{-1}$ where the bimodule action is the usual commutative one. In this particular case, of course $\mathcal{L}^1 \otimes (\mathcal{L}^{-1})^0$ is ample. But $\mathcal{L}^{m_1} \otimes (\mathcal{L}^{-1})^{m_2}$ is not ample for all (m_1, m_2) sufficiently large; just fix m_1 and let m_2 go to infinity.

It is not even necessary for one of the $\mathcal{L}_{(i,\sigma_i)}$ to eventually be ample, since on $\mathbb{P}^1 \times \mathbb{P}^1$, the pair $\mathcal{O}(1,0), \mathcal{O}(0,1)$ is NC-ample, where again these bimodules act only as commutative invertible sheaves.

4.2 Ring theoretic consequences

Unlike the case of only one bimodule, the multi-graded ring B may not be noetherian when $\{\mathcal{L}_{\sigma_i}\}$ is NC-ample. In fact, [C, Example 5.1] gives a simple commutative (and hence not finitely generated) counterexample.

Example 4.2.1. Let C be a smooth integral curve of genus $g > 0$. Let $\mathcal{L}$ be an invertible sheaf of degree 0 such that no power of $\mathcal{L}$ is isomorphic to $\mathcal{O}_C$ and let $\mathcal{M}$

be an invertible sheaf with $\deg \mathcal{M} > g-1$. The pair $\mathcal{L}, \mathcal{M}$ is certainly NC-ample. The ring $B = B(X; \mathcal{L}, \mathcal{M})$ has $B_{(i,0)} = 0$ for $i > 0$ by [H2, p. 295, Lemma 1.2]. By Riemann-Roch for curves, all other graded pieces are non-zero. Let $I = B_{(>0,>0)}$ be the augmentation ideal. Then I/I^2 contains a copy of $\oplus_{i>0} B_{(i,1)}$, which is infinite dimensional. So B cannot be finitely generated.

However, Chan introduces an additional property for an invertible bimodule $\mathcal{L}_\sigma$ on X to guarantee the noetherian condition.

($*$) *There exists a projective scheme Y with automorphism σ and a σ-equivariant morphism $f \colon X \to Y$. That is $\sigma_Y \circ f = f \circ \sigma_X$. There also exists an invertible sheaf $\mathcal{L}'$ on Y such that $\mathcal{L} = f^*\mathcal{L}'$ and such that $\mathcal{L}'_\sigma$ is σ-ample.*

This property ($*$) is saying that for $m \gg 0$, $|\mathcal{L}_\sigma^m|$ is generated by global sections, since it is a pullback of $|(\mathcal{L}')_\sigma^m|$, which is eventually very ample by Proposition 3.2.2(3). Note in particular that if $\mathcal{L}$ is already σ-ample, then $\mathcal{L}_\sigma$ satisfies ($*$) trivially. Using this property, one determines

Theorem 4.2.2. ([C, Theorem 5.2]) *Let X be a projective scheme with commuting invertible bimodules $\mathcal{L}_\sigma, \mathcal{M}_\tau$. Suppose that the pair is NC-ample and each bimodule satisfies ($*$), possibly for different Y. Then $B(X; \mathcal{L}_\sigma, \mathcal{M}_\tau)$ is right noetherian.* □

Then combining Corollary 4.1.13 and the theorem above, we have

Theorem 4.2.3. *Let X be a projective scheme with commuting invertible bimodules $\mathcal{L}_\sigma, \mathcal{M}_\tau$. Suppose that the pair is NC-ample and each bimodule satisfies ($*$), possibly for different Y. Then $B(X; \mathcal{L}_\sigma, \mathcal{M}_\tau)$ is noetherian.* □

Now we can prove that two particularly interesting twisted multi-homogeneous coordinate rings, a Rees ring and a tensor product, are noetherian, strengthening the results of [C, Corollaries 5.7, 5.8]. In the latter case, we may replace his proof, based on spectral sequences, by an easier one since our criterion (4.1.12) simplifies testing the NC-ampleness of the relevant pair of bimodules.

Corollary 4.2.4. *Let $\mathcal{L}_\sigma$ be a σ-ample invertible bimodule on a projective scheme X. Let $B = B(X, \sigma, \mathcal{L})$ be generated in degree one. Then the Rees algebra $B[It] = \oplus_{r=0}^{\infty} I^r t^r$ of B is noetherian, where $I = B_{>0}$ is the augmentation ideal.*

Proof. If B is generated in degree one, then $B[It]$ has bigraded pieces

$$B_{(i,j)} = H^0(X, \mathcal{L}_\sigma^i \mathcal{L}_\sigma^j) t^j$$

since $I^j = \oplus_{l=j}^{\infty} B_l$ when B is generated in degree one. The pair $\mathcal{L}_\sigma, \mathcal{L}_\sigma$ is obviously NC-ample and satisfies ($*$). Thus Theorem 4.2.3 applies. □

Corollary 4.2.5. *Let $\mathcal{L}_\sigma$ be σ-ample on a projective scheme X and let $\mathcal{M}_\tau$ be τ-ample on a projective scheme Y. Then $B(X,\sigma,\mathcal{L}) \otimes B(Y,\tau,\mathcal{M})$ is noetherian.*

Proof. It is argued in [C, Example 4.3] that

$$B(X,\sigma,\mathcal{L}) \otimes B(Y,\tau,\mathcal{M}) \cong B(X \times Y; (\pi_1^*\mathcal{L})_{\sigma\times 1}, (\pi_2^*\mathcal{M})_{1\times\tau}),$$

where the π_i are the natural projections. These two invertible bimodules on $X \times Y$ obviously satisfy $(*)$.

Since $\mathcal{L}_\sigma$ is σ-ample and $\mathcal{M}_\tau$ is τ-ample, there is an m_0 such that $|\mathcal{L}_\sigma^m|$ and $|\mathcal{M}_\tau^m|$ is ample for all $m \geq m_0$. Note that $(\sigma\times 1)^*\pi_1^*\mathcal{L} = \pi_1^*\sigma^*\mathcal{L}$ and a similar formula holds for $\mathcal{M}_\tau$. Then

$$|(\pi_1^*\mathcal{L})_{\sigma\times 1}^{m_1}(\pi_2^*\mathcal{M})_{1\times\tau}^{m_2}|$$

is ample for all $(m_1,m_2) \geq (m_0,m_0)$ by [H2, p. 125, Exercise 5.11].

Now σ is quasi-unipotent and we wish to show $\sigma \times 1$ is as well. It is tempting to think that as a matrix acting on $A^1_{\mathrm{Num}}(X\times Y)$ one has $\sigma\times 1 = \sigma \oplus 1$. However, this may not be the case, since in general $A^1_{\mathrm{Num}}(X\times Y)$ has larger rank than $A^1_{\mathrm{Num}}(X) \oplus A^1_{\mathrm{Num}}(Y)$ [H2, p. 367, Exercise 1.6]. But let $\mathcal{H}_X$ and $\mathcal{H}_Y$ be ample invertible sheaves on X and Y respectively. If $\sigma\times 1$ is not quasi-unipotent, then by Lemma 3.3.2, there exists $r > 1$, $c > 0$, and an integral curve C on $X \times Y$ such that

$$(((\sigma\times 1)^*)^m(\pi_1^*\mathcal{H}_X \otimes \pi_2^*\mathcal{H}_Y).C) \geq cr^m \qquad \text{for all } m \geq 0. \tag{4.2.6}$$

But

$$((\sigma\times 1)^*)^m(\pi_1^*\mathcal{H}_X \otimes \pi_2^*\mathcal{H}_Y) = \pi_1^*(\sigma^*)^m\mathcal{H}_X \otimes \pi_2^*\mathcal{H}_Y.$$

Since σ is quasi-unipotent, the intersection numbers of the right hand side with any curve C must be bounded by a polynomial. This contradicts (4.2.6). So $\sigma\times 1$ must be quasi-unipotent. Similarly, $1\times\tau$ is quasi-unipotent. Thus by Theorem 4.1.12, the pair $(\pi_1^*\mathcal{L})_{\sigma\times 1}, (\pi_2^*\mathcal{M})_{1\times\tau}$ is NC-ample and thus the ring of interest is noetherian. □

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ABSTRACT

Noncommutative ample divisors

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In the noncommutative geometry of Artin, Van den Bergh, and others, the twisted homogeneous coordinate ring is one of the basic constructions. Such a ring is defined by a σ-ample divisor, where σ is an automorphism of a projective scheme X. Many open questions regarding σ-ample divisors have remained.

We derive a relatively simple necessary and sufficient condition for a divisor on X to be σ-ample. As a consequence, we show right and left σ-ampleness are equivalent and any associated noncommutative homogeneous coordinate ring must be noetherian and have finite, integral GK-dimension. We also characterize which automorphisms σ yield a σ-ample divisor. We also generalize our results to the multi-homogeneous case. Through this we see that certain related rings, like the tensor product of twisted homogeneous coordinate rings, are noetherian.