ON THE HEREDITARY PARACOMPACTNESS OF LOCALLY COMPACT, HEREDITARILY NORMAL SPACES

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ABSTRACT. We establish that if it is consistent that there is a supercompact cardinal, then it is consistent that every locally compact, hereditarily normal space which does not include a perfect pre-image of ω_1 is hereditarily paracompact.

This is the fifth in a series of papers ([LTo], [L_2], [FTT], [LT], [T_1] being the logically previous ones) that establish powerful topological consequences in models of set theory obtained by starting with a particular kind of Souslin tree S, iterating partial orders that don't destroy S, and then forcing with S. The particular case of the theorem stated in the abstract when X is perfectly normal (and hence has no perfect pre-image of ω_1) was proved in [LT], using essentially that locally compact perfectly normal spaces are locally hereditarily Lindelöf and first countable. Here we avoid these two last properties by combining the methods of [B_2] and [T_1]. To apply [B_2], we establish the new set-theoretic result that $\mathrm{PFA}^{++}(S)[S]$ implies Fleissner's "Axiom R". This notation is explained below; the model is a strengthening of those used in the previous four papers.

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The results established here were actually proved around 2004, modulo results of Todorcevic announced in 2002 (which now appear in [FTT] and [L_2]) and of the second author [T_1]. We have delayed submission until a correct version of [T_1] existed in preprint form.

Definition. *A continuous map is* **perfect** *if images of closed sets are closed, and pre-images of points are compact.*

It is easy to find locally compact, hereditarily normal spaces which are not paracompact – ω_1 is one such. Non-trivial perfect pre-images of ω_1 may also be hereditarily normal, but are not paracompact. Our result says that consistently, any example must in fact include such a canonical example.

Theorem 1. *If it is consistent that there is a supercompact cardinal, it's consistent that every locally compact, hereditarily normal space that does not include a perfect pre-image of* ω_1 *is (hereditarily) paracompact.*

This is not a ZFC result, since there are many consistent examples of locally compact, perfectly normal spaces which are not paracompact. For example, the Cantor tree over a Q-set, which is the standard example of a locally compact, normal, non-metrizable Moore space – see e.g. [T], which has essentially the same example. Other examples include the Ostaszewski and Kunen lines, as in [FH].

Let us state some axioms we will be using.

PFA^{++}: *Suppose* P *is a proper partial order,* $\{D_\alpha\}_{\alpha<\omega_1}$ *is a collection of dense subsets of* P*, and* $\{\dot{S}_\alpha : \alpha < \omega_1\}$ *is a sequence of terms such that* $(\forall\alpha < \omega_1)_P \Vdash \dot{S}_\alpha$ *is stationary in* ω_1*. Then there is a filter* $G \subseteq P$ *such that*

(i) $(\forall\alpha < \omega_1)\ G \cap D_\alpha \neq 0$,

and (ii) $(\forall\alpha < \omega_1)\ S_\alpha(G) = \{\xi < \omega_1 : (\exists p \in G) p \Vdash \xi \in \dot{S}_\alpha\}$ *is stationary in* ω_1.

Baumgartner [Ba] introduced this axiom and called it "PFA$^+$". Since then, others have called this "PFA^{++}", using "PFA$^+$" for the weaker one-term version. As Baumgartner observed, the usual consistency proof for

PFA, which uses a supercompact cardinal, yields a model for what we are calling PFA^{++}.

Definition. $\Gamma \subseteq [X]^{<\kappa}$ *is* **tight** *if whenever* $\{C_\alpha : \alpha < \delta\}$ *is an increasing sequence from* Γ*, and* $\omega < cf\delta < \kappa$*, then* $\bigcup\{C_\alpha : \alpha < \delta\} \in \Gamma$. **Axiom R:** *if* $\Sigma \subseteq [X]^{<\omega_1}$ *is stationary and* $\Gamma \subseteq [X]^{<\omega_2}$ *is tight and cofinal, then there is a* $Y \in \Gamma$ *such that* $\mathcal{P}(Y) \cap \Sigma$ *is stationary in* $[Y]^{<\omega_1}$. **Axiom R^{++}:** *if* $\Sigma_\alpha(\alpha < \omega_1)$ *are stationary subsets of* $[X]^{<\omega_1}$ *and* $\Gamma \subseteq [X]^{<\omega_2}$ *is tight and cofinal, then there is a* $Y \in \Gamma$ *such that* $\mathcal{P}(Y) \cap \Sigma_\alpha$ *is stationary in* $[Y]^{<\omega_1}$ *for each* $\alpha < \omega_1$.

Fleissner introduced Axiom R in [Fl] and showed it held in the usual model for PFA.

Σ^+: *Suppose* X *is a countably tight compact space,* $\mathcal{L} = \{L_\alpha\}_{\alpha<\omega_1}$ *a collection of disjoint compact sets such that each* L_α *has a neighborhood that meets only countably many* L_β*'s, and* $\mathcal{V}$ *is a family of* $\leq \aleph_1$ *open subsets of* X *such that:*

a) $\bigcup \mathcal{L} \subseteq \bigcup \mathcal{V}$

b) For every $V \in \mathcal{V}$ *there is an open* U_V *such that* $\overline{V} \subseteq U_V$ *and* U_V *meets only countably many members of* $\mathcal{L}$.

Then $\mathcal{L} = \bigcup\limits_{n<\omega} \mathcal{L}_n$*, where each* $\mathcal{L}_n$ *is a discrete collection in* $\bigcup \mathcal{V}$.

Balogh [B_1] proved that MA_{ω_1} implies the restricted version of Σ^+ in which we take the L_α's to be points. We will call that "Σ'".

Definition. *A space is (strongly)* κ*-collectionwise Hausdorff if for each closed discrete subspace* $\{x_d\}_{d\in D}$*,* $|D| \leq \kappa$*, there is a disjoint (discrete) family of open sets* $\{U_d\}_{d\in D}$ *with* $x_d \in U_d$*. A space is (strongly) collectionwise Hausdorff if it is (strongly)* κ*-collectionwise Hausdorff for all* κ*.*

It is easy to see that normal ($\kappa-$) collectionwise Hausdorff spaces are strongly ($\kappa-$) collectionwise Hausdorff.

Balogh [B_2] proved:

Lemma 2. *MA_{ω_1} + Axiom R implies locally compact hereditarily strongly $\aleph_1$-collectionwise Hausdorff spaces which do not include a perfect pre-image of ω_1 are paracompact.*

The consequences of MA_{ω_1} he used are Σ' and Szentmiklóssy's result [S] that *compact spaces with no uncountable discrete subspaces are hereditarily Lindelöf.* Our plan is to find a model in which these two consequences and Axiom R hold, as well as normality implying (strongly) $\aleph_1$-collectionwise Hausdorffness for the spaces under consideration. The model we will consider is of the same genre as those in [LTo], [L_2], [FTT], [LT], and [T_1]. One starts off with a particular kind of Souslin tree S, a *coherent* one, which is obtainable from $\diamondsuit$ or by adding a Cohen real. One then iterates in standard fashion as in establishing MA_{ω_1} or PFA, but omitting partial orders that adjoin uncountable antichains to S. In the PFA case for example, this will establish *PFA(S)*, which is like PFA except restricted to partial orders that don't kill S. In fact it will also establish PFA^{++}(S), which is the corresponding modification of PFA^{++}. We then force with S. For more information on such models, see [Mi] and [L_1]. We use $\mathrm{PFA}^{++}(S)[S]$ *implies* φ to mean that whenever we force over a model of $\mathrm{PFA}^{++}(S)$ with S, φ holds. Similarly for $\mathrm{PFA}(S)[S]$, etc.

In [T_1] it is established that:

Lemma 3. *PFA(S)[S] implies that locally compact normal spaces are $\aleph_1$-collectionwise Hausdorff.*

By doing some preliminary forcing (as in [LT]), one can actually get full collectionwise Hausdorffness, but we won't need that here.

We will assume all spaces are Hausdorff, and use "X^*" to refer to the one-point compactification of a locally compact space X.

There is a bit of a gap in Balogh's proof of Lemma 2. Balogh asserted that:

Lemma 4. *If X is locally compact and does not include a perfect pre-image of ω_1, then X^* is countably tight.*

and referred to [B_1] for the proof. However in [B_1], he only proved this for the case in which X is countably tight. It is not obvious that that hypothesis can be omitted, but in fact it can. We need a definition and lemma.

Definition. *A space Y is* **ω-bounded** *if each separable subspace of Y has compact closure.*

Lemma 5. [G], [Bu]. *If Y is ω-bounded and does not include a perfect pre-image of ω_1, then Y is compact.*

We then can establish Lemma 4 as follows.

Proof. By Lemma 5, every ω-bounded subspace of X is compact. By [B$_1$], it suffices to show X is countably tight. Suppose, on the contrary, that there is a $Y \subseteq X$ which is not closed, but is such that for all countable $Z \subseteq Y$, $\overline{Z} \subseteq Y$. Since X is a k-space, there is a compact K such that $K \cap Y$ is not closed. Then $K \cap Y$ is not ω-bounded, so there is a countable $Z \subseteq K \cap Y$ such that $\overline{Z} \cap K \cap Y$ is not compact. But $\overline{Z} \subseteq Y$, so $\overline{Z} \cap K \cap Y = \overline{Z} \cap K$, which is compact, contradiction.

Lemma 3 takes care of the hereditary strong $\aleph_1$-collectionwise Hausdorffness we need, since if open subspaces are $\aleph_1$-collectionwise Hausdorff, all subspaces are, and open subspaces of locally compact spaces are locally compact. The proposition that

Σ: *in a compact countably tight space, locally countable subspaces of size $\aleph_1$ are σ-discrete.*

is implied by PFA$(S)[S]$ was announced by Todorcevic in the Toronto Set Theory Seminar in 2002.

From Σ it is standard to get the result of Szentmiklóssy quoted earlier: since the compact space has no uncountable discrete subspace, it has countable tightness. If it were not hereditarily Lindelöf, it would have a right-separated subspace of size $\aleph_1$. But Σ implies it has an uncountable discrete subspace, contradiction.

Σ' is established by a minor variation of the forcing for Σ. A proof exists in the union of [L$_2$] and [FTT]. Σ^+, however, is not so clear, and has not yet been proved from PFA$(S)[S]$. Thus, instead of using it to get $\aleph_1$-collectionwise Hausdorffness in locally compact normal spaces with no perfect pre-image of ω_1, as we did in an earlier version of this paper, we are instead quoting Lemma 3, which is a new result of the second author.

Thus all we have to do is prove that $\mathrm{PFA}^{++}(\mathrm{S})[\mathrm{S}]$ implies Axiom R. In order to prove that $\mathrm{PFA}^{++}(\mathrm{S})[\mathrm{S}]$ implies Axiom R, we first note that a straightforward argument using the forcing $\mathit{Coll}\,(\omega_1, X)$ (whose conditions are countable partial functions from ω_1 to X, ordered by inclusion) shows that $\mathrm{PFA}^{++}(\mathrm{S})$ implies Axiom R^{++}.

It then suffices to prove:

Lemma 6. *If Axiom R^{++} holds and S is a Souslin tree, then Axiom R^{++} still holds after forcing with S.*

Proof. First note that if X is a set, P is a c.c.c. forcing and τ is a P-name for a tight cofinal subset of $[X]^{<\omega_2}$, then the set of $a \in [X]^{<\omega_2}$ such that every condition in P forces that a is in the realization of τ is itself tight and cofinal. The tightness of this set is immediate. To see that it is cofinal, let b_0 be any set in $[X]^{<\omega_2}$. Define sets b_α $(\alpha \leq \omega_1)$ and σ_α $(\alpha < \omega_1)$ recursively by letting σ_α be a P-name for a member of the realization of τ containing b_α and letting $b_{\alpha+1}$ be the set of members of X which are forced by some condition in P to be in σ_α. For limit ordinals $\alpha \leq \omega_1$, let b_α be the union of the b_β $(\beta < \alpha)$. Then b_{ω_1} is forced by every condition in P to be in τ.

Since we are assuming that the Axiom of Choice holds, Axiom R^{++} does not change if we require X to be an ordinal. Fix an ordinal γ and let $\rho_\alpha(\alpha < \omega_1)$ be S-names for stationary subsets of $[\gamma]^{<\omega_1}$. Let T be a tight cofinal subset of $[\gamma]^{<\omega_2}$. For each countable ordinal α and each node $s \in S$, let $\tau_{s,\alpha}$ be the set of countable subsets a of γ such that some condition in S extending s forces that a is in the realization of ρ_α. Applying Axiom R^{++}, we have a set $Y \in [\gamma]^{<\omega_2}$ such that each $\mathcal{P}(Y) \cap \tau_{s,\alpha}$ is stationary in $[Y]^{<\omega_1}$.

Since S is c.c.c., every club subset of $[Y]^{<\omega_1}$ that exists after forcing with S includes a club subset of $[Y]^{<\omega_1}$ existing in the ground model. Letting $(\rho_\alpha)_G$ (for each $\alpha < \omega_1$) be the realization of ρ_α, we have by genericity then that after forcing with S, each $\mathcal{P}(Y) \cap (\rho_\alpha)_G$ will be stationary in $[Y]^{<\omega_1}$.

This completes the proof of Theorem 1.

We do not know the answer to the following question; a positive answer would likely enable us to dispense with Axiom R, and possibly with the supercompact cardinal.

Problem. Does MA_{ω_1} imply every locally compact, hereditarily strongly collectionwise Hausdorff space which does not include a perfect pre-image of ω_1 is paracompact?

We also do not know whether in our main result, we can replace "perfect pre-image of ω_1" by "copy of ω_1".

Remark. That PFA$(S)[S]$ does not imply Axiom R is proved in [T_3].

The problem of finding in models of PFA$(S)[S]$ necessary and sufficient conditions for locally compact normal spaces to be paracompact is studied in [T_2] by extending the methods of [B_2] and this note.

References

[B_1] Z. Balogh, *Locally nice spaces under Martin's axiom*, Comment. Math. Univ. Carolin. **24** (1983), 63–87.

[B_2] Z. Balogh, *Locally nice spaces and Axiom R*, Top. Appl. **125** (2002), 335–341.

[Ba] J.E. Baumgartner, *Applications of the proper forcing axiom*, Handbook of Set-theoretic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, pp. 913–959.

[Bu] D. Burke, *Closed mappings*, Surveys in General Topology (G.M. Reed, ed.), Academic Press, 1980, pp. 1–32.

[FH] V. Fedorčuk and K.P. Hart, *Special constructions*, Encyclopedia of General Topology (K.P. Hart, J. Nagata, and J.E. Vaughan, eds.), Elsevier, Amsterdam, pp. 229–232.

[Fl] W.G. Fleissner, *Left separated spaces with point-countable bases*, Trans. Amer. Math. Soc. **294** (1986), 665–677.

[FTT] A. Fischer, F.D. Tall, and S. Todorcevic, *PFA$(S)[S]$ implies there are no compact S-spaces (and more)*, preprint.

[G] G. Gruenhage, *Some results on spaces having an orthobase or a base of subinfinite rank*, Top. Proc. **2** (1977), 151–159.

[H] R. Hodel, *Cardinal functions I*, Handbook of Set-theoretic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984.

[L_1] P. Larson, *An $\mathbb{S}_{\max}$ variation for one Souslin tree*, J. Symbolic Logic **64** (1999), 81–98.

[L_2] P. Larson, *Notes on Todorcevic's Erice lectures on forcing with a coherent Souslin tree*, preprint.

[LT] P. Larson and F.D. Tall, *Locally compact perfectly normal spaces may all be paracompact*, Fund. Math., 210 (2010), 285–300.

[LTo] P. Larson and S. Todorčević, *Katětov's problem*, Trans. Amer. Math. Soc **354** (2002), 1783–1791.

[Mi] T. Miyamoto, ω_1*-Souslin trees under countable support iterations*, Fund. Math. **142** (1993), 257-261.

[S] Z. Szentmiklóssy, *S-spaces and L-spaces under Martin's Axiom*, Topology (A. Császár, ed.), vol. II, North-Holland, 1980, pp. 1139–1146.

[T] F. D. Tall, *Set-theoretic problems concerning Lindelöf spaces*, preprint.

[T_1] F. D. Tall, *PFA(S)[S]: more mutually consistent topological consequences of PFA and V=L*, Canad. J. Math., to appear.

[T_2] F. D. Tall, *PFA*$(S)[S]$ *and locally compact normal spaces*, submitted.

[T_3] F. D. Tall, *PFA*$(S)[S]$ *and the Arhangel'skiĭ-Tall problem*, Top. Proc., to appear.

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