Lévy-Steinitz for countable sets of series

Paul Larson *
Miami University
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Abstract

The Lévy-Steinitz theorem characterizes the values that a conditionally convergent sequence in $\mathbb{R}^n$ can attain under permutations. We use material from [3] to extend this analysis to sequences in $\mathbb{R}^\omega$, under pointwise convergence, reproving a theorem of Stanimir Troyanski [6].

It is shown in [3] that there exists a c.c.c. partial order adding a permutation of $\omega$ making every conditionally convergent real series in the ground model converge to a value not in the ground model. Applying this forcing fact to a countable elementary submodel of a sufficiently large fragment of the universe, one gets the following fact: for any countable set $S$ of conditionally convergent real series, and every countable $X \subseteq \mathbb{R}$, there is a permutation of $\omega$ making each member of $S$ converge to a real number not in $X$. In this note we give a more direct proof of this fact, using the same machinery. The resulting theorem (due to Stanimir Troyanski [6]) is an extension of the Lévy-Steinitz theorem (which characterizes the values that a finite set of series can take under permutations) to countable sets of series. The proof uses the original Lévy-Steinitz theorem, as well as the Polygonal Refinement Theorem, which is used in the original proof of Lévy-Steinitz theorem. It is a simplified version of the proof of the theorem from [3] mentioned above.

We emphasize that our extension of the Lévy-Steinitz theorem applies to $\mathbb{R}^\omega$ under pointwise converge. Corollary 7.2.2 of [4] says that in each infinite-dimensional Banach space there is a series attaining exactly two values under rearrangements.

1 Preliminaries

We start with some material taken from [5], as rewritten in [3].

Given a sequence $\bar{a} = \langle a^i : i < d \rangle$ consisting of real-values series (for some $d \in \omega$), we let $K(\bar{a})$ be the set of $\langle s_i : i < d \rangle \in \mathbb{R}^d$ for which the series $\sum_{i \in d} s_i a^i$ is absolutely convergent. We let $R(\bar{a})$ be the orthogonal complement of $K(\bar{a})$.

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(i.e., the set of vectors in $\mathbb{R}^d$ orthogonal to every element of $K(\bar{a})$). The sets $K(\bar{a})$ and $R(\bar{a})$ are each linear subspaces of $\mathbb{R}^d$, and their dimensions sum to $d$.

We say that a set $I$ consisting of conditionally convergent series is independent if $K(\bar{a}) = \{0\}$ for each finite sequence $\bar{a}$ from $I$. We let $S(\bar{a})$ be the set of values in $\mathbb{R}^d$ of the form $\sum_{n \in \omega} (a_{p(n)}^i : i < d)$ for $p$ a permutation of $\omega$.

The following theorem from 1913 is due to Lévy and Steinitz (see [1, 4, 5]).

**Theorem 1.1 (Lévy-Steinitz).** If $\bar{a} = (a^i : i < d)$ is a finite sequence of conditionally convergent real-valued series, then

$$S(\bar{a}) = \{ (\sum a^i : i < d) + \bar{x} : \bar{x} \in R(\bar{a}) \}.$$

One way to interpret the Lévy-Steinitz Theorem is to note that in the case where $\langle a^i : i < d \rangle$ is independent, it says that every value in $\mathbb{R}^n$ is attainable under some rearrangement. If $\langle a^i : i < d \rangle$ is an arbitrary sequence of conditionally convergent real series, then there exists a set $s \subseteq d$ such that $\langle a^i : i \in s \rangle$ is independent, and such that, for each $j \in d \setminus s$ there exist scalars $k_i (i \in s)$, not all 0 such that $\sum_{i \in s} k_i a^i + a^j$ is absolutely convergent. Given any permutation of $\omega$, then, the value of such an $a^j$ is determined by the values of $a^i (i \in s)$. We carry out the version of this analysis for countable sets in the final section of this paper.

The following is the key lemma in the proof of the Lévy-Steinitz theorem (see [2, 5]).

**Theorem 1.2 (The Polygonal Confinement Theorem; Steinitz).** For each positive integer $d$ there exists a constant $C_d$ such that for each positive $n \in \omega$ and all vectors $v_m (m \in n)$ from $\mathbb{R}^d$, if

$$\sum_{m \in n} v_m = 0$$

and $\|v_m\| \leq 1$ for all $m \in n$, then there is a permutation $p$ of $n \setminus \{0\}$ such that

$$\left\| v_0 + \sum_{m \in k \setminus \{0\}} v_{p(m)} \right\| \leq C_d$$

for every $m \in n + 1$.

The following immediate (and standard) consequence of the Polygonal Confinement Theorem is proved in [3].

**Lemma 1.3.** Let $m$ and $d$ be positive integers, let $\rho$ be a positive real number and let $b$ and $v_i (i \in m)$ be elements of $\mathbb{R}^d$. Suppose that

$$\sum_{i \in m} v_i = b,$$
∥b∥ ≤ ρ and ∥v_i∥ ≤ ρ for all i ∈ m. Then there is a permutation p of m \ {0} such that

\[ \left\| v_0 + \sum_{i \in j \setminus \{0\}} v_{p(i)} \right\| \leq \rho C_d + ∥b∥ \]

for every j ∈ m + 1.

2 Countable independent sets

We prove in this section the version of the Lévy-Steinitz theorem for countable independent sets. The proof is an adaptation of arguments from [3]. The general version is proved in the next section.

Theorem 2.1. Let \( \langle a^i : i < \omega \rangle \) be an independent sequence of conditionally convergent real series and let \( \langle x_i : i < \omega \rangle \) be a sequence of real numbers. Then there is a permutation \( p \) of \( \omega \) such that, for each \( i \in \omega \),

\[ \sum_{j \in \omega} a^i_{p(j)} = x_i. \]

For the rest of this section, fix \( \langle a^i : i < \omega \rangle \) and \( \langle x_i : i < \omega \rangle \) as in the statement of Theorem 2.1, and a nondecreasing sequence of constants \( C_d \) as given by the Polygonal Confinement Theorem. We define a partial order \( P \) from which our desired permutation will be induced by a suitable descending sequence. Conditions in \( P \) are triples \( (f, d, \epsilon) \) such that

- \( f \) is an injection from some \( n \in \omega \) to \( \omega \);
- \( d \) is a positive integer;
- \( \epsilon \) is a positive rational number;
- \[ \left\| \sum_{k < n} (a^i_{f(k)} : i < d) - \langle x_i : i < d \rangle \right\| < \epsilon; \]
- for all \( m \in \omega \setminus \text{Range}(f), \left\| (a^i_m : i < d) \right\| < \epsilon/C_d. \]

The order on \( P \) is defined by:

- \( (g, e, \delta) \leq (f, d, \epsilon) \) if
  - \( g \) extends \( f \);
  - \( e \geq d; \)
  - for all \( m \in \text{Dom}(g) + 1, \left\| \sum_{k \in m \setminus \text{Dom}(f)} (a^i_{g(k)} : i < d) \right\| < 2\epsilon; \)
  - \[ 2\delta + \left\| \sum_{k \in \text{Dom}(g) \setminus \text{Dom}(f)} (a^i_{g(k)} : i < d) \right\| \leq 2\epsilon. \]

Observe that if \( \epsilon \) is greater than both \( |x_0| \) and \( \sup \{ C_1 a^0_m : m \in \omega \} \), then \((\emptyset, 1, \epsilon)\) is a condition in \( P \). If \( \langle (f_n, d_n, \epsilon_n) : n \in \omega \rangle \) is a descending sequence in \( P \) such that

- \( \bigcup_{n \in \omega} f_n \) is a permutation of \( \omega \),
• \( \omega = \bigcup_{n \in \omega} d_n \) and
• \( \lim_{n \to \infty} \epsilon_n = 0 \)

then \( \bigcup_{n \in \omega} f_n \) is as desired. Theorem 2.1 follows then from Lemma 2.2.

**Lemma 2.2.** For each \((f, d, \epsilon) \in P\) and each \(n \in \omega\), there exists a condition \((g, d + 1, \delta) \leq (f, d, \epsilon)\) with \(n \subseteq \text{Dom}(g) \cap \text{Range}(g)\) and \(\delta < 1/n\).

**Proof.** Let \((f, d, \epsilon)\) and \(n\) be given. By the Lévy-Steinitz theorem, there is a permutation \(p\) of \(\omega\) extending \(f\) such that
\[
\sum_{n \in \omega} \langle x_i : i < d + 1 \rangle = \langle x_i : i < d + 1 \rangle.
\]

Let \(\eta < \epsilon\) be such that \(\|\langle a^i_m : i < d \rangle\| < \eta/C_d\) for all \(m \in \omega \setminus \text{Dom}(f)\), and let \(\delta \in \mathbb{Q}^+\) be smaller than both \(1/n\) and \((\epsilon - \eta)/2\). Fix \(n^* \geq n\) such that

• \(n \subseteq \text{Range}(p \upharpoonright n^*)\),
• \(\|\sum_{m \in n^* \setminus \text{Dom}(f)} \langle a^i_{p(m)} : i < d \rangle\| < \epsilon\),
• \(\|\sum_{m \in n^*} \langle a^i_{p(m)} : i < d + 1 \rangle - \langle x_i : i < d + 1 \rangle\| < \delta\) and
• \(\|\langle a^i_m : i < d + 1 \rangle\| < \delta/C_{d+1}\) for each \(m \in \omega \setminus n^*\).

By Lemma 1.3, there is an injection \(g\) from \(n^*\) to \(\omega\) extending \(f\), with the same range as \(p \upharpoonright n^*\), such that
\[
\|\langle a^i_{g(Dom(f))} : i < d \rangle + \sum_{k \in m \setminus \{\text{Dom}(f) + 1\}} \langle a^i_{g(k)} : i < d \rangle\| \leq \frac{\eta}{C_d} C_d + (\epsilon - \delta) < 2\epsilon - 2\delta
\]

for every \(m \in n^* + 1\). Then \((g, A, \delta)\) is as desired. \(\Box\)

### 3 Arbitrary sequences

We adapt the notation introduced in Section 1 to countable sequences. Given a sequence \(\bar{a} = \langle a^i : i < \omega \rangle\) consisting of real-values series, we let \(K(\bar{a})\) be the set of \(\langle s_i : i < \omega \rangle \in \mathbb{R}^\omega\) for which the following hold:

• the set \(\{i \in \omega : d_i \neq 0\}\) is finite;
• the series \(\sum_{i \in \omega} d_i a^i\) is absolutely convergent.

We let \(R(\bar{a})\) be the orthogonal complement of \(K(\bar{a})\) (i.e., the set of vectors in \(\mathbb{R}^\omega\) orthogonal to every element of \(K(\bar{a})\)). The sets \(K(\bar{a})\) and \(R(\bar{a})\) are each linear subspaces of \(\mathbb{R}^\omega\). We let \(S(\bar{a})\) be the set of values in \(\mathbb{R}^\omega\) of the form \(\sum_{n \in \omega} \langle a^i_{p(n)} : i < d \rangle\) for \(p\) a permutation of \(\omega\).

The following natural generalization of the Lévy-Steinitz theorem to countable sequences was first proved by Troyanski [6].
Theorem 3.1 (Lévy-Steinitz for countable sets). If $\bar{a} = \langle a^i : i < \omega \rangle$ is a sequence of conditionally convergent real-valued series, then

$$S(\bar{a}) = \{ \sum_{i} a^i : i < \omega \} + \bar{x} : \bar{x} \in R(\bar{a})\}.$$ 

Proof. For each $i \in \omega$, let $s_i = \sum a^i$. Let $I \subseteq \omega$ be such that $\{a^i : i \in I\}$ is independent, and such that, for each $j \in \omega \setminus i$ there exist $c_j \in \mathbb{R}$ and $d^j_k \in \mathbb{R}$ $(k \in I \cap j)$ such that $\sum_{k \in I \cap j} d^j_k a^k + a^j$ is absolutely convergent, with sum $c_j$.

For one direction of the desired equality, let $\langle x_i : i < \omega \rangle$ be in $R(\bar{a})$. We want to find a permutation $p$ of $\omega$ such that, for each $i \in \omega$, $\sum_{n \in \omega} a^i_{p(n)} = s_i + x_i$. By Theorem 2.1, there is a permutation $p$ such that this equation holds for all $i \in I$. Suppose now that $j$ is in $\omega \setminus I$. Since $\langle x_i : i \in \omega \rangle$ is in $R(\bar{a})$, $\sum_{k \in I \cap j} x_k d^j_k + x_j = 0$. Since $\sum_{k \in I \cap j} d^j_k a^k + a^j$ is absolutely convergent with sum $c_j$, $\sum_{k \in I \cap j} d^j_k s_k + s_j = c_j$ and

$$\sum_{n \in \omega} a^i_{p(n)} = c_j - \sum_{k \in I \cap j} d^j_k \sum_{n \in \omega} a^k_{p(n)}$$

$$= c_j - \sum_{k \in I \cap j} d^j_k (s_k + x_k)$$

$$= (c_j - \sum_{k \in I \cap j} d^j_k s_k) - \sum_{k \in I \cap j} d^j_k x_k$$

$$= s_j + x_j$$

as desired.

For the other direction, let $p$ be a permutation of $\omega$ such that $\sum_{n \in \omega} a^i_{p(n)}$ converges for all $n \in \omega$. For each $i \in \omega$, let $x_i = \sum_{n \in \omega} a^i_{p(n)} - s_i$. We want to see that $\bar{x} = \langle x_i : i \in \omega \rangle$ is in $R(\bar{a})$. To do this, fix $\langle d^i : i < \omega \rangle$ in $\mathbb{R}^\omega$ such that $\{ i \in \omega : d_i \neq 0 \}$ is a finite set $D$, and such that $\sum_{i \in D} d_i a^i$ is absolutely convergent with sum $e$. Then

$$\langle x_i : i < \omega \rangle \cdot \langle d^i : i < \omega \rangle = \sum_{i \in D} x_i d_i$$

$$= \sum_{i \in D} d_i (\sum_{n \in \omega} a^i_{p(n)} - \sum_{n \in \omega} a^i_n)$$

$$= (\sum_{n \in \omega} \sum_{i \in D} d_i a^i_{p(n)}) - (\sum_{n \in \omega} \sum_{i \in D} d_i a^i_n)$$

$$= e - e$$

$$= 0.$$ 

\[\square\]

References


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