Ramsey ultrafilters and Countable-to-one Uniformization

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Abstract

We show that Countable-to-One Uniformization is preserved by forcing with $\mathcal{P}(\omega)/\mathrm{Fin}$ over a model of ZF in which every set of reals is completely Ramsey. We also give an exposition of Todorcevic's theorem that Ramsey ultrafilters are generic for $\mathcal{P}(\omega)/\mathrm{Fin}$ over suitable inner models.

1 Introduction

This paper presents a result on models of the form $M[U]$, where M is an inner model of ZF satisfying certain regularity properties inconsistent with the Axiom of Choice, and U is a Ramsey ultrafilter on the integers. Such extensions have been studied by several authors, notably Henle, Mathias and Woodin [6] and Di Prisco and Todorcevic [2, 3], where the model M is variously taken to be a Solovay model or an inner model of Determinacy in the presence of large cardinals. Our result is that Countable-to-One Uniformization (a weak form of the Axiom of Choice; see the first paragraph of Section 3) is preserved by forcing with $\mathcal{P}(\omega)/\mathrm{Fin}$ over a model of ZF in which every set of reals is completely Ramsey (this includes many standard models of determinacy; see Section 3 and Subsection 1.2). In conjunction with the main result of [13], this fact can be used to show that there is no injection from $\mathcal{P}(\omega)/\mathrm{Fin}$ to $\mathbb{R}$ in models of the form the $M[U]$ considered here (a result previously proved in [3] by other means).

We let Fin denote the ideal of finite subsets of $\omega = \{0, 1, 2, \ldots\}$, and (for subsets x, y of ω) write $x \subseteq^* y$ for $x \setminus y \in \mathrm{Fin}$. It is easy to see that for any $\subseteq^*$-decreasing sequence $\langle x_n : n < \omega\rangle$ consisting of infinite subsets of ω, there is an infinite $y \subseteq \omega$ such that $y \subseteq^* x_n$ for all n. It follows that forcing with the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ over a model of ZF + $\mathrm{DC}_{\mathbb{R}}$ does not add countable subsets of the ground model.[1] Forcing with this Boolean algebra over

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[1]The principle of Dependent Choices (DC) says that every tree of height ω without terminal nodes has an infinite branch; $\mathrm{DC}_{\mathbb{R}}$ is DC restricted to trees on $\mathbb{R}$.

a model M of ZF + $\mathrm{DC}_{\mathbb{R}}$ then produces a model $M[U]$, where the generic filter is naturally interpreted as a nonprincipal ultrafilter U on ω. In fact, the ultrafilter U is a selective (or Ramsey) ultrafilter, which means that for any collection $\{X_n : n \in \omega\} \subseteq U$ there is a set $\{i_n : n \in \omega\}$ (listed in increasing order) in U such that $i_0 \in X_0$ and each $i_{n+1} \in X_{i_n}$.

Ramsey ultrafilters exist if the Continuum Hypothesis holds, and their existence follows from weaker statements such as $cov(\mathcal{M}) = \mathfrak{c}$, where $cov(\mathcal{M})$ is the least cardinality of a collection of meager sets of reals whose union is the entire real line, and $\mathfrak{c}$ denotes the cardinality of the continuum (see Theorem 4.5.6 of [1]). Kunen [10] has shown that consistently there are no Ramsey ultrafilters.

A theorem of Todorcevic (see [4]) implies that in the context of large cardinals every Ramsey ultrafilter is generic over the inner model $L(\mathbb{R})$ for the partial order $\mathcal{P}(\omega)/\mathrm{Fin}$. We give a proof of this theorem in Section 2.

Woodin has shown that under the assumption of a proper class of Woodin cardinals, the theory of $L(\mathbb{R})$ is invariant under set forcing (see [12]). Since $\mathcal{P}(\omega)/\mathrm{Fin}$ is homogeneous, the theory of $L(\mathbb{R})[U]$ is also invariant under set forcing (in this context) when U is taken to be a Ramsey ultrafilter. This should mean that large cardinals give as detailed a theory for $L(\mathbb{R})[U]$ (i.e., answering most natural questions) as they do for the inner model $L(\mathbb{R})$. It remains to be seen whether this is the case. At the present moment many natural questions about this model remain open.

1.1 Notation

Given an infinite set $a \subseteq \omega$, we let $[a]^{\omega}$ denote the set of infinite subsets of a, and we let $[a]^{<\omega}$ denote the set of finite subsets of a (so $\mathrm{Fin} = [\omega]^{<\omega}$). Given $s \in [\omega]^{<\omega}$ and $a \in [\omega]^{\omega}$, we let $[s, a]$ denote the set of infinite subsets of $s \cup a$ with s as an initial segment. Given $s \in [\omega]^{<\omega}$ and a set $a \subseteq \omega$, a/s denotes a in the case that s is the emptyset, and $a \setminus (\max(s) + 1)$ otherwise.

1.2 Selective coideals and Ramsey ultrafilters

A *coideal* C on a set X is a subset of $\mathcal{P}(X)$ such that $\mathcal{P}(X) \setminus C$ is an ideal. Given $a \in C$, we let $C \restriction a$ denote $\{b \in C \mid b \subseteq a\}$. A coideal C on ω is *selective* if it contains all cofinite sets, and if for all $\subseteq$-decreasing sequences $\langle a_n : n \in \omega\rangle$ contained in C, there is a set $\{k_i : i \in \omega\}$ (listed in increasing order) in C such that $k_0 \in a_0$ and each k_{i+1} is in a_{k_i}. As defined above, a Ramsey ultrafilter is a selective ultrafilter on ω.

The following is part of Theorem 4.5.2 of [1].

Theorem 1.1. *A nonprincipal ultrafilter U on ω is Ramsey if and only if either of the following two statements holds.*

- *For every partition $\{y_n : n \in \omega\}$ of ω, either some $y_n \in U$ or there exists an $x \in U$ such that $|x \cap y_n| \leq 1$ for all $n \in \omega$.*
- *For all $a \subseteq [\omega]^2$, there is an $x \in U$ such that $[x]^2 \subseteq a$ or $[x]^2 \cap a = \emptyset$.*

Given $A \subseteq [\omega]^\omega$ and a coideal C on ω, we say that A is *C-Ramsey* (or has the *C-Ramsey property*) if there exists a $b \in C$ such that either $A \cap [b]^\omega = \emptyset$ or $[b]^\omega \subseteq A$. We say that A is *completely C-Ramsey* if for every finite $s \subseteq \omega$ and every $b \in C$, there exists a $d \in C \restriction b$ such that either $A \cap [s, d] = \emptyset$ or $[s, d] \subseteq A$. We drop the prefix C- when C is the coideal of infinite subsets of ω. It follows easily from the definitions that if every set of reals in an inner model M of ZF is C-Ramsey, then every set of reals in M is completely C-Ramsey, even if C is not a member of M. The axioms $\mathrm{AD}_{\mathbb{R}}$ and $\mathrm{AD} + V = L(\mathbb{R})$ each imply that every subset of $[\omega]^\omega$ is completely Ramsey; weakly homogeneously Suslin sets of reals are also completely Ramsey (see pages 382 and 458 of [8]).

Given a coideal C on ω, a set $A \subseteq [\omega]^\omega$ is said to be C-Baire if for every $s \in [\omega]^{<\omega}$ and $b \in C$ there exist $t \in [\omega]^{<\omega}$ and $d \in C \restriction b$ such that $[t, d] \subseteq [s, b]$ and $[t, d] \subseteq A$ or $[t, d] \cap A = \emptyset$. The following is a weakening of Corollary 7.14 of [15] (a corollary to Theorem 2.6 below).

Theorem 1.2. *If C is a selective coideal on ω, then every C-Baire subset of $[\omega]^\omega$ is completely C-Ramsey.*

2 Ramsey ultrafilters are generic

In this section we give a proof of the following theorem of Todorcevic, adapted from [4]. Many of the ideas in this section have their origin in [11].

Theorem 2.1 (Todorcevic)**.** *If there exist infinitely many Woodin cardinals below a measurable cardinal, then every Ramsey ultrafilter is $L(\mathbb{R})$-generic for $\mathcal{P}(\omega)/\mathrm{Fin}$.*

Theorem 2.1 follows from Theorem 2.7 below, via the following definition (which appears on page 206 of [5]) and theorem (which follows from combining arguments given in [5] and [12]).

2.2 Definition. Given a set $A \subseteq {}^\omega\omega$ and an infinite cardinal λ, A is λ-universally Baire if for every topological space X with a regular open base of cardinality at most λ, and for every continuous function $f \colon X \to {}^\omega\omega$, $f^{-1}[A]$ has the property of Baire in X.

Theorem 2.3 (Woodin)**.** *If δ is a limit of Woodin cardinals below a measurable cardinal, all subsets of 2^ω in $L(\mathbb{R})$ are $<\delta$-universally Baire.*

Given a nonprincipal ultrafilter U on ω, the *U-exponential* (or *U-Vietoris* or *U-Ellentuck*) topology on $[\omega]^\omega$ has a base consisting of all sets of the form $[s, a]$, where s is a finite subset of ω and $a \in U$. These sets are regular, as each set of the form $[s, a]$ is clopen.

Let $\pi \colon [\omega]^\omega \to \omega^\omega$ be the function that sends each infinite subset of ω to its increasing enumeration. Letting U be a nonprincipal ultrafilter, π is continuous when its domain is given the U-exponential topology and its range is given the usual product topology. It follows that $A \subseteq [\omega]^\omega$ has the property of Baire in the

U-exponential topology whenever $\pi[A]$ is $\mathfrak{c}$-universally Baire, where $\mathfrak{c}$ denotes $2^{\aleph_0}$, the cardinality of the continuum.

The following lemma follows from Theorem 1.2 above.

Lemma 2.4. *If U is a Ramsey ultrafilter, D is a dense open set in the U-exponential topology, and $[s, a]$ is a basic open set, then there exists a set $a' \subseteq a$ in U such that $[s, a'] \subseteq D$.*

Our proof will also use the following two facts, the first of which follows from Theorem 1.1 and the second of which is a weakening of Lemma 7.12 of [15].

Lemma 2.5. *If U is a Ramsey ultrafilter on ω, and for each finite $s \subseteq \omega$, A_s is a member of U, then there is a set $B \in U$ such that for all $n \in B$, $B/n \subseteq \bigcap_{s \subseteq n} A_{s \cup \{n\}}$.*

Proof. Let E be the set of pairs $i < j$ from ω such that $j \in A_{s \cup \{i\}}$ for all $s \subseteq i$, and let $B \in U$ be such that $[B]^2$ is contained in or disjoint from E. Since U is a filter, fixing $i \in B$ there must be $j \in B/i$ such that $\{i, j\} \in E$, so $[B]^2$ is not disjoint from E. $\square$

Theorem 2.6. *[Selective Galvin Lemma] If $F \subseteq [\omega]^{<\omega}$ and C is a selective coideal, then there exists an $a \in C$ such that $F \cap [a]^{<\omega}$ is either empty or contains an initial segment of every infinite subset of a.*

Theorem 2.1 follows from the following more general fact.

Theorem 2.7. *If U is a Ramsey ultrafilter, $I \subseteq [\omega]^\omega$ is $\supseteq$-dense and I has the property of Baire in the U-exponential topology, then $U \cap I \neq \emptyset$.*

Proof. Let us note first that for any dense open set D in the U-exponential topology, and any $i \in \omega$, D contains a dense open set $D[i]$ which is closed under changes below i. To see this, we check that for all $t \subseteq i$, the set D_i^t consisting of those $b \in [\omega]^\omega$ such that $(b \setminus i) \cup t \in D$ is dense open. Fix t, and note that if $[t_0, c]$ is a basic open set, with $\max(t_0) > i$, then, letting $t_1 = (t_0 \setminus i) \cup t$, there exist a $t_2 \subseteq c/t_1$ and a set $c' \in U \restriction c$ such that $[t_1 \cup t_2, c']$ is contained in $[t_1, c] \cap D$. Then for every $b \in [t_0 \cup t_2, c']$, $(b \setminus i) \cup t$ is in D. Now, let $D[i]$ be the dense open set formed by taking the intersection of all D_i^t, for $t \subseteq i$.

Let $I \subseteq [\omega]^\omega$ be $\supseteq$-dense, and suppose that I has the property of Baire in the U-exponential topology. There exists an open set O such that $O \triangle I$ is meager.

Let us see that O is dense. Fix a basic open set $[s, a]$ and dense open sets D_i $(i \in \omega)$ such that $(O \triangle I) \cap \bigcap_{i \in \omega} D_i = \emptyset$. We may assume that $D_{i+1} \subset D_i$ for each $i \in \omega$. Let $[s_i, a_i]$ $(i \in \omega)$ be such that

- $[s_0, a_0] \subseteq [s, a]$;
- $[t \cup \{\max(s_i)\}, a_i] \subseteq D_i$, for all $t \subseteq \max(s_i)$ and $i \in \omega$ (here we use Lemma 2.4);
- each $[s_{i+1}, a_{i+1}] \subseteq [s_i, a_i]$;

- each s_{i+1} is a proper extension of s_i.

Then $\{\max(s_i) : i \in \omega\}$ is infinite and every infinite subset of it is in each D_i. It has an infinite subset in I, and therefore in O.

We have then that O is dense open in the U-exponential topology, so by adding it to our collection of dense sets if necessary, we may assume that $O = [\emptyset, \omega]$, and fix dense open sets D_i $(i \in \omega)$ such that $\bigcap_{i\in\omega} D_i \subseteq I$. We will be done with the proof once we establish the following claim.

Claim. $U \cap \bigcap_{i\in\omega} D_i \neq \emptyset$.

Replacing each D_i with $D_i[i+1]$ as above, we have that for all $b \in D_i$ and all $t \subseteq (i+1)$, $(b/i) \cup t \in D_i$. We may assume also that $D_j \subseteq D_i$ for all $i < j$ in ω.

For each $i \in \omega$ and each finite $t \subset \omega$, let $a^i_t \subseteq \omega/t$ be an element of U such that $[t, a^i_t] \subseteq D_i$, if such a set exists, otherwise, let $a^i_t = \omega/t$.

For each $i \in \omega$, let

- b_i be an element of U such that for all $n \in b_i$, $b_i/n \subseteq \bigcap\{a^i_{t\cup\{n\}} \mid t \subseteq n\}$ (here we use Lemma 2.5);
- S_i be the set of nonempty finite $t \subset \omega$ such that $[t, a^i_t] \subseteq D_i$;
- c_i be an element of U such that $S_i \cap [c_i]^{<\omega}$ contains an initial segment of every infinite subset of c_i (here we use the Selective Galvin lemma; note that the empty case cannot hold, since D_i is dense open).

Applying Lemma 2.5 again, let $e \in U$ be such that $e/i \subseteq b_i \cap c_i$ for all $i \in e$. We claim then that $e \in D_i$ for all $i \in \omega$. Since the D_i's are shrinking, and e is infinite, it suffices to consider $i \in e$. For each such i, it suffices to see that $e/i \in D_i$. This in turn follows from the fact that $e/i \subseteq c_i$, so some nonempty initial segment s_0 of e/i is in S_i, so $[s_0, a^i_{s_0}] \subseteq D_i$. Since $e/i \subseteq b_i$ and $b_i/s_0 \subseteq a^i_{s_0}$, we have that $e/s_0 \subseteq b_i$ and thus that $e/i \in [s_0, a^i_{s_0}]$. □

As a corollary, we get Mathias's result (in this context) that every selective coideal in $L(\mathbb{R})$ is densely often the coideal of infinite sets.

Corollary 2.8. *Suppose that M is an inner model of ZF containing the reals, and that every set of reals in M is $\mathfrak{c}$-universally Baire in every forcing extension of V by an (ω, ∞)-distributive partial order of cardinality at most $\mathfrak{c}$. Then for every selective coideal C on ω in M, and every $a \in [\omega]^\omega$, there is a $b \in [a]^\omega$ such that $[b]^\omega \subseteq C$.*

Proof. Let $I = \mathcal{P}(\omega) \setminus C$. Since C is selective, a V-generic filter for $\mathcal{P}(a)/I$ gives a Ramsey ultrafilter U which does not intersect I. This ultrafilter U is also M-generic for $\mathcal{P}(a)/\mathrm{Fin}$, which means that there must be a $b \in [a]^\omega \cap U$ such that $[b]^\omega \cap I = \emptyset$. □

This of course implies that there are no infinite maximal antichains in $\mathcal{P}(\omega)/\mathrm{Fin}$.

Corollary 2.9. *If M is an inner model of ZF containing the reals, and every set of reals in M is $\mathfrak{c}$-universally Baire in every forcing extension of V by an (ω, ∞)-distributive partial order of cardinality at most $\mathfrak{c}$, then the partial order $\mathcal{P}(\omega)/\mathrm{Fin}$ contains no infinite maximal antichains in M.*

Proof. If A were such an antichain, let I be the ideal of subsets of ω which are contained mod-finite in a union of finitely many members of A, and let C be the corresponding coideal. Then C is selective, and nowhere equal to Fin. $\square$

3 Countable-to-one Enumeration in models of determinacy

Given sets A and B, $a \in A$ and $X \subseteq A \times B$, we let X_a denote the set of $b \in B$ such that $(a, b) \in X$. *Uniformization* is the statement that for every $X \subseteq \mathbb{R} \times \mathbb{R}$ there is a function $f \subseteq X$ whose domain is the set of $a \in \mathbb{R}$ such that $X_a \neq \emptyset$. *Countable-to-one Uniformization* is Uniformization restricted to the case where each set X_a is countable (in which case we say that X has *countable cross sections*). Finally, *Countable-to-one Enumeration* is the statement that for every $X \subseteq \mathbb{R} \times \mathbb{R}$ having countable cross sections, there is a function F with domain $\mathbb{R}$ such that $F(a)$ is a wellordering of X_a, for each $a \in \mathbb{R}$ (we say that F *uniformly enumerates* X). Countable-to-one Enumeration clearly follows from Uniformization and implies Countable-to-one Uniformization. The first of these implications is not reversable, as we shall see below. We suspect that the second is also not reversable, but don't know of a proof.

It is easy to see that Uniformization is equivalent to determinacy for one-round real games, which of course follows from $\mathrm{AD}_{\mathbb{R}}$. It is also well known that Uniformization fails in models of the form $L(A)$, for A a set of reals (a counterexample is the set of pairs (x, y) such that y is not ordinal definable from x and A; see [14]). In this section we present a proof of Woodin's unpublished theorem that Countable-to-one Enumeration follows from the axiom AD^+, and thus holds in $L(\mathbb{R})$ and other natural models of AD.

3.1 Definition. A set of ordinals S is an ∞-*Borel* code for a set of reals A if for some binary formula ϕ, $A = \{x \in \mathbb{R} \mid L[S, x] \models \phi(S, x)\}$.

The statement that every set of reals has an ∞-Borel code is one of the three statements that make up the axiom AD^+ (see [16] for more details). Recall that for a model M of ZF and sets $x_1, \ldots, x_n$ in M, $\mathrm{HOD}^M_{x_1,\ldots,x_n}$ is the class HOD as defined in M, allowing $x_1, \ldots, x_n$ as parameters. This is always a model of ZFC, and has a natural definable wellordering.

Theorem 3.2 (Woodin)**.** *Countable-to-one Enumeration is a consequence of AD + $DC_{\mathbb{R}}$ + "every set of reals has an ∞-Borel code."*

Before beginning the proof, we note that AD can be replaced by the following consequences, which are are proved in many places, including Chapter 6 of [8].

- (Martin) Every set of Turing degrees either contains or is disjoint from a cone.
- (Mycielski) There is no ω_1-sequence of distinct reals.

Proof of Theorem 3.2. Since all sets of reals are ∞-Borel, it suffices to fix a set of ordinals S and a formula ϕ and show that the set

$$A_S = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid L[S, x, y] \models \phi(S, x, y)\}$$

can be uniformly enumerated, under the assumption that all of its cross sections are countable. We will show that for each $x \in \mathbb{R}$, $(A_S)_x \subseteq \mathrm{HOD}_{S,x}$. From this it follows, using the natural wellordering of $\mathrm{HOD}_{S,x}$, that A_S can be uniformly enumerated. Fix a real x_0. For each $z \in \mathbb{R}$, set

$$H_z = \mathrm{HOD}^{L[S,x_0,z]}_{S,x_0}.$$

Claim. *For a Turing cone of z, $(A_S)_{x_0} \subseteq H_z$.*

Before proving this, we show that the theorem follows. To see this, suppose that the claim holds, and for each z in this Turing cone, let $\langle x^z_\alpha \mid \alpha < \gamma_z\rangle$ be the enumeration of $(A_S)_{x_0}$ in H_z via the natural wellordering of H_z. For each fixed $\alpha < \omega_1$, we get that on a Turing cone of z, x^z_α is a fixed real x^∞_α. The ordinal γ_z must also be the same for a Turing cone of z (call this common value γ_∞); otherwise, we get an ω_1-sequence of distinct reals. So there is a sequence $\langle x^\infty_\alpha \mid \alpha < \gamma_\infty\rangle$ which is equal to $\langle x^z_\alpha \mid \alpha < \gamma_z\rangle$ for a Turing cone of z. Clearly $x^\infty_\alpha \in \mathrm{HOD}_{S,x_0}$ for all $\alpha < \gamma_\infty$. This finishes the proof of the theorem from the claim.

We finish by proving the claim. Since $(A_S)_{x_0}$ is countable, it is a subset of $L[S, x_0, z]$ for a Turing cone of z. Fix any z in this Turing cone. Following Definition 2.3 of [7] (but changing the notation), we let $\mathbb{B}_0$ be the collection of subsets of $\mathcal{P}(\omega)$ in $L[S, x_0, z]$ which are ordinal definable in $L[S, x_0, z]$ from S and x_0. Given a filter $G \subseteq \mathbb{B}_0$ (where $\mathbb{B}_0$ is considered as a partial order under containment), let $y(G)$ be the set of $n \in \omega$ such that $\{y \subseteq \omega \mid n \in y\} \in G$. Then by Vopěnka's Theorem (Theorem 2.4 of [7]), there exist a Boolean algebra $\mathbb{B}_1$ in H_z, a $\mathbb{B}_1$-name $\dot{y} \in H_z$ and an isomorphism $h\colon \mathbb{B}_0 \to \mathbb{B}_1$ such that

1. for every real $y \in L[S, x_0, z]$, $G(y) = h[\{A \in \mathbb{B}_0 \mid y \in A\}]$ is H_z-generic for $\mathbb{B}_1$;
2. if $H \subseteq \mathbb{B}_1$ is H_z-generic and $G = h^{-1}[H]$, then $y(G) = \dot{y}_H$ and, for every trinary formula ψ and every ordinal α,

   $$L_\alpha[S, x_0, y(G)] \models \psi(S, x_0, y(G)) \Leftrightarrow \{y \subseteq \omega \mid L_\alpha[S, x_0, y] \models \psi(S, x_0, y)\} \in G.$$

By (2), densely many conditions in $\mathbb{B}_1$ below

$$\{y \subseteq \omega \mid L[S, x_0, y] \models \phi(S, x_0, y)\}$$

must decide all of $\dot{y}$, since otherwise one can easily construct a real $y(G)$ distinct from all members of the countable set $(A_S)_{x_0}$ (here we use the fact that $\mathcal{P}(\mathbb{B}_1)^{H_z}$ is countable, which follows from the fact that there is no ω_1-sequence of distinct reals). By (1), and the assumption that $(A_S)_{x_0} \subseteq L[S, x_0, z]$, every member of $(A_S)_{x_0}$ is one of these completely determined values of $\dot{y}$, which means that $(A_S)_{x_0} \subseteq H_z$. $\square$

3.3 Remark. A slight modification of the argument just given works just assuming ZF+DC+"there is a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$"; this holds in the Solovay model for Levy collapsing a measurable cardinal to be ω_1.

3.4 Remark. The argument just given shows that under the assumption AD$^+$ + $V = \mathrm{HOD}_{\mathcal{P}(\mathbb{R})}$, one can enumerate subsets of $\mathcal{P}(\mathrm{Ord})\times\mathbb{R}$ which have countable cross-sections.

4 Countable-to-one Uniformization in the $\mathcal{P}(\omega)/\mathrm{Fin}$ extension

In this section we prove the main result of this note. We do not know if the corresponding result holds for Countable-to-one Enumeration.

Theorem 4.1. *Suppose that every set of reals is completely Ramsey, and that Countable-to-one Uniformization holds. Then Countable-to-one Uniformization holds after forcing with $\mathcal{P}(\omega)/\mathrm{Fin}$.*

Before proving Theorem 4.1, we separate out the following lemma, a variation of the results of Section 6 of [11].

Lemma 4.2. *Suppose that every set of reals is completely Ramsey, and let $f\colon [\omega]^\omega \to 2^\omega$ be a partial function whose domain is closed under subsets and finite changes. Then for each $x_0 \in \mathrm{dom}(f)$ there exist $x' \in [x_0]^\omega$ and a collection*

$$\{\tau^n_s : n \in \omega \setminus \{0\}, s \subseteq n\}$$

such that each τ^n_s is in the corresponding set 2^n, and such that for all infinite $x \subseteq \omega$ and all $m \in \omega$, if $x/m \subseteq x'$, then

$$f(x) = \bigcup\{\tau^n_{x\cap n} : n \in (m, \omega) \cap x\}.$$

Proof. Fix $x_0 \in dom(f)$. Find x_n $(n < \omega)$ such that

- each $x_{n+1} \in [x_n]^\omega$;
- for each $n \in \omega$ and $s \subseteq n$, $f(x)\restriction n$ is the same fixed set τ^n_s for all x in $[s \cup \{n\}, x_n]$ (here we use the complete Ramsey property).

Let x' be an infinite subset of x_0 such that, for each $n \in x'$, $x'/n \subseteq x_n$. If $x \subseteq \omega$ is infinite and $m \in \omega$ is such that $x/m \subseteq x'$, then for all $n \in x/m$,

$$x/n \subseteq x'/n \subseteq x_n,$$

so $f(x)\restriction n = \tau^n_{x\cap n}$. Then $f(x) = \bigcup\{\tau^n_{x\cap n} : n \in (m, \omega) \cap x\}$. $\square$

Proof of Theorem 4.1. Let ρ be a $\mathcal{P}(\omega)/\mathrm{Fin}$-name for a subset of $2^\omega \times 2^\omega$ with the property that each cross-section is countable. It suffices to prove the result in the case that each cross section is forced to be nonempty, so we assume this also. Let T be the set of triples (x, y, z) such that $[x]$ forces that (y, z) is in the realization of ρ. By refining T, we may suppose that for each pair (x, y),

$$\{z \mid (x, y, z) \in T\} = \{z \mid \exists w \in [x]^\omega\ (w, y, z) \in T\}$$

whenever the first of these two sets is nonempty (note that it is always countable). To see this, note that since ρ is a name for a set with countable cross-sections, for each y, for densely many $[x]$ there is a sequence of reals that $[x]$ forces to be an enumeration of the cross section of ρ at y, and we may restrict T to triples starting with such pairs (x, y).

Let P_0 be the set of pairs (x, y) for which there exists a z with $(x, y, z) \in T$. Applying Countable-to-one Uniformization, fix a function $Z \colon P_0 \to 2^\omega$ such that for each $(x, y) \in P_0$, $(x, y, Z(x, y)) \in T$.

Let P_1 be the set of pairs $(x, y) \in P_0$ for which there exists a collection

$$\{\tau^n_s : n \in \omega \setminus \{0\}, s \subseteq n\}$$

such that each τ^n_s is in the corresponding 2^n and such that for all infinite $w \subseteq \omega$ and all $m \in \omega$, if $w/m \subseteq x$, then

$$Z(w, y) = \bigcup\{\tau^n_{w\cap n} : n \in (m, \omega) \cap w\}.$$

Applying Lemma 4.2 to the function $Z(x, y)$ (with y fixed), we get the following.

Claim. *For each $y \in {}^\omega 2$ and $x \in [\omega]^\omega$ there exists an $x' \in [x]^\omega$ such that $(x', y) \in P_1$.*

For each pair $([x], y) \in \mathcal{P}(\omega)/\mathrm{Fin} \times 2^\omega$, let $\Sigma^{[x]}_y$ be the set of finite $\sigma \subset \omega$ for which there exists an $x' \in [x]$ such that $Z(w, y)$ is the same for all $w \in [\sigma, x']$. Noting that this constant value must be the same for all such x', we denote it by $Z^*([x], y, \sigma)$. Note that $[x_0] \leq [x_1]$ implies $\Sigma^{[x_1]}_y \subseteq \Sigma^{[x_0]}_y$, so for each y, $\Sigma^{[x]}_y$ is constant below densely many conditions $[x]$.

Claim. *If $(x, y) \in P_1$, then $\Sigma^{[x]}_y \neq \emptyset$.*

To prove the claim, fix $\{\tau^n_s : n \in \omega \setminus \{0\}, s \subseteq n\}$ witnessing that $(x, y) \in P_1$. For each $n \in \omega$ and $s \subset n$ such that $s \cup \{n\} \subseteq x$, try to find $t \cup \{m\}$ and $r \cup \{p\}$, subsets of x end-extending $s \cup \{n\}$, such that τ^m_t and τ^p_r are incompatible (necessarily proper) extensions of τ^n_s. If there always exists such a pair, then there is a perfect set Q consisting of infinite subsets of x such that the values of $Z(w, y)$ for $w \in Q$ are all distinct. This is impossible, by our refinement of T. This proves the claim.

Fixing some enumeration of $[\omega]^{<\omega}$, for each $[x] \in \mathcal{P}(\omega)/\text{Fin}$, let $\sigma_{[x],y}$ denote the least element of $\Sigma_y^{[x]}$, whenever this set is nonempty (and be undefined otherwise). For each $y \in 2^\omega$, for densely many $[x]$, $\sigma_{[x],y}$ is defined and

$$\sigma_{[x'],y} = \sigma_{[x],y}$$

for all $[x'] \leq [x]$. Call this dense set D_y.

Now, suppose that $[a]$ and $[b]$ are two compatible conditions in D_y. Then for any $[c]$ below both $[a]$ and $[b]$, $\sigma_{[c],y}$ is equal to both $\sigma_{[a],y}$ and $\sigma_{[b],y}$. Call this set σ. If $d \in [a]$ and $e \in [b]$ are such that $Z(f, y)$ is the same for all $f \in [\sigma, d]$, and $Z(g, y)$ is the same for all $g \in [\sigma, e]$, then these two constant values are the same, since these two sets are not disjoint. We have then that

- for all $(x, y) \in P_1$, if $[x] \in D_y$, then $(x, y, Z^*([x], y, \sigma_{[x],y})) \in T$;
- for all $(a, y), (b, y) \in P_1$, if $[a], [b] \in D_y$ and $[a]$, $[b]$ are compatible, then $Z^*([a], y, \sigma_{[a],y}) = Z^*([b], y, \sigma_{[b],y})$.

It follows that the set of $(x, y, \sigma_{[x],y})$ for $(x, y) \in P_1$ and $[x] \in D_y$ gives rise to a name for function uniformizing the realization of ρ. $\square$

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