Extensions of the Axiom of Determinacy

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Contents

This draft includes notes to myself, in footnotes. These should probably be ignored. Comments and corrections are more than welcome, although some parts are obviously very rough and these may not be worth commenting on, since they'll probably we rewritten. Ideally, I'll remember to post frequent updates.

0.1 Introduction

The Axiom of Determinacy (AD) is the statement that all length-ω integer games of perfect information are determined. The beginning of Chapter 1 contains a more precise definition, but we expect the reader to be familiar with the classical theory of determinacy, as found in [4, 6], for instance. The axiom AD^+ is a generalization, due to W. Hugh Woodin, of the Axiom of Determinacy. We will define it as the conjunction of the following three statements. This diverges from Woodin's own terminology, as he defines AD^+ to be the conjunction of the latter two statements, but says that the axiom is to be used in the context of $\mathsf{DC}_{\mathbb{R}}$ (which is defined in Section 0.4). The ∞-Borel sets are defined in Definition 9.0.1 for subsets of 2^ω, and in Remark 9.0.5 for subsets of ω^ω. We have renamed the third statement below, which Woodin calls Ordinal Determinacy. The notion of continuity in $<\Theta$-Determinacy refers to the discrete topology on λ, not the interval topology. The restriction of $<\Theta$-Determinacy to the case where π is the identity function on ω^ω is exactly AD.

0.1.1 Definition. The axiom AD^+ is the conjunction of the following three statements.

1. $\mathsf{DC}_{\mathbb{R}}$
2. Every subset of ω^ω is ∞-Borel.
3. ($<\Theta$-Determinacy) For all $\lambda < \Theta$, every $A \subseteq \omega^\omega$ and every continuous function $\pi\colon {}^\omega\lambda \to \omega^\omega$, $\pi^{-1}(A)$ is determined (as a subset of ${}^\omega\lambda$).

It is an open question whether AD implies any or all of the parts of AD^+. It also open whether $\mathsf{AD}_{\mathbb{R}}$ implies AD^+. The issue in this case is whether $\mathsf{AD}_{\mathbb{R}}$ implies $<\Theta$-Determinacy, as $\mathsf{DC}_{\mathbb{R}}$ is easily seen to follow from $\mathsf{AD}_{\mathbb{R}}$, and the second part of AD^+ follows from $\mathsf{AD}_{\mathbb{R}}$ (moreover, from AD + Uniformization) by Theorem 11.2.2.

If $M \subseteq N$ are models of AD with the same reals, and every set of reals in M is Suslin in N, then $M \models \mathsf{AD}^+$.[1] In fact, this is the context which the axiom AD^+ was designed to describe; its original name was "within scales".

Here is a brief[2] outline of what we aim to prove is this book (all due to Woodin):

- In $L(\mathbb{R})$, AD implies AD^+.

[1] Reference the relevant theorems.
[2] and incomplete

- If AD^+ holds then every inner model containing the reals satisfies AD^+.
- If AD^+ holds then the set of Suslin cardinals is closed below Θ. This follows from Theorem 11.3.1.
- Assuming $<\Theta$-Determinacy, $\mathsf{AD}_{\mathbb{R}}$ is equivalent to the assertion that the Solovay sequence has limit length. This is shown in Section 11.4.
- AD^+ implies Σ^2_1 reflection into the Suslin, co-Suslin (i.e., hom) sets
- AD^+ implies that the ultrapower of V by the Turing measure is well-founded.

Theorem 0.1.2 (Woodin)**.** *If* AD*,* $<\Theta$-Determinacy *and* Uniformization *hold, then every subset of* ω^ω *is Suslin.*

Proof. By Theorem 11.2.2, the hypotheses give that every subset of ω^ω is ∞-Borel. The theorem then follows from Corollary 10.1.5 and Theorem 11.1.1. □

Theorem 0.1.3 (Woodin)**.** *Each of the following statements implies the ones below it, and the first two statements are equivalent. If* DC *holds, then all four statements are equivalent.*

1. AD + *"every subset of* ω^ω *is Suslin"*
2. AD^+ + *"every subset of* ω^ω *is Suslin"*
3. $\mathsf{AD}_{\mathbb{R}}$
4. AD + Uniformization

Proof. To see that (3) implies (4), note that $\mathsf{AD}_{\mathbb{R}}$ implies Uniformization, via a game in which each player plays once. That (2) implies (1) follows from applying Ordinal Determinacy in the case $\lambda = \omega$.

As discussed in Chapter 6, Suslin sets can be uniformized, and Uniformization implies $\mathsf{DC}_{\mathbb{R}}$. The implication from (1) to (2) then follows from Theorems 5.0.2 and 7.0.3, and Remark 6.0.9. Theorem 12.0.1 says that (1) implies (3).[3] □

Theorem 0.1.4. *Assume that* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *holds. Let* Γ *be the set of* $A \subseteq \omega^\omega$ *for which* $L(A, \mathbb{R}) \models \mathsf{AD}^+$*. Then*

$$L(\Gamma, \mathbb{R}) \models \mathsf{AD}^+,$$

and, if $\Gamma \neq \mathcal{P}(\omega^\omega)$*, then* $L(\Gamma, \mathbb{R}) \models \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$*.*

The following theorem follows from part (1) of Corollary 10.2.7.

Theorem 0.1.5. *Assuming* $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$*, a subset of* ω^ω *is* ∞*-Borel if and only if there is a set of ordinals* S *such that* $A \in L(S, \mathbb{R})$*.*

[3] We still have to show that (4) implies (1) under DC. This is outlined in section 6.2. The cofinality of Θ being uncountable is the relevant consequence of DC. I should also introduce this theorem properly.

The following is Theorem 11.5.1.

Theorem 0.1.6. *If* AD^+ *holds, then* $\mathsf{AD}_{\mathbb{R}}$ *fails if and only if there is a set of ordinals* T *such that* $L(\mathcal{P}(\mathbb{R})) = L(T, \mathbb{R})$*.*

Theorem 0.1.7. *Assume that* $V = L(\mathcal{P}(\mathbb{R}))$ *and that* AD^+ *holds. Then for all* $A \subseteq \omega^\omega$*, either* $V = L(A, \mathbb{R})$*, or* $A^\#$ *exists.*[4]

This book has a great deal of overlap with many sources, notably [3, 16, 17].

0.2 Notation

We reserve the symbol $\mathbb{R}$ for the real line, which is never used directly. However, we use traditional notation such as $\mathsf{AD}_{\mathbb{R}}$, $\mathsf{DC}_{\mathbb{R}}$, $L(\mathbb{R})$ and so on, as these terms are equivalent to more relevant forms such as $\mathsf{AD}_{\omega^\omega}$, $\mathsf{DC}_{\omega^\omega}$ and $L(\omega^\omega)$. We may also use the word "real" to mean an element of the Baire space ω^ω.

We write Ord for the class of ordinals.

For a set X, we let $\mathrm{TC}(X)$ denote the transitive closure of X.

We write X^Y to mean the set of functions from Y to X, and, when γ is an ordinal $X^{<\gamma}$ to mean $\bigcup_{\alpha\in\gamma} X^\alpha$.

For an ordinal α and a set of ordinals X, $[X]^\alpha$ denotes the collection of subsets of X of ordertype α.

Given a set X, we write $\exists^X$ and $\forall^X$ for existential and universal quantification over X, respectively.

Gödel coding : $\prec\succ$[5]

Given a cardinal κ, $H(\kappa)$ denotes the set of x for which $|\,\mathrm{TC}(x)| < \kappa$.

We sometimes write HF for $H(\aleph_0)$. The members of HF are said to be *hereditarily finite*. A set $x \subseteq \mathrm{HF}$ is *semi-recursive* if x is Σ_1-definable over HF, and *recursive* if x and $\mathrm{HF} \setminus x$ are both semi-recursive

We sometimes write HC for $H(\aleph_1)$. The members of HC are said to be *hereditarily countable*. We say that a set $x \in \omega^\omega$ HC-*codes* a set $y \in \mathrm{HC}$ if $(\omega, \{(n,m) \in \omega\times\omega : x(2^n 3^m) = 0\})$ is isomorphic to $(\mathrm{TC}(\{y\}), \in)$.

0.3 Prerequisites

We expect the reader to be familiar with Zermelo-Fraenkel set theory, relative constructibility, ordinal definability, inner models of the form HOD_X, ultrapowers and forcing. We do not expect the reader to be familiar with everything in [4, 6, 14, 23, 24], but they make good references.

[4] Another statement to consider : The axiom $\mathsf{AD}_{\mathbb{R}}$ implies that the sharp of each subset of ω^ω exists.

[5] ???

0.4 Forms of Choice

Our base theory in this book is Zermelo-Fraenkel set theory (ZF). Additional axioms will be stated as used. Although we will sometimes consider models of the Axiom of Choice (AC), our main interest is in models of the Axiom of Determinacy (AD), which contradicts AC. Weak forms of AC can (and do) hold in models of AD, however.

Given a set X, the principle of Dependent Choice for X (DC_X) is the statement that whenever $T \subseteq {}^{<\omega}X$ is a tree (i.e., a subset of ${}^{<\omega}X$ closed under initial segments) with the property that every element of T has a proper extension in T, there exists an infinite path through T. The principle of Countable Choice for X (CC_X) is the statement that for all countable sets Y, if $\langle A_y : y \in Y\rangle$ is a sequence of nonempty subsets of X, then there exists a function $f \colon Y \to X$ such that $f(y)$ is in A_y, for each $y \in Y$. The principle CC_X follows immediately from DC_X. A classical argument (due to Mycielski; see Remark 1.0.2) shows that $\mathsf{CC}_{\mathbb{R}}$ follows from AD. Whether or not $\mathsf{DC}_{\mathbb{R}}$ follows from AD is an open question. Note however that if $\mathsf{DC}_{\mathbb{R}}$ holds, then any inner model containing $\mathcal{P}(\omega)$ satisfies $\mathsf{DC}_{\mathbb{R}}$.

The axiom of Dependent Choice (DC) asserts that DC_X holds for every set X. Similarly, the axiom of Countable Choice (CC) asserts that CC_X holds for every set X.

0.4.1 Remark. We make frequent use of the standard fact[6] that if every set is definable from a member of X, then DC_X implies DC.

0.5 Partial orders

We list here some classical partial orders which are used as forcing notions throughout the book.

- $\mathrm{Col}(\kappa, X)$, where κ is an infinite cardinal and X is a set. Conditions are functions $f \colon \alpha \to X$, where $\alpha < \kappa$. The order is extension.
- $\mathrm{Col}^*(\kappa, X)$, where κ is an infinite cardinal and X is a set. Conditions are injective functions $f \colon \alpha \to X$, where $\alpha < \kappa$. The order is extension.
- Given a set X consisting of infinite subsets of ω, the classical *almost-disjoint coding* forcing for X (due to Jensen and Solovay [5]) consists of pairs (a, B), where a is a finite subset of ω and B is a finite subset of X, with the order $(a, B) \leq (c, D)$ if $c = a \cap (\max(c) + 1)$, $D \subseteq B$ and $(a \setminus c) \cap r = \emptyset$ for all $r \in D$. This partial order is c.c.c. and adds a $z \subseteq \omega$ having infinite intersection with each element of $\mathcal{P}(\omega)$ not contained mod-finite in the union of a finite subset of B.

[6] citation?

Part I

Preliminaries

Chapter 1

The Axiom of Determinacy

Given a set X, a set $A \subseteq X^\omega$ is *determined* (as a subset of X^ω) if there is a function $\pi \colon X^{<\omega} \to X$ such that one of the two following statements holds.

1. For every $x \in X^\omega$, if $x(2n) = \pi(x\restriction 2n)$ holds for all $n \in \omega$, then $x \in A$.
2. For every $x \in X^\omega$, if $x(2n+1) = \pi(x\restriction(2n+1))$ hold for all $n \in \omega$, then $x \notin A$.

We let AD_X denote the statement that every subset of X^ω is determined (as a subset of X^ω). The Axiom of Determinacy (AD) is the statement AD_ω.

1.0.1 Remark. Let X and Y be sets. The following assertions can be easily verified.

- If there is an injection from X to Y, then AD_Y implies AD_X.
- If X is wellorderable and there is a surjection from X to Y, then AD_X implies AD_Y.
- If AD holds, then so does AD_X for each countable set X.

It is convenient (and conventional) to rephrase determinacy in terms of games. A set $A \subseteq X^\omega$ corresponds to a game $\mathcal{G}_A$ between players I and II, who alternate picking members of X, with I winning if and only if the induced member of X^ω is in A.

I	$x(0)$		$x(2)$		$x(4)$		$\ldots$
II		$x(1)$		$x(3)$		$\ldots$	

The game $\mathcal{G}_A$; I wins if and only if x is in A.

A function $\pi \colon X^{<\omega} \to X$ is then said to be a *strategy* in the game G_A. A function π as in case (1) above is a *winning strategy* for player I; in case (2) it is a winning strategy for player II. If σ is a strategy and x is in X^ω, then we write $\sigma * x$ for combined output of the two players when I plays according to σ and II plays x, that is, the unique $y \in X^\omega$ such that

- $y(2n+1) = x(n)$ for all $n \in \omega$;
- $y(n) = \sigma(y \restriction n)$ for all even $n \in \omega$.

Similarly, we write $x * \sigma$ for the unique $y \in X^\omega$ such that

- $y(2n) = x(n)$ for all $n \in \omega$;
- $y(n) = \sigma(y \restriction n)$ for all odd $n \in \omega$.

The statement that a set $A \subseteq X^\omega$ is determined can then be rephrased as asserting the existence of a strategy σ such that one of the two following statements holds.

- For every $x \in X^\omega$, $\sigma * x$ is in A.
- For every $x \in X^\omega$, $x * \sigma$ is not in A.

1.0.2 Remark. We list four classical consequences of AD, the details of which are presented in Chapter 33 of [4].[1]

1. Jan Mycielski observed that AD implies $\mathsf{CC}_{\mathbb{R}}$. To see this, let A_i $(i \in \omega)$ be nonempty subsets of ω^ω, and consider the game where I plays $i \in \omega$ and then II must list the values of a member of A_i.

2. Morton Davis proved that AD implies that every uncountable subset of ω^ω contains a perfect set. To see this, consider a set $A \subseteq \omega^\omega$, and the game where players I and II collaborate to build an $x \in \omega^\omega$, with player II choosing individual digits as usual, but player I allowed to play finite sequences from ω, with I winning if the concatenation of these moves is in A. Player II has a winning strategy if and only if A is countable, and a winning strategy for I induces a perfect subset of A. It follows from ZF that if every uncountable subset of ω^ω contains a perfect set then there is no injection from ω_1 into ω^ω (such an injection would give a wellordering of ω^ω in ordertype ω_1). The nonexistence of an injection from ω_1 into ω^ω (which we will denote by writing $\aleph_1 \not\leq 2^{\aleph_0}$) is equivalent to the assertion that for any inner model M satisfying Choice (equivalently, for all models of the form $L[S]$, for S a subset of ω_1), and any countable ordinal α, $\mathcal{P}(\alpha) \cap M$ is countable. [2]

3. Stefan Banach proved that AD implies that every subset of ω^ω (similarly, every subset of 2^ω) has the property of Baire, i.e., that for each $A \subseteq \omega^\omega$ there exists an open $U \subseteq \omega^\omega$ such that the symmetric difference $A \triangle U$ is meager. To see this, fix a bijection $\pi \colon \omega \to \omega^{<\omega}$ and, given $A \subseteq \omega^\omega$, let

[1] And Kanamori?

[2] We need to remove instances of the term "inaccessibility of ω_1 to its subsets".

B be the set of $x \in \omega^\omega$ for which the concatenation of the values $\pi(x(i))$ $(i \in \omega)$ is in A. If player I was a winning strategy in $\mathcal{G}_B$, then A is relatively comeager in some open set; if player II does, then A is meager. Wacław Sierpiński that nonpricipal ultrafilters on ω (considered at subsets of 2^ω) do not have the property of Baire. It follows that nonpricipal ultrafilters on ω don't exist under AD, and from this that, under AD every nonprincipal ultrafilter on any set is countably complete.

4. Robert Solovay proved AD implies that the club filter on ω_1 is an ultrafilter. This gives another proof that (under AD) there is no injection from ω_1 into $\mathcal{P}(\omega)$. Since ZF implies the existence of partition of $\mathcal{P}(\omega)$ into $\aleph_1$ many sets, this shows that the statement AD_{ω_1} is inconsistent with ZF.[3] We present a proof that AD implies the measurabilty of ω_1 (not Solovay's original proof) in Remark 1.1.7.

Given $\Gamma \subseteq \mathcal{P}(\omega^\omega)$, we will write $\mathsf{Baire}(\Gamma)$ for the assertion that every element of Γ has the property of Baire.

1.1 Turing determinacy

In this section we prove Martin's theorem that under AD the cone measure on the Turing degrees is an ultrafilter. We give a general version of the theorem, and our proof is slightly more involved than Martin's original proof.

Fixing an enumeration $\langle \sigma_n : n < \omega \rangle$ of $\omega^{<\omega}$, we can associate to each $x \in \omega^\omega$ a function (strategy) $\pi_x \colon \omega^{<\omega} \to \omega$ defined by the formula $\pi_x(\sigma_n) = x(n)$.

Let $\mathrm{even} \colon \omega^\omega \to \omega^\omega$ be the function defined by letting $\mathrm{even}(y)(n) = y(2n)$ for each $n \in \omega$ and let $\mathrm{odd} \colon \omega^\omega \to \omega^\omega$ be the function defined by letting $\mathrm{odd}(y)(n) = y(2n+1)$ for each $n \in \omega$. Given $x \in \omega^\omega$, let

- $I^*_x \colon \omega^\omega \to \omega^\omega$ be the function defined by letting $I^*_x(y) = (\pi_y * x)$ (i.e., the result of playing x for player II against the strategy π_y for player I);
- $II^*_x \colon \omega^\omega \to \omega^\omega$ be the function defined by letting $II^*_x(y) = (x * \pi_y)$ (i.e., the result of playing x for player I against the strategy π_y for player II);
- $\mathcal{F}_x$ be the smallest set of functions on ω^ω which is closed under composition and contains the identity function, even, odd and the functions $I^*_{f(x)}$ and $II^*_{f(x)}$ for each $f \in \mathcal{F}_x$.

Let $\leq_{\mathrm{M}}$ be the reflexive binary relation on ω^ω defined by setting $y \leq_{\mathrm{M}} x$ if $y = f(x)$ for some $f \in \mathcal{F}_x$. Observe that if $y \leq_{\mathrm{M}} x$ then $\mathcal{F}_y \subseteq \mathcal{F}_x$; it follows from this that $\leq_{\mathrm{M}}$ is transitive. The functions even and odd can be used to show that $\leq_{\mathrm{M}}$ is directed. Let $\equiv_{\mathrm{M}}$ be the equivalence relation $\leq_{\mathrm{M}} \cap \geq_{\mathrm{M}}$.

1.1.1 Definition. An *ordered equivalence relation* is a pair $(E, \leq_E)$ where E is an equivalence relation on a set X and $\leq_E$ is a partial order on X such that

[3] We say this somewhere else, too.

$E = \leq_E \cap \geq_E$. We say that $(E, \leq_E)$ is an ordered equivalence relation *on* X. Given an ordered equivalence relation $(E, \leq_E)$ on ω^ω, and $x \in \omega^\omega$, we define the *upward E-cone* of x to be $\mathcal{U}_E(x) = \{[y]_E : y \geq_E x\}$ and the *downward E-cone* of x to be $\mathcal{D}_E(x) = \{[y]_E : y \leq_E x\}$. The *$E$-cone measure* is

$$\{A \subseteq \{[x]_E : x \in X\} : \exists x \in X\, \mathcal{U}_E(x) \subseteq A\}.$$

For the rest of this section, we will let μ_E denote the E-cone measure for an ordered equivalence relation $(E, \leq_E)$. This notation will be modified in later sections, however.

1.1.2 Definition. Given two ordered equivalence relations $(E, \leq_E)$ and $(F, \leq_F)$ on the same underlying set X, say that $(E, \leq_E)$ is *as thick as* $(F, \leq_F)$ if, for all x, y in X,

- for all $x \leq_F y \Rightarrow x \leq_E y$
- for all x, y in X, if $x \leq_E y$, then for some $y' \in [y]_E$, $y' \geq_F x$.

Given the first condition in the definition of "as thick as", the second condition is equivalent to saying that for all $x \in X$,

$$\mathcal{U}_E(x) = \{[y]_E : y \geq_F x\}.$$

1.1.3 Example. The following are examples of ordered equivalence relations which are as thick as $(\equiv_{\mathrm{M}}, \leq_{\mathrm{M}})$.

- $(\equiv_{\mathrm{M}}, \leq_{\mathrm{M}})$.
- The Turing degrees, under Turing reducibility.
- For any set S, the equivalence relation $L(S, x) = L(S, y)$, under the order $x \in L(S, y)$.
- For any set S, the equivalence relation $\mathrm{HOD}_{\{S\}}[x] = \mathrm{HOD}_{\{S\}}[y]$, under the order $x \in \mathrm{HOD}_{\{S,y\}}$.

Assuming $\mathsf{CC}_{\mathbb{R}}$, the cone measures associated to the ordered equivalence relations in Example 1.1.3 are countably complete.

1.1.4 Remark. Suppose that $(E, \leq_E)$ and $(F, \leq_F)$ are ordered equivalence relations on a set X, $(E, \leq_E)$ is as thick as $(F, \leq_F)$, and μ_F is an ultrafilter on the set of F-classes. Then μ_E is an ultrafilter on the set of E-classes. To see this, let A be a set of E-classes, and let B be the set of F-classes contained in a member of A. Since μ_F is an ultrafilter, we can fix an $x \in \omega^\omega$ such that $\mathcal{U}_F(x)$ is either contained in or disjoint from B. Since $\mathcal{U}_E(x) = \{[y]_E : y \geq_F x\}$, $\mathcal{U}_E(x)$ is either contained in or disjoint from A.

Theorem 1.1.5 (Martin)**.** *Suppose that* AD *holds. Then the* $\equiv_{\mathrm{M}}$*-cone measure is a countably ultrafilter on the* $\equiv_{\mathrm{M}}$*-classes.*

Proof. Since AD implies $\mathsf{CC}_{\mathbb{R}}$, the $\equiv_M$-cone measure is countably complete. It suffices then to show that each set of $\equiv_{\mathrm{M}}$-classes contains or is disjoint from an upward cone. Let A be a set of $\equiv_{\mathrm{M}}$-classes. Consider the game G_{A^*}, where A^* is the set of $x \in \omega^\omega$ such that $[x]_{\equiv_{\mathrm{M}}}$ is in A. Let π be a winning strategy (for either player), and let $x \in \omega^\omega$ be such that $\pi = \pi_x$. Let $y \in \omega^\omega$ be such that $y \geq_{\mathrm{M}} x$. If π is a winning strategy for player I, then $[\pi_x * y]_{\equiv_{\mathrm{M}}}$ is in A, and $[y]_{\equiv_{\mathrm{M}}} = [\pi_x * y]_{\equiv_{\mathrm{M}}}$. If π is a winning strategy for player II, then $[y * \pi_x]_{\equiv_{\mathrm{M}}}$ is not in A, and $[y]_{\equiv_{\mathrm{M}}} = [y * \pi_x]_{\equiv_{\mathrm{M}}}$. □

Remark 1.1.4 and Theorem 1.1.5 give the following.

Corollary 1.1.6. *If* AD *holds and* $(E, \leq_E)$ *is an ordered equivalence relation on* ω^ω *which is as thick as* $\equiv_{\mathrm{M}}$*, then* μ_E *is a countably complete ultrafilter on* $\{[x]_E : x \in \omega^\omega\}$.

The following remark gives Solovay's theorem that, assuming AD, ω_1 is measurable.

1.1.7 Remark. Suppose that μ is a countably complete ultrafilter on $\mathcal{P}_{\aleph_1}(\omega^\omega)$ which is *fine* (i.e., for each $x \in \omega^\omega$ the set $\{\sigma \in \mathcal{P}_{\aleph_1}(\omega^\omega) : x \in \sigma\}$ is in μ). For each $\alpha < \omega_1$, let X_α be the set of $x \in \omega^\omega$ which HC-code α. For each $\sigma \in \mathcal{P}_{\aleph_1}(\omega^\omega)$, let α_σ be $\sup\{\alpha < \omega_1 : X_\alpha \cap \sigma \neq \emptyset\}$. Let U be the set of $A \subseteq \omega_1$ for which $\{\sigma \in \mathcal{P}_{\aleph_1}(\omega^\omega) : \alpha_\sigma \in A\} \in \mu$. Then U is a countably complete ultrafilter on ω_1.

1.1.8 Example. Let S be a set of ordinals, let $\leq_S$ be the binary relation on ω^ω given by the binary relation $x \in \mathrm{HOD}_{\{S,y\}}$, let $\equiv_S$ be the corresponding equivalence relation and let μ_S be the $\equiv_S$-cone measure. Assume that μ_S is an ultrafilter. It follows then by Remark 1.1.7 that ω_1 is inaccessible to its subsets (we defined this term in Remark 1.0.2). For each $x \in \omega^\omega$, there exist y and z in ω^ω such that y is generic for Sacks forcing over $L[S,x]$ and z codes a wellordering of ω in ordertype $\omega_1^{L[S,x]}$. By standard facts about Sacks forcing,[4] $x \oplus y$ is an $\leq_S$ successor of x. It follows then that the set of $\leq_S$-successors contains a set in μ_S. Let $f\colon \omega^\omega \to \omega_1$ be the S-invariant function defined by setting $f(x)$ to be $\omega_1^{L[S,x]}$. Since Sacks forcing preserves ω_1, we have that for every $x \in \omega^\omega$, there exist y, z in ω^ω both $\leq_S$-above x, such that $f(x) = f(y)$ and $f(x) < f(z)$.

[4] citation!

Chapter 2

The Wadge hierarchy

2.1 Wadge determinacy

Given sets $A, B \subseteq \omega^\omega$, we say that A is *Wadge reducible* to B (and write $A \leq_{\mathrm{W}} B$) if there is a continuous function $f\colon \omega^\omega \to \omega^\omega$ such that $A = f^{-1}[B]$. We say that A is *Lipschitz reducible* to B (and write $A \leq_{\mathrm{L}} B$) if there is a function $f\colon \omega^\omega \to \omega^\omega$ with the following properties.

- For all $x, y \in \omega^\omega$ and $n \in \omega$, if $x \restriction n = y \restriction n$, then $f(x)\restriction n = f(y)\restriction n$.
- $A = f^{-1}[B]$

Clearly, $A \leq_{\mathrm{L}} B$ implies $A \leq_{\mathrm{W}} B$.

2.1.1 Remark. Since $A = f^{-1}[B]$ implies that $(\omega^\omega \setminus A) = f^{-1}[\omega^\omega \setminus B]$, $A \leq_{\mathrm{W}} B$ implies $(\omega^\omega \setminus A) \leq_{\mathrm{W}} (\omega^\omega \setminus B)$ and $A \leq_{\mathrm{L}} B$ implies $(\omega^\omega \setminus A) \leq_{\mathrm{L}} (\omega^\omega \setminus B)$.

The relations $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{L}}$ are easily seen to be reflexive and transitive. We write $=_{\mathrm{W}}$ and $=_{\mathrm{L}}$ respectively for the equivalence relations $\leq_{\mathrm{W}} \cap \geq_{\mathrm{W}}$ and $\leq_{\mathrm{L}} \cap \geq_{\mathrm{L}}$, whose equivalence classes are respectively called *Wadge classes* and *Lipschitz classes*. Given $A \subseteq \omega^\omega$, we write $[A]_{\mathrm{W}}$ for the Wadge class of A, and $[A]_{\mathrm{L}}$ for the Lipschitz class of A. We say that a Wadge class or Lipschitz class is *selfdual* if it contains a pair of complements (in which case it is closed under complements, by Remark 2.1.1), and *nonselfdual* otherwise. We write $<_{\mathrm{W}}$ for $\leq_{\mathrm{W}} \setminus =_{\mathrm{W}}$ and $<_{\mathrm{L}}$ for $\leq_{\mathrm{L}} \setminus =_{\mathrm{L}}$.

2.1.2 Definition. Let A, B be subsets of ω^ω. We write $A \otimes B$ for the set of $\langle x_i : i \in \omega\rangle$ for which $\langle x_{2i} : i \in \omega\rangle \in A$ if and only if $\langle x_{2i+1} : i \in \omega\rangle \in B$.

2.1.3 Remark. The relation $A \leq_{\mathrm{L}} B$ is equivalent to the assertion that player II wins the game $\mathcal{G}_{A\otimes(\omega^\omega \setminus B)}$; it also follows from player I winning $\mathcal{G}_{B\otimes A}$.

This observation leads to the following fundamental fact.

Theorem 2.1.4 (Wadge)**.** *If A, B are subsets of ω^ω. If player I has a winning strategy in $\mathcal{G}_{A\otimes B}$, then $B \leq_{\mathrm{L}} A$. If player II has a winning strategy in $\mathcal{G}_{A\otimes B}$, then $A \leq_{\mathrm{L}} (\omega^\omega \setminus B)$.*

Wadge's theorem (along with Remarks 2.1.1 and 2.1.3) has the following consequences. Proposition 2.1.5 shows that the ordering on the Lipschitz degrees induced by $\leq_{\mathrm{L}}$ is almost linear, the exceptions being pairs of the form $[A]_{\mathrm{L}}$, $[\omega^\omega \setminus A]_{\mathrm{L}}$, for nonselfdual classes $[A]_{\mathrm{L}}$. [1]

Proposition 2.1.5. *Let A, B be subsets of ω^ω such that $A \otimes B$ and $B \otimes A$ are both determined. If $A \not\leq_{\mathrm{L}} B$ and $B \not\leq_{\mathrm{L}} A$, then $A =_{\mathrm{L}} (\omega^\omega \setminus B)$.*

Proposition 2.1.6. *Let A, B be subsets of ω^ω such that $B \otimes A$ and $(\omega^\omega \setminus B) \otimes A$ are both determined. If $A <_{\mathrm{L}} B$, then player I wins both $G_{B\otimes A}$ and $G_{(\omega^\omega\setminus B)\otimes A}$.*

Proof. In the case of $B \otimes A$, one gets otherwise that $A <_{\mathrm{L}} B \leq_{\mathrm{L}} (\omega^\omega \setminus A)$, and therefore by Remark 2.1.1 that $A =_{\mathrm{L}} (\omega^\omega \setminus A)$, giving a contradiction. □

Proposition 2.1.7. *Let A, B be subsets of ω^ω such that $A \otimes B$ and $B \otimes A$ are both determined. If $A <_{\mathrm{W}} B$ then $A <_{\mathrm{L}} B$.*

Proof. Since $B \not\leq_{\mathrm{W}} A$, $B \not\leq_{\mathrm{L}} A$, so by Proposition 2.1.5, either $A =_{\mathrm{L}} (\omega^\omega \setminus B)$ (which is impossible, as then $A \leq_{\mathrm{W}} B$ would imply $B \leq_{\mathrm{W}} A$) or $A \leq_{\mathrm{L}} B$, which gives the desired result. □

We let $\mathsf{Wadge\ Determinacy}$ be the statement that $A \otimes B$ is determined for all subsets A, B of ω^ω. By Proposition 2.1.7, the following theorem shows (under the hypotheses of the theorem) that $<_{\mathrm{W}}$ is wellfounded as well. Theorem 2.2.8 shows that $<_{\mathrm{L}}$ is wellfounded if and only if $<_{\mathrm{W}}$ is.

Theorem 2.1.8 (Martin)**.** *If $\mathsf{Wadge\ Determinacy} + \mathsf{CC}_{\mathbb{R}} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$ holds, then there does not exist a sequence $\langle A_i : i \in \omega\rangle$ consisting of subsets of ω^ω, such that $A_{i+1} <_{\mathrm{L}} A_i$ for each $i \in \omega$.*

Proof. Suppose towards a contradiction that such a sequence $\langle A_i : i < \omega\rangle$ does exist. Applying Proposition 2.1.6 and $\mathsf{CC}_{\mathbb{R}}$ we can fix winning strategies f_i^0, f_i^1 ($i \in \omega$) for player I in the games $G_{A_i\otimes A_{i+1}}$ and $G_{(\omega^\omega\setminus A_i)\otimes A_{i+1}}$, respectively.

For each $x \in {}^\omega 2$, define $y_i(x) \in \omega^\omega$ ($i \in \omega$) by letting $y_i(x)(k)$ be $f_i^{x(i)}(y_{i+1}\restriction k)$ for each even $k \in \omega$, and letting $y_i(x)(k) = y_{i+1}(x)(k-1)$ for each odd k. Then for each $i \in \omega$, if $x(i) = 0$ then $y_i(x) \in A_i \Leftrightarrow y_{i+1}(x) \in A_{i+1}$, and if $x(i) = 1$ then $y_i(x) \in A_i \Leftrightarrow y_{i+1}(x) \notin A_{i+1}$.

Let Z be the set of $x \in 2^\omega$ for which $y_0(x) \in A_0$. It suffices to show that whenever $x, x' \in 2^\omega$ differ at exactly one point, $x \in Z$ if and only if $x' \notin Z$, since no subset of 2^ω with the property of Baire can have this property.

First observe that if $x, x' \in {}^\omega 2$ and $i \in \omega$ are such that $x(j) = x'(j)$ for all $j \geq i$, then $y_i(x) = y_i(x')$. If i_0 is the unique $i \in \omega$ such that $x(i) \neq x'(i)$, then

[1] The second sentence of the statement of Proposition 2.1.5 is called the Semi-Linear Ordering Principle for Lipschitz maps by Andretta.

$y_{i_0+1}(x) = y_{i_0+1}(x')$. Since $x(i_0) \neq x'(i_0)$, it follows that $y_{i_0}(x) \in A_{i_0}$ if and only if $y_{i_0}(x') \notin A_{i_0}$. Since $x(i) = x'(i)$ for all $i < i_0$, it follows that $y_i(x) \in A_i$ if and only if $y_i(x') \notin A_i$ for all such i. □

Corollary 2.1.9 (Martin). *If* $\mathsf{DC}_{\mathbb{R}}$ + Wadge Determinacy + Baire($\mathcal{P}(\omega^\omega)$) *holds, then* $<_{\mathrm{L}}$ *is wellfounded.*

Proof. If $<_{\mathrm{L}}$ is illfounded, then $\mathsf{DC}_{\mathbb{R}}$ gives a sequence $\langle A_i : i \in \omega\rangle$ consisting of subsets of ω^ω, such that $A_{i+1} <_{\mathrm{L}} A_i$ for each $i \in \omega$. □

As noted in Section 0.4, it is open whether AD implies $\mathsf{DC}_{\mathbb{R}}$.[2] Moreover, it is an open question whether $\mathsf{DC}_{\mathbb{R}}$ is needed for Corollary 2.1.9 (in particular whether it can be replaced with $\mathsf{CC}_{\mathbb{R}}$).

The *Lipschitz rank* LR(A) (respectively, *Wadge rank* WR(A)) of a set $A \subseteq \omega^\omega$ is recursively defined to the be the least ordinal greater than the Lipschitz (Wadge) rank of every $B \subseteq \omega^\omega$ with $B <_{\mathrm{L}} A$ ($B <_{\mathrm{W}} A$), if this is defined. We let $\mathcal{W}$ be the set of $A \subseteq \omega^\omega$ for which WR(A) is defined. By Martin's theorem, if $\mathsf{DC}_{\mathbb{R}}$ and Wadge determinacy hold, and every subset of 2^ω has the property of Baire, then LR(A) and WR(A) are defined for each $A \subseteq \omega^\omega$. By Proposition 2.1.5, for each ordinal α, the subsets of ω^ω of Lipschiz rank (Wadge rank) α (if there are any) consist either of a single Lipschitz (Wadge) class or a pair of Lipschitz (Wadge) classes corresponding to complements. Theorem 2.2.8 says more about the relationship between the two sets of classes.

2.2 Lipschitz degrees and Wadge degrees

Following [29], we give some more details on the structure of the Lipschitz and Wadge hierarchies, using Wadge Determinacy, $\mathsf{CC}_{\mathbb{R}}$ and the assumption that all sets of reals have the Baire property, but not $\mathsf{DC}_{\mathbb{R}}$. We begin by noting that $\emptyset$ and ω^ω are the only subsets of ω^ω of Lipchitz or Wadge rank 0, and that they are Wadge (and thus Lipschitz) inequivalent.

Proposition 2.2.1 defines the least Lipschitz class above a given selfdual Lipschitz class.

Proposition 2.2.1. *Suppose that* Wadge Determinacy *holds, and that* $A \subseteq \omega^\omega$ *is such that* $[A]_{\mathrm{L}}$ *is selfdual. Fix* $i \in \omega$*, and let* B_i *be the set of* $x \in \omega^\omega$ *for which* $x(0) = i$ *and* $\langle x(1), x(2), x(3), \ldots\rangle \in A$*. Then* $[B_i]_{\mathrm{L}}$ *is selfdual, and* $[B_i]_{\mathrm{L}}$ *is the* $\leq_{\mathrm{L}}$*-least Lipschitz class above* $[A]_{\mathrm{L}}$*.*

Proof. It is easy to check that $A \leq_{\mathrm{L}} B_i$ and that $[B_i]_{\mathrm{L}}$ is selfdual. To see that $A <_{\mathrm{L}} B_i$, note that whenever $f\colon \omega^\omega \to \omega^\omega$ has the property that for each $n \in \omega$, $f(x)\restriction n$ depends only on $x\restriction n$, there is an $x \in \omega^\omega$ such that $x(0) = i$ and $f(x) = \langle x(1), x(2), x(3), \ldots\rangle$. It follows that no such f witnesses that $(\omega^\omega \setminus A) \leq_{\mathrm{L}} B_i$. Finally, if $C \subseteq \omega^\omega$ is such that $A <_{\mathrm{L}} C$, then player I wins $G_{C\otimes A}$, by Proposition 2.1.7. Any strategy witnessing this can be used to show that $B_i \leq_{\mathrm{L}} C$ (using the fact that $C \neq \omega^\omega$). □

[2] Woodin has shown that one gets a model of $\mathsf{AD}_{\mathbb{R}}$ from a counterexample?

2.2.2 Remark. For any $A \subseteq \omega^\omega$, if B_i is defined as in the statement of Proposition 2.2.1, then $[A]_{\mathrm{W}} = [B_i]_{\mathrm{W}}$.

2.2.3 Remark. An argument similar to the proof of Proposition 2.2.1 shows that $[\bigcup_{i\in\omega} B_i]_{\mathrm{L}}$ is also the $\leq_{\mathrm{L}}$-least Lipschitz class above $[A]_{\mathrm{L}}$ (again assuming Wadge Determinacy).

2.2.4 Remark. Let $\pi\colon \omega \times \omega \to \omega$ be a bijection, and suppose that A_i $(i \in \omega)$ are subsets of ω^ω. Let $C \subseteq \omega^\omega$ be the set of functions of the form

$$\langle \pi(i, x(0)), x(1), x(2), \ldots \rangle$$

for $i \in \omega$ and $x \in A_i$. Then $A_i \leq_{\mathrm{L}} C$ for all $i \in \omega$. If $\mathsf{CC}_{\mathbb{R}}$ holds, then for all $D \subseteq \omega^\omega$, if $A_i \leq_{\mathrm{L}} D$ for all $i \in \omega$ then $C \leq_{\mathrm{L}} D$. If $\{A_i : i \in \omega\}$ does not have a $\leq_{\mathrm{L}}$-maximal element, then $A_i <_{\mathrm{L}} C$ for all $i \in \omega$.

If Wadge Determinacy holds, then $[C]_{\mathrm{L}}$ is nonselfdual exactly in the case where $\{A_i : i \in \omega\}$ has a $\leq_{\mathrm{L}}$-maximal element whose Lipschitz class is nonselfdual. To see this, note first of all that $[C]_{\mathrm{L}}$ is clearly selfdual in the case where $\{A_i : i \in \omega\}$ has a $\leq_{\mathrm{L}}$-maximal element whose Lipschitz class is selfdual, as $[C]_{\mathrm{L}}$ is equal to this class (similarly, if $\{A_i : i \in \omega\}$ has a $\leq_{\mathrm{W}}$-maximal element, then $[C]_{\mathrm{W}}$ is equal to this class). In the remaining case, $\{[A_i]_{\mathrm{L}} : i \in \omega\}$ has the same supremum as

$$\{[A_i]_{\mathrm{L}} : i \in \omega\} \cup \{[\omega^\omega \setminus A_i]_{\mathrm{L}} : i \in \omega\}.$$

Running the construction of C above with the set $\{A_i : i \in \omega\} \cup \{\omega^\omega \setminus A_i : i \in \omega\}$ clearly gives a selfdual Lipschitz class (equal to $[C]_{\mathrm{L}}$). This argument gives the following facts, for any $A \subseteq \omega^\omega$.

- If $[A]_{\mathrm{L}}$ is nonselfdual, then the pair $[A]_{\mathrm{L}}, [\omega^\omega \setminus A]_{\mathrm{L}}$ has a $\leq_{\mathrm{L}}$-least upper bound, and this upper bound is selfdual.
- If $[A]_{\mathrm{L}}$ is the $\leq_{\mathrm{L}}$-supremum of a countable set of Lipschitz classes strictly below it, and Wadge Determinacy+$\mathsf{CC}_{\mathbb{R}}$ holds, then $[A]_{\mathrm{L}}$ is selfdual (Proposition 2.2.5 below gives the converse, for non-successor classes).

From Proposition 2.2.1 and Remarks 2.2.2 and 2.2.4 it follows that for all $A \subseteq \omega^\omega$ such that $[A]_{\mathrm{L}}$ is selfdual, the first ω_1 many Lipschitz classes above $[A]_{\mathrm{L}}$ are all selfdual and contained in $[A]_{\mathrm{W}}$. The following proposition shows that the nonselfdual Lipschitz classes are exactly those whose Lipschitz rank is either 0 or an ordinal of uncountable cofinality.

Proposition 2.2.5. *Suppose that* $\mathsf{CC}_{\mathbb{R}}$ *and* Wadge Determinacy *hold. Let $A \subseteq \omega^\omega$ be such that $[A]_{\mathrm{L}}$ is selfdual, and $[A]_{\mathrm{L}}$ is not the $\leq_{\mathrm{L}}$-least Lipschitz class above any other class. Then $[A]_{\mathrm{L}}$ is the $\leq_{\mathrm{L}}$-supremum of a countable set of Lipschitz classes strictly below it.*

Proof. Let $\pi\colon \omega \times \omega \to \omega$ be a bijection. For each $i \in \omega$, let B_i be the set of sequences of the form $\langle \pi(n, x(0)), x(1), x(2), \ldots \rangle$ such that either n is even and $\langle i, x(0), x(1), x(2), \ldots \rangle$ is in A, or n is odd and $\langle i, x(0), x(1), x(2), \ldots \rangle$ is not in

A. Then, for each $i \in \omega$, B_i is selfdual and (using the fact that A is selfdual), $B_i \leq_{\mathrm{L}} A$.

For each $i \in \omega$, let C_i be the set of $x \in \omega^\omega$ such that $x(0) = i$ and $\langle x(1), x(2), \ldots\rangle \in B_i$. By Proposition 2.2.1, each $[C_i]_{\mathrm{L}}$ is the $<_{\mathrm{L}}$-successor of the corresponding $[B_i]_{\mathrm{L}}$. Since $[A]_{\mathrm{L}}$ is not a successor class, it follows that $B_i <_{\mathrm{L}} C_i <_{\mathrm{L}} A$ holds, for each $i \in \omega$. Let D be the set constructed from $\{C_i : i \in \omega\}$ as in Remark 2.2.4. Then $[D]_L$ is the supremum of $\{[C_i]_L : i < \omega\}$, and it suffices to see that $A \leq_{\mathrm{L}} D$.

Let $g\colon {}^{<\omega}\omega \to \omega$ be such that for all $n \in \omega \setminus 2$,

$$g(\langle i_0, \ldots, i_n\rangle) = \langle \pi(i_0, i_0), \pi(0, i_1), i_2, \ldots, i_n\rangle.$$

Let $g^*\colon \omega^\omega \to \omega^\omega$ be the Lipschitz function induced by g. Let us see that g^* witnesses that $A \leq_{\mathrm{L}} D$. Fix $x \in \omega^\omega$, and let $g^*(x)$ have the form

$$\langle \pi(i_0, y(0)), y(1), y(2), \ldots\rangle$$

for some $i_0 \in \omega$ and some $y \in \omega^\omega$. Then $i_0 = x(0) = y(0)$, $y(1) = \pi(0, x(1))$ and $y(i) = x(i)$ for all $i \in \omega \setminus 2$. We want to see that $x \in A$ if and only if $g^*(x) \in D$. Now, $g^*(x)$ is in D if and only if $y \in C_{i_0}$, which in turn happens if and only if $\langle \pi(0, x(1)), x(2), x(3), \ldots\rangle$ is in B_{i_0}, which happens if and only if x is in A. □

The following proposition completes the analysis of which Wadge classes are selfdual classes, as well as the relationship between the Lipschitz classes and the Wadge classes.

Theorem 2.2.6 (Steel, Van Wesep)**.** *Suppose that*

$$\mathsf{Wadge\ Determinacy} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$$

holds. For all $A \subseteq \omega^\omega$, if $[A]_{\mathrm{W}}$ is selfdual, then so is $[A]_{\mathrm{L}}$.

Proof. Suppose towards a contradiction that $[A]_{\mathrm{W}}$ is selfdual but $[A]_{\mathrm{L}}$ is not. By Wadge Determinacy, if $[A]_{\mathrm{L}}$ is not selfdual, then player I wins $G_{A \otimes A}$. Let g_2 be a strategy witnessing this.

Consider the following game between players I and II. For each $i \in \omega$, player I plays a value $x(i)$, and player II either passes or plays a value $y(j)$, for j the least $k \in i+1$ for which a value for $y(k)$ has not yet been chosen. If at the end of the game there is a $j \in \omega$ for which $y(j)$ has not been chosen, then II loses. Otherwise II wins if and only if $x \in A \Leftrightarrow y \notin A$. The statement that $A \leq_{\mathrm{W}} (\omega^\omega \setminus A)$ is equivalent to the existence of a winning strategy for II. Let g_1 be such a strategy.

For each positive $n \in \omega$ and each sequence $\bar{c} = \langle c_m : m < n\rangle \in 3^n$, there is a unique (possibly partial) function $s_{\bar{c}}$ on $n \times \omega$ satisfying the following conditions.

- For each $m < n$, the set of i for which $(m, i) \in \mathrm{dom}(s_{\bar{c}})$ is an ordinal $\alpha_m \in \omega + 1$. We let t_m be the function with domain α_m such that $t_m(i) = s_{\bar{c}}(m, i)$ for all $i < \alpha_m$.

- The largest $m < n$ for which $\alpha_m > 0$ is the largest $m < n$ for which $c_m = 2$, and for this m, $\alpha_m = 1$ and $s_{\bar{c}}(m, 0) = g_2(\langle\rangle)$.
- For each $m < n-1$ such that $c_m = 2$, $\alpha_m = \alpha_{m+1} + 1$, and for each $i < \alpha_m$, $s_{\bar{c}}(m, i)$ is the response given by g_2 when player II plays $t_m \restriction i$.
- For each $m < n-1$ such that $c_m = 0$, $\alpha_m = \alpha_{m+1}$, and $t_m = t_{m+1}$.
- For each $m < n-1$ such that $c_m = 1$, t_m is the longest sequence of nonpassing moves made by g_1 in response to t_{m+1}.

Choose integers i_k ($k \in \omega$) so that $i_0 = 0$, $i_{k+1} > i_k + 1$ for all $k \in \omega$ and, for each sequence $\bar{c} = \langle c_m : m < i_{k+1}\rangle$ as above, if $c_m = 2$ for all

$$m \in i_{k+1} \setminus \{i_p : p \leq k\},$$

then the corresponding value α_j is at least k for each $j \leq i_k$.

For each $x \in {}^{\omega}2$ and each $m \in \omega$, let c^x_m be $x(k)$ if $m = i_k$ for some $k \in \omega$, and let c^x_m be 2 otherwise. For each such x there is a unique sequence $\langle y^x_m : m \in \omega\rangle \in (\omega^\omega)^\omega$ such that for each $m \in \omega$,

- if $c^x_m = 2$, then y^x_m is the sequence produced by g_2 when player II plays y^x_{m+1};
- if $c^x_m = 0$, then $y^x_m = y^x_{m+1}$;
- if $c^x_m = 1$, then y^x_m is the sequence produced by g_1 when player I plays y^x_{m+1}.

Then as in the proof of Theorem 2.1.8, $x, x' \in {}^{\omega}2$ and $k \in \omega$ are such that $x(p) = x'(p)$ for all $p > k$, then $y^x_m = y^{x'}_m$ for all $m > i_k$. So again, if we let Z be the set of x for which $y^x_0 \in A$, Z cannot have the property of Baire, since whenever x and x' disagree at exactly one point, exactly one of them will be in Z. $\square$

Theorem 2.2.6 has the following corollary.

Corollary 2.2.7. *Suppose that* Wadge Determinacy + Baire($\mathcal{P}(\omega^\omega)$) *holds. Let A be a subset of ω^ω. If $[A]_{\mathrm{L}}$ is nonselfdual, then $[A]_{\mathrm{L}} = [A]_{\mathrm{W}}$.*

Proof. Supposing otherwise, fix $B \in [A]_{\mathrm{W}} \setminus [A]_{\mathrm{L}}$. By Theorem 2.1.4, either $(\omega^\omega \setminus A) \leq_{\mathrm{L}} B$ or $(\omega^\omega \setminus B) \leq_{\mathrm{L}} A$. Each of these implies that $[A]_{\mathrm{W}}$ is selfdual. $\square$

Summarizing, we have the following.

Theorem 2.2.8. *Suppose that* Wadge Determinacy + $\mathsf{CC}_{\mathbb{R}}$ + Baire($\mathcal{P}(\omega^\omega)$) *holds.*

1. *The minimal Wadge classes consist of the singletons $\{\emptyset\}$ and $\{\omega^\omega\}$.*
2. *Each selfdual Wadge class contains $\aleph_1$ many selfdual Lipschitz classes, and these are ordered in ordertype ω_1 by $\leq_{\mathrm{L}}$.*

3. *Each nonselfdual Wage class is equivalent to the corresponding Lipschitz class, and has a selfdual class as an immediate successor.*

4. *A Wadge class which is neither minimal nor a successor is selfdual if and only if it is the supremum of a countable set of classes strictly below it.*

As we show in Proposition 2.5.4, a straightforward diagonal argument (adapted from [25]) shows that there is no largest Wadge degree, if Wadge Determinacy holds.

2.2.9 Remark. The material in this section does not give a definition for the least Wadge class above a given selfdual class, or show (with assuming $\mathsf{DC}_{\mathbb{R}}$) that such a class exists. Adding $\mathsf{DC}_{\mathbb{R}}$ to the hypotheses of Theorem 2.2.8, Proposition 2.2.5 gives that each seldfual Wadge class $[A]_W$ has a pair of non-selfdual classes as immediate successors, the Wadges classes corresponding to the $<_L$-least Lipschitz classes above of the Lipschitz classes contained in $[A]_W$.

2.3 Pointclasses

The notion of Wadge reducibility naturally generalizes to other topological spaces. In general, one could say that for any pair of topological spaces X and Y, and any sets $A \subseteq X$ and $B \subseteq Y$, that $A \leq_{\mathrm{W}} B$ if there is a continuous function $f\colon X \to Y$ such that $A = f^{-1}[B]$. We will be concerned only with topological spaces of the form $X_1 \times \cdots \times X_n$ for some positive $n \in \omega$, where at least one X_i is ω^ω, and each X_i is either ω^ω or ω. Let $\mathcal{X}$ be the collection of such spaces; these spaces are all homeomorphic with ω^ω. The notions of Wadge reducibilty and Wadge rank carry over naturally to subsets of spaces in $\mathcal{X}$. We use the word *pointclass* to denote a collection of subsets of spaces in $\mathcal{X}$. A *boldface pointclass* is a pointclass closed under continuous preimages, i.e., the union of an initial segment of the Wadge hierarchy.

Given a finite sequence $s \in {}^{<\omega}\omega$, we let $[s] = \{x \in \omega^\omega \mid x\restriction|s| = s\}$. A *basic open interval* of a space $X_1 \times \cdots X_n$ in $\mathcal{X}$ is a product of the form $a_1 \times \cdots \times a_n$, where, for each $i \in \{1, \ldots, n\}$,

- a_i is of the form $[s]$, for some $s \in {}^{<\omega}\omega$, if $X_i = \omega^\omega$;
- a_i is either $\emptyset$, ω or $\{m\}$ for some $m \in \omega$ if $X_i = \omega$.

We say that continuous functions $f\colon X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_m$ between spaces in $\mathcal{X}$ is *recursive* if the set of pairs of basic open intervals $U \subseteq X$, $V \subseteq Y$ for which $f[U] \subseteq V$ is recursive (i.e., Δ_1-definable over HF). A *lightface pointclass* is a pointclass closed under preimages of continuous functions which are recursive. Under these definitions boldface pointclasses are also lightface.

Given a set $A \subseteq X$ (for some $X \in \mathcal{X}$), we write $\check{A}$ for $X \setminus A$. Given a pointclass Γ, we write $\check{\Gamma}$ for $\{\check{A} : A \in \Gamma\}$. We say that a pointclass Γ is *selfdual* if $\Gamma = \check{\Gamma}$; otherwise it is *nonselfdual*. Observe that Γ is a boldface pointclass if and only if $\check{\Gamma}$ is.

2.3.1 Example. The collection of analytic subsets of spaces in $\mathcal{X}$ (i.e., $\utilde{\Sigma}^1_1$) is a nonselfdual boldface pointclass, as are the projective pointclasses $\utilde{\Sigma}^1_n$, $\utilde{\Pi}^1_n$ and $\utilde{\Delta}^1_n$, for all $n \in \omega$.

The following is an immediate, but useful, corollary of Theorem 2.1.4 (and Remark 2.1.1).

Corollary 2.3.2. *Assume that* Wadge Determinacy+Baire($\mathcal{P}(\omega^\omega)$) *holds. Let* Γ *be a nonselfdual boldface pointclass, and suppose that* Λ *is a boldface pointclass properly containing* Γ. *Then* $\check{\Gamma} \subseteq \Lambda$.

A member A of a pointclass Γ is *complete* for Γ (or Γ-*complete*) if every member of Γ is a continuous preimage of (i.e., Wadge below) A.

Proposition 2.3.3. *Suppose that* Wadge Determinacy + Baire($\mathcal{P}(\omega^\omega)$) *holds. If* Γ *is a boldface pointclass, then every member of* $\Gamma \setminus \check{\Gamma}$ *is* Γ*-complete.*

Proof. Fix a set
$$A \in \mathcal{P}(\omega^\omega) \cap (\Gamma \setminus \check{\Gamma}).$$
Then $[A]_L$ is nonselfdual, so $[A]_L = [A]_W$, by Corollary 2.2.7. Suppose towards a contradiction that there is a $B \in \Gamma$ such that $B \not\leq_{\mathrm{W}} A$. Then, by Theorem 2.1.4, $A \leq_{\mathrm{W}} (\omega^\omega \setminus B)$, contradicting our assumption that $A \notin \check{\Gamma}$. $\square$

We write $\omega^\omega \times \mathcal{X}$ for the set of $X \in \mathcal{X}$ of the form
$$\omega^\omega \times X_1 \times \cdots \times X_n,$$
for some $X_1, \ldots, X_n$ (all equal to either ω or ω^ω). Given $X \in \omega^\omega \times \mathcal{X}$ and $A \subseteq X$, we write $\exists^{\omega^\omega} A$ for the set
$$\{(x_1, \ldots, x_n) : \exists x_0 \in \omega^\omega \, (x_0, \ldots, x_n) \in A\},$$
and $\forall^{\omega^\omega} A$ for the set
$$\{(x_1, \ldots, x_n) : \forall x_0 \in \omega^\omega \, (x_0, \ldots, x_n) \in A\}.$$
Given a pointclass Γ, we write $\exists^{\omega^\omega}\Gamma$ for
$$\{\exists^{\omega^\omega} A : A \in \Gamma \wedge \exists X \in \omega^\omega \times \mathcal{X} \, A \subseteq X\}$$
and $\forall^{\omega^\omega}\Gamma$ for
$$\{\forall^{\omega^\omega} A : A \in \Gamma \wedge \exists X \in \omega^\omega \times \mathcal{X} \, A \subseteq X\}.$$
Similarly, for any ordinal δ, we write $\bigcup_\delta \Gamma$ for the collection of sets of the form $\bigcup_{\alpha<\delta} A_\alpha$, where each A_α is in Γ, and the A_α's are all subsets of the same element of $\mathcal{X}$.

A pointclass Γ is $\exists^{\omega^\omega}$-*closed* if $\Sigma^1_1 \subseteq \Gamma$ and $\exists^{\omega^\omega}\Gamma \subseteq \Gamma$, and $\forall^{\omega^\omega}$-*closed* if $\Pi^1_1 \subseteq \Gamma$ and $\forall^{\omega^\omega}\Gamma \subseteq \Gamma$.[3]

[3] We need to avoid trivialities like the Wadge class of the emptyset. Ideally, our usage is consistent.

Given a pointclass Γ, we write $o(\Gamma)$ for the ordertype of $(\Gamma \cap \mathcal{P}(\omega^\omega), \leq_W)$, identifying this with the corresponding ordinal in the case that the restriction of $\leq_W$ to $\Gamma \cap \mathcal{P}(\omega^\omega)$ is wellfounded. In this case $o(\Gamma)$ is the supremum of the Wadge ranks of the members of Γ.

Proposition 2.3.4. *Assume that $\mathsf{CC}_{\mathbb{R}}$ holds. If Γ is an $\exists^{\omega^\omega}$-closed boldface pointclass with a complete set, then Γ is closed under countable unions. If Δ is a selfdual $\exists^{\omega^\omega}$-closed boldface pointclass and $o(\Delta)$ has uncountable cofinality, then Δ is closed under countable unions and countable intersections.*

Proof. We prove the first part first. Let $A \subseteq \omega^\omega$ be Γ-complete, and let $B_i \subseteq \omega^\omega$ $(i \in \omega)$ be elements of Γ. Applying $\mathsf{CC}_{\mathbb{R}}$, for each i, let f_i be a continuous function such that $B_i = f_i^{-1}[A]$. Let C be $\{(x, y) \in \omega^\omega \times \omega^\omega : y \in B_{x(0)}\}$. Then $C \leq_W A$, via the continuous function which send each pair (x, y) to $f_{x(0)}(y)$. It follows that C is in Γ. Since $\bigcup_{i \in \omega} B_i = \{y : \exists x \in \omega^\omega (x, y) \in C\}$, we are done.

For the second part, the assumptions imply that every countable subset of Δ is contained in a boldface pointclass Γ_0 having a complete set. Then $\exists^{\omega^\omega}\Gamma$ is contained in Δ and satisfies the assumptions of the first part. $\square$

2.3.5 Remark. The proof of Proposition 2.3.4 shows the following, without the assumption of $\mathsf{CC}_{\mathbb{R}}$

- If Γ is an $\exists^{\omega^\omega}$-closed boldface pointclass with a complete set, then Γ is closed under unions.
- If Δ is a selfdual $\exists^{\omega^\omega}$-closed boldface pointclass and $o(\Delta)$ has uncountable cofinality, then Δ is closed under unions and intersections.

2.3.6 Remark. Let Γ be a boldface pointclass. If Γ is closed under unions, then so are $\exists^{\omega^\omega}\Gamma$ and $\forall^{\omega^\omega}\Gamma$. Similarly, if Γ is closed under countable unions, then so are $\exists^{\omega^\omega}\Gamma$ and $\forall^{\omega^\omega}\Gamma$.

2.4 Universal sets

Given an integer $n \in \omega \setminus 2$, a set $A \subseteq (\omega^\omega)^n$ is *universal* for a pointclass Γ (or Γ-*universal*) if $A \in \Gamma$, and for every $B \subseteq (\omega^\omega)^{n-1}$ in Γ there is an $x \in \omega^\omega$ such that

$$B = \{(y_1, \ldots, y_{n-1}) \mid (x, y_1, \ldots, y_{n-1}) \in A\}$$

(we call this set A_x). If Γ is closed under Wadge-equivalence, and n is in $\omega \setminus 2$, then there exists a universal subset of $(\omega^\omega)^n$ in Γ if and only if there is a universal subset of $(\omega^\omega)^2$ in Γ.

2.4.1 Example. Fixing an enumeration $\langle \sigma_n : n \in \omega \rangle$ of $\omega^{<\omega}$, the set of (x, y) in $(\omega^\omega)^2$ such that $y \in [\sigma_n]$ for some $n \in x^{-1}[\{0\}]$ is a universal open set. It follows that there is a universal closed set $C \subseteq (\omega^\omega)^3$, and universal analytic and coanalytic subsets of $(\omega^\omega)^2$.

Proposition 2.4.2. *If Δ is a selfdual boldface pointclass, then Δ does not have a universal set.*

Proof. Given a set $A \subseteq (\omega^\omega)^2$, let B be the set of $x \in \omega^\omega$ for which $(x, x) \notin A$. Then for any $x \in \omega^\omega$, $x \in B$ if and only if $x \notin A_x$. ☐

Theorem 2.4.3. *Suppose that* Wadge Determinacy + Baire($\mathcal{P}(\omega^\omega)$) *holds. If Γ is a boldface pointclass, then Γ has a universal set if and only if it is nonselfdual.*

Proof. The selfdual case follows from Proposition 2.4.2.

For the other case, suppose that Γ is nonselfdual, and fix a set

$$A \in \mathcal{P}(\omega^\omega) \cap (\Gamma \setminus \check{\Gamma}).$$

Then $[A]_L$ is nonselfdual, so $[A]_L = [A]_W$, by Corollary 2.2.7. Then A is Γ-complete by Proposition 2.3.3. Fix a bijection $\rho\colon \omega^{<\omega} \to \omega$. For each $x \in \omega^\omega$, define $f_x\colon \omega^\omega \to \omega^\omega$ by setting $f_x(y)(n)$ to be $x(\rho(y\restriction(n+1)))$.

Then each f_x is Lipschitz, and each Lipschitz function from ω^ω to ω^ω is equal to f_x for some $x \in \omega^\omega$. Now define the set $U \subseteq (\omega^\omega)^2$ by setting $(x, y) \in U$ if and only if $f_x(y) \in A$. Then $U =_L A$, and U is Γ-universal. ☐

2.4.4 Remark. Suppose that Γ is a boldface pointclass and $U \subseteq (\omega^\omega)^n$ is Γ-universal, for some $n \in \omega \setminus 3$. Then

$$\{(z_1, \ldots, z_{n-2}) \in (\omega^\omega)^{n-2} : \exists y \in \omega^\omega\ (x, y, z_1, \ldots, z_{n-2}) \in U\}$$

is universal for $\exists^{\omega^\omega}\Gamma$ and

$$\{(z_1, \ldots, z_{n-2}) \in (\omega^\omega)^{n-2} : \forall y \in \omega^\omega\ (x, y, z_1, \ldots, z_{n-2}) \in U\}$$

is universal for $\forall^{\omega^\omega}\Gamma$. It follows from Theorem 2.4.3 that if Γ is nonselfdual, then so are $\exists^{\omega^\omega}\Gamma$ and $\forall^{\omega^\omega}\Gamma$ (although either or both of these pointclasses could be equal to Γ).

We will need sequences of universal sets which relate well to one another.

2.4.5 Definition. Let $\bar{U} = \langle U_n : n \in \omega \setminus \{0\}\rangle$ be such that each U_n is a subset of $(\omega^\omega)^{n+1}$.

- The sequence $\bar{U}$ has the *s-m-n property* if for each pair of positive integers $n < m$, there exists a continuous $s_{m,n}\colon (\omega^\omega)^{n+1} \to \omega^\omega$ such that, for all $x, y_1, \ldots, y_m \in \omega^\omega$,
$$(x, y_1, \ldots, y_m) \in U_m$$
if and only if $(s_{m,n}(x, y_1, \ldots, y_n), y_{n+1}, \ldots, y_m) \in U_{m-n}$.
- The sequence $\bar{U}$ has the *recursion property* (with respect to a pointclass Γ) if for each $n \in \omega \setminus \{0\}$ and each
$$A \in \mathcal{P}((\omega^\omega)^{n+1}) \cap \Gamma,$$

there exists an $x \in \omega^\omega$ such that for all $y_1, \ldots, y_n \in \omega^\omega$,

$$(x, y_1, \ldots, y_n) \in U_n$$

if and only if $(x, y_1, \ldots, y_n) \in A$.

We will omit the phrase "with respect to Γ" when talking about sequences $\bar{U}$ with the recursion property, since Γ is recoverable from $\bar{U}$. Similarly, when we say that $\langle U_n : n \in \omega \setminus \{0\} \rangle$ is a sequence of sets with the s-m-n property, we will mean that each U_n is a subset of $(\omega^\omega)^{n+1}$.

Theorems 2.4.6 and 2.4.7 below show that if Γ is a boldface pointclass with a universal set, then there exists a sequence of Γ-universal set with the s-m-n and recursion properties. The statement of Theorem 2.4.6, and its proof, are taken from [3].[4]

Theorem 2.4.6. *If Γ is a bolface pointclass with a universal set then there exists a sequence of Γ-universal sets with the s-m-n property.*

Proof. Fix homeomorphisms $\pi_n \colon \omega^\omega \to (\omega^\omega)^n$ for each $n \in \omega \setminus 2$, and let $\pi_{m,n} \colon \omega^\omega \to \omega^\omega$ ($n < m \in \omega \setminus 2$) be such that

$$\pi_m(x) = (\pi_{m,0}(x), \ldots, \pi_{m,m-1}(x))$$

for all $x \in \omega^\omega$, so that $\pi_{m,n}(\pi_m^{-1}(x_0, \ldots, x_{m-1})) = x_n$ for all $x_0, \ldots, x_{m-1} \in \omega^\omega$. Let $U \subseteq (\omega^\omega)^2$ be a universal set for Γ. For each $n \in \omega \setminus \{0\}$, define $U_n \subseteq (\omega^\omega)^{n+1}$ by setting $(x, y_1, \ldots, y_n) \in U_n$ if and only if

$$(\pi_{2,0}(x), \pi_{n+1}^{-1}(\pi_{2,1}(x), y_1, \ldots, y_n)) \in U.$$

Let us check that each U_n is Γ-universal. Fix $n \in \omega \setminus \{0\}$ and $A \subseteq (\omega^\omega)^n$ in Γ. We want to find an $x \in \omega^\omega$ such that $U_{n,x} = A$. Fixing any $z \in \omega^\omega$, we have that $\pi_{n+1}^{-1}[\{(z, y_1, \ldots, y_n) : (y_1, \ldots, y_n) \in A\}]$ is in Γ. Since U is universal, there is a $w \in \omega^\omega$ such that $U_w = \pi_{n+1}^{-1}[\{(z, y_1, \ldots, y_n) : (y_1, \ldots, y_n) \in A\}]$. Let $x = \pi_2^{-1}(w, z)$. Then for all $y_1, \ldots, y_n \in \omega^\omega$, $(x, y_1, \ldots, y_n)$ is in U_n if and only if $(w, \pi_{n+1}^{-1}(z, y_1, \ldots, y_n))$ is in U, which holds if and only if $(y_1, \ldots, y_n)$ is in A.

To check that the s-m-n property holds, fix $m > n$ in ω. Let W be the set of $w \in \omega^\omega$ for which

$$(u(w), \pi_{m+1}^{-1}(v(w), r_1(w), \ldots, r_n(w), t_1(w), \ldots, t_{m-n}(w))) \in U,$$

where

- $u(w) = \pi_{2,0}(\pi_{n+1,0}(\pi_{m-n+1,0}(w)))$;
- $v(w) = \pi_{2,1}(\pi_{n+1,0}(\pi_{m-n+1,0}(w)))$;
- $r_i(w) = \pi_{n+1,i}(\pi_{m-n+1,0}(w))$ for $i \in \{1, \ldots, n\}$;

[4] But to whom are they due?

- $t_j(w) = \pi_{m-n+1,j}(w)$ for $j \in \{1, \dots, m-n\}$.

Then $W \leq_W U$, and, as U is universal for Γ, there exists a $z \in \omega^\omega$ such that $U_z = W$.

Define $s_{m,n} \colon (\omega^\omega)^{n+1} \to \omega^\omega$ by setting

$$s_{m,n}(x, y_1, \dots, y_n) = \pi_2^{-1}(z, \pi_{n+1}^{-1}(x, y_1, \dots, y_n))$$

for all $x, y_1, \dots, y_n \in \omega^\omega$. Now fix $x, y_1, \dots, y_m \in \omega^\omega$. Then $(x, y_1, \dots, y_m) \in U_m$ if and only if

$$(\pi_{2,0}(x), \pi_{m+1}^{-1}(\pi_{2,1}(x), y_1, \dots, y_m)) \in U,$$

and $(s_{m,n}(x, y_1, \dots, y_n), y_{n+1}, \dots, y_m) \in U_{m-n}$ if and only if

$$(\pi_{2,0}(s_{m,n}(x, y_1, \dots, y_n)), \pi_{m-n+1}^{-1}(\pi_{2,1}(s_{m,n}(x, y_1, \dots, y_n)), y_{n+1}, \dots, y_m)) \in U.$$

Now, $\pi_{2,0}(s_{m,n}(x, y_1, \dots, y_n) = z$ and

$$\pi_{2,1}(s_{m,n}(x, y_1, \dots, y_n)) = \pi_{n+1}^{-1}(x, y_1, \dots, y_n).$$

Since $U_z = W$, we have that

$$(z, \pi_{m-n+1}^{-1}(\pi_{n+1}^{-1}(x, y_1, \dots, y_n), y_{n+1}, \dots, y_m)) \in U$$

if and only if $\pi_{m-n+1}^{-1}(\pi_{n+1}^{-1}(x, y_1, \dots, y_n), y_{n+1}, \dots, y_m) \in W$, which, letting w be $\pi_{m-n+1}^{-1}(\pi_{n+1}^{-1}(x, y_1, \dots, y_n), y_{n+1}, \dots, y_m)$, holds if and only if

$$(u(w), \pi_{m+1}^{-1}(v(w), r_1(w), \dots, r_n(w), t_1(w), \dots, t_{m-n}(w))) \in U,$$

which holds if and only if

$$(\pi_{2,0}(x), \pi_{m+1}^{-1}(\pi_{2,1}(x), y_1, \dots, y_m)) \in U,$$

the two sequences being the same. □

Finally, we observe that a sequence of Γ-universal sets with the s-m-n property has the $\bar{U}$ has the recursion property.

Theorem 2.4.7. *If Γ is a pointclass and $\bar{U} = \langle U_n : n \in \omega \setminus \{0\}\rangle$ is a sequence of Γ-universal sets with the s-m-n property, then $\bar{U}$ has the recursion property.*

Proof. Fix $n \in \omega \setminus \{0\}$ and $A \in \mathcal{P}((\omega^\omega)^{n+1}) \cap \Gamma$. Applying the assumption that U_{n+1} is Γ-universal, let $y \in \omega^\omega$ be such that for all $w, z_1, \dots, z_n$ in ω^ω, $(y, w, z_1, \dots, z_n) \in U_{n+1}$ if and only if $(s(w, w), z_1, \dots, z_n) \in A$, where $s \colon (\omega^\omega)^2 \to \omega^\omega$ witnesses the s-m-n property for $\bar{U}$ in the role of $s_{n+1,1}$. Then for all $w, z_1, \dots, z_n \in \omega^\omega$,

$$(s(y, w), z_1, \dots, z_n) \in U_n$$

if and only if

$$(y, w, z_1, \dots, z_n) \in U_{n+1}$$

if and only if

$$(s(w, w), z_1, \dots, z_n) \in A.$$

Then $x = s(y, y)$ is as desired. □

Theorem 2.4.8 below is known as the Recursion Theorem. A function $f\colon \omega^\omega \to \omega^\omega$ is in $\underset{\sim}{\Sigma}^1_1$ if it is $\underset{\sim}{\Sigma}^1_1$ as a subset of $\omega^\omega \times \omega^\omega$ (equivalently, if the f-preimage of each open set is in $\underset{\sim}{\Sigma}^1_1$.

Theorem 2.4.8 (Kleene)**.** *Suppose that Γ is a boldface pointclass and that $\bar{U} = \langle U_n : n \in \omega \setminus \{0\}\rangle$ is a sequence of Γ-universal sets with the recursion property. If $f\colon \omega^\omega \to \omega^\omega$ is in $\underset{\sim}{\Sigma}^1_1$, then there is an $x \in \omega^\omega$ such that $U_{2,x} = U_{2,f(x)}$.*

Proof. Since Γ is a boldface pointclass, the set

$$A = \{(x, y, z) \in (\omega^\omega)^3 : (f(x), y, z) \in U_2\}$$

is in Γ. Then for each $x \in \omega^\omega$, $A_x = U_{2,f(x)}$. Applying the recursion property, we get an $x \in \omega^\omega$ such that $U_{2,x} = A_x$. □

Using a recursive bijection $\pi\colon \omega \to \omega \times \omega$, we can associate to each $y \in \omega^\omega$ an ω-sequence $\langle (y)_n : n \in \omega\rangle$ of elements of ω^ω by setting $(y)_n(m)$ to be $y(\pi^{-1}(n, m))$. Roughly following 7D.7 (page 430) of [23], we say that a set $B \subseteq X$ (for some X in $\mathcal{X}$) is *pos-*$\underset{\sim}{\Sigma}^1_1(A)$ (for some $A \subseteq \omega^\omega$) if

$$B = \{x \in X : \exists y \in \omega^\omega((\forall n \in \omega\, (y)_n \in A) \wedge (x, y) \in S)\},$$

for some $\underset{\sim}{\Sigma}^1_1$ set $S \subseteq \omega^\omega \times X$. If $A_1, \ldots, A_n$ are subsets of ω^ω, we write pos-$\underset{\sim}{\Sigma}^1_1(A_1, \ldots, A_n)$ for pos-$\underset{\sim}{\Sigma}^1_1(A)$, where A is the disjoint union of $A_1, \ldots, A_n$, i.e., the set of reals of the form $\langle \pi(i, x(0)), x(1), x(2), \ldots\rangle$ for $x \in A_i$ and $i \in \{1, \ldots, n\}$ (we call this set $A_1 \oplus \cdots \oplus A_n$).

2.4.9 Remark. For any $X \in \mathcal{X}$ and $A \subseteq \omega^\omega$, A is in pos-$\underset{\sim}{\Sigma}^1_1(A)$, and pos-$\underset{\sim}{\Sigma}^1_1(A)$ is a boldface pointclass closed under $\exists^{\omega^\omega}$, $\wedge$ and $\vee$. The existence of universal sets for $\underset{\sim}{\Sigma}^1_1$ (as shown in Example 2.4.1) implies that each pointclass of the form pos-$\underset{\sim}{\Sigma}^1_1(A)$ has a universal set.

Given $A \subseteq \omega^\omega$, we write $\underset{\sim}{\Sigma}^1_1(A)$ for pos-$\underset{\sim}{\Sigma}^1_1(A, \omega^\omega \setminus A)$ and $\underset{\sim}{\Sigma}^1_1(A_1, \ldots, A_n)$ for pos-$\underset{\sim}{\Sigma}^1_1(A_1, \ldots, A_n, \omega^\omega \setminus A_1, \ldots, \omega^\omega \setminus A_n)$. Given a positive $n \in \omega$, $\underset{\sim}{\Pi}^1_n(A)$ is the set of complements of sets in $\underset{\sim}{\Sigma}^1_1(A)$, and $\underset{\sim}{\Sigma}^1_{n+1}(A)$ is the set of continuous images of sets in $\underset{\sim}{\Pi}^1_n(A)$. We say that a set B is *projective in* A if it is in $\bigcup_{n\in\omega} \underset{\sim}{\Sigma}^1_{n+1}(A)$. We say that a pointclass Δ is *projectively closed* if for each $A \in \Delta \cap \mathcal{P}(\omega^\omega)$, every set projective in A is in Δ.

2.4.10 Remark. Let $\bar{U} = \langle U_n : n < \omega\rangle$ be a sequence of universal sets for $\underset{\sim}{\Sigma}^1_1$ with the s-m-n property. For any $A \subseteq \omega^\omega$, let $U_n(A)$ be the set

$$\{x \in (\omega^\omega)^{n+1} : \exists y \in \omega^\omega((\forall n \in \omega\, (y)_n \in A) \wedge (x, y) \in U_{n+1})\}.$$

Then each $U_n(A)$ is universal for pos-$\underset{\sim}{\Sigma}^1_1(A)$. Furthermore, if the functions $s_{m,n}$ ($n < m < \omega$) witness the s-m-n property for $\bar{U}$, then for all $n < m < \omega$, function $s_{m+1,n}$ witnesses the s-m-n property for $\bar{U}(A) = \langle U_n(A) : n < \omega\rangle$ for m and n. Similarly, given a function $f\colon \omega^\omega \to \omega^\omega$ and $n \in \omega \setminus \{0\}$, if $U_{n,x} = U_{n,f(x)}$, then $U_{n,x}(A) = U_{n,f(x)}(A)$ for all $A \subseteq \omega^\omega$ (writing $U_{n,x}(A)$ for $U_n(A)_x$, i.e., the cross-section of $U_n(A)$ at x).

For $\bar{U}$ as Remark 2.4.10, $n \in \omega$ and $A_1, \ldots, A_m \subseteq \omega^\omega$, we write $U_n(A_1, \ldots, A_m)$ for $U_n(A_1 \oplus \cdots \oplus A_m)$.

2.5 The cardinal Θ

This short section presents a result of Solovay on the height of the Wadge hierarchy, taken from [25].

2.5.1 Definition. The ordinal Θ is defined to be the least ordinal which is not a surjective image of $\mathcal{P}(\omega)$.

It follows immediately from this definition that Θ is a cardinal.

2.5.2 Remark. We will often making use of the fact that continuous functions on ω^ω can be coded by members of ω^ω. While there are many ways of doing this, we fix one for concreteness and convenience. Recall from Example 2.4.1 that there is a universal set $C \subseteq (\omega^\omega)^3$ for the pointclass of closed subsets of spaces in $\mathcal{X}$. For each continuous function $f\colon \omega^\omega \to \omega^\omega$, there is an $x \in \omega^\omega$ such that $C_x = f$. Let $\mathcal{F}^{\mathrm{c}}$ be the set of x for which C_x is a continuous function from ω^ω to ω^ω, and for each $x \in \mathcal{F}^{\mathrm{c}}$ let f^{c}_x denote the set C_x. Then $\mathcal{F}^{\mathrm{c}}$ is in $\forall^{\omega^\omega}\exists^{\omega^\omega}\Gamma$, where Γ is the pointclass of closed sets. Using universal closed sets $C_k \subseteq (\omega^\omega)^{k+3}$ ($k \in \omega$), we can in a similar fashion fix for each pair $(n, m) \in \omega \times \omega$ a set $\mathcal{F}^{\mathrm{c},n,m}$ of codes x for all continuous functions $f^{\mathrm{c},n,m}_x\colon (\omega^\omega)^n \to (\omega^\omega)^m$.

2.5.3 Remark. The usual diagonal argument shows that it is not possible to have an association of each $x \in \omega^\omega$ to a continuous function $f^*_x\colon \omega^\omega \to \omega^\omega$ in such a way that the map $(x, y) \mapsto f^*_x(y)$ is continuous on $(\omega^\omega)^2$. To see this, consider the function $g\colon \omega^\omega \to \omega^\omega$ defined by setting $g(x)(n) = f^*_x(x)(n) + 1$ for each $x \in \omega^\omega$, for any such map $x \mapsto f^*_x$.

Theorem 2.5.4 (Solovay)**.** *If* Wadge Determinacy *holds, then there there a function $j\colon \mathcal{P}(\omega^\omega) \to \mathcal{P}(\omega^\omega)$ such that, for all $A \subseteq \omega^\omega$, $A <_{\mathrm{W}} j(A)$.*

Proof. Fix $A \subseteq \omega^\omega$. We define $j(A)$ in two cases, depending on whether $[A]_W$ is selfdual or not. In the nonselfdual case, let $j(A)$ be the set of reals of the form $\langle \pi(i, x(0)), x_1, x_2, \ldots \rangle$ where x is in A if and only if i is even. Then $j(A) >_W A$ as in Remark 2.2.4.

In the selfdual case, let $j(A)$ be the set of $x \in \mathcal{F}^{\mathrm{c}}$ such that $x \notin (f^{\mathrm{c}}_x)^{-1}[A]$. $\square$

2.5.5 Remark. The proof of Theorem 2.5.4 shows that if $A \subseteq \omega^\omega$ and Δ is the smallest selfdual boldface pointclass containing A, then $j(A)$ is in $\forall^{\omega^\omega}\Delta$. This is clear in the case where $[A]_{\mathrm{W}}$ is nonselfdual, as then in fact $[j(A)]_{\mathrm{W}} = \Delta$. In the other case, $x \in j(A)$ if and only if $x \in \mathcal{F}^{\mathrm{c}}$ and for all $y \in \omega^\omega$, if $f^{\mathrm{c}}_x(x) = y$ then $y \notin A$. Replacing $\forall y$ with $\exists y$ shows that if $\undertilde{\Pi}^1_1 \subseteq \Delta$, then $j(A)$ is in $\exists^{\omega^\omega}\Delta$ also

2.5.6 Question. Must j map $\mathcal{W}$ into $\mathcal{W}$?

Recall that we let $\mathcal{W}$ denote the set of $A \subseteq \omega^\omega$ for which $W(A)$ is defined. The main result of this section is the following fact. One direction is proved in Propositions 2.5.8, the other in Theorem 2.5.9.

Theorem 2.5.7 (Solovay). *If* Wadge Determinacy *holds then* Θ *is the supremum of the set of ordinals* γ *for which there exists a* $<_{\mathrm{W}}$*-increasing sequence*

$$\langle A_\alpha : \alpha < \gamma \rangle.$$

If one assumes in addition that $\mathcal{P}(\omega^\omega) = \mathcal{W}$ (equivalently, that $<_{\mathrm{W}}$ is well-founded), then one gets that Θ is the supremum of $\{W(A) : A \subseteq \omega^\omega\}$ (Corollary 2.5.11).

The following proposition shows that if $\langle A_\alpha : \alpha \leq \gamma \rangle$ is a $\leq_W$-increasing sequence, then there is a surjection from ω^ω to $\gamma+1$ defined by mapping $x \in \omega^\omega$ to α if

$$[A_\alpha]_{\mathrm{W}} = [(f^{\mathrm{c}}_x)^{-1}[A_\gamma]]_{\mathrm{W}}$$

(if there exists such an α, and 0 otherwise). It follows that $\gamma < \Theta$. It shows moreover that for each $A \subseteq \omega^\omega$, the set $\{W(B) : B \in \mathcal{W},\ B \leq_W A\}$ is a bounded subset of Θ.

Proposition 2.5.8. *Assume that* Wadge Determinacy *holds, let* A *be a subset of* ω^ω*, and let* Δ *be the smallest selfdual boldface pointclass containing* A *and* $\mathcal{F}^{\mathrm{c}}$*. Let* $\leq$ *be the order on* $\mathcal{F}^{\mathrm{c}}$ *defined by setting* $x \leq y$ *if and only if*

$$(f^{\mathrm{c}}_x)^{-1}[A] \leq_{\mathrm{W}} (f^{\mathrm{c}}_x)^{-1}[A].$$

Then $(\mathcal{F}^{\mathrm{c}}, \leq)$ *is isomorphic to* $(\{B \subseteq \omega^\omega : B \leq_{\mathrm{W}} A\}, \leq_{\mathrm{W}})$*, and* $\leq$ *is in* $\exists^{\omega^\omega}\forall^{\omega^\omega}\Delta$.

Proof. That $(\mathcal{F}^{\mathrm{c}}, \leq)$ is isomorphic to $(\{B \subseteq \omega^\omega : B \leq_{\mathrm{W}} A\}, \leq_{\mathrm{W}})$ is immediate from the definitions. That $\leq$ is in $\forall^{\omega^\omega}\Delta$ follows noting that $x \leq y$ if and only if $\{x, y\} \subseteq \mathcal{F}^{\mathrm{c}}$ and there exists a $u \in \omega^\omega$ such that for all z, w, v, t in ω^ω, if $w = f^{\mathrm{c}}_x(z)$, $v = f^{\mathrm{c}}_u(z)$ and $t = f^{\mathrm{c}}_y(v)$, then $w \in A$ if and only if $t \in A$ (that is, $z \in (f^{\mathrm{c}}_x)^{-1}[A]$ if and only if $f^{\mathrm{c}}_u(z) \in (f^{\mathrm{c}}_y)^{-1}[A]$). □

Theorem 2.5.9 (Solovay). *If* Wadge Determinacy *holds,* γ *is an ordinal and* $g\colon \omega^\omega \to \gamma$ *is a surjection, then there is a* $<_{\mathrm{W}}$*-increasing sequence* $\langle A_\alpha : \alpha < \gamma \rangle$ *definable from* g.

Proof. Let $g\colon \omega^\omega \to \gamma$ be a surjection, for some ordinal γ. Recursively define the sequence $\langle A_\alpha : \alpha < \gamma \rangle$ by setting each A_α to be

$$j(\pi[\{(x, y) \in \omega^\omega \times \omega^\omega \mid g(y) < \alpha \text{ and } x \in A_{g(y)}\}]),$$

where j is as in Theorem 2.5.4. □

2.5.10 Remark. Let Δ be a nonempty $\exists^{\omega^\omega}$-closed selfdual pointclass, let γ be a ordinal such that $\bigcup_\gamma \Delta \subseteq \Delta$ and let $f\colon \omega^\omega \to \gamma$ be a surjection such that $f^{-1}[\{\alpha\}] \in \Delta$ for all $\alpha < \gamma$. The proof of Theorem 2.5.9, along with Remark 2.5.5, shows that Δ contains a set of Wadge rank at least γ.

It follows that if $\mathcal{P}(\omega^\omega) = \mathcal{W}$ then Θ is the supremum of the Wadge ranks of the elements of $\mathcal{P}(\omega^\omega)$.

Corollary 2.5.11 (Solovay)**.** *Suppose that* Wadge Determinacy *holds, and that* $W(A)$ *is defined for every* $A \subseteq \omega^\omega$*. Then* $\Theta = \{W(A) \mid A \subseteq \omega^\omega\}$*.*

Chapter 3

Coding Lemmas

Theorem 3.0.1 is one form of the Moschovakis Coding Lemma. We follow [17]. Given a strategy $\sigma\colon \omega^{<\omega} \to \omega$, and $x \in \omega^\omega$, we let $\sigma \circ x$ (similarly, $x \circ \sigma$) be the sequence of moves played by player I (II) when he plays according to σ and player II (I) plays x.

Theorem 3.0.1 (The Coding Lemma; Moschovakis)**.** *Assume that* AD *holds. Let*

- *X be a subset of ω^ω;*
- *Z be a subset of $X \times \omega^\omega$;*
- *f be a function from X to the ordinals;*
- *$=_f$ be $\{(y,z) \in X^2 : f(y) = f(z)\}$;*
- *$<_f$ be $\{(y,z) \in X^2 : f(y) < f(z)\}$.*

For each $y \in X$, let $[y]_f$ denote $\{z \in X : f(y) = f(z)\}$. Then there exists a pos-$\undertilde{\Sigma}^1_1(=_f, <_f)$ set $A \subseteq Z$ such that for all $y \in X$,

$$A \cap ([y]_f \times \omega^\omega) = \emptyset \text{ if and only if } Z \cap ([y]_f \times \omega^\omega) = \emptyset.$$

Proof. It suffices to consider the case where the range of f is an ordinal α^*, and, proving the theorem by induction, we may assume that it holds for all smaller ordinals. As pos-$\undertilde{\Sigma}^1_1(=_f, <_f)$ is closed under unions and contains all finite subsets of ω^ω, we may assume that α^* is a limit ordinal. Applying Theorem 2.4.6 and Remark 2.4.10, let $\bar{U} = \langle U_n : n \in \omega \setminus \{0\}\rangle$ be a sequence of universal sets for pos-$\undertilde{\Sigma}^1_1(=_f, <_f)$ with the s-m-n property. We seek an $x \in \omega^\omega$ such that

1. $U_{2,x} \subseteq Z$;
2. for all $y \in X$, if $Z \cap ([y]_f \times \omega^\omega) \neq \emptyset$ then $U_{2,x} \cap ([y]_f \times \omega^\omega) \neq \emptyset$.

Let $Y = \{x \in \omega^\omega : U_{2,x} \subseteq Z\}$. For each $x \in \omega^\omega$, let α_x be the least value $f(y)$ for $y \in X$ witnessing the failure of item (2) for x, if there exists such a y; otherwise, let $\alpha_x = \alpha^*$. Consider the game between players I and II where player I builds $x_1 \in \omega^\omega$, player II builds $x_2 \in \omega^\omega$, and I wins if and only if $x_1 \in Y$ and either $x_2 \notin Y$ or $\alpha_{x_1} \geq \alpha_{x_2}$. We will show that a winning strategy for either player gives the desired conclusion.

First, suppose first that σ is a winning strategy for player I. Then for all $x_2 \in \omega^\omega$, $U_{2,\sigma\circ x_2} \subseteq Z$. The set $\bigcup_{x_2 \in \omega^\omega} U_{2,\sigma\circ x_2}$ is in pos-$\undertilde{\Sigma}^1_1(=_f, <_f)$. By our assumption that the theorem holds for all ordinals smaller than α^*, there exist for each $\alpha < \alpha^*$ an $x_2 \in Y$ such that $\alpha_{x_2} \geq \alpha$. It follows then that $\bigcup_{x_2 \in \omega^\omega} U_{2,\sigma\circ x_2}$ is as desired.

Now suppose that τ is a winning strategy for player II. For each $y \in X$, let $[<y]_f$ denote the set of $z \in X$ with $f(z) < f(y)$. The set of $(x, y, z, w) \in (\omega^\omega)^4$ for which $(x, z, w) \in U_2$, $(y, z) \in X^2$ and $z \in [<y]_f$ is in pos-$\undertilde{\Sigma}^1_1(=_f, <_f)$. Fix $a_0 \in \omega^\omega$ such that this set is U_{4,a_0}. Let $s_{4,2} \colon (\omega^\omega)^3 \to \omega^\omega$ be a continuous function witnessing the s-m-n property of $\bar{U}$ for $n = 2$ and $m = 4$, and let $h_0 \colon (\omega^\omega)^2 \to \omega^\omega$ be defined by setting $h_0(x, y) = s_{4,2}(a_0, x, y)$. Then for all $(x, y) \in \omega^\omega \times X$,

$$U_{2,h_0(x,y)} = U_{2,x} \cap ([<y]_f \times \omega^\omega). \tag{3.1}$$

Similarly, the set of $(x, y, z) \in (\omega^\omega)^3$ for which there exists a $w \in [y]_f$ with $(h_0(x, w) \circ \tau, y, z) \in U_2$ and $y \in [w]_f$ is in pos-$\undertilde{\Sigma}^1_1(=_f, <_f)$. Fix $a_1 \in \omega^\omega$ such that this set is U_{3,a_1}. Let $s_{3,1} \colon (\omega^\omega)^2 \to \omega^\omega$ be a continuous function witnessing the s-m-n property of $\bar{U}$ for $n = 1$ and $m = 3$, and let $h_1 \colon \omega^\omega \to \omega^\omega$ be defined by setting $h_1(x) = s_{3,1}(a_1, x)$. Then for all $x \in \omega^\omega$,

$$U_{2,h_1(x)} = \bigcup_{w \in X} (U_{2,h_0(x,w)\circ\tau} \cap ([w]_f \times \omega^\omega)). \tag{3.2}$$

By Theorem 2.4.8, there exists an $x_1 \in \omega^\omega$ such that $U_{2,x_1} = U_{2,h_1(x_1)}$. We show now that x_1 satisfies conditions (1) and (2). For condition (1), suppose towards a contradiction that the condition fails, and consider $(y, z) \in U_{2,x_1} \setminus Z$ with $f(y)$ minimal. Since $U_{2,x_1} = U_{2,h_1(x_1)}$, there exists a $w \in X$ such that

$$(y, z) \in U_{2,h_0(x_1,w)\circ\tau} \cap ([w]_f \times \omega^\omega).$$

Then $f(y) = f(w)$ and, by equation (3.1), for all $(a, b) \in U_{2,h_0(x_1,w)}$, (a, b) is in U_{2,x_1} and $f(a) < f(w)$. By the minimality of $f(y)$, it follows that $(a, b) \in Z$ and therefore that $h_0(x_1, w) \in Y$. Since τ is a winning strategy for II, $h_0(x_1, w)\circ\tau \in Y$, which means that $(y, z) \in Z$, giving a contradiction.

For condition (2), suppose towards a contradiction that $\alpha_{x_1} < \alpha^*$. Let $y \in X$ be such that $f(y) = \alpha_{x_1}$. By equation (3.1), the fact that x_1 is in Y and the definition of α_{x_1}, $h_0(x_1, y) \in Y$, and $\alpha_{h_0(x_1,y)} = \alpha_{x_1}$. Since τ is a winning strategy for II, $\alpha_{h_0(x_1,y)\circ\tau} > \alpha_{h_0(x_1,y)} = \alpha_{x_1}$, which is impossible, as

$$(U_{2,h_0(x_1,y)\circ\tau} \cap ([y]_f \times \omega^\omega)) \subseteq U_{2,x_1},$$

by equation (3.2) and the choice of x_1. This completes the proof. □

As there is a surjection from ω^ω to $\utilde{\Sigma}^1_1$, Theorem 3.0.1 gives the following immediate corollary.

Corollary 3.0.2. *If* AD *holds, then each of the following hold.*

1. *For each $\lambda < \Theta$ there is a surjection from $\mathcal{P}(\omega)$ to $\mathcal{P}(\lambda)$.*
2. *Θ is a limit cardinal.*

We now prove a uniform version Theorem 3.0.1 which is used in the proof of Theorem 5.1.3, which in turn is used to prove that the strong partition cardinals are cofinal below Θ. We fix a sequence $\bar{U} = \langle U_n : n < \omega \rangle$ of universal sets for $\utilde{\Sigma}^1_1$ with the s-m-n property (and functions $s_{m,n}$ $(n < m < \omega)$ witnessing this), and for any $A \subseteq \omega^\omega$, let $\langle U_n(A) : n < \omega \rangle$ be as in Remark 2.4.10.

Theorem 3.0.3 (The Uniform Coding Lemma)**.** *Assume that* AD *holds. Let*

- *X be a subset of ω^ω;*
- *Z be a subset of $X \times \omega^\omega$;*
- *f be a function from X to the ordinals.*

For each $y \in X$, let $[y]_f$ denote $\{z \in X : f(y) = f(z)\}$, let $[< y]_f$ denote $\{z \in X : f(z) < f(y)\}$ and let

$$C_y = \omega^\omega \setminus ([y]_f \cup [< y]_f).$$

Then there exists an $x \in \omega^\omega$ such that for all $y \in X$,

1. *$U_{2,x}([y]_f, C_y) \subseteq Z \cap ([y]_f \times \omega^\omega)$,*
2. *$U_{2,x}([y]_f, C_y) \neq \emptyset$ if and only if $Z \cap ([y]_f \times \omega^\omega) \neq \emptyset$.*

Proof. It suffices to consider the case where the range of f is an ordinal α^*, and, proving the theorem by induction, we may assume that it holds for all smaller ordinals. As $\utilde{\Sigma}^1_1$ is closed under unions and contains all finite subsets of ω^ω, we may assume that α^* is a limit ordinal. Let Y be the set

$$\{x \in \omega^\omega : \forall y \in X \, U_{2,x}([y]_f, C_y) \subseteq Z \cap ([y]_f \times \omega^\omega)\}.$$

For each $x \in \omega^\omega$, let α_x be the least value $f(y)$ for a $y \in X$ witnessing the failure of conclusion (2) of the theorem, with respect to x, if there exists such a y; otherwise, let $\alpha_x = \alpha^*$. Consider the game between players I and II where player I builds $x_1 \in \omega^\omega$, player II builds $x_2 \in \omega^\omega$, and I wins if and only if $x_1 \in Y$ and either $x_2 \notin Y$ or $\alpha_{x_1} \geq \alpha_{x_2}$. We will show that a winning strategy for either player gives the desired conclusion.

First, suppose first that σ is a winning strategy for player I. Let $z_\sigma \in \omega^\omega$ be such that $U_{2,z_\sigma} = \bigcup_{x \in \omega^\omega} U_{2,\sigma \circ x}$. Then for all $P_1, P_2 \subseteq \omega^\omega$ (in particular, in the case $P_1 = [y]_f$, $P_2 = C_y$, for some $y \in X$),

$$U_{2,z_\sigma}(P_1, P_2) = \bigcup_{x \in \omega^\omega} U_{2,\sigma \circ x}(P_1, P_2).$$

Since σ is a winning strategy for I, z_σ is in Y. By our assumption that the theorem holds for all ordinals smaller than α^*, there exist for each $\alpha < \alpha^*$ an $x_2 \in Y$ such that $\alpha_{x_2} \geq \alpha$. It follows then that z_σ is as desired.

Now suppose that τ is a winning strategy for player II. For each $i \in \{1,2\}$, let Q_i be the set of pairs $(y,r) \in \omega^\omega \times \omega^\omega$ such that

$$(r)_0 = \langle \pi(i, y(0)), y(1), y(2), \ldots \rangle$$

(this corresponds to how we defined the class pos-$\underset{\sim}{\Sigma}^1_1(A_1, A_2)$ in terms of our definition of pos-$\underset{\sim}{\Sigma}^1_1(A)$). Then for any $P, P' \subseteq \omega^\omega$ and $z \in \omega^\omega$,

- $z \in P$ if and only if $(z,r) \in Q_1$ holds for some $r \in P \oplus P'$.
- $z \in P'$ if and only if $(z,r) \in Q_2$ holds for some $r \in P \oplus P'$.

For each $r \in \omega^\omega$, let $r^- \in \omega^\omega$ be such that $(r^{-1})_n = (r)_{n+1}$ for all $n \in \omega$.

Let $a_0 \in \omega^\omega$ be such that U_{5,a_0} is the set of $(x,y,z,w,r) \in (\omega^\omega)^5$ for which $(x,z,w,r^-) \in U_3$ and $(y,r) \in Q_2$. Let $s_{5,2} \colon (\omega^\omega)^3 \to \omega^\omega$ be a continuous function witnessing the s-m-n property of $\bar{U}$ for $n = 2$ and $m = 5$, and let $h_0 \colon (\omega^\omega)^2 \to \omega^\omega$ be defined by setting $h_0(x,y) = s_{5,2}(a_0,x,y)$. Then for all $(x,y) \in \omega^\omega \times \omega^\omega$, all $P, P' \subseteq \omega^\omega$ and all $(z,w) \in \omega^\omega \times \omega^\omega$,

$$(z,w) \in U_{2,h_0(x,y)}(P,P')$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (h_0(x,y), z, w, r) \in U_3)$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (a_0, x, y, z, w, r) \in U_5)$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (x,z,w,r^-) \in U_3 \wedge (y,r) \in Q_2)$$

if and only if

$$(z,w) \in U_{2,x}(P,P') \wedge y \in P'.$$

That is, we have the following fact (**) : for all $(x,y) \in \omega^\omega \times \omega^\omega$, and all $P, P' \subseteq \omega^\omega$ (in particular, sets of the form C_y), $U_{2,h_0(x,y)}(P,P')$ is $U_{2,x}(P,P')$ if y is in P' and $\emptyset$ otherwise.

Now let $a_1 \in \omega^\omega$ be such that U_{4,a_1} is the set of $(x,z,w,r) \in (\omega^\omega)^4$ such that, for some $y \in \omega^\omega$, $(z,w,r^-) \in U_{3,h_0(x,y)\circ\tau}$ and $(y,r) \in Q_1$ holds. Let $s_{4,1} \colon (\omega^\omega)^2 \to \omega^\omega$ be a continuous function witnessing the s-m-n property of $\bar{U}$ for $n = 1$ and $m = 3$, and let $h_1 \colon \omega^\omega \to \omega^\omega$ be defined by setting $h_1(x) = s_{4,1}(a_1, x)$.

Then for all $x \in \omega^\omega$, all $P, P' \subseteq \omega^\omega$ and all $(z,w) \in \omega^\omega \times \omega^\omega$,

$$(z,w) \in U_{2,h_1(x)}(P,P')$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (h_1(x), z, w, r) \in U_3)$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (a_1, x, z, w, r) \in U_4)$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge \exists y \in \omega^\omega((z, w, r^-) \in U_{3,h_0(x,y)\circ\tau} \wedge (y, r) \in Q_1))$$

if and only if

$$\exists y \in P\, (z, w) \in U_{2,h_0(x,y)\circ\tau}(P, P').$$

Then for all $x \in \omega^\omega$ and all $P, P' \subseteq \omega^\omega$,

$$U_{2,h_1(x)}(P, P') = \bigcup_{y \in P} U_{2,h_0(x,y)\circ\tau}(P, P'). \tag{3.3}$$

By Theorem 2.4.8 and Remark 2.4.10, there exists an $x_1 \in \omega^\omega$ such that $U_{2,x_1}(P, P') = U_{2,h_1(x_1)}(P, P')$ for all $P, P' \subseteq \omega^\omega$. We show now that x_1 satisfies both conditions in the conclusion of the theorem. For condition (1) (i.e., the assertion that $x_1 \in Y$), fix $y^* \in X$ with

$$U_{2,x_1}([y^*]_f, C_{y^*}) \not\subseteq Z \cap ([y^*]_f \times \omega^\omega)$$

and $f(y^*)$ minimal. Since

$$U_{2,x_1}([y^*]_f, C_{y^*}) = U_{2,h_1(x_1)}([y^*]_f, C_{y^*}),$$

there exists a $y' \in [y^*]_f$ such that

$$U_{2,h_0(x_1,y')\circ\tau}([y^*]_f, C_{y^*}) \not\subseteq Z \cap ([y^*]_f \times \omega^\omega).$$

We want to see that $h_0(x_1, y') \in Y$, since then, as τ is a winning strategy for II, we will have that $h_0(x_1, y') \circ \tau \in Y$, giving a contradiction. For each $y \in X$, we have by (**) that

$$U_{2,h_0(x_1,y')}([y^*]_f, C_y) = U_{2,x}([y^*]_f, C_y) \subseteq Z \cap ([y]_f \times \omega^\omega)$$

if $f(y) < f(y')$, by the minimality of $f(y^*)$, and $U_{2,h_0(x_1,y')}([y^*]_f, C_y) = \emptyset$ otherwise. In either case, $U_{2,h_0(x_1,y')}([y^*]_f, C_y) \subseteq Z \cap ([y]_f \times \omega^\omega)$, which shows that $h_0(x_1, y') \in Y$ as desired.

For condition (2), suppose toward a contradiction that $\alpha_{x_1} < \alpha^*$. Let $y^* \in X$ be such that $f(y^*) = \alpha_{x_1}$. By (**), the fact that x_1 is in Y and the definition of α_{x_1}, $h_0(x_1, y^*) \in Y$, and $\alpha_{h_0(x_1,y^*)} = \alpha_{x_1}$. Since τ is a winning strategy for II, $\alpha_{h_0(x_1,y^*)\circ\tau} > \alpha_{h_0(x_1,y^*)} = \alpha_{x_1}$, which is impossible, as

$$(U_{2,h_0(x_1,y^*)\circ\tau}([y^*]_f, C_{y^*}) \cap ([y^*]_Q \times \omega^\omega)) \subseteq U_{2,x_1}([y^*]_f, C_{y^*}),$$

by equation (3.3) and the choice of x_1. This completes the proof.

□

Chapter 4

Properties of pointclasses

4.1 Separation and reduction

Given a pointclass Δ, and disjoint subsets A and B of the same space in $\mathcal{X}$, we say that A and B are Δ-separable if there exists a set $C \in \Delta \cap \check{\Delta}$ such that $A \subseteq C$ and $B \cap C = \emptyset$. We generally use this property only with selfdual pointclasses Δ. We say that a pointclass Γ satisfies the *separation property* if all pairs of disjoint subsets of ω^ω in Γ are $(\Gamma \cap \check{\Gamma})$-separable. We write $\mathsf{Sep}(\Gamma)$ to indicate that Γ has the separation property.

We follow [28]. Let us say that a function $f\colon \omega^\omega \to \omega^\omega$ is *strongly Lipschitz* if for all $x, y \in \omega^\omega$ and $n \in \omega$, if $x \restriction n = y \restriction n$, then $f(x)(n) = f(y)(n)$. Strategies for Player I induce strongly Lipschitz functions.

Lemma 4.1.1 (Steel). *Assume that* AD *holds, and let* Γ *be a boldface pointclass. Let* (A_0, A_1) *be an inseparable pair of* Γ *sets. Let* B_0 *and* B_1 *be disjoint sets, both in* Γ*, or both in* $\check{\Gamma}$*. Then there is a strongly Lipschitz map* f *so that* $f[B_0] \subseteq A_0$ *and* $f[B_1] \subseteq A_1$.

Proof. Consider the game where I plays x, II plays y, and I wins if $y \in B_0$ implies $x \in A_0$, and $y \in B_1$ implies $x \in A_1$. It suffices to see that II cannot have a winning strategy. Supposing that σ were a winning strategy for II, let g be the function $x \mapsto x \circ \sigma$. Then g is continuous, and $g^{-1}[B_0]$ is in Δ, since the range of g is contained in $B_0 \cup B_1$. Furthermore, $g^{-1}[B_0]$ separates A_0 and A_1. □

Theorem 4.1.2 (Steel). *Assume that* AD *holds. If* Γ *is a nonselfdual boldface pointclass, then at least one of* Γ *and* $\check{\Gamma}$ *satisfies the separation property.*[1]

Proof. Suppose toward a contradiction that the pairs (A_0, A_1) and (C_0, C_1) form counterexamples to $\mathsf{Sep}(\Gamma)$ and $\mathsf{Sep}(\check{\Gamma})$, respectively. By Lemma 4.1.1, there is a strongly Lipschitz function f mapping A_0 into C_0 and A_1 into C_1. Let $B_0 = f^{-1}[C_0]$ and let $B_1 = f^{-1}[C_1]$. Then B_0 and B_1 are disjoint and both

[1] Prove at most, too? This is apparently in the Van Wesep paper. What are we assuming?

in $\check{\Gamma}$, and $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$. Applying Lemma 4.1.1 again, there exist strongly Lipschitz functions f_0, f_1 and f_2 such that

- $f_0[A_0] \subseteq A_1$;
- $f_0[A_1] \subseteq A_0$;
- $f_1[A_0] \subseteq A_0$;
- $f_1[\omega^\omega \setminus B_0] \subseteq A_1$;
- $f_2[A_1] \subseteq A_1$;
- $f_2[\omega^\omega \setminus B_1] \subseteq A_0$.

Since f_0, f_1 and f_2 are all strongly Lipschitz, for each $r \in 3^\omega$ there is a unique sequence $\langle x^r_n : n < \omega\rangle$ of elements of ω^ω such that $x^r_n = f_{r(n)}(x^r_{n+1})$ for all $n \in \omega$. Let 3^ω have the topology induced by the Baire topology on ω^ω. By $\mathsf{Baire}(\mathcal{P}(\omega^\omega))$, every subset of 3^ω has the property of Baire.

Now suppose that $\{r \in 3^\omega : x^r_0 \notin A_0 \cup A_1\}$ is nonmeager. By the Baire property, we have an $s \in 3^{<\omega}$ such that $\{r \in 3^\omega : x^{s^\frown r}_0 \notin A_0 \cup A_1\}$ is comeager. However, for each $s \in 3^{<\omega}$ and $r \in 3^\omega$, if $x^r_0 \in A_0 \cup A_1$ then $x^{s^\frown r}_0 \in A_0 \cup A_1$. It follows then that $x^r_0 \notin A_0 \cup A_1$, for comeagerly many $r \in 3^\omega$. Since $B_0 \cap B_1 = \emptyset$, we may fix an $i^* \in \{0,1\}$ such that $x^r_0 \notin B_{i^*}$, for nonmeagerly many $r \in 3^\omega$. Inspecting f_0, f_1 and f_2, we see that for each $i \in \{0,1\}$ and each $r \in 3^\omega$, if $x^r_0 \notin B_i$ then $x^{\langle i+1\rangle^\frown r}_0 \in A_{1-i}$. It follows then that $x^r_0 \in A_0 \cup A_1$ for nonmeagerly many $r \in 3^\omega$, which gives a contradiction.

Suppose on the other hand that $\{r \in 3^\omega : x^r_0 \in A_0 \cup A_1\}$ is comeager. Choose $i^* \in \{0,1\}$ so that $\{r \in 3^\omega : x^r_0 \in A_{i^*}\}$ is nonmeager. Choose $s \in 3^{<\omega}$ so that $\{r \in 3^\omega : x^{s^\frown r}_0 \in A_{i^*}\}$ is comeager. Then for $r \in 3^\omega$ such that $x^r_0 \in A_{i^*}$, $x^{s^\frown\langle 0\rangle^\frown r}_0 \in A_{1-i^*}$. So $x^{s^\frown r}_0 \in A_{1-i^*}$ for nonmeagerly many $r \in 3^\omega$, giving another contradiction. □

A pointclass Γ is said to have the *reduction property* if for all $A, B \subseteq \omega^\omega$ in Γ, there exist disjoint $A', B' \in \Gamma$ such that $A' \subseteq A$, $B' \subseteq B$ and $A' \cup B' = A \cup B$. We write $\mathsf{Red}(\Gamma)$ to indicate that the pointclass Γ has the reduction property. It follows easily from the definitions that $\mathsf{Red}(\Gamma)$ implies $\mathsf{Sep}(\check{\Gamma})$.

4.1.3 Remark. Let A, B, C and D be subsets of ω^ω such that

$$D \cap (A \setminus B) = C \cap (B \setminus A) = C \cap D = \emptyset$$

and $A \cup B \subseteq C \cup D$. Then $A \setminus D$ and $B \setminus C$ satisfy the conclusion of the reduction property with respect to A and B.

Theorem 4.1.4. *Assume that* Wadge Determinacy + $\mathsf{Baire}(\mathcal{P}(\omega^\omega))$ *holds. If* Γ *is a nonselfdual boldface pointclass and* $\mathsf{Red}(\Gamma)$ *holds, then* $\mathsf{Sep}(\Gamma)$ *does not hold.*

Proof. By Theorem 2.4.3 there exist a Γ-universal set $U \subseteq (\omega^\omega)^2$. Let $\pi\colon \omega^\omega \to (\omega^\omega)^2$ be a continuous bijection, and let π_0, π_1 be functions on ω^ω such that $\pi(x) = (\pi_0(x), \pi_1(x))$ for all $x \in \omega^\omega$.[2] Let A be the set of $x \in \omega^\omega$ such that

[2] We're following page 204 of Moschovakis.

$(\pi_0(x), x) \in U$ and let B be the set of $x \in \omega^\omega$ such that $(\pi_1(x), x) \in U$. Now suppose that A' and B' in Γ witness the reduction property for A and B (so $A' \subseteq A$, $B' \subseteq B$, $A' \cap B' = \emptyset$ and $A' \cup B' = A \cup B$). Finally, suppose that $C \in \Gamma \cap \check{\Gamma}$ witnesses the separation property for A' and B' (i.e., C contains A' and is disjoint from B'). Let $x \in \omega^\omega$ be such that $U_{\pi_0(x)} = \omega^\omega \setminus C$ and $U_{\pi_1(x)} = C$. If $x \in C$, then $x \in B \setminus A$, so $x \in B'$, giving a contradiction. Similarly, if $x \notin C$, then $x \in A \setminus B$, so $x \in A'$, giving another contradiction. □

4.1.5 Remark. Theorems 4.1.2 and 4.1.4 together imply, assuming AD, that if Γ is a nonselfdual boldface pointclass, then at most one of Γ and $\check{\Gamma}$ has the reduction property. Theorem 4.1.6, which is Theorem 5.5 of [29],[3] shows that if in addition Γ is closed under finite unions and intersections, then the separation property holds for exactly one of Γ and $\check{\Gamma}$, and the reduction property holds for the other.

Theorem 4.1.6 (Van Wesep)**.** *If Γ is a boldface pointclass closed under finite intersections, and $\mathsf{Sep}(\check{\Gamma})$ holds, then $\mathsf{Red}(\Gamma)$ holds.*

The proof of Theorem 4.1.6 uses the following lemma.

Lemma 4.1.7. *Suppose that AD holds, and that Γ is a boldface pointclass closed under finite intersections. If $\mathsf{Red}(\Gamma)$ does not hold, then for all A, B in $\mathcal{P}(\omega^\omega) \cap \check{\Gamma}$ there exist $C, D \in \Gamma$ such that*

- $A \setminus B \subseteq C \setminus D$;
- $B \setminus A \subseteq D \setminus C$;
- $C \cup D = \omega^\omega$.

Proof. Let (E, F) be subsets of ω^ω witnessing $\neg\mathsf{Red}(\Gamma)$, and fix A, B in $\mathcal{P}(\omega^\omega) \cap \check{\Gamma}$. Consider the game $\mathcal{G}(E, F, A, B)$ in which player I produces $x \in \omega^\omega$, player II produces $y \in \omega^\omega$ and player II wins if and only if the following conditions are met:

- $y \in E \cup F$;
- $x \in A \setminus B \Rightarrow y \in E \setminus F$;
- $x \in B \setminus A \Rightarrow y \in F \setminus E$.

Suppose toward a contradiction that σ is a winning strategy for player I. Let $E'' = \{y \in \omega^\omega : \sigma \circ y \notin A\}$ and let $F'' = \{y \in \omega^\omega : \sigma \circ y \notin B\}$. Then we have the following:

- $E \cup F \subseteq E'' \cup F''$;
- $(E \cup F) \cap (E'' \cap F'') = \emptyset$;
- $E \setminus F \subseteq E''$;

[3] At least in the new version

- $F \setminus E \subseteq F''$;
- $E'', F'' \in \Gamma$.

Let $E' = E \cap E''$ and let $F' = F \cap F''$. Then E' and F' are disjoint sets in Γ, $E \cup F = E' \cup F'$, $E \setminus F \subseteq E'$ and $F \setminus E \subseteq F'$. This contradicts the assumption that E and F witness the failure of $\mathsf{Red}(\Gamma)$.

It follows then that there is a winning strategy τ for player II. Let $C = \{x \in \omega^\omega : x \circ \tau \in E\}$ and let $D = \{x \in \omega^\omega : x \circ \tau \in F\}$. Then C and D are as desired. □

Proof of Theorem 4.1.6. Supposing that $\mathsf{Red}(\Gamma)$ fails, we show that $\mathsf{Red}(\check{\Gamma})$ holds, contradicting Theorem 4.1.4. Let A and B be subsets of ω^ω in Γ. Let C and D be as given by Lemma 4.1.7. Applying $\mathsf{Sep}(\check{\Gamma})$, let $E \in \Gamma \cap \check{\Gamma}$ be such that $\omega^\omega \setminus C \subseteq E$ and $E \cap (\omega^\omega \setminus D) = \emptyset$. Let $C' = C \setminus E$ and let $D' = D \cap E$. Then

$$D' \cap (A \setminus B) = C' \cap (B \setminus A) = C' \cap D' = \emptyset$$

and $A \cup B \subseteq C' \cup D'$, so $A \setminus D'$ and $B \setminus C'$ satisfy the conclusion of the reduction property with respect to A and B, as in Remark 4.1.3. □

The following result is sometimes called the 0th Periodicity Theorem. It appears in.[4]. Recall that $x \leq_{\mathrm{T}} y$ means .... and that $x \equiv_{\mathrm{T}} y$ means[5]

Theorem 4.1.8 (Kechris)**.** *Assume that* AD *holds.*[6] *Suppose that* Γ *is a boldface pointclass closed under countable intersections and countable unions, and that* $\mathsf{Red}(\Gamma)$ *holds.*

- *If* $\exists^{\omega^\omega}\Gamma \subseteq \Gamma$*, then* $\mathsf{Red}(\forall^{\omega^\omega}\Gamma)$ *holds.*
- *If* $\forall^{\omega^\omega}\Gamma \subseteq \Gamma$*, then* $\mathsf{Red}(\exists^{\omega^\omega}\Gamma)$ *holds.*[7]

Proof. Let A and B be subsets of $(\omega^\omega)^2$ in Γ. For the first part we will find a reduction for $\{x : \forall y \in \omega^\omega A(x,y)\}$ and $\{x : \forall y \in \omega^\omega B(x,y)\}$. Let

$$A' = \{(x,z) \in (\omega^\omega)^2 : \forall y \leq_{\mathrm{T}} z\, A(x,y)\}$$

and let

$$B' = \{(x,z) \in (\omega^\omega)^2 : \forall y \leq_{\mathrm{T}} z\, B(x,y)\}.$$

Since Γ is closed under countable intersections, A' and B' are in Γ, so there exist C and D in Γ reducing them. Let

$$C' = \{(x,z) \in (\omega^\omega)^2 : \exists z' \equiv_{\mathrm{T}} z\, C(x,z')\}$$

[4] Kechris 1977B

[5] ....

[6] and more?

[7] We may need a lot of basic stuff about the reduction property, like its relation to PWO, and whether selfdual classes can satisfy it.

and let

$$D' = \{(x,z) \in (\omega^\omega)^2 : \forall z' \equiv_{\mathrm{T}} z\, D(x,z')\}.$$

Then C' and D' are in Γ, since Γ is closed under countable intersections and countable unions, and they reduce A' and B'. Let $\mathcal{G}_A(x)$ (for $x \in \omega^\omega$) be the game where players I and II collaborate to build $z \in \omega^\omega$, where I wins if and only if $(x,z) \in C'$, and let E be the set of x for which I has a winning strategy. Let $\mathcal{G}_B(x)$ (for $x \in \omega^\omega$) be the game where players I and II collaborate to build $z \in \omega^\omega$, where I wins if and only if $(x,z) \in D'$, and let F be the set of x for which I has a winning strategy. Then E and F are in $\forall^{\omega^\omega}\Gamma$ (via the formulation that player II does not have a winning strategy, and the assumption that $\exists^{\omega^\omega}\Gamma \subseteq \Gamma$), and they reduce $\{x : \forall y \in \omega^\omega A(x,y)\}$ and $\{x : \forall y \in \omega^\omega B(x,y)\}$. To see this, fix $x \in \omega^\omega$ and suppose that at least one of $\forall y \in \omega^\omega A(x,y)$ and $\forall y \in \omega^\omega B(z,y)$ holds. In the former case, $\forall z \in \omega^\omega A'(x,z)$ holds, and in the latter case $\forall z \in \omega^\omega B'(x,z)$ holds. Then for each $z \in \omega^\omega$, (x,z) is either in C' or D', and one of the sets $\{z \in \omega^\omega : (x,z) \in C'\}$ and $\{z \in \omega^\omega : (x,z) \in D'\}$ contains a Turing cone. Then player I has a winning strategy in one of the games $\mathcal{G}_A(x)$ and $\mathcal{G}_B(x)$. We leave it to the reader to check the conditions from the conclusion of the reduction property.

For the second part we find a reduction for $\{x : \exists y \in \omega^\omega A(x,y)\}$ and $\{x : \exists y \in \omega^\omega B(x,y)\}$. Let

$$A' = \{(x,z) \in (\omega^\omega)^2 : \exists y \leq_{\mathrm{T}} z\, A(x,y)\}$$

and let

$$B' = \{(x,z) \in (\omega^\omega)^2 : \exists y \leq_{\mathrm{T}} z\, B(x,y)\}.$$

Since Γ is closed under countable unions, A' and B' are in Γ, so there exist C and D in Γ reducing them. Let

$$C' = \{(x,z) \in (\omega^\omega)^2 : \exists z' \equiv_{\mathrm{T}} z\, C(x,z')\}$$

and let

$$D' = \{(x,z) \in (\omega^\omega)^2 : \forall z' \equiv_{\mathrm{T}} z\, D(x,z')\}.$$

Then C' and D' are in Γ, since Γ is closed under countable intersections and countable unions, and they reduce A' and B'. Let $\mathcal{G}_A(x)$ (for $x \in \omega^\omega$) be the game where players I and II collaborate to build $z \in \omega^\omega$, where I wins if and only if $(x,z) \in C'$, and let E be the set of x for which I has a winning strategy. Let $\mathcal{G}_B(x)$ (for $x \in \omega^\omega$) be the game where players I and II collaborate to build $z \in \omega^\omega$, where I wins if and only if $(x,z) \in D'$, and let F be the set of x for which I has a winning strategy. Then E and F are in $\exists^{\omega^\omega}\Gamma$ (via the formulation that player I has a winning strategy, and the assumption that $\forall^{\omega^\omega}\Gamma \subseteq \Gamma$), and they reduce $\{x : \forall y \in \omega^\omega A(x,y)\}$ and $\{x : \forall y \in \omega^\omega B(x,y)\}$. To see this, fix $x \in \omega^\omega$ and suppose that at least one of $\exists y \in \omega^\omega A(x,y)$ and $\exists y \in \omega^\omega B(z,y)$ holds. Then again one of the sets $\{z \in \omega^\omega : (x,z) \in C'\}$ and $\{z \in \omega^\omega : (x,z) \in D'\}$ contains a Turing cone. Then player I has a winning strategy in one of the games $\mathcal{G}_A(x)$ and $\mathcal{G}_B(x)$. We again leave it to the reader to check the remaining details. □

I'm not sure we're using any of what follows.....

4.1.9 Remark. Theorems 5.2 and 5.3 of [29] show that for any nonselfdual pointclass Γ, at most one of Γ and $\check{\Gamma}$ have the separation property.[8]

Theorem 1.1(b) of [27].

Theorem 4.1.10 (Kechris)**.** *Assume that* Wadge Determinacy + Baire($\mathcal{P}(\omega^\omega)$) *holds. Suppose that* Γ *is a nonselfdual boldface pointclass, and that* $\mathsf{Sep}(\Gamma)$ *holds. Let* $\Delta = \Gamma \cap \check{\Gamma}$*. If* Δ *is closed under* $\exists^{\omega^\omega}$*, then so is* Γ*.*

Proof. The pointclass $\exists^{\omega^\omega}\Gamma$ is a boldface pointclass containing Γ. If it is not contained in Γ, then by Remark 2.2.4 and Theorem 2.2.6 it contains a selfdual boldface pointclass properly containing Γ, and in particular there exists a set in $\mathcal{P}(\omega^\omega) \cap (\exists^{\omega^\omega}\Gamma \setminus \Gamma)$ whose complement is also in $\exists^{\omega^\omega}\Gamma \setminus \Gamma$. It suffices then to show that if A and B are disjoint subsets of ω^ω in $\exists^{\omega^\omega}\Gamma$, then there is a set C in Δ which contains A and is disjoint from B.

Letting A and B be as given, fix P and Q in Γ such that

$$A = \{x : \exists y \in \omega^\omega P(x, y)\}$$

and

$$B = \{x : \exists y \in \omega^\omega Q(x, y)\}.$$

Let $P' = \{(x, y, z) \in (\omega^\omega)^3 : P(x, y)\}$ and $Q' = \{(x, y, z) \in (\omega^\omega)^3 : Q(x, z)\}$. Then P' and Q' are in Γ, and they are disjoint since A and B are. Applying $\mathsf{Sep}(\Gamma)$, there exists a $D \in \Delta$ containing P' and disjoint from Q'. Let

$$C = \{x : \exists y \in \omega^\omega \forall z \in \omega^\omega D(x, y, z)\}.$$

Since $\exists^{\omega^\omega}\Delta \subseteq \Delta$ and Δ is selfdual, C is in Δ. Finally, $A \subseteq C$ and $C \cap B = \emptyset$. □

4.2 The prewellordering property

Given a subset P of a space in $\mathcal{X}$, a *norm* on P is a function from P to the ordinals. A norm is *regular* if its range is an ordinal. We write $|\phi|$ for the range of a regular norm ϕ.

A *prewellordering* on a set X is a binary relation $\leq$ on which is reflective, transitive, total (so for all $x, y \in X$, at least one of $x \leq y$ and $y \leq x$ holds) and wellfounded. Equivalently, a prewellordering on a set P is a set of the form $\{(x, y) \in P : f(x) \leq f(y)\}$, where f is a norm on P. Given a pointclass Γ, we let $\delta(\Gamma)$ be the supremum of the lengths of the prewellorderings in $\Gamma \cap \check{\Gamma}$.

Given a pointclass Γ, a space X in $\mathcal{X}$ and a set $P \subseteq X$, a Γ-*norm* on P is norm ϕ on P for which each of the following sets are in Γ :

- $\leq^*_\phi$, the set of pairs $(x, y) \in X \times X$ such that $x \in P$ and either $y \notin P$ or $\phi(x) \leq \phi(y)$;

[8] Maybe we want to prove this after all?

- $<^*_\phi$, the set of pairs $(x,y) \in X \times X$ such that $x \in P$ and either $y \notin P$ or $\phi(x) < \phi(y)$.

4.2.1 Remark. Suppose that Γ is a boldface pointclass, and ϕ is a Γ-norm on a subset P of a space X in $\mathcal{X}$. Let Y be a space in $\mathcal{X}$ and let $f\colon Y \to X$ be a continuous function. Then $\phi \circ f$ is a Γ-norm on $f^{-1}[P]$. In particular, if P is Γ-complete, and P has a Γ-norm, then every member of Γ has a Γ-norm.

4.2.2 Remark. Let Γ be a pointclass, let P be an element of Γ contained in a space X in $\mathcal{X}$. Let ϕ be a Γ-norm on P. For points x, y in X, the negation of $x \leq^*_\phi y$ is equivalent to the statement

$$x \notin P \vee (y \in P \wedge \phi(y) < \phi(x))$$

and the negation of $x <^*_\phi y$ is equivalent to the statement

$$x \notin P \vee (y \in P \wedge \phi(y) \leq \phi(x)).$$

4.2.3 Remark. It follows from Remark 4.2.2 that if Γ is a boldface pointclass, ϕ is a Γ norm on a set P and $z \in P$, then $\{(x,y) : \phi(x) \leq \phi(y) < \phi(z)\}$ is a prewellordering in $\Gamma \cap \check{\Gamma}$.

Lemma 4.2.4. *Suppose that Γ is boldface pointclass closed under unions and universal quantification over ω^ω, and that ϕ is a Γ-norm on a set $P \in \Gamma \setminus \check{\Gamma}$. Let Q be a subset of P in $\check{\Gamma}$. Then there exists a $y \in P$ such that $\phi(x) < \phi(y)$ for all $x \in Q$.*

Proof. Supposing otherwise, $P \in \check{\Gamma}$, as $y \in P$ if and only if there exists an $x \in Q$ such that $x <^*_\phi y$ fails. □

A pointclass Γ has the *prewellordering property* if every member of Γ has a Γ-norm. We write $\mathsf{PWO}(\Gamma)$ to mean that Γ has the prewellordering property.

Theorem 4.2.5. *If Γ is a boldface pointclass closed under unions and satisfying the prewellordering property, then $\mathsf{Red}(\Gamma)$ holds.*

Proof. Let A and B be subsets of ω^ω in Γ. For each $i \in 2$, let $f_i\colon \omega^\omega \to \omega^\omega$ be such that, for each $x \in \omega^\omega$, $f_i(x)(0) = i$ and $f_i(x)(n+1) = x(n)$ for all $n \in \omega$. Let ϕ be a Γ-norm on $f_0[A] \cup f_1[B]$. Let $A' = \{x \in \omega^\omega : f_0(x) \leq^*_\phi f_1(x)\}$ and let $B' = \{x \in \omega^\omega : f_1(x) <^*_\phi f_0(x)\}$. Then A' and B' reduce A and B. □

4.2.6 Remark. The pointclass $\underset{\sim}{\Pi}^1_1$ has the prewellordering property (see 4B.1 of [23]). Adapting an argument of Novikov, Moschovakis showed that if Γ is a boldface pointclass closed under $\forall^{\omega^\omega}$ and $\mathsf{PWO}(\Gamma)$ holds, then $\mathsf{PWO}(\exists^{\omega^\omega}\Gamma)$ also holds. The First Periodicity Theorem (due to Martin and Moschovakis) says that, assuming AD, if Γ is a boldface pointclass closed under $\exists^{\omega^\omega}$ and $\mathsf{PWO}(\Gamma)$ holds, then $\mathsf{PWO}(\forall^{\omega^\omega}\Gamma)$ also holds. See 4B.1 and 6B.3 of [23].[9]

[9] Get the reference right. These references are from the new version of the book.

4.2.7 Remark. Suppose that Γ is a boldface pointclass, and P is an element of $\Gamma \setminus \check{\Gamma}$, and ϕ is a Γ-norm on P with range γ. Then ϕ gives rise to a γ-sequence of sets in $\Gamma \cap \check{\Gamma}$ whose union is P. In particular, $\check{\Gamma}$ is not closed under γ-unions.

Given a set $A \subseteq \omega^\omega$, a $\utilde{\Sigma}^2_1(A)$ set is a set of the form

$$\{(x_1, \ldots, x_n) \in (\omega^\omega)^n : \exists B \subseteq \omega^\omega\ \phi(A, B, x_1, \ldots, x_n, y)\},$$

where the quantifiers in ϕ range over the hereditarily countable sets, and y is an element of ω^ω. If $\phi(A, B, x_1, \ldots, x_n, y)$ holds for suitable A, B $x_1, \ldots, x_n$ and y, we say that the formula $\exists B \subseteq \omega^\omega\ \phi(A, B, x_1, \ldots, x_n, y)$ is *witnessed* by B. The pointclass $\utilde{\Sigma}^2_1(A)$ is the smallest boldface pointclass containing each $\utilde{\Sigma}^2_1(A)$ subset of ω^ω.

4.2.8 Remark. A set $C \subseteq \omega^\omega$ is $\utilde{\Sigma}^2_1(A)$ if, for some first order formula ϕ and some $y \in \omega^\omega$, C is the set of $x \in \omega^\omega$ for which there is an ω-structure M containing $\omega^\omega \cup \{A\}$ such that $(M, E) \models \phi(A, y, x)$, where E is the $\in$-relation of M. Modifying this formulation one can define a universal $\utilde{\Sigma}^2_1(A)$ set from the set of $(n, x, y) \in \omega \times \omega^\omega \times \omega^\omega$ for which there exists an ω-structure M containing $\omega^\omega \cup \{A\}$ and a set T (naturally coded by a set of reals) such that T is the theory of M (in parameters from M, using the Gödel numbering for first order formulas) and n is the Gödel number of a formula ϕ such that $M \models \phi(A, x, y)$. It follows $\utilde{\Sigma}^2_1(A)$ is nonselfdual. One can also verify in this way that $\utilde{\Sigma}^2_1(A)$ is closed under $\exists^{\omega^\omega}$ and $\wedge$.

We write $\Pi^2_1(A)$ for $\check{\utilde{\Sigma}}^2_1(A)$ (the class of complements of members of $\utilde{\Sigma}^2_1(A)$), and $\utilde{\Delta}^2_1$ for $\utilde{\Sigma}^2_1(A) \cap \utilde{\Pi}^2_1(A)$. We write $\utilde{\delta}^2_1(A)$ for $\delta(\utilde{\Sigma}^2_1(A))$. Proposition 2.5.8 shows that for each $A \in \mathcal{W}$, $\utilde{\delta}^2_1(A)$ is at least the Wadge rank of A.

4.2.9 Remark. Assuming that $\mathcal{W} = \mathcal{P}(\omega^\omega)$, for any set $A \subseteq \omega^\omega$ the pointclass $\Sigma^2_1(A)$ has the prewellordering property. To see this, fix a $\Sigma^2_1(A)$ set X of the form

$$\{x \in \omega^\omega : \exists B \subseteq \omega^\omega\ \psi(A, B, x)\}.$$

Orders $\leq^*$ and $<^*$ witnessing that X has the prewellordering property can be defined using the Wadge rank of the witness B. That is, define $x \leq^* y$ to hold if there exists a $B \subseteq \omega^\omega$ such that $\psi(A, B, x)$ holds, and for every continuous function $f \colon \omega^\omega \to \omega^\omega$ such that $f^{-1}[B] <_{\mathrm{W}} A$, $\psi(A, f^{-1}[B], y)$ fails. Similarly, define $x <^* y$ to hold if there exists a $B \subseteq \omega^\omega$ such that $\psi(A, B, x)$ holds, and for every continuous function $f \colon \omega^\omega \to \omega^\omega$, $\psi(A, f^{-1}[B], y)$ fails.

Other examples of pointclasses with the prewellordering property are given by the following theorem.

Theorem 4.2.10 (Martin)**.** *Suppose that* Wadge Determinacy + Baire($\mathcal{P}(\omega^\omega)$) *holds. Let Δ be a selfdual bolface pointclass closed under unions but not wellordered unions, and let ρ be the least ordinal γ such that $\bigcup_\gamma \Delta \not\subseteq \Delta$. Then $\bigcup_\rho \Delta$ has the prewellordering property.*

Proof. Since Δ is closed under unions, ρ is a limit ordinal. Fix $F \in \bigcup_\rho \Delta \setminus \Delta$. By the minimality of ρ, there exists a function $f \colon \rho \to \Delta$ such that

- for all $\alpha < \beta < \rho$, $f(\alpha) \subseteq f(\beta)$;
- for all limit ordinals $\beta < \rho$, $f(\beta) = \bigcup_{\alpha<\beta} f(\alpha)$;
- $F = \bigcup_{\alpha<\rho} f(\alpha)$.

Define $\phi \colon F \to \rho$ by setting $\phi(x)$ to be the least $\alpha < \rho$ such that $x \in f(\alpha)$. Then ϕ is a $\bigcup_\rho \Delta$-norm on F. To see this, note first that, for all x, y in ω^ω, $x <^*_\phi y$ if and only if there exists an $\alpha < \rho$ such that $x \in f(\alpha)$ and $y \notin f(\alpha)$, and this set is in $\bigcup_\rho \Delta$. Similarly, $x \leq^*_\phi y$ if and only if there exists an $\alpha < \rho$ such that $x \in f(\alpha + 1)$ and $y \notin f(\alpha)$, and this set is also in $\bigcup_\rho \Delta$. □

4.2.11 Remark. Assuming that Wadge Determinacy and Baire($\mathcal{P}(\omega^\omega)$) hold, Theorem 4.2.10 implies that if Δ is a selfdual boldface pointclass containing a countable Wadge-cofinal subset, then $\bigcup_\omega \Delta$ has the prewellordering property. This holds in particular whenever Δ is the pointclass of sets projective in a fixed subset of ω^ω.

The following theorem shows that at most one member of a complementary pair of nonselfdual bolface pointclasses can have the prewellordering property.

Theorem 4.2.12. *If Γ is a boldface pointclass with a universal set, then Γ and $\check{\Gamma}$ do not both have the prewellordering property.*

Proof. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal Γ-set, and let ϕ be a Γ-norm on U. Let $\pi \colon \omega^\omega \to \omega^\omega \times \omega^\omega$ be a recursive bijection, and let π_0 and π_1 be such that $\pi(x) = (\pi_0(x), \pi_1(x))$ for all $x \in \omega^\omega$. Let B be the set of (x, y) such that

$$(\pi_0(x), y) <^*_\phi (\pi_1(x), y),$$

and let C be the set of (x, y) such that

$$(\pi_1(x), y) <^*_\phi (\pi_0(x), y).$$

Then B and C are disjoint and in Γ.

Let $\rho \colon \omega \times 2 \to \omega$ be bijection. For each $i \in s$, let ρ^* be the function on $\omega^\omega \times \omega^\omega$ defined by setting $\rho^*_i(x, y)$ to be $(\langle \pi(x(0), i), x_1, x_2, \ldots \rangle, y)$. Let P be the set

$$(\rho^*_0[(\omega^\omega \times \omega^\omega) \setminus B] \cup \rho^*_1[(\omega^\omega \times \omega^\omega) \setminus C].$$

Then P is in $\check{\Gamma}$. Let ψ be a $\check{\Gamma}$-norm on P. Let E be the set of $(x, y) \in \omega^\omega \times \omega^\omega$ such that

$$\rho^*_0(x, y) <^*_\psi \rho^*_1(x, y).$$

Since $B \cap C = \emptyset$, E is also the set of (x, y) such that

$$\neg(\rho^*_1(x, y) \leq^*_\psi \rho^*_0(x, y)),$$

so E is in Δ.

We derive a contradiction to Theorem 2.4.2 by showing that E is universal for Δ. Fixing $D \in \Delta$, let x be such that $\omega^\omega \setminus D = U_{x_0}$ and $D = U_{x_1}$. Then $\omega^\omega \setminus D = B_x$ and $D = C_x$, since, for all $y \in \omega^\omega$,

$$y \in D \Leftrightarrow ((x_1, y) \in U \wedge (x_0, y) \notin U).$$

Finally, for each $y \in \omega^\omega$,

$$(x, y) \in C \Rightarrow (x, y) \in E$$

and

$$(x, y) \in B \Rightarrow (x, y) \notin E.$$

It follows that $E_x = D$. □

Finally, the following theorem shows that for certain pointclasses, every witness to the prewellordering property has the same length. Part of the proof of Theorem 4.2.13 is reused for Theorem 4.3.3.

Theorem 4.2.13 (Moschovakis)**.** *Suppose that Γ is a $\forall^{\omega^\omega}$-closed boldface pointclass with a universal set which is closed under unions, and let $\langle U_n : n \in \omega \setminus \{0\}\rangle$ be sequence of Γ-universal sets with the recursion property. Then for every $n \in \omega \setminus \{0\}$ and every regular Γ-norm ϕ on U_n, $|\phi| = \delta(\Gamma)$.*

Proof. Fix $n \in \omega$, and let ϕ be a Γ-norm on U_n. Let $\prec$ be a strict prewellordering on a set $Y \subseteq (\omega^\omega)^n$, with $\prec \in \check{\Gamma}$. Since any two spaces in $\mathcal{X}$ are homeomorphic, it suffices to show that the rank of $\prec$ is at most $|\phi|$. Let Q be the set of $(x, y) \in \omega^\omega \times (\omega^\omega)^n$ such that for all $z \in (\omega^\omega)^n$, $z \prec y$ implies $(x, z) <^*_\phi (x, y)$. Then $Q \in \Gamma$. By Theorem 2.4.7, there exists an $x^* \in \omega^\omega$ such that $U_{n,x^*} = Q_{x^*}$.

Then one can verify by induction on the $\prec$-rank of y that for all $y \in Y$, $y \in U_{n,x^*}$ and, for all $z \in (\omega^\omega)^n$, $z \prec y$ implies $\phi(x^*, z) < \phi(x^*, y)$. To do this, fix $y \in Y$, and suppose that for all $z \in Y$ with $z \prec y$, z is in U_{n,x^*} and, for all $w \in (\omega^\omega)^n$ $w \prec z$ implies $\phi(x^*, w) < \phi(x^*, z)$. To see that y is in U_{n,x^*}, we need to see that for all $z \in (\omega^\omega)^n$, if $z \prec y$ then $(x^*, z) <^*_\phi (x^*, y)$. Fixing $z \in (\omega^\omega)^n$ such that $z \prec y$, have that the failure of $(x^*, z) <^*_\phi (x^*, y)$ implies that either $(x^*, z) \notin U_n$ or $(x^*, y) \in U_n$ and $\phi(x^*, y) \leq \phi(x^*, z)$). Since (x^*, z) is in U_n, we have that (x^*, y) is in U_n. Finally, fix once again a $z \in (\omega^\omega)^n$ such that $z \prec y$. Since $(x^*, y) \in U_n$, we have that $(x^*, z) <^*_\phi (x^*, y)$, and since (x^*, y) is in U_n, this means that $\phi(x^*, z) < \phi(x^*, y)$, as desired. □

4.2.14 Remark. It follows from Theorem 4.2.13 and Remark 4.2.1 and 4.2.3 that if Wadge determinacy holds, Γ is a $\forall^{\omega^\omega}$-closed boldface pointclass with a universal set which is closed under unions, P is in $\Gamma \setminus \check{\Gamma}$ and ϕ is a regular Γ-norm on P, then $|\phi| = \delta(\Gamma)$.

4.3 Prewellorderings and wellfounded relations

Theorems 4.3.1 and 4.3.2 connect the length of prewellorderings in a bolface pointclass Γ with the closure of Γ under wellfounded unions.

Theorem 4.3.1 (Martin)**.** *Assume that* AD *holds. Let Γ be a nonselfdual boldface pointclass closed under $\exists^{\omega^\omega}$ and $\wedge$. Let ρ be an ordinal. Suppose that there exists a prewellordering in Γ of length ρ. Then $\bigcup_\rho \Gamma \subseteq \Gamma$.*

Proof. Fix $f\colon \rho \to \Gamma$, and let $F = \bigcup_{\alpha<\rho} f(\alpha)$. We want to see that F is in Γ. Let $\preceq$ be a prewellordering in Γ of length ρ. Let X be the domain of $\prec$, and for each $x \in X$ let α_x denote the $\preceq$-rank of x. By Theorem 2.4.3, we may let $U \subseteq (\omega^\omega)^2$ be a Γ-universal set. Let R be the set of $(x, y) \in X \times \omega^\omega$ for which $U_y = f(\alpha_x)$. By the Coding Lemma (Theorem 3.0.1), there is a set $S \subset R$ in pos-$\underset{\sim}{\Sigma}^1_1(\preceq)$ (which is contained in Γ, since Γ is $\exists^{\omega^\omega}$-closed) such that for all $x \in \omega^\omega$, if $R_x \neq \emptyset$ then $S_x \neq \emptyset$. Then for all $z \in \omega^\omega$, $z \in F$ if and only if there exist $x \in X$ and $y \in \omega^\omega$ such that $S(x, y)$ and $U(y, z)$. It follows that F is in Γ. □

Theorem 4.3.2 (Martin)**.** *Assume that* AD *holds. Let Γ be a nonselfdual pointclass closed under $\forall^{\omega^\omega}$ and $\vee$, and assume that Γ has the prewellordering property. Let $\Delta = \Gamma \cap \check{\Gamma}$. Then Δ is closed under $< \delta(\Gamma)$-length unions and intersections.*

Proof. Since Δ is closed under complements, it suffices to see that it is closed under unions of length less than $\delta(\Gamma)$. Supposing otherwise, let $\rho < \delta(\Gamma)$ be minimal such that there is exists a function $f\colon \rho \to \Delta$ with $\bigcup_{\alpha<\rho} f(\alpha) \notin \Delta$. Since Δ is closed under unions, ρ is a limit ordinal. By Theorem 4.2.10, the pointclass $\bigcup_\rho \Delta$ has the prewellordering property. By Theorem 4.3.1, $\bigcup_\rho \Delta = \check{\Gamma}$. We then have a contradiction to Theorem 4.2.12. □

A binary relation $\prec$ on a set X is said to be *wellfounded* if there is a function $f\colon X \to \mathrm{Ord}$ such that $f(x) < f(y)$ whenever $x \prec y$. The least ordinal containing the range of such a function is said to the rank of $\prec$. The following theorem relates the lengths of prewellorderings in certain pointclasses to the ranks of the wellfounded relations in the same class.

Theorem 4.3.3 (Moschovakis)**.** *Suppose that*

$$\mathsf{Wadge\ Determinacy} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$$

holds. Let Γ be a nonselfdual boldface pointclass closed under $\forall^{\omega^\omega}$ and $\vee$, and suppose that PWO(Γ) *holds. Then any wellfounded relation in $\check{\Gamma}$ has length less than some prewellordering in $\Gamma \cap \check{\Gamma}$.*

Proof. Let $\bar{U} = \langle U_n : n < \omega \rangle$ be a sequence of universal sets for Γ with the s-m-n property. Let ϕ be a regular Γ-norm on U_2. Then the range of ϕ is some $\delta \leq \delta(\Gamma)$, by Remark 4.2.3. Let $\prec$ be a wellfounded relation in $\check{\Gamma}$ on a set $X \in \check{\Gamma}$. We will show that the rank of $\prec$ is at most δ. This will suffice, since there are wellfounded relations in $\check{\Gamma}$ whose rank is more than that of $\prec$ (one more, for instance), and the argument will apply to these relations also.

Let A be the set of $(x,y) \in \omega^\omega \times \omega^\omega$ such that, for all $z \in X$, if $z \prec y$ then $(x,z) <^*_\phi (x,y)$. By Theorem 2.4.7, there is an $x \in \omega^\omega$ such that $A_x = U_{2,x}$.

We verify first by induction on the $\prec$-rank of $y \in \omega^\omega$ that $(x,y) \in U_2$. First note that if $(x,y) \notin U_2$ and (x,z) is in U_2 for all $z \prec y$, then (x,y) is in A, which implies that in fact (x,y) is in U_2. It follows then that for all $y, z \in X$, if $z \prec y$ then $\phi(x,z) < \phi(x,y)$, as desired. □

In conjunction with Theorem 4.3.5, the following theorem shows that the supremum of the Wadge ranks of the sets in a pointclass of the form $\underset{\sim}{\Sigma}^2_1(A)$ (assuming that the Wadge hierarchy is wellfounded) is a regular cardinal.

Theorem 4.3.4. *Assume that* AD *holds. Let* Γ *be a nonselfdual boldface pointclass closed under* $\exists^{\omega^\omega}$ *and* $\wedge$. *Then the supremum of the ranks of the wellfounded relations in* Γ *is a regular cardinal.*[10]

Proof. Let κ be the supremum of the ranks of the wellfounded relations in Γ. Clearly, κ is a limit ordinal. Let ρ be the cofinality of κ, suppose that $\rho < \kappa$, and let $f \colon \rho \to \kappa$ be cofinal. Let $\prec$ be a wellfounded relation in Γ, on a set $X \subseteq \omega^\omega$, of rank ρ, and let $\preceq$ be the non-strict part of $\prec$. For each $x \in X$, let α_x be the $\prec$-rank of x. Applying Theorem 2.4.3, let $U \subseteq (\omega^\omega)^3$ be a universal Γ-set, and let R be the set of $(x,y) \in X \times \omega^\omega$ for which U_y a wellfounded relation of length at least $f(\alpha_x)$. By Theorem 3.0.1, there exists a pos-$\underset{\sim}{\Sigma}^1_1(\preceq)$-set $A \subseteq R$ such that, for all $x \in X$, if $R_x \neq \emptyset$ then $A_x \neq \emptyset$. Then A is in Γ. Now define the relation $<$ on $(\omega^\omega)^3$ by setting $(x,y,z) < (a,b,c)$ if and only if $(x,y) \in A$, $(x,y) = (a,b)$ and $(y,z,c) \in U$. Then $<$ is a wellfounded relation in Γ of rank κ, giving a contradiction. □

A *projective algebra* is a selfdual boldface pointclass containing all closed subsets of spaces in $\mathcal{X}$ and closed under real quantification. The following is a weak version of a theorem from [13].

Theorem 4.3.5. *Suppose that* AD *holds, and that* $\mathcal{W} = \mathcal{P}(\omega^\omega)$. *Let* Γ *be a nonselfdual boldface pointclass closed under* $\forall^{\omega^\omega}$ *and* $\vee$, *and suppose that* PWO(Γ) *holds. Let* $\Delta = \Gamma \cap \check{\Gamma}$, *and suppose that* Δ *is a projective algebra. Then the following three ordinals are equivalent:*

- $o(\Delta)$*;*
- $\delta(\Gamma)$*;*
- *The supremum of the ranks of the wellfounded relations in* Δ.

Proof. Let κ be the supremum of the ranks of the wellfounded relations in Δ. That $\delta(\Gamma) \leq \kappa$ follows immediately; that $\delta(\Gamma) = \kappa$ follows from Theorem 4.3.3. That $o(\Delta) \leq \delta(\Gamma)$ follows from Proposition 2.5.8. That $\delta(\Gamma) \leq o(\Delta)$ follows from Theorems 2.5.9 and 4.3.2, and Remark 2.5.10. □

[10] To whom is this due?

4.4 Closure under wellordered unions

We include an application of the material in Section 4.2 and 4.3 which will be used in Chapter 6.[11]

The following is extracted from the proof of Lemma 2.18 of [3]

Lemma 4.4.1. *Suppose that* AD *holds, and that* Γ *is a nonselfdual boldface pointclass closed under* $\exists^{\omega^\omega}$ *and finite intersections, but not wellordered unions. Let* Γ_1 *be* $\exists^{\omega^\omega}\check{\Gamma}$ *and let* κ *be the least ordinal* γ *such that* $\bigcup_\gamma \Gamma \not\subseteq \Gamma$. *Then* $\bigcup_\kappa \Gamma_1 = \Gamma_1$, *from which it follows that* $\bigcup_\kappa \Gamma = \Gamma_1$.

Proof. By Corollary 2.3.2, $\check{\Gamma} \subseteq \bigcup_\kappa \Gamma$. Since $\bigcup_\kappa \Gamma$ is closed under $\exists^{\omega^\omega}$, the pointclass $\exists^{\omega^\omega}\check{\Gamma}$ (which we will call Γ_1) is also contained in $\bigcup_\kappa \Gamma$. We will show first there is a prewellordering of length κ in Γ_1. From this and Theorem 4.3.1 it will follow that $\bigcup_\kappa \Gamma_1 \subseteq \Gamma_1$, from which the rest of the lemma follows.

Let $A \subseteq \omega^\omega$ be any set in $\check{\Gamma} \setminus \Gamma$. Let $S \subseteq (\omega^\omega)^2$ be a universal $\utilde{\Sigma}^1_1$ set, and let $B = \{x \in \omega^\omega : S_x \subseteq A\}$. Since $\check{\Gamma}$ is closed under $\forall^{\omega^\omega}$ and $\vee$, $B \in \check{\Gamma}$, so B can be written as $\bigcup_{\alpha<\kappa} B_\alpha$, where each B_α is in Γ. For each $\alpha < \kappa$, let $A_\alpha = \{y \in \omega^\omega : \exists x \in B_\alpha\, y \in S_x\}$. Then $A = \bigcup_{\alpha<\kappa} A_\alpha$, and every $\utilde{\Sigma}^1_1$ subset of A (being S_x for some $x \in \omega^\omega$) is contained in some A_α.

Let $U \subseteq (\omega^\omega)^2$ be a universal Γ set. Let $\mathcal{G}$ be the game where I plays $x \in \omega^\omega$, II plays y and z, and II wins if and only if $x \notin A$ or

$$\exists\alpha > |x|(U_y = A_\alpha \wedge z \in A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta),$$

where $|x|$ denotes the least $\gamma < \kappa$ such that $x \in A_\gamma$. Since every $\utilde{\Sigma}^1_1$ subset of A is contained in some A_α, I cannot have a winning strategy in $\mathcal{G}$. Let τ be a winning strategy for II. Given $x \in \omega^\omega$, let $\tau_y(x)$ and $\tau_z(x)$ denote, respectively, the y and z parts of the response to x (played by I) given by τ. Define the binary relation $\prec$ on ω^ω by setting $x_0 \prec x_1$ if and only if $\{x_0, x_1\} \subseteq A$ and $\tau_z(x_1) \notin U_{\tau_y(x_0)}$. Then for all $x_0, x_1 \in A$, $x_0 \prec x_1$ if and only if, for some $\alpha < \kappa$, $\tau_z(x_0) \in A_\alpha$ and $\tau_z(x_1) \notin A_\alpha$. It follows that $\prec$ is a strict prewellordering in $\check{\Gamma}$ of length κ. The relation of being in the same $\prec$-class is then the intersection of a relation in Γ with one in $\check{\Gamma}$, and therefore in Γ_1. $\square$

Theorem 4.4.2. *Suppose that* AD *holds, and that* $\mathcal{W} = \mathcal{P}(\omega^\omega)$. *Suppose that* Δ *is a projective algebra such that* $\bigcup_\omega \Delta \not\subseteq \Delta$. *Then* $\bigcup_\omega \Delta$ *is nonselfdual, has the prewellordering property and is closed under wellordered unions.*

Proof. Since Δ is a projective algebra and $\bigcup_\omega \Delta \not\subseteq \Delta$, there exist $B_i \subseteq \omega^\omega$ ($i \in \omega$) which are collectively Wadge cofinal in Δ. Let $\pi\colon \omega^\omega \to (\omega^\omega)^\omega$ be a continuous bijection, and let π_i ($i \in \omega$) be the functions on ω^ω such that $\pi(x) = \langle \pi_i(x) : i \in \omega\rangle$ for all $x \in \omega^\omega$. Recalling our coding in Remark2.5.2 of continuous functions f^c_x by $x \in \mathcal{F}^c \subseteq \omega^\omega$, for each $i \in \omega$ let C_i be the set of $(x, y) \in (\omega^\omega)^2$ such that $\pi_i(x) \in \mathcal{F}^c$ and $f^c_{\pi_i(x)}(y) \in B_i$. Since Δ is a projective

[11] this has to be improved

algebra, each C_i is in Δ. Let $C = \bigcup_{i\in\omega} C_i$. We claim that C is universal for the pointclass $\bigcup_\omega \Delta$. To verify this, consider a set $\bigcup_{i\in\omega} A_i$, where each A_i is in Δ. Applying $\mathsf{CC}_{\mathbb{R}}$, we may find E_i and x_i ($i \in \omega$) such that each x_i is in $\mathcal{F}^c$, each E_i is $(f^c_{x_i})^{-1}[B_i]$ and $\bigcup_{i\in\omega} A_i = \bigcup_{i\in\omega} E_i$. Letting $g\colon \omega^\omega \to (\omega^\omega)^2$ be the function sending y to $(\pi^{-1}(\langle x_i : i \in \omega\rangle), y)$, we have that $\bigcup_{i\in\omega} E_i = g^{-1}[C]$. Since $\bigcup_\omega \Delta$ has a universal set, it is nonselfdual, by Proposition 2.4.2.

Let $\Gamma = \bigcup_\omega \Delta$. Then Γ is closed under $\exists^{\omega^\omega}$ and $\wedge$. It follows now by Theorem 4.2.10 that Γ has the prewellordering property. By the First Periodicity Theorem (see Remark 4.2.6), $\forall^{\omega^\omega}\Gamma$ does as well. We will now show (extracting the relevant part of the proof of Lemma 2.18 of [3]) that Γ is closed under wellordered unions.

Supposing otherwise, let $\Gamma_1 = \exists^{\omega^\omega}\check{\Gamma}$ and let κ be the least ordinal γ such that $\bigcup_\gamma \Gamma \not\subseteq \Gamma$. By Lemma 4.4.1, $\bigcup_\kappa \Gamma = \bigcup_\kappa \Gamma_1 = \Gamma_1$. Let $\Delta_1 = \Gamma_1 \cap \check{\Gamma}_1$. Let ρ be the least ordinal γ (necessarily at most κ) such that $\bigcup_\gamma \Delta_1 \not\subseteq \Delta_1$. Since $\bigcup_\rho \Delta_1$ properly contains Δ_1 and is contained in Γ_1, $\Gamma_1 = \bigcup_\rho \Delta_1$. It follows from Theorem 4.2.10 that Γ_1 has the prewellordering property. Since $\check{\Gamma}_1$ has the prewellordering propery (as observed above) this contradicts Theorem 4.2.12. □

Lemma 2.4 of [13].

Proposition 4.4.3. *Assume*[12]*. Let* Γ *be a* $\exists^{\omega^\omega}$*-closed boldface pointclass which is closed under countable unions and intersections but not* $\forall^{\omega^\omega}$*-closed. If* $\mathsf{Red}(\Gamma)$ *holds, then* Γ *is closed under wellordered unions.*[13]

Proof. Working toward a contradiction, let κ be the least ordinal γ such that for some $f\colon \gamma \to \Gamma$, $\bigcup_{\alpha<\gamma} f(\alpha) \notin \Gamma$. Then κ is a regular uncountable cardinal. Fix such a function f, and let $F = \bigcup_{\alpha<\kappa} f(\alpha)$. By Remark 2.2.4 and Theorem 2.2.6, the boldface pointclass $\bigcup_\kappa \Gamma$ contains both Γ and $\check{\Gamma}$. Letting $\Gamma_1 = \exists^{\omega^\omega}\check{\Gamma}$, we have also that $\Gamma \subseteq \Gamma_1 \subseteq \bigcup_\kappa \Gamma$, in the latter case because Γ is $\exists^{\omega^\omega}$-closed. Letting $\Delta_1 = \Gamma_1 \cap \check{\Gamma}_1$, we have that $\Gamma \subseteq \Delta_1$, so $\bigcup_\kappa \Delta_1$ contains Γ_1, which properly contains Δ_1. Let ρ be minimal such that $\bigcup_\rho \Delta_1 \not\subseteq \Delta_1$. Then $\bigcup_\rho \Delta_1$ has the prewellordering property, by Proposition 4.2.10.

By Lemma 4.4.1, $\Gamma_1 = \bigcup_\kappa \Gamma_1$. As above it equals $\bigcup_\rho \Delta_1$.[14]

Let δ_1 be the supremum of the ranks of the wellfounded relations in Γ_1. By Proposition 4.3.3, δ_1 is at most the supremum of the lengths of the prewellorderings in Δ_1. It follows that $\rho \geq \delta_1$, since otherwise, by Proposition 4.2.10, Γ_1 has the prewellordering property, contradicting the fact that $\check{\Gamma}_1$has the reduction property, by the 0th Periodicity theorem (Theorem 4.1.8).

Let δ be the supremum of the ranks of the wellfounded relations in Γ. Let $U \subseteq (\omega^\omega)^3)$ be a Γ-universal set, and let W be the set of $x \in \omega^\omega$ for which U_x is a wellfounded relation. Then the relation $<$ on $(\omega^\omega)^2$ defined by setting $(x, y) < (w, z)$ if and only if $x = w$, $x \in W$ and $(y, z) \in U_x$ is in Γ_1, which shows

[12] at least $\mathsf{Wadge\ Determinacy}$ and $\mathsf{Baire}(\mathcal{P}(\omega^\omega))$
[13] You may have to say/show that Red follows from PWO, or something else.
[14] We need to see that PWO implies reduction.

(since Γ_1 is closed under unions which implies that δ_1 is a limit ordinal) that $\delta < \delta_1$.[15] It follows that $\kappa > \delta$.

Now let $\prec$ be a wellfounded relation in Γ_1. Since $\Gamma_1 \subseteq \bigcup_\kappa \Gamma$, we can write $\prec$ as $\bigcup_{\alpha<\kappa} \prec_\alpha$, where each $\prec_\alpha$ is in Γ. By the minimality of κ we may assume that $\alpha < \beta < \kappa$ implies that $\prec_\alpha \subseteq \prec_\beta$. For each x in the field of $\prec$, and each $\alpha < \kappa$, let $g_x(\alpha)$ be 0 if x is not in the field of $\prec_\alpha$, and its rank in $\prec_\alpha$ otherwise. Then each function g_x is nondecreasing. Since $\delta < \kappa$ and κ is regular, each g_x has an eventually constant value ξ_x. If $x \prec y$, then $\xi_x < \xi_y$, which means that the rank of $\prec$ is at most δ, showing that $\delta_1 \leq \delta$, giving a contradiction.

□

[15] this argument appears in Kechris '74

Chapter 5

Strong Partition Cardinals

Given an ordinal α and a set X of ordinals, $[X]^\alpha$ denotes the collection of subsets of X of ordertype α. Given ordinals α, β, δ and γ the formula $\alpha \rightarrow (\beta)^\delta_\gamma$ asserts that for every partition of $[\alpha]^\delta$ into γ many pieces, there exists an $X \in [\alpha]^\beta$ such that $[X]^\delta$ is contained in one piece. We write $\alpha \rightarrow (\beta)^\delta$ when $\gamma = 2$. A *strong partition cardinal* is a regular uncountable cardinal κ such that $\kappa \rightarrow (\kappa)^\kappa_\mu$ holds for all $\mu < \kappa$.

5.0.1 Remark. Some authors define a strong partition cardinal using the ostensibly weaker property $\kappa \rightarrow (\kappa)^\kappa$. We do not know whether ZF implies that these two definitions are equivalent. However, if κ is a regular uncountable cardinal such that $\kappa \rightarrow (\kappa)^\kappa$, then $\forall \mu < \kappa(\kappa \rightarrow (\kappa)^\mu_\mu)$ holds. This can be seen, for instance, by considering, given an $f\colon [\kappa]^\mu \rightarrow \mu$, the function $f'\colon [\kappa]^\kappa \rightarrow 2$ defined by setting $f'(X) = 0$ if and only if $f(a) = f(b)$, where a is the initial segment of X of ordertype μ, and b is the initial segment of $X \setminus a$ of ordertype μ. The property $\forall \mu < \kappa(\kappa \rightarrow (\kappa)^\mu_\mu)$ is enough for our application of strong partition cardinals in Theorem 7.0.3.

Our aim in this chapter is to prove the following theorem.

Theorem 5.0.2. *([9]) If AD holds then the strong partition cardinals are cofinal below Θ.*

We do not know whether AD implies that cardinals of the form $\undertilde{\delta}^2_1(A)$ are strong partition cardinals. One problem is that we do not know how to show that the pointclass $\undertilde{\Sigma}^2_1(A)$ has the prewellordering property without some additional assumption. In Remark 4.2.9 we show this using the additional assumption that the Wadge hierarchy is wellfounded. A second problem is that we do know whether AD implies that $\undertilde{\Sigma}^2_1(A)$ is $\forall^{\omega^\omega}$-closed, although it is in many natural models of AD. Adding these two properties as hypotheses, we get the following.

Theorem 5.0.3. *If AD holds and $\mathcal{W} = \mathcal{P}(\omega^\omega)$, then for each $A \subseteq \omega^\omega$ for which $\undertilde{\Sigma}^2_1(A)$ is $\forall^{\omega^\omega}$-closed, $\undertilde{\delta}^2_1(A)$ is a strong partition cardinal.*

To get around these issues, we relativize $\utilde{\Sigma}^2_1(A)$ to the inner model $L(A,\omega^\omega)$. We write loc-$\utilde{\Sigma}^2_1(A)$ for

$$(\utilde{\Sigma}^2_1(A))^{L(A,\omega^\omega)},$$

loc-$\utilde{\Delta}^2_1(A)$ for $(\utilde{\Delta}^2_1(A))^{L(A,\omega^\omega)}$ and loc-$\utilde{\delta}^2_1(A)$ for $(\utilde{\delta}^2_1(A))^{L(A,\omega^\omega)}$. As with $\utilde{\Sigma}^2_1(A)$, each pointclass loc-$\utilde{\Sigma}^2_1(A)$ is nonselfdual, has a universal set and closed under $\exists^{\omega^\omega}$ and $\wedge$.

5.0.4 Remark. To see that the pointclasses loc-$\utilde{\Sigma}^2_1(A)$ satisfy the prewellordering property (without assuming that $\mathcal{W} = \mathcal{P}(\omega^\omega)$), fix a $\Sigma^2_1(A)$ set X of the form

$$\{x \in \omega^\omega : L(A,\omega^\omega) \models \exists B \subseteq \omega^\omega\ \psi(A,B,x)\}.$$

Orders $\leq^*$ and $<^*$ witnessing that X has the prewellordering property can be defined using the least ordinal α such that $L_\alpha(A,\omega^\omega)$ contains a set B as in the definition of X. The Solovay Basis Theorem (Theorem 5.0.5) shows that this gives a loc-$\utilde{\Sigma}^2_1(A)$-norm on X.

The Solovay Basis Theorem (Theorem 5.0.5 below) says that for each $A \subseteq \omega^\omega$, loc-$\utilde{\Delta}^2_1(A)$ is Σ_1-elementary in $\mathcal{P}(\omega^\omega) \cap L(A,\omega^\omega)$. The proof of the theorem, which we leave to the reader, follows from the fact that in models of the form $L(A,\omega^\omega)$, every set is ordinal definable from A and a member of ω^ω, which implies that every set has an elementary submodel which is the surjective image of ω^ω. This means that there is a minimal (relative to a parameter from ω^ω) witness to each true $\utilde{\Sigma}^2_1(A)$ statement, and that that minimal witness is in (loc-$\utilde{\Delta}^2_1(A))^{L(A,\omega^\omega)}$. In particular, membership in loc-$\utilde{\Sigma}^2_1(A)$ sets is witnessed by loc-$\utilde{\Delta}^2_1(A)$ sets. Using the Solovay Basis Theorem, it is not hard to show that pointclasses of the form loc-$\utilde{\Sigma}^2_1(A)$ are $\forall^{\omega^\omega}$-closed.

Theorem 5.0.5 (Solovay Basis Theorem)**.** *Let A be a subset of ω^ω and let x and y be an element of ω^ω. Let ϕ be a formula whose quantifiers range over the hereditarily countable sets. If there exists a set B in $\mathcal{P}(\omega^\omega) \cap L(A,\omega^\omega)$ such that $\phi(A,B,x,y)$ holds, then there exists such a set B in loc-$\utilde{\Delta}^2_1(A)$.*

The following gives Theorem 5.0.2.

Theorem 5.0.6. *If* AD *holds then loc-$\utilde{\delta}^2_1(A)$ is a strong partition cardinal, for each $A \subseteq \omega^\omega$.*

Our proofs of Theorems 5.0.3 and 5.0.2 diverge at just one point, establishing that each of the relevant pointclasses satisfies the prewellordering property (see Section 4.2). The prewellordering property is part of a very deep and general theory of boldface pointclasses under AD, but we are mostly interested in taking the shortest path to proving the existence of strong partition cardinals. Readers interested in this general theory are directed to [10, 11] for an introduction. Much of this chapter is taken from [3] and conversations with its author.

5.1 The Martin Conditions

We complete the proof of Theorem 5.0.6 (so also Theorem 5.0.2) by presenting an argument of Martin connecting closure properties of pointclasses with partition properties of their associated ordinals. The following definition (a slightly modified version of Definition 2.33 on page 1783 of [3]) is due to Martin; the properties that it lists are known as the Martin conditions.

5.1.1 Definition. Let $\lambda \leq \kappa$ be ordinals. We say that κ is λ-*reasonable* if there exist a non-selfdual boldface pointclass Γ closed under $\forall^{\omega^\omega}$ and a function

$$\phi \colon \omega^\omega \to \mathcal{P}(\lambda \times \kappa)$$

such that, letting $\Delta = \Gamma \cap \check{\Gamma}$,

1. for each function $F \colon \lambda \to \kappa$ there is an $x \in \omega^\omega$ such that $\phi(x) = F$;

2. for each pair $(\beta, \gamma) \in \lambda \times \kappa$, the set

$$R_{\beta,\gamma} = \{x \in \omega^\omega : \{\eta < \kappa : (\beta, \eta) \in \phi(x)\} = \{\gamma\}\}$$

   is in Δ;

3. for each $\beta < \lambda$ and each A in $\exists^{\omega^\omega} \Delta \cap \mathcal{P}(R_\beta)$ there exists a $\gamma_0 < \kappa$ such that

$$A \subseteq \bigcup_{\gamma < \gamma_0} R_{\beta,\gamma},$$

   where $R_\beta = \bigcup\{R_{\beta,\gamma} : \gamma < \kappa\}$.

Given a function ϕ and sets R_β ($\beta < \lambda$) as in Definition 5.1.1, $\phi(x)$ is a function from λ to κ for each $x \in \bigcap_{\beta<\lambda} R_\beta$.

In Theorem 5.1.2 and 5.1.3 below, we assume that Γ has the prewellordering property, which may make these theorems less than optimal. Versions without this additional hypothesis appear in [3].[1]

Theorem 5.1.2 (Martin)**.** *Suppose that* AD *holds. Let* Γ *be a nonselfdual bolface pointclass closed under* $\forall^{\omega^\omega}$ *and* $\vee$*, and assume that* Γ *has the prewellordering property. Let* $\Delta = \Gamma \cap \check{\Gamma}$*, and suppose that* Δ *is a projective algebra. Let* κ *be* $\delta(\Gamma)$ *and suppose that* κ *is a regular cardinal. Let* λ *be an ordinal such that* $\omega \cdot \lambda \leq \kappa$*. If* Γ *witnesses that* κ *is* $\omega \cdot \lambda$*-reasonable, then* $\kappa \to (\kappa)^\lambda$*.*

Proof. Let ϕ, $R_{\beta,\gamma}$ ($\beta < \lambda, \gamma < \kappa$) and R_β ($\beta < \lambda$) be as in Definition 5.1.1, with respect to $\omega \cdot \lambda$, κ and Γ. Fix $P \colon [\kappa]^\lambda \to 2$. Consider the integer game in which player I plays the values of $x \in \omega^\omega$ and player II plays the values of $y \in \omega^\omega$.

---

[1] Really?

I	$x(0)$		$x(1)$		$x(2)$		$\ldots$
II		$y(0)$		$y(1)$		$\ldots$	

If there is a least ordinal $\beta < \omega \cdot \lambda$ such that $\{x, y\} \not\subseteq R_\beta$, then II wins provided that $x \notin R_\beta$. Otherwise, letting $f_{x,y} : \lambda \to \kappa$ be defined by setting $f_{x,y}(\beta)$ to be

$$\sup\{\max(\phi(x)(\eta), \phi(y)(\eta)) : \eta < \omega \cdot (\beta + 1)\},$$

II wins if and only if $P(f_{x,y}) = 1$.

Suppose first that τ is a winning strategy for player II. Fix for this paragraph a pair (β, γ) in $(\omega \cdot \lambda) \times \kappa$. Let $S_{\beta,\gamma} = \bigcap_{\delta \leq \beta} \bigcup_{\eta \leq \gamma} R_{\delta,\eta}$. Then $S_{\beta,\gamma}$ is in Δ by Theorem 4.3.2. Therefore, the set $S_{\beta,\gamma} \circ \tau = \{x \circ \tau : x \in S_{\beta,\gamma}\}$ is in Δ, as Δ is a projective algebra. We have that $S_{\beta,\gamma} \subseteq \bigcap_{\delta \leq \beta} R_\beta$, which implies that $S_{\beta,\gamma} \circ \tau \subseteq R_\beta$. Letting $\theta(\beta, \gamma)$ be $\sup\{\phi(y)(\beta) : y \in S_{\beta,\gamma} \circ \tau\}$, we have by part (3) of Definition 5.1.1 that $\theta(\beta, \gamma) < \kappa$.

Let C_0 be the set of elements of κ closed under θ, and C be any cofinal subset of C_0 (for the present theorem it suffices to work with C_0, but the fact that the rest of the argument works with any cofinal $C \subseteq C_0$ will be used later in this section). Let C' be the set of $\gamma \in C$ for which $\mathrm{ot}(\gamma \cap C)$ has the form $\eta + \omega$, for some ordinal η. Suppose now that $F\colon \lambda \to C'$ is increasing; we want to see that $P(F) = 1$. Applying part 1 of Definition reasonabledef, let x be such that $\phi(x)$ is an increasing function, with range contained in C, such that $F(\beta) = \sup\{\phi(x)(\delta) : \delta < \omega \cdot (\beta + 1)\}$ for all $\beta < \omega \cdot \lambda$. Let $y = x \circ \tau$; then y is in $\bigcap_{\beta < \omega \cdot \lambda} R_\beta$. It suffices to see that $\phi(y)(\beta) \leq \phi(x)(\beta + 1)$ for all $\beta < \omega \cdot \lambda$, since then $F = f_{x,y}$, so $P(F) = 1$.

Fix $\beta < \omega \cdot \lambda$. We have that x is in $S_{\beta,\phi(x)(\beta)}$, so y is in $S_{\beta,\phi(x)(\beta)} \circ \tau$. Then $\phi(y)(\beta) \leq \theta(\beta, \phi(x)(\beta)) < \phi(x)(\beta+1)$, since $\phi(x)$ is increasing and $\phi(x)(\beta+1)$, being in C, is closed under θ.

A winning strategy for player I can be used as a winning strategy for player II in the game where the values of P are reversed. It follows that if there is a winning strategy for player I, then there is homogeneous set for value 0. □

As Remarks 4.2.8 and 5.0.4 and Theorems 4.3.3, 4.3.4 and 4.3.5 together show that the pointclasses loc-$\underset{\sim}{\Sigma}^2_1(A)$ satisfy the hypotheses of Theorem 5.1.3 (and Remark 4.2.9 shows that the pointclasses $\underset{\sim}{\Sigma}^2_1(A)$ do, if the Wadge hierarchy is assumed to be wellfounded), Theorem 5.1.3 gives the versions of Theorems 5.0.3 and 5.0.6 for the weaker notion of strong partition cardinal discussed in Remark 5.0.1.

Theorem 5.1.3. *Assume that* AD *holds. Let Γ be a nonselfdual bolface pointclass with the prewellordering property, closed under $\forall^{\omega^\omega}$ and $\vee$. Let $\Delta = \Gamma \cap \check{\Gamma}$, and suppose that Δ is a projective algebra. Let κ be $\delta(\Gamma)$ and suppose that κ is a regular cardinal. Then $\kappa \to (\kappa)^\kappa$.*

Proof. By Theorem 5.1.2, it suffices to see that κ is $\omega{\cdot}\kappa$-reasonable. By Theorem 4.3.5, $o(\Delta) = \delta(\Gamma)$. By Theorem 4.2.13 and Remark 4.2.3, there exist a set $X \subseteq \omega^\omega$ in $\Gamma \setminus \check{\Gamma}$ and a surjection $f\colon X \to \kappa$ such that, for each $x \in X$, the set

$$\{y \in X : f(y) \leq f(x)\}$$

is in Δ. Let, for each $x \in X$,

$$[x]_f = \{y \in X : f(y) = f(x)\},$$

$$[< x]_f = \{y \in X : f(y) < f(x)\}$$

and

$$C_x = \omega^\omega \setminus ([x]_f \cup [< x]_f).$$

By Theorem 4.3.2, each of these sets is in Δ.

Fix a sequence $\bar{U} = \langle U_n : n < \omega \setminus \{0\}\rangle$ of $\underset{\sim}{\Sigma}^1_1$-universal sets with the s-m-n property. Define $\phi\colon \omega^\omega \to \mathcal{P}(\kappa{\times}\kappa)$ by setting, for each $z \in \omega^\omega$ and $(\beta, \gamma) \in \kappa{\times}\kappa$, $(\beta, \gamma) \in \phi(z)$ if and only if there exist x, y in X with

- $f(x) = \beta$,
- $f(y) = \gamma$ and
- $(x, y) \in U_{2,z}([x]_f, C_x)$.
- for all $(v, w) \in U_{2,z}([x]_f, C_x)$, $f(w) = \gamma$.

The Uniform Coding Lemma (Theorem 3.0.3) implies that part (1) of Definition 5.1.1 is satisfied. To see this, given $F\colon \kappa \to \kappa$, let Z_F be

$$\{(x, y) \in X \times X : F(f(x)) = f(y)\}.$$

By the Uniform Coding Lemma, there is a $z \in \omega^\omega$ such that, for all $x \in X$,

1. $U_{2,z}([x]_f, C_x) \subseteq Z_F \cap ([x]_f \times \omega^\omega)$,
2. $U_{2,z}([x]_f, C_x) \neq \emptyset$ if and only if $Z_F \cap ([x]_f \times \omega^\omega) \neq \emptyset$.

It follows from this that $\phi(z) = F$.

To see that part (2) of Definition 5.1.1 is satisfied, fix $(\beta, \gamma) \in \kappa \times \kappa$. By the definition of $R_{\beta,\gamma}$, for each $z \in \omega^\omega$, $z \in R_{\beta,\gamma}$ implies $(\beta, \gamma) \in \phi(z)$. The reverse implication follows from last condition on the definition of ϕ. It suffices then to see that $\{z \in \omega^\omega : (\beta, \gamma) \in \phi(z)\}$ is in Δ. This can be seen by fixing $x^*, y^* \in X$ such that $f(x^*) = \beta$ and $f(y^*) = \gamma$, and applying the fact that Δ is a projective algebra.

For part (3), fix $\beta < \kappa$ and $A \subseteq R_\beta$ with A in $\exists^{\omega^\omega}\Delta$ (which is the same as Δ). Fix $x^* \in X$ with $f(x^*) = \beta$. Let D be the set of $y \in \omega^\omega$ for which there exist $z \in A$ and $x \in [x^*]_f$ with $(x, y) \in U_{2,z}([x^*]_f, C_{x^*})$. Then $D \subseteq X$ and $D \in \Delta$, so $f[D]$ is a bounded subset of κ, by Lemma 4.2.4. □

We now finally finish the proofs of Theorems 5.0.3 and 5.0.6. Given ordinals μ, κ, let us define (μ, κ)-*club directedness* to be the statement that if $\langle \mathcal{C}_\alpha : \alpha < \mu \rangle$ is a sequence of nonempty sets of club subsets of κ, then there is a club subset of κ contained in at least one member of each $\mathcal{C}_\alpha$. The proof of Theorem 5.1.2 (i.e., the fact that the proof worked with any cofinal $C \subseteq C_0$) shows that, for each $\mu < \kappa$, the statement $\kappa \to (\kappa)^\lambda_\mu$ follows from the hypotheses of that theorem, if one adds to the hypotheses the statement that (μ, κ)-club directedness holds. We derive this latter property from the Martin conditions and the Moschovakis Coding Lemma.

Theorem 5.1.4. *Suppose that* AD *holds. Let* Γ *be a nonselfdual bolface pointclass closed under* $\forall^{\omega^\omega}$ *and* $\vee$, *and assume that* Γ *has the prewellordering property. Let* κ *a regular uncountable cardinal. If* Γ *witnesses that* κ *is* $\omega \cdot \kappa$-*reasonable, then, for all* $\mu < \kappa$, (μ, κ)-*club directedness holds.*

Proof. Let ϕ, $R_{\beta,\gamma}$ $(\beta < \lambda, \gamma < \kappa)$ and R_β $(\beta < \lambda)$ be as in Definition 5.1.1, with respect to $\omega \cdot \kappa$, κ and Γ. Fix $\mu < \kappa$, and let $\langle \mathcal{C}_\alpha : \alpha < \mu \rangle$ be a sequence of nonempty sets of club subsets of κ. Let X be $\bigcup_{\alpha<\mu} R_{0,\alpha}$, and let $f \colon X \to \mu$ be such that $x \in R_{0,f(x)}$ for all $x \in X$. By Theorem 4.3.2, the corresponding relations $=_f$ and $<_f$ from the statement of the Moschovakis Coding Lemma (Theorem 3.0.1) are in Δ. Let Z be the set of $(x, y) \in X \times \omega^\omega$ such that $\phi(y)$ is an increasing function whose range is in $\mathcal{C}_{f(x)}$. Let A be as in the conclusion of Theorem 3.0.1, with respect to X, Z, and f. Then A is in $\exists^{\omega^\omega}\Delta$, and so is $A_1 = \{y : \exists x(x, y) \in A\}$. Furthermore, $A_1 \subseteq \bigcap_{\beta<\kappa} R_\beta$. If follows from condition (3) of Definition 5.1.1 that there is a function $g \colon \kappa \to \kappa$ such that

$$A \subseteq \bigcap_{\beta<\kappa} \bigcup_{\gamma<g(\beta)} R_{\beta,\gamma}.$$

Then the closure points of g form a club subset of κ contained in at least one member of each $\mathcal{C}_\alpha$. $\square$

Chapter 6

Suslin sets and Uniformization

Recall that a *tree* T on a set X is a set of finite sequences from X which is closed under initial segments. We write $[T]$ for the set of infinite sequences whose finite initial segments are all in T. If T is a tree on a product of two sets $X \times Y$, and s is a finite sequence of elements of X, then T_s denotes the set of $t \in Y^{<\omega}$ for which $(s,t) \in T$. For each $f \in X^\omega$, we write T_f for $\bigcup_{n\in\omega} T_{f\restriction n}$. The *projection* of T, $\mathrm{p}[T]$, is the set of $f \in x^\omega$ for which T_f is illfounded, i.e., for which there exists a $g \in Y^\omega$ such that $(f,g) \in [T]$ (identifying, for notational simplicity, pairs of sequences with sequences of pairs).

If T is an illfounded tree on a set X, and $\leq$ is a wellordering of X, the *leftmost branch* of T *relative to* $\leq$ ($\mathrm{lb}_{\leq}(T)$) is the unique $f \in X^\omega$ such that, for each $n \in \omega$, $f(n)$ is the $\leq$-least member of X for which the tree

$$\{\sigma \in X^{<\omega} : (f\restriction n)^\frown\sigma \in T\}$$

is illfounded. When X is an ordinal or a finite product of ordinals we use the ordinal ordering or the corresponding lexicographical ordering for $\leq$ and write $\mathrm{lb}(T)$.

6.0.1 Definition. Given ordinals γ and η, a set $A \subseteq \eta^\omega$ is γ-*Suslin* if there exists a tree $T \subseteq (\eta \times \gamma)^{<\omega}$ such that $A = \mathrm{p}[T]$. A set $A \subseteq \eta^\omega$ is *Suslin* if it is γ-Suslin for some ordinal γ.

6.0.2 Remark. Every Suslin subset of ω^ω is γ-Suslin for some $\gamma < \Theta$. To see this, suppose that α is an ordinal and T is tree on $\omega \times \alpha$, and observe that the set $\{\mathrm{lb}(T_x)(n) : x \in \mathrm{p}[T], n \in \omega\}$ is a surjective image of ω^ω, and therefore has ordertype less than Θ. Alternately one can use the fact that (assuming only ZF) if β is an ordinal with $T \in L_\beta(T,\mathbb{R})$, then $L_\beta(T,\mathbb{R})$ has an elementary submodel which contains ω^ω and is a surjective image of ω^ω.

Given an ordinal γ, we write $\mathcal{S}_\gamma$ for the smallest boldface pointclass containing each γ-Suslin subsets of ω^ω and $\mathcal{S}_{<\gamma}$ for $\bigcup_{\alpha<\gamma} \mathcal{S}_\alpha$. Then $\mathcal{S}_\gamma$ and $\mathcal{S}_{<\gamma}$ are boldface pointclasses closed under $\exists^{\omega^\omega}$, intersections and unions.

6.0.3 Remark. If γ is less than Θ, and AD holds, then, by the Coding Lemma, $\mathsf{CC}_{\mathcal{P}(\gamma)}$ holds, so $\mathcal{S}_\gamma$ is closed under countable intersections and countable unions.

The following definition is due to Kechris.

6.0.4 Definition. A cardinal κ is *Suslin* if it is infinite and there exists $A \subseteq \omega^\omega$ which is κ-Suslin but not γ-Suslin for any $\gamma < \kappa$.

In symbols, κ is Suslin if and only if $\mathcal{S}_\kappa \setminus \mathcal{S}_{<\kappa} \neq \emptyset$. By Remark 6.0.2, every Suslin cardinal is less than Θ. By Remark 6.0.3, AD implies that a limit of Suslin cardinals of countable cofinality is a Suslin cardinal if it is below Θ (more is true, as well shall see in Section 11.3).

The following theorem will be used in the proof of Theorem 6.0.7 below.

Theorem 6.0.5. *Assume that* AD *holds, and let* Γ *be a boldface selfdual pointclass closed under* $\exists^{\omega^\omega}$*, and let* κ *be an infinite cardinal such that there exists a prewellordering of* ω^ω *of length* κ *in* Γ*. Then* $\mathcal{S}_\kappa \subseteq \Gamma$.

Proof. We fix an $A \in \mathcal{S}_\kappa \cap \mathcal{P}(\omega^\omega)$ and show that $A \in \Gamma$. Replacing κ with a smaller cardinal if necessary we may assume that A is not λ-Suslin for any $\lambda < \kappa$. Let $\preceq$ be a prewellordering of ω^ω in Γ of length κ. Using a bijection between $\kappa \times \omega$ and ω if necessary, we may fix a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ such that $A = p[T]$ and such that there is a function $u \colon \kappa \to T$ such that for each $\alpha \in \kappa$, $u(\alpha)$ is the unique node (s,t) of T for which α is the last value taken by t. Let $f \colon X \to \kappa$ be the function which sends each member of X to its $\preceq$-rank. Let $\pi \colon \omega^\omega \to (\omega^{<\omega} \times (\omega^\omega)^{<\omega})$ be a recursive bijection. Let Z be the set of $(x,y) \in X \times \omega^\omega$ such that $\pi(y)$ is a tuple $(s, z_0, \ldots, z_{|s|-1})$ for some nonempty $s \in \omega^{<\omega}$, such that

- s is the first coordinate of $u(f(x)))$,
- for all positive $n < |s|$, $s \restriction n$ is the first coordinate of $u(f(z_{n-1})))$.

By the Coding Lemma, there is a set $R \subseteq Z$ such that for all x, if there exists a y with $(x,y) \in Z$, then there exist $(x',y') \in R$ with $f(x) = f(y)$. Then A is the set of $w \in \omega^\omega$ for which there exist $\langle q_i : i \in \omega\rangle \in (\omega^\omega)^\omega$ such that for all $n \in \omega$ there exist $(x,y) \in R$ such that, letting $\pi(y) = (s, z_0, \ldots, z_{|s|-1})$, $s = w \restriction n$ and, for each $m < n$, $f(z_m) = f(q_m)$. This shows that A is in Γ. $\square$

6.0.6 Definition. Let γ be an ordinal and let T be a tree on $\omega \times \gamma$. The *minimization* of T is the tree

$$\{(x \restriction n, \mathrm{lb}(T_x) \restriction n) : x \in \mathrm{p}[T], n \in \omega\}.$$

We say that T is *minimal* if it is equal to its minimization and γ-*full* if

$$\{\langle (0,\alpha)\rangle : \alpha < \gamma\} \subseteq T.$$

We note that the minimization of a tree T as in Definition 6.0.6 is minimal and has the same projection as T.

Theorem 6.0.7 (Kechris [7])**.** *Assume that* AD *holds.*[1] *If κ is a Suslin cardinal, then $\mathcal{S}_\kappa$ is not closed under complements.*[2]

Proof. Assume toward a contradiction that $\mathcal{S}_\kappa$ is closed under complements. Since $\mathcal{S}_\kappa$ is (always) $\exists^{\omega^\omega}$-closed, it follows from this assumption that it is also $\forall^{\omega^\omega}$-closed. Since $\mathcal{S}_\kappa$ is closed under countable unions, it follows from Remark 4.2.11 that it then has the prewellordering property. By Theorem 4.3.5 and the assumption that $\mathcal{S}_\kappa$ is closed under complements, the supremum of the Wadge ranks of the members of $\mathcal{S}_\kappa$ is the same as the supremum of the lengths of the prewellorderings in $\mathcal{S}_\kappa$. Call this ordinal λ.

For each $A \subseteq \omega^\omega$ in $\mathcal{S}_\kappa$, $\underset{\sim}{\Sigma}^1_1(A)$ and $\underset{\sim}{\Sigma}^1_2(A)$ are properly contained in $\mathcal{S}_\kappa$. By Remark 4.1.5 and Theorems 4.1.8 and 4.4.3, one of these two pointclasses (for each A) is closed under wellordered unions. It follows then that for each $\rho < \lambda$, any welloredered union of sets from $\mathcal{S}_\kappa$ of Wadge rank less than λ is in $\mathcal{S}_\kappa$.

Let A be a subset of ω^ω in $\mathcal{S}_\kappa \setminus \mathcal{S}_{<\kappa}$ and let Γ be the member of $\{\underset{\sim}{\Sigma}^1_1(A), \underset{\sim}{\Sigma}^1_2(A)\}$ which is closed under wellordered unions. Then $\mathcal{S}_{<\kappa}$ is contained in Γ, since every member of $\mathcal{S}_{<\kappa}$ is Wadge-reducible to A. If the cofinality of κ were uncountable, it would follow that $\mathcal{S}_\kappa \subseteq \Gamma$, giving a contradiction. To see this, let T be a tree on $\omega \times \kappa$, and for each $\alpha < \kappa$, let $T\restriction\alpha$ denote $T \cap (\omega \times \alpha)^{<\omega}$. If κ has uncountable cofinality, then $\mathrm{p}[T] = \bigcup_{\alpha<\kappa} \mathrm{p}[T\restriction\alpha]$, and the latter is a wellordered union of sets in $\mathcal{S}_{<\kappa}$.

To complete the proof, we show that κ has uncountable cofinality. Since $\mathcal{S}_\kappa$ is closed under countable unions, λ has uncountable cofinality. It suffices then to show that $\lambda = \kappa$. By Theorem 6.0.5, $\lambda \leq \kappa$. To show that $\kappa \leq \lambda$, let T be a minimal tree on $\omega \times \kappa$ projecting to A. Then T has cardinality κ. For each $\sigma \in T$, let A_σ be the set of $x \in A$ for which σ is an initial segment of $(x, \mathrm{lb}(T_x))$. For each $\rho < \lambda$, let $T\restriction\rho$ be the set of $\sigma \in T$ for which the Wadge rank of A_σ is less than ρ. For each $n \in \omega$ and $\rho < \lambda$, let $<_{n,\rho}$ be the lexicographical order on the members of $T\restriction\rho$ of length n. We claim that each $<_{n,\rho}$ has ordertype at most λ. The claim completes the proof of the theorem, since it implies that T is a union of λ many sets of cardinality at most λ. To prove the claim, fix $n \in \omega$ and $\rho < \lambda$. The minimality of T implies that for each $\sigma \in T\restriction\rho$ of length n, the set $B_\sigma = A_\sigma \setminus \bigcup\{A_\tau : \tau <_{n,\rho} \sigma\}$ is nonempty. The union of all the sets $B_\tau \times B_\sigma$ for $\tau <_n \sigma$ is in $\mathcal{S}_\kappa$, and is a prewellordering with the same length as $<_{n,\rho}$, which must therefore be less than λ. □

6.0.8 Remark. If Wadge Determinacy and Baire($\mathcal{P}(\omega^\omega)$) hold, and $\kappa < \lambda$ are Suslin cardinal, then, by Corollary 2.3.2, $\check{\mathcal{S}}_\kappa \subseteq \mathcal{S}_\lambda$. It follows (under the same assumption) that if κ is a limit of Suslin cardinals, then $\mathcal{S}_{<\kappa}$ is a projective algebra. If AD holds, λ is a Suslin cardinal and Δ is a projective algebra contained in S_λ, then $\mathcal{S}_\lambda$ contains $\bigcup_\omega \Delta$ (see Remark 6.0.3). If in addition Δ is not closed under countable unions then, by Theorem 4.4.2, every wellordered union of sets in Δ is in $\mathcal{S}_\lambda$.

[1] And more, e.g., $\mathcal{W} = \mathcal{P}(\omega^\omega)$?

[2] Does this argument show that for every $A \in \mathcal{S}_\kappa \setminus \mathcal{S}_{<\kappa}$, $\omega^\omega \setminus A \notin \mathcal{S}_\kappa$?

6.0.9 Remark. By the Moschovakis Coding Lemma, if AD holds then for each $\kappa < \Theta$ is there is a subset of ω^ω which is not κ-Suslin. It follows that if every subset of ω^ω is Suslin, then the Suslin cardinals are cofinal in Θ. The converse is shown in Theorem 6.0.13 below.

Lemma 6.0.11 below shows that if λ is a Suslin cardinal, then there exists a strictly $\subseteq$-increasing λ-sequence of Suslin sets, induced by a single tree on $\omega \times \lambda$.[3]

Let $<^2_\ell$ be the following version of the lexicographical order on finite sequences of pairs of ordinals : $(s_0, t_0) <^2_\ell (s_1, t_1)$ if and only if there exists an $i \in \omega$ such that the following hold:

- $(s_0 \restriction i, t_0 \restriction i) = (s_1 \restriction i, t_1 \restriction i)$;
- $s_0(i) < s_1(i) \vee (s_0(i) = s_1(i) \wedge t_0(i) < t_1(i))$.

If A is a set of pairwise incompatible sequences, and $n \in \omega$ is such that each member of A has length at most n, then the restriction of $<^2_\ell$ to A wellorders A.

Lemma 6.0.10. *Let γ and η be ordinals, let T be a minimal tree on $\omega \times \gamma$, and let $A \subseteq T$ be an antichain such that*

- *every element of $[T]$ intersects A;*
- *the restriction of $<^2_\ell$ to A has ordertype η.*

Then there is a minimal η-full tree T' on $\omega \times \max\{\gamma, \eta\}$, definable from A and T, such that $\mathrm{p}[T] = \mathrm{p}[T']$.

Proof. Enumerate A in $<^2_\ell$-order as $\langle (s_\alpha : t_\alpha) : \alpha < \eta \rangle$. For each $\alpha < \eta$, let

- T_α be the set of $(s, t) \in T$ which are compatible with (s_α, t_α);
- $f_\alpha \colon \gamma^{<\omega} \to \max\{\gamma, \eta\}^{<\omega}$ be defined by setting
  - $|f_\alpha(t)| = |t|$;
  - for all $i \in |t| \cap |t_\alpha|$, $f_\alpha(t)(i) = \alpha$;
  - for all $i \in |t| \setminus |t_\alpha|$, $f_\alpha(t)(i) = t(i)$;
- T'_α be $\{(s, f_\alpha(t)) : (s, t) \in T_\alpha\}$.

Let $T' = \bigcup_{\alpha < \eta} T_\alpha$. Then T' is η-full, and since $\mathrm{p}[T_\alpha] = \mathrm{p}[T'_\alpha]$ for all $\alpha < \eta$, and every element of $[T]$ intersects A, $\mathrm{p}[T'] = \mathrm{p}[T]$. To see that T' is minimal, fix $(s', t') \in T'$ and $\alpha < \eta$ such that $(s', t') \in T'_\alpha$. It suffices to consider the case where $|s'| \geq |s_\alpha|$. Let $(s, t) \in T_\alpha$ be such that $s' = s$ and $t' = f_\alpha(t)$. Since T is minimal, there exists an $x \in \mathrm{p}[T_\alpha]$ such that (s, t) is an initial segment of $(x, \mathrm{lb}(T_x))$. Applying the definition of $<^2_\ell$ we have that for all $\beta < \alpha$, $x \notin \mathrm{p}[T_\beta]$, so $x \notin \mathrm{p}[T'_\beta]$. Then $\mathrm{lb}(T'_x) = \mathrm{lb}(T'_{\alpha,x})$, and t' is an initial segment of $\mathrm{lb}(T'_x)$ as desired. □

[3] Irrelevant question : does the conclusion of Lemma 6.0.11 imply that λ is a Suslin cardinal?

Lemma 6.0.11. *If λ is a Suslin cardinal, there is a minimal λ-full tree on $\omega \times \lambda$.*

Proof. Let T_0 be a tree on $\omega \times \lambda$ projecting to a subset of ω^ω which is not γ-Suslin for any $\gamma < \lambda$, and such that each member of $\mathrm{p}[T_0]$ takes value 0 at 0. Let T_1 be the minimization of T_0. Since T_0 witnesses that λ is Suslin, $|T_1| = \lambda$.

$$\{(x\restriction n, \mathrm{lb}(T_{0,x})\restriction n) : x \in \mathrm{p}[T_0], n \in \omega\}.$$

Then $|T_1| = \lambda$ and T_1 is minimal. If some level of T_1 has $<^2_\ell$-ordertype at least λ, let A be the length-λ $<^2_\ell$-initial segment of such a level, and let T_1' be the set of nodes of T_1 compatible with a member of A. Then T_1' is minimal, and the lemma follows by applying Lemma 6.0.10 with T_1' and A.

Suppose then that T_1 does not contain such an antichain. For each $n \in \omega$, let κ_n be the cardinality of the set of nodes of T_1 of length n. Then each κ_n is less than λ, and $\sup_{n\in\omega} \kappa_n = \lambda$. We can then complete the proof by using (for each $n \in \omega$) the nth level to produce a minimal κ_n-full tree on $\omega \times \kappa_n$, modifying the trees to make their domains disjoint, and combining these trees to make a minimal λ-full tree on $\omega \times \lambda$. We give the details below.

For each $n \in \omega$, let A_n be the length-κ_n $<^2_\ell$-initial segment of the nth level of T_1, and let $T_{1,n}$ be the set of nodes of $T_{1,n}$ compatible with a member of A_n. Then $T_{1,n}$ is minimal; let $T_{2,n}$ be the result of applying Lemma 6.0.10 to $T_{1,n}$ and A_n.

Let $\pi\colon \omega \times \omega \to \omega$ be an injection, and for each $n \in \omega$ let $\pi^*_{0,n}$ be the length-preserving function on $\omega^{<\omega}$ defined by setting $\pi^*_{0,n}(s)(0)$ to be $s(0)$ (in the case where s is nonempty) and $\pi^*_{0,n}(s)(i) = \pi(n,i)$ for all nonzero $i \in \mathrm{dom}(s)$. Again for each $n \in \omega$, let $\gamma_n = \sum_{m<n} \kappa_m$ (using ordinal addition) and let $\pi^*_{1,n}$ be the length-preserving function on $\gamma^{<\omega}$ defined by setting $\pi^*_{0,n}(t)(0)$ to be $\gamma_n + t(0)$ (in the case where s is nonempty) and $\pi^*_{0,n}(t)(i) = t(i)$ for all nonzero $i \in \mathrm{dom}(t)$. Finally, let T_2 be

$$\{(\pi^*_{0,n}(s), \pi^*_{1,n}(t)) : n \in \omega, (s,t) \in T_{2,n}\}.$$

Then T_2 is as desired. □

6.0.12 Remark. Suppose that AD holds, λ is an ordinal and that T is a minimal λ-full tree on $\omega \times \lambda$. For each $\alpha < \lambda$, let

- T_α be the set of $\sigma \in T$ which are either empty or have $(0, \beta)$, for some $\beta < \alpha$ as their first element;
- $A_\alpha = \mathrm{p}[T_\alpha]$.

Then $\langle A_\alpha : \alpha < \lambda\rangle$ is a strictly $\subseteq$-increasing sequence of λ-Suslin sets. Moreover, for each pair $\alpha \leq \beta$ below λ, $A_\alpha \times A_\beta$ is λ-Suslin. Let $\leq = \bigcup_{\alpha\leq\beta<\lambda} A_\alpha \times A_\beta$. Then $\leq$ is a prewellordering of length λ. Let Δ be a projective algebra containing $\mathcal{S}_\lambda$ such that $\bigcup_\omega \Delta \not\subseteq \Delta$.[4] If AD holds, then, by Theorem 4.4.2, $\leq$ is in $\bigcup_\omega \Delta$.

[4]By Kechris's theorem that $\mathcal{S}_\kappa$ is not closed under complements, the smallest projective algebra containing S_λ works.

Still assuming AD, if κ is a Suslin cardinal and $\Delta \subseteq \mathcal{S}_\kappa$ (which holds by Remark 6.0.8 if there are infinitely many Suslin cardinals between λ and κ), then $\leq$ is in $\mathcal{S}_\kappa$, by Remark 6.0.3.

Theorem 6.0.13. *If* AD *holds, and the Suslin cardinals are cofinal in* Θ*, then every subset of* ω^ω *is Suslin.*

Proof. Suppose towards a contradiction that there is a subset of ω^ω which is not Suslin. It follows then by Wadge Determinacy that there is a set $A \subseteq \omega^\omega$ such that every Suslin set is Wadge reducible to A. Let Δ be the smallest projective algebra with A as a member. Then $\bigcup_\omega \Delta$ is also a proper initial segment of the Wadge hierarchy, and $\bigcup_\omega \Delta \not\subseteq \Delta$.[5] It follows from the first paragraph of Remark 6.0.12 that for each Suslin cardinal κ there is a prewellordering in $\bigcup_\omega \Delta$ of length κ, contradicting the definition of Θ. □

6.1 Uniformization

Given sets X, Y and $A \subseteq X \times Y$, we say that a function $f\colon X \to Y$ *uniformizes* A if $\mathrm{dom}(f) = \{x \in X \mid A_x \neq \emptyset\}$, and $(x, f(x)) \in A$ for all $x \in \mathrm{dom}(f)$. We say then that A is *uniformized.* If $A \subseteq \alpha^\omega \times \beta^\omega$ is the projection of a tree T on $\alpha \times \beta \times \gamma$, for some ordinals α, β and γ, then a uniformizing function for A can be constructed from T by considering T as a tree on $\alpha \times (\beta \times \gamma)$ and using the function $x \mapsto \mathrm{lb}(T_x)$ for $x \in \mathrm{p}[T]$. This gives the following fact.

Theorem 6.1.1. *Every Suslin subset of* $(\omega^\omega)^2$ *is uniformized.*

We let Uniformization be the statement that every subset of $(\omega^\omega)^2$ has a uniformizing function. Note that Uniformization easily implies $\mathsf{DC}_\mathbb{R}$.

6.1.2 Definition. Given a set X, a set $A \subseteq X^\omega$ is *quasi-determined* (as a subset of X^ω) if there is a function $\pi\colon X^{<\omega} \to \mathcal{P}(X) \setminus \{\emptyset\}$ such that one of the two following statements holds.

1. For every $x \in X^\omega$, if $x(2n) \in \pi(x \restriction 2n)$ holds for all $n \in \omega$, then $x \in A$.
2. For every $x \in X^\omega$, if $x(2n+1) \in \pi(x \restriction (2n+1))$ hold for all $n \in \omega$, then $x \notin A$.

A function $\pi\colon X^{<\omega} \to \mathcal{P}(X) \setminus \{0\}$ is then said to be a *quasi-strategy* in the game G_A. Such a function π as in case (1) above is a *winning quasi-strategy* for player I; in case (2) it is a winning quasi-strategy for player II. We let quasi-AD_X denote the statement that every subset of X^ω is quasi-determined, as a subset of X^ω.

If X is wellorderable, then quasi-AD_X and AD_X are equivalent. Remark 6.1.3 shows that a form of uniformization suffices to prove this.

[5] Say why, for both parts of the sentence?

6.1.3 Remark. If X is a set, and every subset of $X^{<\omega} \times X$ is uniformized, then every quasi-determined game on X is determined. As quasi-$\mathsf{AD}_{\mathbb{R}}$ implies Uniformization (via a game where I plays a real, and II plays the infinite string of coordinates of a partner real), quasi-$\mathsf{AD}_{\mathbb{R}}$ and $\mathsf{AD}_{\mathbb{R}}$ are equivalent.

Given sets $t_1, \ldots, t_n$, we let $\mathrm{OD}_{t_1,\ldots,t_n}$ denote the class of sets which are definable from $t_1, \ldots, t_n$ and finite sequence of ordinals (i.e., which are ordinal definable from $t_1, \ldots, t_n$). For each finite sequence $t_1, \ldots, t_n$, there is a definable class-length wellordering of $\mathrm{OD}_{t_1,\ldots,t_n}$.

Theorem 6.1.4 (Kechris-Solovay)**.** *Suppose that there is a set Z such that every subset of ω^ω is ordinal definable from Z and a member of ω^ω, and that for all $x \in \omega^\omega$, $\omega^\omega \not\subseteq \mathrm{OD}_{Z,x}$. Then the set $\{(x, y) \in (\omega^\omega)^2 \mid y \notin \mathrm{OD}_{Z,x}\}$ is not uniformizable.*

Proof. Any uniformizing function would be ordinal definable from Z and some $x \in \omega^\omega$, which would mean that $f(x) \in \mathrm{OD}_{Z,x}$. □

The following theorem will be used in the proof of Corollary 9.1.7.[6]

Theorem 6.1.5 (Martin-Steel[21])**.** *In $L(\mathbb{R})$, every $\utilde{\Sigma}^2_1$ set is Suslin.*[7]

Theorem 6.1.4 can be used to show that $\utilde{\Pi}^2_1$ sets are not Suslin in $L(\mathbb{R})$.

6.2 The Kunen-Martin property

Every $\utilde{\Sigma}^1_1$ prewellordering of ω^ω has countable length (see [23, 15], for instance). Applying this fact in a forcing extension by $\mathrm{Col}(\omega, \kappa)$, we get the following.

Theorem 6.2.1 (Kunen-Martin)**.** *Every κ-Suslin prewellordering has length less than κ^+.*

Suppose that Δ is a boldface pointclass, and that $\{A_i : i < \omega\} \subseteq \Delta$ is Wadge-cofinal in Δ. Let A^* be the set of $x \in \omega^\omega$ such that $\langle x(1), x(2), \ldots\rangle \in A_{x_0}$. We say that Δ has the *Kunen-Martin property* if $w(A)^+ = \delta^1_1(A)$.[8]

Becker [1] showed that if Δ is a projectively closed pointclass with countable Wadge cofinality, every set in Δ is uniformized by a set in Δ, and Δ satisfies the Kunen-Martin property, then δ_Δ is a Suslin cardinal. Woodin showed that the Kunen-Martin property followed from the other hypotheses in this implication.

6.3 A pointclass fact

Theorem 6.3.1. *Assume that* AD *holds and let Δ be a boldface selfdual pointclass closed under $\exists^{\omega^\omega}$, and let κ be a cardinal less than* $\mathrm{cof}(o(\Delta))$*. Then $\mathcal{S}_\kappa \subseteq \Delta$.*

[6] Isn't it also used to reflect Ordinal Determinacy?
[7] Assuming $\mathsf{DC}_{\mathbb{R}}$? Say more?
[8] Change to our notation.

Proof. Let A be a κ-Suslin subset of ω^ω. We may assume that A is not λ-Suslin for any $\lambda < \kappa$. By Proposition 2.5.8, we may fix a prewellordering $\preceq$ in Δ of length κ on a set X. Using a bijection between $\kappa \times \omega$ and ω if necessary, we may fix a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ such that $A = p[T]$ and such that there is a function $u \colon \kappa \to T$ such that for each $\alpha \in \kappa$, $u(\alpha)$ is the unique node (s, t) of T for which α is the last value taken by t. Let $f \colon X \to \kappa$ be the function which sends each member of X to its $\preceq$-rank. Let $\pi \colon \omega^\omega \to (\omega^{<\omega} \times (\omega^\omega)^{<\omega})$ be a recursive bijection. Let Z be the set of $(x, y) \in X \times \omega^\omega$ such that $\pi(y)$ is a tuple $(s, z_0, \ldots, z_{|s|-1})$ for some nonempty $s \in \omega^{<\omega}$, such that

- s is the first coordinate of $u(f(x)))$,
- for all positive $n < |s|$, $s{\restriction}n$ is the first coordinate of $u(f(z_{n-1})))$.

By the Coding Lemma, there is a set $R \subseteq Z$ such that for all x, if there exists a y with $(x, y) \in Z$, then there exist $(x', y') \in R$ with $f(x) = f(y)$. Then A is the set of $w \in \omega^\omega$ for which there exist $\langle q_i : i \in \omega\rangle \in (\omega^\omega)^\omega$ such that for all $n \in \omega$ there exist $(x, y) \in R$ such that, letting $\pi(y) = (s, z_0, \ldots, z_{|s|-1})$, $s = w{\restriction}n$ and, for each $m < n$, $f(z_m) = f(q_m)$. This shows that A is in Δ. □

6.4 The Solovay sequence

6.4.1 Definition. The *Solovay sequence* is the unique sequence $\langle \theta_\alpha : \alpha \leq \Omega\rangle$ satisfying the following conditions.

- θ_0 is the least ordinal which is not the surjective image of ω^ω via an ordinal definable function.
- For each $\alpha < \Omega$, $\theta_{\alpha+1}$ is the least ordinal which is not the surjective image of ω^ω via a function which is ordinal definable from an element of $\mathcal{P}(\omega^\omega)$ of Wadge rank θ_α.
- For limit ordinals $\beta \leq \Theta$, $\theta_\beta = \sup_{\beta<\gamma} \theta_\alpha$.
- $\theta_\Omega = \Theta$.

The ordinal Ω is called the *length of the Solovay sequence*.[9]

6.4.2 Remark. While it is possible for nonlimit elements of the Solovay sequence to be singular[10], if the length of the Solovay sequence is not a limit ordinal, then Θ is regular, since in this case there is an $A \subset \omega^\omega$ such that every element of $\mathcal{P}(\omega^\omega)$ is ordinal definable from A and an element of ω^ω.

For each $\xi < \Theta$, let $\mathcal{W}_\xi$ be the collection of subsets of ω^ω of Wadge rank less than ξ. It follows almost immediately from the definition of the Solovay sequence that for each θ on the Solovay sequence $\mathcal{P}(\omega^\omega) \cap L(\mathcal{W}_\theta) = \mathcal{W}_\theta$, since

[9] There are two other possible definitions of the Solovay sequence. Comment on the choice of the set of Wadge rank θ_α?
[10] reference

every element of $\mathcal{P}(\omega^\omega) \cap L(\mathcal{W}_\theta)$ is ordinal definable from an element of $\mathcal{W}_\theta$. It follows from this that if $\mathsf{AD}_\mathbb{R}$ holds, and the length of the Solovay sequence has uncountable cofinality, then $L(\mathcal{W}_\theta) \models \mathsf{AD}_\mathbb{R}$ for club many θ on the Solovay sequence.

Theorem 2.5.9 and 6.1.4 give the following.[11]

Theorem 6.4.3. *If* Wadge Determinacy + Uniformization + $\aleph_1 \not\leq 2^{\aleph_0}$ *holds, then the length of the Solovay sequence is a limit ordinal.*

Solovay [25] showed that $\mathsf{AD}_\mathbb{R}$ + $V{=}L(\mathcal{P}(\omega^\omega))$, DC holds if and only if the length of the Solovay sequence has uncountable cofinality. For the easier direction, note that Wadge Determinacy + $\mathsf{DC}_{\mathcal{P}(\omega^\omega)}$ implies that $\mathrm{cof}(\Theta)$ is uncountable, since otherwise there exists an ω-sequence of sets of reals whose Wadge ranks are unbounded in Θ, and thereby a set of reals of Wadge rank at least Θ contracting Corollary 2.5.11.[12] The following theorem gives the harder direction.

Theorem 6.4.4 (Solovay [25])**.** *If* AD + $\mathsf{DC}_\mathbb{R}$ *holds,* $V{=}\mathrm{HOD}_{\mathcal{P}(\omega^\omega)}$ *and* $\mathrm{cof}(\Theta)$ *is uncountable, then* DC *holds.*

Proof. By Remark 0.4.1, it suffices to show that $\mathsf{DC}_{\mathcal{P}(\omega^\omega)}$ holds. Let T be a tree on $\mathcal{P}(\omega^\omega)$ without terminal nodes. We want to see that T has an infinite path. For each $\xi < \Theta$, let T_ξ be the set of $t \in T$ such that the range of t is contained in $\mathcal{W}_\xi$. For each $t \in T$, let $w(t)$ be the least Wadge rank of a set $A \subseteq \omega^\omega$ such that $t^\frown\langle A\rangle$ is in T.

First suppose that for each $\xi < \Theta$, $w[T_\xi]$ is bounded in Θ. Define $s\colon \Theta \to \Theta$ by setting $s(\xi)$ to be the supremum of $w[T_\xi]$. Then since $\mathrm{cof}(\Theta) > \aleph_0$ there is a $\zeta < \Theta$ closed under w. Then T_ζ is a subtree of T without terminal nodes, and since $\mathsf{DC}_\mathbb{R}$ holds we have that T_ζ contains an infinite path, so we are done.

We continue the proof under the assumption that for some $\xi_* < \Theta$, $w[T_{\xi_*}]$ is cofinal in Θ. Since T_{ξ_*} is a surjective image of ω^ω, the ordertype of $w[T_{\xi_*}]$ must be less than Θ. It follows that Θ is singular. Let λ be its cofinality and let $c\colon \lambda \to \Theta$ be increasing, continuous and cofinal.

Toward a contradiction, suppose that T does not contain an infinite path. Then each T_ξ is also wellfounded. Let $s\colon \omega^\omega \to \lambda$ be a surjection. For each $x \in \omega^\omega$, let $s'(x) = \sup\{s(y) : y \in \omega^\omega,\ y \leq_\mathrm{T} x\}$. Then s' is Turing-invariant. Since $\mathrm{cof}(\lambda) > \aleph_0$, $s'(x) < \lambda$ for each $x \in \omega^\omega$. For each $t \in T$, let r_t be the function on ω^ω defined by setting $r_t(x)$ to be 0 if some member of t has Wadge rank at least $s'(x)$, and the rank of t in $T_{s'(x)}$ otherwise. Then each r_t is Turing invariant, and $r_t < r_{t'}$ on a Turing cone whenever $t < t'$ in T. Let $R = \{r_t : t \in T\}$ and for each $\alpha < \lambda$ let $R_\alpha = \{r_t : t \in T_{c(\alpha)}\}$. Since $\mathsf{DC}_\mathbb{R}$ holds and each T_ξ is wellfounded, each ultrapower R_α/μ_T is wellfounded. Since T is illfounded, R/μ_T is illfounded. Since R is an increasing λ-union of the

[11] Note somewhere that Θ is regular if the Solovay sequence has successor (or zero) length.

[12] Maybe we need to say something about ultrapowers by μ_T, and even say what μ_T is. We might also point out that part of this argument is used later to show that $\mathsf{DC}_\mathbb{R}$ follows from the wellfoundedness of the Turing ultrapower.

sets R_α, and λ has uncountable cofinality, we get a contradiction by picking an increasing sequence of α_n's such that each $R_{\alpha_{n+1}}/\mu_{\mathrm{T}}$ contains a strict lower bound for $R_{\alpha_{n+1}}/\mu_{\mathrm{T}}$.

□

Part II

AD^+

Chapter 7

$<\Theta$-determinacy

Let λ be an ordinal, and let $\pi\colon \lambda^{<\omega} \to \omega^{<\omega}$ be such that $s \subseteq t$ implies $\pi(s) \subseteq \pi(t)$ and $\mathrm{dom}(s) = \mathrm{dom}(\pi(s))$ (we say that a function with this property is *extension-preserving*). Given $A \subseteq \omega^\omega$, let $G_{\pi,A}$ be the game in which players I and II alternate choosing ordinals $\alpha_i \in \lambda$ ($i \in \omega$), where I wins if and only if $\bigcup_{n\in\omega} \pi(\langle \alpha_0, \ldots, \alpha_n \rangle) \in A$.

I	α_0		α_2		α_4		$\ldots$
II		α_1		α_3		$\ldots$	

The game $G_{\pi,A}$

We let λ-Determinacy denote the statement that for all $\pi\colon \lambda^{<\omega} \to \omega^{<\omega}$ as above, $G_{\pi,A}$ is determined. Given an ordinal γ, we let $<\gamma$-Determinacy denote the statement that λ-Determinacy holds for all $\lambda < \gamma$.

7.0.1 Remark. We are primary interested in $<\Theta$-Determinacy, which is usually called Ordinal Determinacy. In the usual definition of Ordinal Determinacy one has an ordinal λ, a continuous function $f\colon \lambda^\omega \to \omega^\omega$ (with respect to the product topology on λ^ω) and a payoff set $A \subseteq \omega^\omega$. The corresponding game is a defined as above, except that I wins if and only if $f(\langle \alpha_i : i < \omega \rangle) \in A$. The difference is that the usual definition allows continuous functions, whereas the function mapping runs of the game into ω^ω in our definition is Lipschitz. We leave it to the reader to verify that the two definitions are equivalent.[1]

7.0.2 Remark. Consider the game of length ω in which player I plays $\alpha \in \omega_1$ (and then makes no other moves) and then player II plays (digit by digit) an element x of 2^ω, with II winning if and only if x codes (under some fixed coding) a wellordering of ω of ordertype α. By part (2) of Remark 1.0.2, if AD holds then there is no injection from ω_1 into ω^ω, so this game is not determined.[2]

[1] Say more?
[2] Say more?

The *Lusin-Sierpiński order* (sometimes called the Brouwer-Kleene order) is the ordering on wellordered-sequences of ordinals defined by setting $s <_{\mathrm{LS}} t$ if either s properly extends t or $s(\alpha) < t(\alpha)$ for n minimal such that $s(\alpha) \neq t(\alpha)$. Then $<_{\mathrm{LS}}$ is a (as defined, class-sized) linear order. Given a set T on a set of ordinals, $<_{\mathrm{LS}}$ is wellfounded if and only if T is.

As noted in Remark 5.0.1, strong partition cardinals satisfy the conditions on δ in the statement of Theorem 7.0.3

Theorem 7.0.3 (Moschovakis, Woodin). *Assume that* $\mathsf{DC}_{\mathbb{R}}$ *holds. Suppose that* $\kappa < \delta$ *are infinite cardinals, and that* δ *is regular cardinal such that*

$$\forall \mu < \delta(\delta \rightarrow (\delta)^{\mu}_{\mu})$$

holds. Let $A \subseteq \omega^{\omega}$ *be such that* A *and* $\omega^{\omega} \setminus A$ *are both* κ*-Suslin. Let* $\lambda < \delta$ *be an ordinal, and let* $\pi \colon \lambda^{<\omega} \rightarrow \omega^{<\omega}$ *be extension-preserving. Then* $G_{\pi,A}$ *is determined.*

Proof. Fix trees T and S on $\omega \times \kappa$ such that $A = \mathrm{p}[T]$ and $\omega^{\omega} \setminus A = \mathrm{p}[S]$.

For each $u \in \omega^{<\omega}$, recalling that T_u denotes the set $\{s : (u, s) \in T\}$, let

$$T^*_u = \bigcup \{T_{u\restriction n} : n \leq |u|\}$$

and let

$$S^*_u = \bigcup \{S_{u\restriction n} : n \leq |u|\}.$$

For each such u, $<_{\mathrm{LS}} \restriction T^*_u$ and $<_{\mathrm{LS}} \restriction S^*_u$ are wellorderings; let ζ_u and ξ_u denote their respective ordertypes. Then $\max\{\zeta_u, \xi_u\} < \kappa^+ < \delta$. For all $x \in \omega^{\omega}$, $x \in A$ if and only if

$$<_{\mathrm{LS}} \restriction (\bigcup_{n\in\omega} T_{x\restriction n})$$

is illfounded, and $x \notin A$ if and only if

$$<_{\mathrm{LS}} \restriction (\bigcup_{n\in\omega} S_{x\restriction n})$$

is illfounded.

Let OP^T_u denote the set of functions from $T^*_u$to δ for which, for all $s, t \in T^*_u$, if $s <_{\mathrm{LS}} t$ then $f(s) < f(t)$. Let OP^S_u denote the set of functions from S^*_u to δ for which, for all $s, t \in S^*_u$, if $s <_{\mathrm{LS}} t$ then $f(s) < f(t)$.

We define two games of length ω, G^T and G^S. In G^T, player I plays ordinals $\alpha_i \in \lambda$ (for $i \in \omega$ even) and player II plays pairs (α_i, f_i) (for $i \in \omega$ odd) such that each α_i is in λ and each f_i is a function in $OP^T_{\pi(\langle\alpha_0,\ldots,\alpha_i\rangle)}$. Player II wins a run G^T if and only if $f_i \subseteq f_j$ for all $i < j$ in ω.

I	α_0		α_2		α_4		$\ldots$
II		α_1, f_1		α_3, f_3		$\ldots$	

The game G^T

In G^S, player II plays ordinals $\alpha_i \in \lambda$ (for $i \in \omega$ odd) and player I plays pairs (α_i, f_i) (for $i \in \omega$ even) such that each α_i is in λ and each f_i is a function in $OP^S_{\pi(\langle\alpha_0,\ldots,\alpha_i\rangle)}$. Player I wins a run of G^S if and only if $f_i \subseteq f_j$ for all $i < j$ in ω.

I	α_0, f_0		α_2, f_2		α_4, f_4		$\ldots$
II		α_1		α_3		$\ldots$	

The game G^S

The games G^T and G^S are each closed. Therefore, in G^T, either player I has a winning strategy, or player II has a winning quasi-strategy (not necessarily a strategy, as the set of possible moves for II may not be wellorderable), and in G^S, either player II has a winning strategy, or player I has a winning quasi-strategy. It cannot be, however, that player II has a winning quasi-strategy τ_{II} in G^T and player I has a winning quasi-strategy τ_I in G^S. Supposing that such quasi-strategies existed, using $\mathsf{DC}_{\mathbb{R}}$, the Moschovakis Coding Lemma, the fact that $\delta < \Theta$ and the fact that each function f_i as above can be coded by a subset of δ, we could find $\langle(\alpha_i, f_i) : i \in \omega\rangle$ such that

$$\langle\alpha_0, (\alpha_1, f_1), \alpha_2, (\alpha_3, f_3), \ldots\rangle$$

is according to τ_{II} and

$$\langle(\alpha_0, f_0), \alpha_1, (\alpha_2, f_2), \alpha_3, \ldots\rangle$$

is according to τ_I. Then $\bigcup_{i\in\omega} f_{2i}$ would witness that $\pi(\langle\alpha_i : i \in \omega\rangle) \notin A$, and $\bigcup_{i\in\omega} f_{2i+1}$ would witness that $\pi(\langle\alpha_i : i \in \omega\rangle) \in A$, giving a contradiction.

The proof is then completed by proving the following claims.

Claim 1. *If I has a winning strategy in G^T, then I has a winning strategy in $G_{\pi,A}$.*

Claim 2. *If II has a winning strategy in G^S, then II has a winning strategy in $G_{\pi,A}$.*

We will prove Claim 1 and leave the proof of Claim 2, which is a routine modification, to the reader (Claim 2 can also be proved directly from Claim 1 by modifying the payoff set A). Recall that for an ordinal α and a set of ordinals X, $[X]^\alpha$ denotes the collection of subsets of X of ordertype α. For notational convenience, we identify each element of $[X]^\alpha$ with the corresponding order-preserving function on α which enumerates it.

Fix for this paragraph an ordinal $\alpha < \delta$. For each $g \in [\delta]^{\omega\alpha}$ (for some $\alpha \leq \delta$), let g^* be the function on α defined by setting $g^*(\xi)$ to be $\sup\{g(\omega\cdot\xi+n) : n \in \omega\}$ for each $\xi < \alpha$. For each $X \subseteq \delta$, let $X(\alpha)$ denote $\{g^* : g \in [X]^{\omega\alpha}\}$. Let μ_α be the set of $\mathcal{A} \subseteq [\delta]^\alpha$ for which $C(\alpha) \subseteq \mathcal{A}$, for some club $C \subseteq \delta$.

Subclaim. *For each $\alpha < \delta$, μ_α is a δ-complete ultrafilter on $[\delta]^\alpha$.*

To prove the subclaim, fix $\rho < \delta$ and suppose that $[\delta]^\alpha$ is the union of sets $\mathcal{A}_\beta$ ($\beta < \rho$). For each $\beta < \rho$, let $\mathcal{B}_\beta$ be the set of $g \in [\delta]^{\omega\alpha}$ for which $g^* \in \mathcal{A}_\beta$. Since $\delta \to (\delta)^{\omega\alpha}_\rho$ holds, there exist an unbounded $H \in [\delta]^\delta$ and a $\beta < \rho$ such that $[H]^{\omega\alpha} \subseteq \mathcal{B}_\beta$. Let C be the set of limit points of H. Then C is a club subset of δ. Given $h \in [C]^{\omega\alpha}$, define $g \in [H]^{\omega\alpha}$ by setting $g(\gamma) = \min(H \setminus h(\gamma))$ for all $\gamma < \omega\alpha$. Then $h^* = g^*$, which is in $\mathcal{A}_\beta$. It follows that $C(\alpha) \subseteq \mathcal{A}_\beta$ as desired. This ends the proof of the subclaim.

For each $u \in \omega^{<\omega}$, let Q_u be the bijection from $[\delta]^{\zeta_u}$ to OP^T_u with the property that $Q_u(g)$ and g have the same range, for all $g \in [\delta]^{\zeta_u}$. Let ν_u be $\{Q_u[\mathcal{A}] : \mathcal{A} \in \mu_{\zeta_u}\}$. Then ν_u a δ-complete ultrafilter on OP^T_u.

Fix a winning strategy Σ for I in G^T. We define a strategy σ for I in $G_{\pi,A}$. We let $\sigma(\langle\rangle) = \Sigma(\langle\rangle)$. For each positive $n \in \omega$, we let $\sigma(\langle\alpha_0, \ldots, \alpha_{2n-1}\rangle)$, be the unique value α such that, letting $u = \pi(\langle\alpha_0, \ldots, \alpha_{2n-1}\rangle)$, for ν_u-many $f \in OP^T_u$,

$$\Sigma(\alpha_0, (\alpha_1, f{\restriction}T^*_{u\restriction 2}), \alpha_2, (\alpha_3, f{\restriction}T^*_{u\restriction 4}), \ldots, \alpha_{2n-2}, (\alpha_{2n-1}, T^*_u)) = \alpha.$$

Now suppose that $\langle\alpha_i : i \in \omega\rangle$ is the result of a run of $G_{\pi,A}$ according to σ, and that I has lost, so that $\pi(\langle\alpha_i : i \in \omega\rangle)$ is not in A. Let $x = \pi(\langle\alpha_i : i \in \omega\rangle)$. Then $x \notin p[T]$, so $<_{\mathrm{LS}} {\restriction} \bigcup_{n\in\omega} T_{x\restriction n}$ is wellfounded and there exists a function $f \colon \bigcup_{n\in\omega} T_{x\restriction n} \to \delta$ preserving $<_{\mathrm{LS}}$, i.e., such that $f{\restriction}T^*_{x\restriction n} \in OP^T_{x\restriction n}$ for all $n \in \omega$.

For each positive even $n \in \omega$, let $\mathcal{A}_n$ be the set in $\nu_{x\restriction n}$ used to choose α_n. Using $\mathsf{DC}_{\mathbb{R}}$, the Moschovakis Coding Lemma and the fact that $\delta < \Theta$, we can find a sequence $\langle C_n : n \in \omega \text{ even}\rangle$ of club subsets of δ such that each C_n witnesses (via the corresponding function $Q_{x\restriction n}$) that the corresponding set $\mathcal{A}_n$ is in ν_n (so, for all $g \in [C_n]^{\omega\zeta_{x\restriction n}}$, $Q_{x\restriction n}(g^*)$ is in $\mathcal{A}_n$). Let C be the intersection of the C_n's, and let C' be the set of $\beta \in C$ such that the ordertype of $C \cap \beta$ is $\gamma + \omega$, for some ordinal γ. Let g be an order-preserving function from the range of f into C'. Then for all even $n \in \omega$, $Q^{-1}_{x\restriction n}((g \circ f){\restriction}T^*_{x\restriction n})$ is equal to h^* for some $h \in [C]^{\omega\zeta_{x\restriction n}}$, so $(g \circ f){\restriction}T^*_{x\restriction n}$ is in $\mathcal{A}_n$. Letting $f_{n-1} = (g \circ f){\restriction}T^*_{x\restriction n}$ for each even $n \in \omega \setminus \{0\}$, we have that the run of G^T given by

$$\alpha_0, (\alpha_1, f_1), \alpha_2, (\alpha_3, f_3), \ldots$$

is according to Σ and yet losing for player I, giving a contradiction. □

Again, the Moschovakis Coding Lemma gives us that determinacy for continuous games on ${}^\omega\lambda$ is local.[3] Similarly, the Martin-Steel theorem gives us that, assuming AD, $<\Theta$-determinacy holds in $L(\mathbb{R})$.[4]

[3] explain

[4] and explain

Chapter 8

Cone measures

Given an ordered equivalence relation $(E, \leq_E)$, we let WF_E denote the assertion that the ultrapower $\mathrm{Ord}^{\mathcal{C}_E}/\mu_E$ is wellfounded, where $\mathcal{C}_E$ denotes the set of E-classes. Note that if $(E, \leq_E)$ is as thick as an ordered equivalence relation $(F, \leq_F)$, then $\mathrm{Ord}^{\mathcal{C}_E}/\mu_E$ embeds into $\mathrm{Ord}^{\mathcal{C}_F}/\mu_F$, so WF_F implies WF_E.

We record the following basic fact about ultrapowers of this type, which appears as Lemma 3.3 of [16].

Theorem 8.0.1 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Let S be a set of ordinals and let $\leq_S$ be the binary relation on ω^ω defined by setting $x \leq_S y$ if and only if $x \in \mathrm{HOD}_{\{S,y\}}$. Let $\equiv_S$ be the corresponding equivalence relation, let $\mathcal{S}$ be the corresponding set of equivalence classes and let μ_S be the $\equiv_S$-cone measure. Let f be the function on ω^ω defined by setting $f(x)$ to be $\omega_1^{L[S,x]}$. Then f is $\equiv_S$-invariant and f represents ω_1 in $\mathrm{Ord}^{\mathcal{S}}/\mu_S$.*

Proof. It is easy to see that f is $\equiv_S$-invariant, and that for each $\alpha < \omega_1$, $\{[x]_{\equiv_S} : x \in \omega^\omega,\ f(x) > \alpha\}$ is in μ_S. Suppose now that g is a $\equiv_S$-invariant function on ω^ω such that $\{[x]_{\equiv_S} : x \in \omega^\omega,\ f(x) > g(x)\} \in \mu_S$. We would like to see that there is an $\alpha < \omega_1$ such that $\{[x]_{\equiv_S} : x \in \omega^\omega,\ g(x) = \alpha\} \in \mu_S$. By the countable completeness of μ_S (which in turn follows from Corollary 1.1.6), this would follow from the existence of an $\alpha < \omega_1$, $\{[x]_{\equiv_S} : x \in \omega^\omega,\ g(x) \leq \alpha\} \in \mu_S$. Suppose toward a contradiction that no such α exists.

Let WO be the set of z in ω^ω coding wellorderings of ω.[1] By[2], for each analytic subset A of WO there is a countable ordinal α such that every member of A codes a wellordering of length less than α.[3] Let $\mathcal{G}$ be the game in which player I produces $x \in \omega^\omega$, II produces y, z in ω^ω, and II wins if and only if $x \leq_S y$ and z codes a wellordering of length $f(y)$. We derive a contradiction by showing that neither player has a winning strategy. Each strategy σ for I can be defeated by playing a y such that $\sigma \in L[S, y]$ and a $z \in \mathrm{WO}$ coding $f(y)$. On the other hand, if σ is a winning strategy for II, then the set A consisting of the

[1] Under some fixed coding...

[2] citation

[3] this fact should be fixed somewhere else

z's produced by σ by playing against all $x \in \omega^\omega$ is an analytic subset of WO. Letting α be a countable ordinal greater than the length of all the wellorderings coded by elements of A, and letting x be such that $f(y) > \alpha$ for all $y \geq_S x$, we get a winning play for I against σ. $\square$

Let us say that $(E, \leq_E)$ is *locally countable* if for all $x \in \omega^\omega$, $\bigcup \mathcal{D}_E(x)$ is countable. Recall that $\mathsf{DC}_\mathbb{R}$ is the statement that whenever T is a tree of finite sequences from $\mathbb{R}$ without terminal nodes, T has an infinite branch. A special case of the following theorem (with the same proof) appears as Lemma 2.10 of [2].[4] Woodin has shown that AD^+ implies WF_T, where T is Turing equivalence. [5]

Theorem 8.0.2. *Let $(E, \leq_E)$ be a locally countable ordered equivalence relation on ω^ω. If the E-cone measure is an ultrafilter, and WF_E holds, then $\mathsf{DC}_\mathbb{R}$ holds.*

Proof. Let T be a tree of finite sequences from ω^ω without an infinite path. We will find a ranking function for T. For each $\bar{x} \in T$ and each $z \in \omega^\omega$, let $f_{\bar{x}}([z]_E)$ be 0 if the members of $\bar{x}$ are not all in $\mathcal{D}_E(z)$, and the rank of $T \restriction \mathcal{D}_E(z)$ below $\bar{x}$ otherwise (which exists, since, as $\mathcal{D}_E(z)$ is countable, $\mathsf{DC}_{\mathcal{D}_E(z)}$ holds). Then if $\bar{x}$ extends $\bar{y}$ in T, $f_{\bar{x}}([z]_E) < f_{\bar{y}}([z]_E)$ for all $[z]_E$ containing both $\bar{x}$ and $\bar{y}$. The map $\bar{x} \mapsto [f_{\bar{x}}]_{\mu_E}$ then ranks E. $\square$

A similar argument gives Theorem 8.0.3 below, which is similar to Theorem 4.3.5. Given a set $A \subseteq \omega^\omega$, we let $\Delta_\omega(A)$ be the smallest projective algebra with A as an element, and say that the members of $\Delta_\omega(A)$ are *projective in* A. We let $\delta_\omega(A)$ be the supremum of the lengths of the prewellorderings in $\Delta_\omega(A)$.

Theorem 8.0.3. *Suppose that there exists a locally countable ordered equivalence relation $(E, \leq_E)$ on ω^ω such that the E-cone measure is an ultrafilter and WF_E holds. Then for each $A \subseteq \omega^\omega$, $\delta_\omega(A)$ is the supremum of the ranks of the wellfounded transitive relations on ω^ω in $\Delta_\omega(A)$.*

Proof. It suffices to fix a wellfounded transitive relation R on ω^ω in $\Delta_\omega(A)$ and show that its rank is less than $\delta_\omega(A)$. Let $\mathcal{E}$ be the set of E-equivalence classes, and let μ_E be the E-cone measure. Given $x \in \omega^\omega$, define a function $\pi_x \colon \mathcal{E} \to \omega_1$ by setting $\pi_x(e)$ to be the rank of $R_x \restriction e$, where R_x is $\{(y, z) \in \omega^\omega \times \omega^\omega : xRyRz\}$.

Define the relation $\leq^*$ on ω^ω by setting $x \leq^* y$ if and only if

$$\{e \in \mathcal{E} : \pi_{\bar{x}}(e) \leq \pi_{\bar{y}}(e)\} \in \mu_E.$$

Since μ_E is an ultrafilter and WF_E holds, $\leq^*$ is a prewellordering. Furthermore, $\leq^*$ is projective in A. Since R is transitive, if xRy, then for all $e \in \mathcal{E}$, $\pi_y(e) \leq \pi_x(e)$. It follows that the rank of R is less than or equal to the length of $\leq^*$. $\square$

We give one additional application of the results of this section.[6]

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[4] It's attributed to Woodin there. In the notes from Woodin's lectures, it's attributed to Solovay.

[5] We need to get a proof of this, and figure out the general form. In particular, does it work for $\equiv_\mathrm{M}$?

[6] Maybe you can write something better than this.

Theorem 8.0.4. *Let S be a set of ordinals, and let $(E, \leq_E)$ be the ordered equivalence relation on ω^ω corresponding to the order $x \in L[S, y]$. If μ_E is an ultrafilter and there is no injection from ω_1 into ω^ω then there is an $x_0 \in \omega^\omega$ such that for all $y \in \omega^\omega$ with $[y]_E \geq [x_0]_E$ and all $\gamma < \omega_1^V$ which are infinite cardinals in $L[S, y]$, $L[S, y] \models 2^\gamma = \gamma^+$.*

Proof. By Remark 1.1.7, μ_S being an ultrafilter implies that ω_1^V is measurable. Since μ_E is an ultrafilter, there is an $x_0 \in \omega^\omega$ such that either

- $L[S, y] \models \mathsf{CH}$ for all $y \in \omega^\omega$ with $x_0 \in L[S, y]$ or
- $L[S, y] \models \neg\mathsf{CH}$ for all $y \in \omega^\omega$ with $x_0 \in L[S, y]$.

We show that the former holds, by finding one such $y \geq_E x_0$.

Let P be the set of ordered pairs from $\omega^{<\omega}$, and let $\pi\colon P \to \omega$ be a recursive bijection. Let

$$p\colon \omega^\omega \times \omega^\omega \to \mathcal{P}(\omega)$$

be defined by setting $p(x, y)$ to be $\pi[\{(x \cap n, y \cap n) : n \in \omega\}]$. Then p sends distinct pairs from ω^ω to infinite almost disjoint subsets of ω.

Recall that the partial order $\mathrm{Col}^*(\omega_1, \omega^\omega)$ consists of injections from countable ordinals to ω^ω, ordered by extension, and adds a bijection $g\colon \omega_1 \to \omega^\omega$ (see Section 0.5). Let $\dot{X}$ be a $\mathrm{Col}^*(\omega_1, \omega^\omega)$-name for the p-image of the set of $(x, y) \in \omega^\omega \times \omega^\omega$ such that $g(x) < g(y)$. Let $Q_{\dot{X}}$ be a $\mathrm{Col}^*(\omega_1, \omega^\omega)$-name for the Jensen-Solovay almost disjoint coding forcing for the realization of $\dot{X}$ (see Section 0.5).

Suppose that M is a transitive model of ZFC, and that (g, z) is M-generic for $(\mathrm{Col}^*(\omega_1, \omega^\omega) * Q_{\dot{X}})^M$. Then g is an element of $M[z]$ (see Remark A.0.5 of [18] for a discussion of why $M[z]$ is a generic extension of M). Moreover, since $Q_{\dot{X}_g}$ is c.c.c. and has cardinality $\aleph_1$ in $M[g]$ (which satisfies CH), CH holds in $M[g][z]$ (which is the same as $M[z]$).

Since ω_1^V is measurable, and since $L[S, x_0] \models \mathsf{AC}$, the partial order

$$(\mathrm{Col}^*(\omega_1, \omega^\omega) * Q_{\dot{X}})^{L[S, x_0]}$$

is countable, and there exists a pair (g, z) which is $L[S, x_0]$-generic for this partial order. The argument just given shows that $L[S, x_0, z] \models \mathsf{CH}$, so any real in $L[S, x_0, z]$ coding the pair (x_0, z) works as our desired y.

Again using the measurability of ω_1^V, $\mathcal{P}(\alpha) \cap L[S, x_0]$ is countable for each $\alpha < \omega_1^V$ so there exist $L[S, x_0]$-generic filters for each corresponding partial order $\mathrm{Col}(\omega, \alpha)$ (see Section 0.5 again). Since these extensions must also satisfy CH, we have that $L[S, x_0] \models 2^\gamma = \gamma^+$ for all $\gamma < \omega_1^V$ which are infinite cardinals in $L[S, x_0]$. The same considerations apply to each $y \in \omega^\omega$ such that $[y]_E \geq [x_0]_E$. □

8.1 Measurable cardinals

We give here an application of Theorem 8.0.3, a weak consequence of which will be used in Section 9.1.

Theorem 8.1.1 ($\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$). *For each $A \subseteq \omega^\omega$, $\delta_\omega(A)$ is a limit of measurable cardinals.*

Proof. Given $B \subseteq \omega^\omega$, we let ρ_B denote the supremum of the ranks of the wellfounded relations in $\utilde{\Sigma}^1_1(B, \omega^\omega \setminus B)$, and we let κ_B be the supremum of the ranks of the $\utilde{\Delta}^1_1(B, \omega^\omega \setminus B)$ prewellorderings. By Theorem 8.0.3, $\delta_\omega(A) = \sup\{\rho_B : B \in \Delta_\omega(A)\}$.

Let $B \subseteq \omega^\omega$ be projective in A, let $U \subseteq (\omega^\omega)^3$ be a complete $\utilde{\Sigma}^1_1(B, \omega^\omega \setminus B)$-set, and let B^* be the set of x for which U_x is wellfounded. Define a prewellordering $\leq^*$ on B^* by setting $x \leq^* y$ if and only if the rank of $\mathrm{rk}(U_x) \leq \mathrm{rk}(U_y)$, where $\mathrm{rk}(U_x)$ denotes the rank of U_x. This order is not necessarily projective in A.

Claim 1 follows from the fact that a union of a collection of pairwise disjoint wellfounded transitive relations is wellfounded and transitive.

Claim 1. *If $Z \subseteq B^*$ is in $\utilde{\Sigma}^1_1(B, \omega^\omega \setminus B)$, then $\{\mathrm{rk}(U_x) : x \in Z\}$ is bounded in ρ_B.*

We adapt Solovay's original proof of the measurability of ω_1 under AD to find a measure on ρ_B. For each $X \subseteq \rho_B$ we consider the game $\mathcal{G}_X$, where player I builds $x_i \in \omega^\omega$ for even $i \in \omega$ and player II builds $x_i \in \omega^\omega$ for odd $i \in \omega$. We use the way of doing this illustrated in the following diagram.

I	$x_0(0)$		$x_0(1), x_2(0)$		$x_0(2), x_2(1), x_4(0)$		$\ldots$
II		$x_1(0)$		$x_1(1), x_3(0)$		$\ldots$	

The game $\mathcal{G}_X$.

Given x_i ($i \in \omega$) produced by a run of this game, the winner is decided as follows.

- If there is an $i \in \omega$ such that $x_i \notin B^*$, then I wins if the least such i is odd, and II wins if the least such i is even.
- If $x_i \in B^*$ for all $i \in \omega$, and there is an $i \in \omega$ such that $\neg(x_i \leq^* x_{i+1})$, then I wins if the least such i is even, and II wins if the least such i is odd.
- If $x_i \in B^*$ and $x_i \leq^* x_{i+1}$ holds for all $i \in \omega$, then I wins if and only if $\sup\{\mathrm{rk}(U_{x_i}) : i \in \omega\} \in X$.

Since a winning strategy for player II in $\mathcal{G}_X$ (for some $X \subseteq \rho_B$) can be converted into a winning strategy for player I in $\mathcal{G}_{\rho_B \setminus X}$, for each $X \subseteq \rho_B$, player I has a winning strategy in at least one of the games $\mathcal{G}_X$ and $\mathcal{G}_{\rho_B \setminus X}$. Let W be the set of $X \subseteq \rho_B$ for which I has a winning strategy.

Claim 2 shows that W is a countably complete ultrafilter. Its completeness, which by the claim is at least κ_B, is a measurable cardinal. The theorem them follows from Claim 2 and the fact that $\{\kappa_B : B \in \Delta_\omega(A)\}$ is cofinal in $\delta_\omega(A)$ (by the definition of $\delta_\omega(A)$).

Claim 2. *W is closed under intersections of cardinal less than κ_B.*

We finish by proving the claim, using the Moschovakis Coding Lemma plus boundedness. Fix $\gamma < \kappa_B$ and a prewellordering $\leq_R$ in $\utilde{\Delta}^1_1(B, \omega^\omega \setminus B)$ of a set $R \subseteq \omega^\omega$ such that $\leq_R$ has length γ. Let $=_R$ be $\leq_R \cap \geq_R$, and let $<_R$ be $\leq_R \setminus =_R$. For each $a \in R$, let $\mathrm{rk}_R(a)$ denote the rank of a in $\leq_R$. Suppose that $\langle X_\alpha : \alpha < \gamma\rangle$ is a sequence of sets in W. It suffices to show that the intersection $\bigcap\{X_\alpha : \alpha < \gamma\}$ is nonempty. Let Z be the set of pairs (a, b) such that $a \in R$ and, for some $\alpha < \gamma$, $\mathrm{rk}_R(a) \geq \alpha$ and b is a winning strategy for player I in $\mathcal{G}_{X_\alpha}$ (more formally, b is a code for a strategy using some fixed bijection between ω and $\omega^{<\omega}$). By the Coding Lemma, there is a pos-$\utilde{\Sigma}^1_1(=_R, <_R)$ set $Z^* \subseteq Z$ such that for each $\alpha < \gamma$ there is an $(a, b) \in Z^*$ with $\mathrm{rk}_R(a) = \alpha$. Let $T = \{b : \exists a\, (a, b) \in Z^*\}$. Then T is a collection of (codes for) winning strategies and T is in $\utilde{\Sigma}^1_1(B, \omega^\omega \setminus B)$.

Let Σ be the set of finite sequences $\langle y_i : i < j_*\rangle$ from B^* such that, for each $j < j_*$, whenever $\langle x_i : i < \omega\rangle$ is a run of $\mathcal{G}_\emptyset$ (the payoff set is irrelevant here) where I plays according to a strategy from T and $x_{2i+1} = y_i$ for all $i < j_*$, $\mathrm{rk}(U_{x_{2i}}) < \mathrm{rk}(U_{x_{2i+1}})$. By Claim 1, each sequence in Σ has a proper extension in Σ. By $\mathsf{DC}_\mathbb{R}$, there is a a sequence $\langle y_i : i \in \omega\rangle$ whose finite initial segments are all in Σ. Since $\langle y_i : i < \omega\rangle$ forms a losing play for player II against any of the strategies from T, it follows that $\sup\{\mathrm{rk}(U_{y_i}) : i \in \omega\}$ is in X_α for all $\alpha < \gamma$, as desired. □

8.2 Steel's theorem and a Mathias-like poset

This section gives an application of Turing Determinacy, due to Steel, which will be used somewhere.[7]

Recall that we let HF denote the set of hereditary finite sets, and for $x, y \subseteq$ HF we say that x is *Turing reducible* to y if x is Δ_1 definable over HF in a predicate for y. We write $x \leq_\mathrm{T} y$ to indicate that x is Turing reducible to y, $\equiv_\mathrm{T}$ for the induced equivalence relation, and $[x]_\mathrm{T}$ for the $\equiv_\mathrm{T}$-equivalence class of x. We call the $\equiv_\mathrm{T}$-equivalence classes Turing degrees. A *Turing cone* is a set of the form $\{x : x \geq_\mathrm{T} x_0\}$ for some $x_0 \subseteq$ HF.

By Theorem 1.1.5, assuming AD, if X is a set of Turing degrees, then there is an $x_0 \subseteq$ HF such that $\{[x]_\mathrm{T} : x \geq_\mathrm{T} x_0\}$ is either contained in or disjoint from X (i.e., $\bigcup X$ is either contained in or disjoint from a Turing cone).

Theorem 8.2.1 (Steel)**.** *Assume.*[8] *Let S_1 and S_2 be sets of ordinals. If*

$$\omega^\omega \cap L[S_s, x] \not\subseteq L[S_1, x]$$

for a Turing cone of $x \in \omega^\omega$, then, for all $\alpha < \omega_1^V$, for a Turing cone of $x \in \omega^\omega$, $\mathcal{P}(\alpha) \cap L[S_1, x]$ is a countable set in $L[S_2, x]$.

[7] Where?

[8] something

To prove this, we use a variation of Mathias forcing relative to an ultrafilter on ω. Our partial order is a slight simplification of the one used originally by Steel[9], and is in fact forcing-equivalent to it. Let $\mathcal{S}$ be the set of nonempty finite subsets of ω and let $\mathcal{F}$ be the set of functions from ω to U. The domain of $\mathbb{P}_U$ is $\mathcal{S} \times \mathcal{F}$. The order on $\mathbb{P}_U$ is defined as follows : $(s', F') \leq (s, F)$ if the following hold (where $\max(s)$ denotes the largest element of a set s):

- $s' \cap (\max(s) + 1) = s$;
- for each $k \in s' \setminus s$, $k \in F(\max(s \cap k))$;
- for each $i \in \omega$, $F'(t) \subseteq F(t)$.

The second condition above distinguishes $\mathbb{P}_U$ from the usual Mathias forcing relative to an ultrafilter. This condition enables Lemmas 8.2.2 and 8.2.3 below. The corresponding facts hold for Mathias forcing, but only for Ramsey ultrafilters; the ultrafilters we use below are not Ramsey.

Let us say that a function $F \in \mathcal{F}$ is *refining* if $F(i) \subseteq F(j)$ whenever $j < i \in \omega$. For densely many $\mathbb{P}_U$-conditions (s, F), F is refining. Given $F \in \mathcal{F}$, say that an infinite $a \subseteq \omega$ is *F-fast* if, for all $i \in a$, $a \setminus (i + 1) \subseteq F(i)$. If F is refining, then every infinite subset of an F-fast set is F-fast. Every infinite $a \subseteq \omega$ (in the ground model or any outer model) generates a filter $G_a \subseteq \mathbb{P}_U$ consisting of those $(s, F) \in \mathbb{P}_U$ such that s is an initial segment of a and $a \setminus \max(s)$ is F-fast.

Lemma 8.2.2 (The Genericity Condition)**.** *Assume that $\mathsf{CC}_{\mathbb{R}}$ holds. Let M be a transitive model of ZF, and let U be a nonprincipal ultrafilter on ω in U. Let a be a subset of ω. Then G_a is M-generic for $\mathbb{P}_U$ if and only if, for each $F \in \mathcal{F}$ in M, there is an $i \in a$ such that $a \setminus i$ is F-fast.*

Proof. The forward direction follows immediately from the fact that for every $s \in \mathcal{S}$, and any two functions $F, F' \in \mathcal{F}$, (s, F) and (s', F) are compatible. For the reverse direction, fix a with the given property and let D be a dense open subset of $\mathbb{P}_U$ in M. Working in M, we recursively define a ranking function ρ_D on $\mathcal{S}$, letting $\rho_D(s)$ be

- 0 if there exists an $F \in \mathcal{F}$ such that $(s, F) \in D$;
- α if $\neg(\rho_D(s) < \alpha)$ and $\{i \in \omega : \rho_D(s \cup \{i\}) < \alpha\} \in U$;
- ω_1 if $\neg(\rho_D(s) < \omega_1)$.

Observe that, for each $s \in \mathcal{S}$, if $\{i \in \omega : \rho_D(s \cup \{i\}) < \omega_1\} \in U$, then $\rho_D(s)$ is at most

$$\sup\{\rho_D(s \cup \{i\}) : i \in \omega,\, \rho_D(s \cup \{i\}) < \omega_1\},$$

which is less than ω_1. It follows that, by $\mathsf{CC}_{\mathbb{R}}$, we can choose, for each $s \in \mathcal{S}$ a function $F_s \colon \omega \to U$ such that

[9] Woodin

- if $\rho_D(s) = 0$, then $(s, F_s) \in D$;
- if $\rho_D(s) < \omega_1$, then for all $i \in F_s(\max(s))$, $i > \max(s)$ and
$$\rho_D(s \cup \{i\}) < \rho_D(s);$$
- if $\rho_D(s) = \omega_1$, then for all $i \in F_s(\max(s))$, $i > \max(s)$ and
$$\rho_D(s \cup \{i\}) = \omega_1.$$

Define $F^*\colon \omega \to U$ by setting $F^*(s)$ to be $\bigcap\{F_t(s) : t \in \mathcal{P}(\max(s)+1) \setminus \{\emptyset\}\}$.

We claim that $\rho_D(s) < \omega_1$ for all $s \in \mathcal{S}$. To see this, suppose that $s \in \mathcal{S}$ is such that $\rho_D(s) = \omega_1$. By the choice of F_s, and the definition of F^*, we have that $\rho_D(s') = \omega_1$ whenever $(s', F') \in \mathbb{P}_U$ is such that $(s', F') \leq (s, F)$. This contradicts our assumption that D is dense.

Finally, there is an $i \in a$ such that $a \setminus i$ is F^*-fast. Then the sequence of values $\langle \rho_D(a \cap j) : j \in \omega \setminus i \rangle$ must decrease until reaching 0, showing that $G_a \cap D \neq \emptyset$. □

The Genericity Condition, plus the fact that F is refining for densely many conditions (s, F), gives the following.

Lemma 8.2.3. *Let M be a transitive model of ZF, let U be a nonprincipal ultrafilter on ω in U, and suppose that $a \subseteq \omega$ is such that G_a is an M-generic filter for $\mathbb{P}_U$. Then for any infinite $a' \subseteq a$, $G_{a'}$ is an M-generic filter for $\mathbb{P}_U$.*

Recall that a set is *recursive* if it is Δ_1 over HF. Let us say that a function $\pi\colon 2^{<\omega} \to [\omega]^{<\omega}$ is *diffuse* if, letting $\pi^*(x)$ denote $\cup\{\pi(t \restriction i) : i < \omega\}$ for each $x \in 2^\omega$, for all distinct $x_0, \ldots, x_{n-1} \in 2^\omega$, if $\sigma_i \in \{\pi^*(x_i), \omega \setminus \pi^*(x_i)\}$ for each $i < n$ then $\bigcap_{i<n} \sigma_i$ is infinite. We leave it to the reader to check that recursive diffuse functions exist (the desired infinite sets can be chosen such a way that they contain infinite sets depending only only the initial segments of x_i $(i < n)$ witnessing distinctness, and the the set of choices between each set $p(x_i)$ and its complement).

Given a function $\pi\colon 2^{<\omega} \to [\omega]^{<\omega}$, say that a set $a \subseteq \omega$ is π-*weak* if there exist infinitely many $x \in 2^\omega$ such that at least one of $a \cap \pi^*(x)$ and $a \setminus \pi^*(x)$ is finite.

Lemma 8.2.4. *If $\pi\colon 2^{<\omega} \to [\omega]^{<\omega}$ is diffuse, n is a positive integer and a_i $(i < n)$ are π-weak subsets of ω, then $\bigcup_{i<n} a_i \neq \omega$.*

Proof. Supposing otherwise, there exist distinct $x_i \in 2^\omega$ $(i < n)$ and $\sigma_i \in \{\pi^*(x_i), \omega \setminus \pi^*(x_i)\}$ $(i < n)$ such that, for each $i < n$, $a_i \setminus \pi^*(x_i)$ is finite. Letting $\sigma'_i = \omega \setminus \sigma_i$ for each $i < n$, we get a contradiction to the diffuseness of π. □

Proof of Theorem 8.2.1. Let $x \in \omega^\omega$, $y \in 2^\omega$ be such that $y \in 2^\omega \cap L[S_2, x] \setminus L[S_1, x]$. Let $\pi\colon 2^{<\omega} \to [\omega]^{<\omega}$ be a recursive diffuse function.

Let U be, in $L[S_1, x]$, an ultrafilter on ω not containing any π-weak subset of ω. By Σ^1_1 absoluteness, for each $a \in U$, every $z \in 2^\omega$ such that at least one of $a \cap \pi^*(z)$ and $a \setminus \pi^*(z)$ is finite is in $L[S_1, x]$. It follows that $\pi^*(y)$ and $\omega \setminus \pi^*(y)$ have infinite intersection with each member of U. Furthermore, $\pi^*(y)$ is in $L[S_2, x]$.

By Lemma 8.2.3, there exists an $L[S_1, x]$-generic set g such that $g \cap \pi^*(y)$ and $g \setminus \pi^*(y)$ are both infinite. To see this, enumerate all functions $F \colon [U]^{<\omega} \to U$ with $F \in L[S_1, x]$ as F_0, F_1, F_2, ... Choose $n_0 \in F_0(\emptyset) \cap \sigma$, $n_1 \in F_0(\langle n_0 \rangle) \cap F_1(\langle n_0 \rangle) \setminus \emptyset$, etc. Fix such a g, and let $b \subseteq \omega$ code an enumeration of $\omega^\omega \cap L[S_1, x]$. Let g^* be an infinite subset of g, enumerated in increasing order by $\langle m_i : i < \omega \rangle$, such that, for each $i \in \omega$, $m_i \in \pi^*(y)$ if and only if $i \in b$. Then g^* is $L[S_1, x]$-generic.

Since $\pi^*(y) \in L[S_2, x]$, b is in $L[S_2, x, g^*]$. Hence $\omega^\omega \cap L[S_1, x] \subseteq L[S_2, x, g^*]$, so $\omega^\omega \cap L[S_1, x][g^*] \subseteq L[S_2, x, g^*]$. Now by Turing Determinacy we are done. ☐

The same proof shows that Turing Determinacy implies the following: Suppose that S_1, S_2 are sets of ordinals, and, on a cone $\omega^\omega \cap L[S_1, x] \subsetneq L[S_2, x]$ then on a cone $\omega^\omega \cap L[S_1, x]$ is a countable set in $L[S_2, x]$, and in fact, for a Turing cone of $x \in \omega^\omega$, $\mathcal{P}(\alpha) \cap L[S_1, x]$ is a countable set in $L[S_2, x]$ for all $\alpha < \omega_1^{L[S_2,x]}$. In this case, $\omega_1^{L[S_2,x]}$ is strongly inaccessible in $L[S_1, x]$ (regularity follows from Remark 8.2.5).

8.2.5 Remark. A classical forcing argument due to Solovay shows that if M is a model of ZF, P is a partial order in M, $\mathcal{P}(P) \cap M$ is countable and x is a set which is in $M[g]$ for every M-generic filter $g \subseteq P$, then x is in M. Roughly, the proof proceeds by analyzing the assumed inability to construct a generic filter g such that no P-name from M realizes to be x. The same argument, using the partial order $\mathrm{Col}(\omega, \alpha)$, shows that if S and T sets of ordinals such that $\omega^\omega \cap L[S, x] \subseteq L[T, x]$ for a Turing cone of $x \in \omega^\omega$, then, for each $\alpha < \omega_1^V$, $\mathcal{P}(\alpha) \cap L[S, x] \subseteq L[T, x]$ for a Turing cone of $x \in \omega^\omega$.

8.3 Pointed trees

In this section we prove a result due to Martin which will be used in Section 8.4. Given a set of ordinals S, we say that

- $\leq_S$ is the binary relation on ω^ω defined by setting $x \leq_S y$ if and only if $x \in L[S, y]$;
- an S-*cone* is a set of the form $\{y \in \omega^\omega : x \in L[S, y]\}$, for some $x \in \omega^\omega$;
- a set $A \subseteq \omega^\omega$ is S-*positive* if it intersects every S-cone;
- a tree $a \subseteq \omega^{<\omega}$ is S-*pointed* if a is perfect (i.e., has a incompatible pair of extensions for each node) and, for every $x \in [a]$, $a \in L[S, x]$;

The following theorem appears in [8].

Theorem 8.3.1 (Martin)**.** *Suppose that* AD *holds. Let S be a set of ordinals and let A be a subset of ω^ω. Then $A \subseteq \omega^\omega$ is S-positive if and only if there is an S-pointed perfect tree a such that $[a] \subseteq A$.*

Proof. The reverse implication (which does not use the assumption of AD) follows from the fact that for any perfect tree $a \subseteq \omega^{<\omega}$ and any $x \in \omega^\omega$ there is a $y \in [a]$ such that $x \in L[a, y]$.

For the forward implication, consider the game $\mathcal{G}$ where I and II respectively build x and y in ω^ω, and II wins if and only if $x \leq_S y$ and $y \in A$. If σ is a strategy for I and $y \in A$ is such that σ is in $L[S, y]$, then y is a winning play for II against σ. This shows that if A is S-positive then I cannot have a winning strategy. Suppose then that σ is a winning strategy for II. For each $s \in \omega^{<\omega}$, let $\sigma(s)$ denote the sequence of ($|s|$-many) moves made according to σ in response to s. Let $y \in 2^\omega$ be such that $\sigma \in L[S, y]$, and let Y be the set of $t \in \omega^{<\omega} \cup \omega^\omega$ such that $t(2n) = y(n)$ whenever $n \in \omega$ and $2n \in \mathrm{dom}(t)$. Working in $L[S, y]$ we can choose for each $s \in 2^{<\omega}$, recursively in $|s|$, a t_s in $Y \cap \omega^{<\omega}$, in such a way that, for all $s, s' \in 2^{<\omega}$,

- if $|s| = |s'|$ then $|t_s| = |t_{s'}|$;
- if $s \perp s'$ then $\sigma(t_s) \perp \sigma(t_{s'})$.

The achievability of the second condition above follows from the fact that σ is a winning strategy for II, which implies that, for any $t \in Y$, the set

$$\{x \circ \sigma : x \in \omega^\omega \cap Y, t \subseteq x\}$$

is S-positive, and therefore has size at least 2. Let a be the tree of initial segments of the members of $\{t_s : s \in 2^{<\omega}\}$. We claim that a is as desired. To see this, fix $z \in [a]$ and $x \in \omega^\omega \cap Y$ such that $z = x \circ \sigma$. Since σ is a winning strategy for II, z is in A, and x is in $L[S, z]$. This implies that y is in $L[S, z]$ and therefore that a is in $L[S, z]$ as desired. □

8.4 Coding ultrapowers

This section uses the notation introduced in Section 8.3, for a set of ordinals S. In addition, we define

- $\equiv_S$ to be the equivalence relation on ω^ω defined by setting $x \equiv_S y \Leftrightarrow L[S, x] = L[S, y]$;
- $[x]_S$, for $x \in \omega^\omega$ to be the $\equiv_S$-equivalence class of x;
- $\mathcal{D}_S$ to be the set of $\equiv_S$-equivalence classes;
- μ_S to be the collection of all $A \subseteq \mathcal{D}_S$ for which there exists an $x \in \omega^\omega$ with $[y]_S \in A$ for all $y \geq_S x$.

A function on ω^ω is said to be S-*invariant* if it is constant on each $\equiv_S$-equivalence class.

We work under the assumption that μ_S is an ultrafilter (which AD implies that it is, by Corollary 1.1.6) on $\mathcal{D}_S$, and let δ^∞_S be $\prod \omega_2^{L[S,x]}/\mu_S$, assuming that this ultraproduct is wellfounded (as is it, assuming DC). We aim to put an upper bound on δ^∞_S, under the assumption that, for some set of ordinals T, the set of $x \in \omega^\omega$ for which $L[S,x] \cap \omega^\omega$ is a countable set in $L[T,x]$ contains an S-cone. For this set of ordinals T, we define the relation $<^{\mathrm{c}}_T$ on $(\omega^\omega)^2$ by setting $(x,y) <^{\mathrm{c}}_T (z,w)$ to hold if and only if

- $x = z$;
- y and w are in $L[T,x]$;
- y comes before w in the constructibility order in $L[T,x]$ using T and x as predicates.

In the statement of Theorem 8.4.1 the set $\leq_S \times <^{\mathrm{c}}_T$ is used for conveneince as a set coding both $\leq_S$ and $<^{\mathrm{c}}_T$.

Theorem 8.4.1. *Assume that* AD *holds, and let S and T be sets of ordinals. Suppose that f is an S-invariant function such that, for an S-cone of x, $f(x)$ is a transitive set in $H(\aleph_1)^{L[T,x]}$. Then $\prod f(x)/\mu_S$ is isomorphic to a relation which is projective in $\leq_S \times <^{\mathrm{c}}_T$.*

Proof. Let R_f be the set of (x,y) such that $y \in \omega^\omega \cap L[T,x]$ and y HC-codes $f(x)$, and let R^*_f be the set of $<^{\mathrm{c}}_T$-minimal pairs in R_f. Then R^*_f is a uniformizing function for R_f. Note however that R^*_f need not be S-invariant.

Let E be the set of $y \in \omega^\omega$ which HC-code a transitive set. Then E is coanalytic. If $y \in E$ HC-codes a transitive set M, then, since M is rigid, for each $k \in \omega$ there is a unique p such that there is an isomorphism between

$$(\omega, \{(n,m) \in \omega \times \omega : y(2^n 3^m) = 0\})$$

and $(\mathrm{TC}(\{M\}), \in)$ sending k to p. We call this element $p_{y,k}$. The following set are projective:

- $\{(y,y',k,k') \in E^2 \times \omega^2 : p_{y,k} = p_{y',k'}\}$;
- $\{(y,y',k,k') \in E^2 \times \omega^2 : p_{y,k} \in p_{y',k'}\}$;

For any S-invariant h such that $h(x) \in f(x)$ on an S-cone there is a $k \in \omega$ such that the set $A_{h,k} = \{x \in \omega^\omega : p_{R^*_f(x),k} = h(x)\}$ is S-positive. Since R^*_f may not be S-invariant, there may be more than one such k. The pair $(k, A_{h,k})$ codes h on an S-cone, however, in the sense that, for an S-cone of $z \in \omega^\omega$,

$$h(z) = h(x) = p_{R^*_f(x),k}$$

for any $x \in [z]_S \cap A_{h,k}$.

By Theorem 8.3.1, every S-positive set contains the (S-positive) set of paths through some S-pointed tree. Let $\langle \sigma_i : i < \omega \rangle$ be a recursive listing of $\omega^{<\omega}$, and let C_f be the set of pairs $(k, b) \in \omega \times 2^\omega$ such that $a_b = \{\sigma_i : b(i) = 1\}$ is an S-pointed perfect tree and, for all $x, y \in [a_b]$, if $x \equiv_S y$ then $p_{R^*_f(x),k} = p_{R^*_f(y),k}$.

For each $(k, b) \in C_f$ and $z \in \omega^\omega$, set $h_{(k,b)}([z]_S)$ to be the common value of $p_{R^*_f(x),k}$ for all $x \in [z]_S \cap [a_b]$ (if there is such a value, and $\emptyset$ otherwise). Define an equivalence relation $\sim$ on C_f by setting

$$(k, b) \sim (k', b')$$

if $h_{(k,b)}$ and $h_{(k',b')}$ agree on an S-cone, and let $[k, b]_f$ denote the $\sim$-class of (k, b). Define $[k, b]_f \in_f [k', b']_f$ to hold whenever

$$h_{(k,b)}([z]_S) \in h_{(k',b')}([z]_S)$$

for an S-cone of $z \in \omega^\omega$. Then $(C_f/\sim, \in_f)$ is isomorphic to $\prod f(x)/\mu_S$. Since C_f, $\sim$ and the relation on C_f induced by $\in_f$ are projective in $\leq_S \times <^{\mathrm{c}}_T$ we are done. □

8.5 Forcing with positive sets

Given a set S consisting of ordinals, let $\mathbb{P}_S$ be the partial order of S-positive subsets of ω^ω and let $\mathbb{S}_S$ be the partial order of S-pointed perfect trees, each ordered by $\subseteq$. By Theorem 8.3.1, each $\mathbb{P}_S$ condition contains the set of infinite paths through some S-pointed tree, which is in turn S-positive. It follows that V-generic filter for either partial order generates one for the other.

A V-generic filter $G \subseteq \mathbb{P}_S$ induces an ultrapower $\prod_{x\in\omega^\omega} V/G$, whose elements are represented by functions with domain ω^ω. The generic $x_G \in \omega^\omega$ added by $\mathbb{P}_S$ is represented by the identity function in this ultrapower. The following theorem shows that the corresponding ultrapower of any set of ordinals, in particular the ultrapower of S itself, is the same in the ultraproducts with respect to G and μ_S. The latter ultraproduct of course is computed in V, while the former is not.

Theorem 8.5.1. *Let S be a set of ordinals and let $G \subseteq \mathbb{P}_S$ be a V-generic filter. Let k be the map sending $[f]_S$ to $[f]_G$, for each S-invariant function f from ω^ω to* Ord. *Then k maps $\prod_S \mathrm{Ord}/\mu_S$ isomorphically to $\prod \mathrm{Ord}/G$.*

Proof. Since each S-positive set has S-positive intersection with each S-cone, k is injective (mod μ_S) and order preserving. We show that it is onto. Let $g\colon A \to \mathrm{Ord}$ be a function in V with S-positive domain. Define f on

$$\bigcup\{[x]_S : x \in A\}$$

by letting $f(x)$ be $\min(g[[x]_S \cap A])$. Then f is S-invariant. Let C be

$$\{x \in A : f(x) = g(x)\}.$$

It suffices (by the genericity of G) to show that C is S-positive, since C forces that $[g]_G = [f]_G = k([f]_S)$. Supposing otherwise, let $x \in \omega^\omega$ be such that $\{y \in \omega^\omega : y \geq_S x\}$ is disjoint from C. Since A is S-positive, there is a $y \geq_S x$ in A. Then there is a $y' \cong_S y$ in A such that $g(y) = f(y')$. Then $y' \in C$ and $y \geq_S x$, giving a contradiction. □

In the following corollary we let j_G denote the embedding of V into $\prod_{x\in\omega^\omega} V/G$ sending each element of V to the class represented by its corresponding constant function, assuming that G is a V-generic filter for $\mathbb{P}_S$. We let j_S be the corresponding embedding of V into $\prod_{x\in\omega^\omega} V/\mu_S$.

Corollary 8.5.2. *If S is a set of ordinals and $G \subseteq \mathbb{P}_S$ is a V-generic filter, then for every set of ordinals T, $j_G(T) = j_S(T)$.*

Proof. Each ordinal in each ultrapower is represented by a function whose range is either contained in or disjoint from T. □

The follow is an immediate consequence of Theorem 8.5.1 and the corollary above.

Corollary 8.5.3. *Let S be a set of ordinals, let S^* be the μ_S-ultrapower of S, let $G \subseteq \mathbb{P}_S$ be a V-generic filter and let x_G be the corresponding generic element of ω^ω. Then $\prod_{x\in\omega^\omega} L[S,x]/G$ is isomorphic to $L[S^*, x_G]$.*

The following remark will be useful in Section 11.5.

8.5.4 Remark. For all $x, y \in \omega^\omega$, $y \in L[S^\infty, x]$ if and only if $y \in L[S, x]$ for an S-cone of z, i.e., if and only if $y \in L[S, x]$. So $\mathbb{S}_S$ and $\mathbb{S}_{S^\infty}$ are isomorphic as partial orders. If x_G is V-generic for $\mathbb{P}_S$, then, it is also $\mathbb{S}_{S^\infty}$-generic over $L(S^\infty, \omega^\omega)$.

Chapter 9

∞-Borel sets

For an infinite cardinal κ, let $\mathcal{L}_{\kappa,0}$ be the language with propositional variables P_n ($n \in \omega$), closed under negation and wellordered conjunctions of cardinality less than κ.[1] Formally, we require the conjunctions and disjunctions to be indexed by sets of ordinals. As the conjunctions and disjunctions are wellordered, we can associate to each sentence in $\mathcal{L}_{\kappa,0}$ a unique code in $\mathcal{P}(\kappa)$ via a definable (injective) pairing function $\prec \cdot, \cdot \succ$ on the ordinals, and write ϕ_S for the sentence coded by the set S. Fixing one such coding, we let

- $\phi_{\{\prec 0,n\succ\}}$ be P_n for each $n \in \omega$,
- $\phi_{\{\prec 1,\alpha\succ : \alpha \in S\}}$ be $\neg\phi_S$, and
- $\phi_{\{\prec 2+\alpha,\beta\succ : \alpha<\gamma, \beta \in S_\alpha\}}$ be $\bigwedge_{\alpha<\gamma} \phi_{S_\alpha}$.

We let $\mathcal{L}_{\infty,0}$ be the class of sentences which are in $\mathcal{L}_{\kappa,0}$ for some infinite cardinal κ. Each $x \in 2^\omega$ can be thought of as a $\mathcal{L}_{\infty,0}$-structure, where, for each $n \in \omega$, P_n is interpreted as true if and only if $x(n) = 1$. For a sentence ϕ of $\mathcal{L}_{\infty,0}$, we let A_ϕ be the set of $x \in 2^\omega$ such that $x \models \phi$, and say that ϕ is an ∞-*Borel code* (or a κ-Borel code, if $\phi \in \mathcal{L}_{\kappa,0}$) for A_ϕ. We also say that a set of ordinals S is an ∞-Borel code or κ-Borel code if the corresponding formula ϕ_S is.

9.0.1 Definition. Given a cardinal κ, a subset of 2^ω is κ-*Borel* if it is equal to A_ϕ for some $\phi \in \mathcal{L}_{\kappa,0}$, and $<\kappa$-*Borel* if it is γ-Borel for some cardinal $\gamma < \kappa$. A subset of 2^ω is ∞-*Borel* if it is κ-Borel for some infinite cardinal κ.

9.0.2 Remark. For each infinite cardinal κ, the class of κ-Borel sets is closed under continuous preimages, and thus Wadge reducibility. To see this, fix $\phi \in \mathcal{L}_{\kappa,0}$ and a continuous function $f\colon 2^\omega \to 2^\omega$. To express "$f(x) \models \phi$" it suffices to find, for each $n \in \omega$, an expression for "$f(x)(n) = 1$", since then one can form

[1] It is claimed somewhere that ∞-Borel sets first appear in [12]. The smallest class containing the clopen sets and closed under wellordered unions and complements may be bigger, though, so I should check.

a suitable ϕ' by replacing each instance of each P_n in ϕ with the corresponding expression. The existence of these expressions follows from the continuity of f (and, in the case $\kappa = \omega$, the compactness of 2^ω).

9.0.3 Remark. The collection of ∞-Borel subsets of 2^ω is not necessarily closed under wellfounded unions, as there is no procedure for picking an ∞-Borel code for a given ∞-Borel set.

Theorem 9.0.4 below gives a number of convenient reformulations of the notion of ∞-Borel set.[2] Item (2) below is especially useful.[3]

For α an ordinal, $x \in 2^\omega$ and $C \subseteq {}^\alpha\omega \times 2^\omega$, let $G(\alpha, x, C)$ be the length-ω game on α such that player I wins if $(\vec{\beta}, x) \in C$, where $\vec{\beta}$ is the sequence produced by the run of the game.

Theorem 9.0.4 (ZF)**.** *The following are equivalent, for a set $A \subseteq 2^\omega$.*

1. *A is ∞-Borel.*
2. *For some set of ordinals S and some first-order formula θ,*

$$A = \{x \in 2^\omega : L[S, x] \models \theta(S, x)\}.$$

3. *For some ordinal α and some clopen $C \subseteq {}^\omega\alpha \times 2^\omega$, A is the set of $x \in 2^\omega$ for which Player I has a winning strategy in $G(\alpha, x, C)$.*
4. *For some ordinal α and some closed $C \subseteq {}^\omega\alpha \times 2^\omega$, A is the set of $x \in 2^\omega$ for which Player I has a winning strategy in $G(\alpha, x, C)$.*

Proof. That (1) implies (2) follows from the fact that sentences in $\mathcal{L}_{\infty,0}$ can be coded by sets of ordinals, and the absoluteness of the relation $x \models \phi$. For the direction (2) $\Rightarrow$ (1), fix an ordinal γ such that for all $x \in 2^\omega$, $L_\gamma[S, x] \models \theta(S, x)$ if and only if $L[S, x] \models \theta(S, x)$. For each ordinal $\alpha \leq \gamma$,

- let $\mathcal{L}^*_\alpha$ be the extension of the first-order language of set theory given by adding constant symbols $\dot{S}$, $\dot{x}$ and $\dot{\beta}$ for each $\beta \in \alpha$;
- let T_α denote the collection of $\mathcal{L}^*_\alpha$ sentences of the form

$$\forall x_1 \ldots \forall x_n((x_1 = X^{\beta_1}_{\phi_1} \wedge \ldots \wedge x_n = X^{\beta_n}_{\phi_n}) \rightarrow \psi)$$

  where

  - each β_i is in α;
  - each ϕ_i is a unary formula in the corresponding $\mathcal{L}^*_{\beta_i}$;
  - each expression of the form $x_i = X^{\beta_i}_{\phi_i}$ denotes the formula saying that x_i is the set of sets satisfying ϕ_i in the structure

$$(L_{\dot{\beta}_i}[\dot{S}, \dot{x}]; \dot{S}, \dot{x}, \dot{\delta}\,(\delta \in d_i), \in),$$

    where d_i is the set of symbols from $\{\dot{\delta} : \delta < \beta_i\}$ appearing in ϕ_i;

[2] Do it cardinal by cardinal?

[3] Something about subsets of ω^ω.

– ψ is a $\mathcal{L}^*_\alpha$ formula with free variables $x_1, \ldots, x_n$ (so each sentence of $\mathcal{L}^*_\alpha$ is itself an element of T_α).

Since each expression of the form $x_i = X_{\phi_i}$ above asserts that x_i is defined by ϕ_i, each sentence in T_α holds in the same structures as the corresponding sentence

$$\exists x_1 \ldots \exists x_n (x_1 = X^{\beta_1}_{\phi_1} \wedge \ldots \wedge x_n = X^{\beta_n}_{\phi_n} \wedge \psi).$$

We refer to this below as the *existential form* of the sentence.

Working by recursion on $\alpha \leq \gamma$, we describe a procedure which associates each sentence $\theta \in T_\alpha$ a sentence $\rho_{\theta,\alpha}$ of $\mathcal{L}_{|\gamma|^+,0}$ such that for all $x \in 2^\omega$, $L_\alpha[S,x] \models \theta$ (with the symbols $\dot{\beta}$ ($\beta < \alpha$), $\dot{S}$ and $\dot{x}$ given their natural interpretations) if and only if $x \models \rho_{\phi,\alpha}$. For finite α this follows from the fact that the $\mathcal{L}^*_\alpha$-theory of $(L_\alpha[S,x]; S, x, \beta\,(\beta \in \alpha), \in)$ depends only on $x \cap \alpha$ (as S is fixed) so the desired $\rho_{\theta,\alpha}$ can consist of a disjunction of conjunctions describing $x \cap \alpha$ exactly (and since there is a canonical finite set of such sentences our procedure can pick the least one in a suitable ordering).

For the limit and successor cases, we induct on the complexity of ψ in the representation of the members of T_α given above. For limit α, the induction hypothesis gives us the desired sentences $\rho_{\theta,\alpha}$ when ψ is a Δ_0 formula, and the steps corresponding to $\wedge$, $\vee$ and $\neg$ are handled by combining the formulas $\rho_{\theta,\alpha}$ in the same manner (to see that this works in the case of $\vee$ and $\neg$, use the fact that each sentence in T_α has an equivalent existential form). If ψ has the form $Qx\psi'$, for Q either $\forall$ or $\exists$, we let $\rho_{\theta,\alpha}$ be the conjunction (when $Q = \forall$) or the disjunction (when $Q = \exists$) of all the sentences $\rho_{\theta^\beta_\phi,\alpha}$, where $\beta < \alpha$, ϕ is a unary formula in $\mathcal{L}^*_\beta$ and θ^β_ϕ is formed from θ by adding $\forall x$ to the beginning of θ and $x = X^\beta_\phi$ at the appropriate place. The point here is that $\alpha \times \mathcal{L}^*_\alpha$ is wellorderable, and each set in the range of the quantifier Q is defined by some formula in $\bigcup_{\beta<\alpha} \mathcal{L}^*_\beta$.

The successor step from α to $\alpha + 1$ is similar, except that, given a sentence $\theta \in T_{\alpha+1}$ as above, with ψ a Δ_0 formula, we must deal with the possibility that some of the formulas ϕ_i define subsets of $L_\alpha[S,x]$, so that we cannot simply apply the induction hypothesis. Each such $T_{\alpha+1}$ sentence θ is equivalent to a sentence θ' in T_α, formed by moving the introduction of the corresponding variables and their definitions into the ψ part of θ'. For instance, if $x_i = X^\alpha_{\phi_i}$ appears in θ, then it does not appear and θ', and ψ' (the ψ part of θ') asserts about the set of things satisfying ϕ_i in

$$(L_{\dot{\alpha}}[\dot{S}, \dot{x}]; \dot{S}, \dot{x}, \dot{\delta}\,(\delta \in d_i), \in)$$

what ψ says about x_i. Then we can let $\rho_{\theta,\alpha+1}$ be $\rho_{\theta',\alpha}$. The rest of this step is the same as the limit case. This completes the proof of (2) $\Rightarrow$ (1).

That (3) implies (4) is immediate. The direction (4) $\Rightarrow$ (2) follows from the absoluteness[4] of the existence of winning strategies in closed games. For the direction (1) $\Rightarrow$ (3), consider a witness ϕ to (1) as a tree. Players I and II

[4] make this clearer

choose a path from the root of the tree to a terminal node, with I choosing successors to $\vee$'s, II choosing successors to $\wedge$'s, and I winning if and only if $x(n) = 0$, for $P(n)$ the terminal node reached by the run of the game. If $x \models \phi$, then I can play to maintain that x satisfies each sub-sentence visited during the run of the game, and conversely for II. □

[5]

9.0.5 Remark. Part (2) clearly makes sense for subsets of ω^ω (or sets of subsets of any set in L), so we will say that a set $A \subseteq \omega^\omega$ is ∞-*Borel* is there exists a set S of ordinals such that $A = \{x \in \omega^\omega : L[S, x] \models \phi(S, x)\}$, for some binary formula ϕ. One could also revise the infinitary language given above to describe subsets of $\mathcal{P}(\omega \times \omega)$, starting with variables $P_{n,m}$ $(n, m \in \omega)$ interpreted so that $x \models P_{n,m}$ if and only if $x(n) = m$. Under this formulation everything goes through as above (trivially, as we are merely relabeling a countable set). In this sense, however, ω^ω is $\aleph_1$-Borel as a subset of $\mathcal{P}(\omega \times \omega)$, but not $\aleph_0$-Borel. When we describe subsets of ω^ω in this revised sense (i.e., when we say that some set of ordinals is an ∞-Borel code for a subset of ω^ω) then we restrict our attention to κ-Borel sets for uncountable κ.

Given a set of ordinals S and a binary formula θ from the first order language of set theory, we say that the pair (S, θ) is an ∞-*Borel** code for the set

$$\{x \in 2^\omega : L[S, x] \models \theta(S, x)\}.$$

If in addition S is a subset of an uncountable ordinal γ, we say that the pair (S, θ) is a γ-*Borel** code for the set

$$\{x \in 2^\omega : L_\gamma[S, x] \models \theta(S, x)\}.$$

9.0.6 Remark. The proof of Theorem 9.0.4 shows that, for any uncountable cardinal γ, every γ-Borel subset of 2^ω has a γ^+-Borel* code, and every subset of 2^ω which has a γ-Borel* code is γ^+-infinity Borel.

9.0.7 Remark. If κ is a cardinal and T is a tree on $\omega \times \kappa$ such that $\mathrm{p}[T]$ is nonempty, then $\mathrm{p}[T]$ is nonempty in $L[T]$. However, it is possible to have $\phi \in \mathcal{L}_{\kappa,0}$ such that A_ϕ is nonempty in V but empty in $L[\phi]$ (for instance, there is a ϕ in L such that A_ϕ is the set of Cohen reals over L).

It follows from Theorem 9.0.4 that Suslin subsets of 2^ω are ∞-Borel. It can happen that all subsets of 2^ω are ∞-Borel without all such sets being Suslin. This happens in $L(\mathbb{R})$, for instance, by Corollary 9.1.7, Theorem 6.1.4 and the fact that Suslin sets are uniformized.[6]

Given a cardinal κ, we say that a subset of topological space X is *weakly* κ-*Borel* if it is in the smallest collection of subsets of X containing the closed

[5]Maybe we can use this: the implication from (2) to (1) in the proof is carried out in $L[S]$, aside from the choice of γ.

[6]Assuming AD, or just no ω_1-sequence of reals?

sets and closed under unions and intersections of size less than κ. A set is *weakly ∞-Borel* if it is weakly κ-Borel for some cardinal κ. Some authors call these sets "∞-Borel" and use "effectively ∞-Borel" for our notion of ∞-Borel.

9.0.8 Remark. Suppose that AD holds, and that Δ is a projective algebra which is not closed under countable unions. By Theorem 4.4.2, $\bigcup_\omega$ has the prewellordering property. By the first Periodicity Theorem (see Remark 4.2.6), $\forall^{\omega^\omega}\bigcup_\omega \Delta$ does as well. Let $\Gamma = \forall^{\omega^\omega}\bigcup_\omega \Delta$ and let $\Delta' = \Gamma \cap \check{\Gamma}$. By Theorem 4.3.2, every weakly $\delta(\Delta)'$-Borel subset of ω^ω is in Δ'.

We will make use of the following lemma in Section 11.3.[7]

Lemma 9.0.9. *Assume that* AD *holds, and suppose that κ is a limit of Suslin cardinals and that κ has uncountable cofinality. Then every set which is $<\kappa$-Borel is also $<\kappa$-Suslin.*

Proof. We make use of Remark 6.0.3 and the comments immediately before. Let $\gamma < \kappa$ be a limit of Suslin cardinals of countable cofinality. By Remark 6.0.8, $\mathcal{S}_{<\gamma}$ is a projective algebra which is not closed under countable unions. Letting $\Gamma = \forall^{\omega^\omega}\bigcup_\omega \mathcal{S}_{<\gamma}$ and $\Delta' = \Gamma \cap \check{\Gamma}$, we have by Remark 9.0.8 that every $<\gamma$-Borel set is in Δ'. Since $\mathcal{S}_\gamma$ is closed under countable unions, it contains $\bigcup_\omega S_{<\gamma}$. If $\gamma' > \gamma$ is a Suslin cardinals greater than γ, then $\mathcal{S}_{\gamma'}$ is $\exists^{\omega^\omega}$-closed and contains the complement of every set in $\mathcal{S}_\gamma$, so $\Delta' \subseteq \mathcal{S}_{\gamma'}$. □

9.1 Local ∞-Borel codes

Given a set $A \subseteq (\omega^\omega)^n$ (for some $n \in \omega$) let δ_A denote the supremum of the lengths of the prewellorderings P such that either $P \leq_W A$ or $P \leq_W \check{A}$. Recalling from Remark 2.5.2 that $\mathcal{F}^{c,1,b}$ is the set of $x \in \omega^\omega$ such that $f^{c,1,n}_x$ is a continuous function from ω^ω to $(\omega^\omega)^n$, let $\leq_A$ be the relation on $(\omega^\omega)^3$ defined by setting $(x_0, y_0, z_0) \leq_A (x_1, y_1, z_1)$ if $x_0 = x_1$, $(f^{c,1,n}_{x_0})^{-1}[A]$ is a prewellordering and $((y_0, z_0), (y_1, z_1)) \in (f^{c,1,n}_{x_0})^{-1}[A]$. Then $\leq_A$ is a wellfounded relation projective in A, and the rank of $\leq_A$ is δ_A. If μ_{T} is an ultrafilter on the Turing degrees, then, by Theorem 8.0.3, $\delta_A < \delta_\omega(A)$. By Theorem 8.1.1, assuming $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$, there is a regular cardinal in the interval $(\delta_A, \delta_\omega(A))$. Let δ^*_A be the least such regular cardinal. In this section we show (assuming $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$) that if A is ∞-Borel then it is δ^*_A-Borel.

We define the *rank* $\mathrm{rk}(\phi)$ of a sentence in $\mathcal{L}_{\infty,0}$ by setting

- $\mathrm{rk}(P_n) = 0$ for each propositional variable P_n;
- $\mathrm{rk}(\neg\phi) = \mathrm{rk}(\phi)$ for all $\phi \in \mathcal{L}_{\infty,0}$;
- $\mathrm{rk}(\bigvee_{\alpha\in Y}\phi_\alpha) = \mathrm{rk}(\bigwedge_{\alpha\in Y}\phi_\alpha) = \sup\{\mathrm{rk}(\phi_\alpha) + 1 : \alpha \in Y\}$ for all sets $Y \subseteq \mathrm{Ord}$ and all formulas ϕ_α ($\alpha \in Y$) in $\mathcal{L}_{\infty,0}$,

[7] The lemma is not optimal.

For each $\phi \in \mathcal{L}_{\infty,0}$ we define the formula $\phi^* \in \mathcal{L}_{\infty,0}$ by recursion on the rank of ϕ as follows.

- $P_n^* = P_n$, for each each propositional variable P_n;
- $(\neg\phi)^* = \neg\phi^*$, for all $\phi \in \mathcal{L}_{\infty,0}$;
- for all sets $Y \subseteq \mathrm{Ord}$, and all $\mathcal{L}_{\infty,0}$-sentences ϕ_α $(\alpha \in Y)$,
  - $(\bigvee_{\alpha\in Y} \phi_\alpha)^* = \bigvee_{\alpha\in X} \phi^*_\alpha$, where $X = \{\alpha \in Y : A_{\phi_\alpha} \not\subseteq \bigcup_{\beta\in\alpha\cap Y} A_{\phi_\beta}\}$;
  - $(\bigwedge_{\alpha\in Y} \phi_\alpha)^* = \bigwedge_{\alpha\in X} \phi^*_\alpha$, where $X = \{\alpha \in Y : A_{\phi_\alpha} \not\supseteq \bigcap_{\beta\in\alpha\cap Y} A_{\phi_\beta}\}$,

Then for all $\phi \in \mathcal{L}_{\infty,0}$, $A_\phi = A_{\phi^*}$, $\mathrm{rk}(\phi) \geq \mathrm{rk}(\phi^*)$ and $\phi^{**} = \phi^*$.

Let κ_{B} be the supremum of the ordinals γ for which there exists a γ-sequence of $\mathcal{L}_{\infty,0}$ sentences defining nonempty disjoint sets. Each sentence of the form ϕ^* as above is in $\mathcal{L}_{\kappa_{\mathrm{B}},0}$. Letting χ_{B} be the least cardinal χ such that every ∞-Borel set is χ-Borel, it follows that $\chi_{\mathrm{B}} \leq \kappa_{\mathrm{B}}$.

9.1.1 Remark. The Moschovakis Coding Lemma implies, assuming $\mathsf{ZF}+\mathsf{AD}$, that $\chi_{\mathrm{B}} = \Theta$ if all subsets of 2^ω are ∞-Borel. The converse also holds (see Remark 9.1.4).

9.1.2 Definition. Given $n \in \omega \setminus \{0,1\}$, and a function

$$\pi\colon \omega \to n \times \omega,$$

let $\pi_0\colon \omega \to n$ and $\pi_1\colon \omega \to \omega$ be such that $\pi(k) = (\pi_0(k), \pi_1(k))$ for all $k \in \omega$, and let $\sigma_\pi\colon (2^\omega)^n \to 2^\omega$ be such that

$$\sigma_\pi(x_0,\ldots,x_{n-1})(k) = x_{\pi_0(k)}(\pi_1(k))$$

for all $x_0,\ldots,x_{n-1}$ in 2^ω.

A set $A \subseteq (2^\omega)^n$ is said to be ∞-Borel (likewise, κ-Borel, for some infinite cardinal κ) if for some bijection $\pi\colon n\times\omega \to \omega$ and some ∞-Borel (κ-Borel) $B \subseteq 2^\omega$, A is the set of $(x_0,\ldots,x_{n-1}) \in (2^\omega)^n$ such that $\sigma_\pi(x_0,\ldots,x_{n-1}) \in B$.

9.1.3 Remark. Suppose again that $S = \langle \phi_\alpha : \alpha < \eta\rangle$ is a sequence of $\mathcal{L}_{\infty,0}$-sentences (for some ordinal η). Define a prewellordering $\leq_S$ on $\bigcup_{\alpha<\delta_A} A_{\phi_\alpha}$ defined by setting $x \leq_S y$ if the least θ such that $x \in A_{\phi_\theta}$ is less than or equal to the least θ such that $y \in A_{\phi_\theta}$. Then $\leq_S$ is ∞-Borel, and in fact χ^+-Borel, where χ is the least cardinal γ such that each ϕ_α is in $\mathcal{L}_{\gamma,0}$. Letting λ_{B} be the supremum of of the lengths of the ∞-Borel prewellorderings, we get that $\kappa_{\mathrm{B}} \leq \lambda_{\mathrm{B}}$.

9.1.4 Remark. Suppose that $A \subseteq 2^\omega$ is not ∞-Borel. If P is an ∞-Borel prewellordering, then, by Remark 9.0.2, A is not Wadge below P. It follows then that P is Wadge reducible to either A or $\check{A}$, and thus that the length of P is less than δ_A. In particular, if Wadge Determinacy holds and there is a subset of 2^ω which is not ∞-Borel, then $\lambda_{\mathrm{B}} < \Theta$.

A similar argument show that $\delta^*_A \geq \chi_{\mathrm{B}}$ for every ∞-Borel set A.

Theorem 9.1.5. *Suppose that* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *holds, and let* $A \subseteq 2^\omega$ *be* ∞*-Borel. Then* A *is* $(\delta^*_A)^+$*-Borel.*

Proof. If $\delta^*_A \geq \chi_{\mathrm{B}}$ we are done. Supposing otherwise, using a sentence ϕ in $\mathcal{L}_{\infty,0}$ of minimal rank such that $\phi^* \notin \mathcal{L}_{\delta^*_A,0}$, we may find (using the regularity of δ^*_A) a sequence $S = \langle \phi_\alpha : \alpha < \delta^*_A \rangle$, consisting of $\mathcal{L}_{\delta^*_A,0}$ sentences, such that the sets A_{ϕ_α} are nonempty and pairwise disjoint. Then $\leq_S$ (as in Remark 9.1.3) is $(\delta^*_A)^+$-Borel, and, as it has length δ_A, is not Wadge below either A or $\check{A}$. By Wadge Determinacy, then, A is Wadge below $\leq_S$, which means that A is $(\delta^*_A)^+$-Borel, by Remark 9.0.2. □

Suppose (for this paragraph) that $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ holds, and let $A \subseteq 2^\omega$ be ∞-Borel. Let $B \subseteq \omega^\omega$ have Wadge rank δ^*_A. By Proposition 2.5.8, there is a prewellordering of $\mathcal{F}^c$ of length δ_A which is projective in B. By the Moschovakis Coding Lemma, every subset of δ^*_A is coded (relative to this prewellordering) by a set of reals projective in B. Putting this all together with Theorem 9.1.5 gives the following corollaries.

Corollary 9.1.6. *Suppose that* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *holds, and let* $A \subseteq 2^\omega$ *be* ∞*-Borel. Then* $A = A_\phi$ *for some* $\phi \in \mathcal{L}_{(\delta^*_A)^+,0} \cap L(A, \mathbb{R})$.

Corollary 10.2.7 below is a stronger version of the following corollary, and its proof does not use Theorem 6.1.5.

Corollary 9.1.7. *Suppose that* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *holds. Then, in* $L(\mathbb{R})$*, every subset of* 2^ω *is* ∞*-Borel.*

Proof. We work in $L(\mathbb{R})$, noting that the hypothesis imply that AD and $\mathsf{DC}_{\mathbb{R}}$ hold there. Supposing that some subset of 2^ω is not ∞-Borel, there exists an ordinal γ such that

- in $L_\gamma(\mathbb{R})$, Θ (as defined in $L_\gamma(\mathbb{R})$) is a limit cardinal;
- for every $A \in \mathcal{P}(\omega^\omega) \cap L_\gamma(\mathbb{R})$, $\underset{\sim}{\Sigma}^1_1(A) \subseteq L_\gamma(\mathbb{R})$;
- for some $A \in \mathcal{P}(2^\omega) \cap L_\gamma(\mathbb{R})$, A is not ∞-Borel in $L_\gamma(\mathbb{R})$.

Since $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ is Σ_1-elementary in $L(\mathbb{R})$, there is a $\gamma_0 < \underset{\sim}{\delta}^1_2$ satisfying these properties. By the Moschovakis Coding Lemma, $L_{\gamma_0}(\mathbb{R})$ is correct about cardinals below its version of Θ. By Theorem 6.1.5, all sets in $L_{\underset{\sim}{\delta}^1_2}(\mathbb{R})$ are Suslin, which by Theorem 9.1.5 implies that in $L_{\gamma_0}(\mathbb{R})$ all sets are ∞-Borel, giving a contradiction. □

9.2 Strong ∞-Borel codes

Given a sentence ϕ of $\mathcal{L}_{\infty,0}$ and a set of ordinals σ, we define the sentence $\phi \restriction \sigma$ recursively on the complexity of ϕ as follows:

- $P_n \restriction \sigma = P_n$, for each $n \in \omega$;
- $(\neg\phi)\restriction\sigma = \neg(\phi\restriction\sigma)$;
- $(\bigwedge_{\alpha<\gamma} \phi_\alpha)\restriction\sigma = \bigwedge_{\alpha\in\gamma\cap\sigma}(\phi_\alpha\restriction\sigma)$.

For us the importance of this definition is the following : if δ is an infinite cardinal, S is a subset of δ coding a sentence ϕ_S in $\mathcal{L}_{\delta,0}$, $X \subseteq 2^\omega$ and Z is an elementary submodel of $L_{\delta^+}(X,S)$, then, for all $x \in X$, $x \in A_{\phi_S}$ if and only if $x \in A_{\phi\restriction(Z\cap\gamma)}$. It does not follow, however, that $A_{\phi\restriction(Z\cap\gamma)} \subseteq A_{\phi_S}$.

9.2.1 Definition. Suppose that δ is an infinite cardinal and that ϕ is a sentence in $\mathcal{L}_{\delta,0}$. Associate to ϕ a game $\mathcal{G}_{\phi,\delta}$ on δ, where I and II collaborate to build a countable set $\sigma \subseteq \delta$, and I wins if $A_{\phi\restriction\sigma} \subseteq A_\phi$. We say that ϕ is a *strong ∞-Borel code* (or *strong δ-Borel code*) for A_ϕ if I has a winning strategy in $\mathcal{G}_{\phi,\delta}$. If S is the set of ordinals such that $\phi = \phi_S$, we also say that S is a strong ∞-Borel code or strong δ-Borel code for A_ϕ. [8]

For any ordinal δ, δ-Determinacy implies the determinacy of the game in the definition of strong ∞-Borel code, as one can continuously associate to σ an element of ω^ω coding the transitive collapse of an elementary submodel of L_{δ^+} and, letting $\bar{S}$ be the image of S under this transitive collapse, ask if $A_{\phi_{\bar{S}}}$ (which is the same as $A_{\phi_S\restriction\sigma}$) is a subset of A_{ϕ_S}.

Theorem 9.2.3. *If δ is an infinite cardinal and $\phi \in \mathcal{L}_{\delta,0}$ is a strong ∞-Borel code, then A_ϕ is δ-Suslin.*

Proof. Fix a winning strategy τ for player I in $\mathcal{G}_{\phi,\delta}$. Then for all $x \in \omega^\omega$, $x \in A_\phi$ if and only if there is a countable $\sigma \subseteq \delta$ produced by a run of $\mathcal{G}_S$ where I plays according to τ and $x \in A_{\phi\restriction\sigma}$. To see this, note first of all that, since a τ is winning strategy for player I, if there is a play against τ producing σ with x in the corresponding set $A_{\phi\restriction\sigma}$, then $x \in A$. For the other direction, fix $S \subseteq \delta$ such that $\phi = \phi_S$. Then for each $x \in A$ there exists a σ resulting from a run of $\mathcal{G}_{\phi,\delta}$ in which I plays according to τ such that, for some countable $Z \prec L_{\delta^+}[S,x]$, $x \in Z$ and $Z \cap \delta = \sigma$, in which case $x \in A_{\phi\restriction\sigma}$.

There is a tree T on $\omega \times \delta \times \omega$ whose branches are the triples (x,f,y) where $x \in \omega^\omega$, $f \in \delta^\omega$ is a run of $\mathcal{G}_{\phi,\delta}$ where I plays according to τ, and $y \in \omega^\omega$ codes the transitive collapse M of a countable elementary submodel Z of $L_{\delta^+}[S,x]$ with $x \in Z$ and $Z \cap \delta = \mathrm{range}(f)$ such that, letting $\bar{S}$ be the image of S under the transitive collapse of Z, $M \models x \in A_{\phi_{\bar{S}}}$. Finally, $A_\phi = \mathrm{p}[T]$.[9] □

Strong ∞-Borel codes are preserved under disjunctions, given the appropriate form of determinacy.

[8] There is an alternate definition, which requires a measure μ.

9.2.2 Definition (Strong ∞-Borel code). Suppose that $A \subseteq \omega^\omega$, S is a strong ∞-Borel code if (suppose $S \subseteq \delta$) the set of $\sigma \in \mathcal{P}_{\aleph_1}(\delta)$ such that $s_\sigma \cong \mathrm{collapse}(\sigma(\sigma \cap S))$ defines a Borel subset of A is of measure 1 for μ, where μ is the measure on $\mathcal{P}_{\aleph_1}(\delta)$.

[9] Maybe this proof can be cleaned up.

Theorem 9.2.4. *Assume that* $\mathsf{DC}_{\mathbb{R}}$ *holds. Let* κ *be an infinite cardinal below* Θ *such that* κ-$\mathsf{Determinacy}$ *holds, and let* $\langle \phi_\alpha : \alpha < \kappa \rangle$ *be a sequence of* $\mathcal{L}_{\kappa,0}$ *sentences which are strong* ∞*-Borel codes. Then* $\bigvee_{\alpha<\kappa} \phi_\alpha$ *is a strong* ∞*-Borel code.*

Proof. We can work in $L(A, \mathbb{R})$ for some set $A \subseteq \omega^\omega$ of Wadge rank greater than κ, and use the fact that DC holds in $L(A, \mathbb{R})$. Let $\phi = \bigvee_{\alpha<\kappa} \phi_\alpha$. It suffices to see that player II cannot have a winning strategy in $\mathcal{G}_{\phi,\kappa}$. Fixing a strategy τ for player II, we can find by DC a countable $\sigma \subseteq \kappa$ and winning strategies ρ_α for player I in the games $\mathcal{G}_{\phi_\alpha,\kappa}$ $(\alpha \in \sigma)$ such that σ is the result of a run of $\mathcal{G}_{\phi,\kappa}$ where II has played by τ and also, for each $\alpha \in \sigma$, the result of a run of $\mathcal{G}_{\phi_\alpha,\kappa}$ where player I has played by ρ_α. Then

$$A_{\phi\restriction\sigma} = \bigcup_{\alpha\in\sigma} A_{\phi_\alpha\restriction\sigma} \subseteq \bigcup_{\alpha\in\sigma} A_{\phi_\alpha} \subseteq A_\phi,$$

showing that I wins this run of the game. $\square$

Chapter 10

Vopěnka algebras

Recall that for a set X, OD_X is the class of all sets which are ordinal definable from a finite sequence from X. We let HOD_X denote the class of all sets in OD_X whose transitive closures are contained in $\mathrm{OD}_{\mathrm{TC}(\{X\})}$, where $\mathrm{TC}(X)$, the transitive closure of X, is the smallest transitive set containing X. One can find an introduction to HOD on pages 194-195 of [4], including a proof that it satisfies ZFC.[1] The presentation there relativizes to HOD_X, except for the fact that HOD_X is a model of ZF and satisfies the Axiom of Choice if and only if it contains a wellordering of $\mathrm{TC}(X)$.

Vopěnka's theorem (see page 249 of [4]) says that every set of ordinals is set-generic over HOD. Theorem 10.0.1 below is a relativized version of this fact. Relativizing to the sets X and Z there requires only minor modifications of the proof. The rest of this chapter contains variations of Vopěnka's theorem due to Woodin.

Suppose that X and Z are sets such that Z is in $\mathrm{OD}_{\mathrm{TC}(X)}$. Let κ be the least cardinal λ such that $X \subseteq V_\lambda$, Z is in $\mathrm{OD}^{V_\lambda}_{\mathrm{TC}(X)}$ (i.e., Z is definable in V_λ from a finite sequence from X and a finite sequence from λ) and every subset of $\mathcal{P}(Z)$ in $\mathrm{OD}_{\mathrm{TC}(X)}$ is in $\mathrm{OD}^{V_\lambda}_{\mathrm{TC}(X)}$. Let $P_{Z,X}$ be the set of triples $(n, \bar{x}, \bar{\alpha})$ such that

- n is the Gödel number of a ternary formula ϕ,
- $\bar{x}$ is a finite sequence of elements of $\mathrm{TC}(X)$,
- $\bar{\alpha}$ is a finite sequence of elements of κ and
- the set $B_{n,\bar{x},\bar{\alpha}} = \{C \subseteq Z : V_\kappa \models \phi(C, \bar{x}, \bar{\alpha})\}$ is nonempty.

The set $P_{Z,X}$ is in HOD_X, as is the reflexive and transitive ordering $\leq_{Z,X}$ on $P_{Z,X}$ defined by setting $(n, \bar{x}, \bar{\alpha}) \leq_{Z,X} (m, \bar{y}, \bar{\beta})$ if $B_{n,\bar{x},\bar{\alpha}} \subseteq B_{m,\bar{y},\bar{\beta}}$. Let $\equiv_{Z,X}$ be the induced equivalence relation on $P_{Z,X}$, and for each $(n, \bar{x}, \bar{\alpha}) \in P_{Z,X}$, let $[n, \bar{x}, \bar{\alpha}]_{Z,X}$ denote the corresponding equivalence class. We let $\mathbb{V}_{Z,X}$ (the

[1] Refer the reader to Kunen also for the basics of HOD?

Vopěnka algebra for X on Z) be the partial order whose domain is the set of $\equiv_{Z,X}$-classes of $P_{Z,X}$, with the order inherited from $\leq_{Z,X}$. We write $\mathbb{V}_Z$ when $X = \emptyset$.

Theorem 10.0.1. *Let X be a set, and let γ be an ordinal. For each $A \subseteq \gamma$, there is a* HOD_X*-generic filter* $G \subseteq \mathbb{V}_{\gamma,X}$ *such that* $\mathrm{HOD}_{X\cup\{A\}} = \mathrm{HOD}_X[G]$.

Proof. Fix $A \subseteq \gamma$. If D is a subset of $P_{\gamma,X}$ and an element of HOD_X, then

$$\bigcup\{B_{n,\bar{x},\bar{\alpha}} : (n,\bar{x},\bar{\alpha}) \in D\}$$

is $\mathrm{OD}_{\mathrm{TC}(X)}$. It follows that if this union is not all of $\mathcal{P}(\gamma)$, then there is a triple $(m,\bar{y},\bar{\beta})$ in $P_{\gamma,X}$ such that

$$B_{m,\bar{y},\bar{\beta}} = \mathcal{P}(\gamma) \setminus \bigcup\{B_{n,\bar{x},\bar{\alpha}} : (n,\bar{x},\bar{\alpha}) \in D\}.$$

From this it follows that the set of equivalences classes of triples $(n,\bar{x},\bar{\alpha}) \in P_{\gamma,X}$ for which $A \in B_{n,\bar{x},\bar{\alpha}}$ is a HOD_X-generic filter for $\mathbb{V}_{\gamma,X}$. Let G_A denote this filter. The definition just given shows that G_A is in $\mathrm{HOD}_{X\cup\{A\}}$ and therefore that $\mathrm{HOD}_{X\cup\{G_A\}}$ is contained in $\mathrm{HOD}_{X\cup\{A\}}$.

The set K consisting of those pairs $(\delta,(n,\bar{x},\bar{\alpha})) \in \gamma \times P_{\gamma,X}$ for which

$$B_{n,\bar{x},\bar{\alpha}} = \{C \subseteq \gamma : \delta \in C\}$$

is also a member of HOD_X. Since A is equal to

$$\{\delta : \exists\,[n,\bar{x},\bar{\alpha}]_{Z,X} \in G_A\,(\delta,(n,\bar{x},\bar{\alpha})) \in K\},$$

it follows that A is in $\mathrm{HOD}_X[G_A]$. Since $\mathrm{HOD}_X[G_A] \subseteq \mathrm{HOD}_{X\cup\{G_A\}}$, it follows also that A is in $\mathrm{HOD}_{X\cup\{G_A\}}$, $\mathrm{HOD}_{X\cup\{A\}} = \mathrm{HOD}_{X\cup\{G_A\}}$ and that each of these models contains $\mathrm{HOD}_X[G_A]$.

To see that $\mathrm{HOD}_{X\cup\{A\}} \subseteq \mathrm{HOD}_X[G_A]$, it suffices to see that every subset of $X^{<\omega} \times \mathrm{Ord}$ in $\mathrm{HOD}_{X\cup\{A\}}$ is in $\mathrm{HOD}_X[G_A]$, since, in $\mathrm{HOD}_{X\cup\{A\}}$, every set is a surjective image of $X^{<\omega} \times \zeta$, for some ordinal ζ. Fix then an ordinal ζ and a set $Q \subseteq X^{<\omega} \times \zeta$ in $\mathrm{HOD}_{X\cup\{A\}}$. Fix a quaternary formula ϕ, a finite tuple $\bar{x}$ from $\mathrm{TC}(X)$ and a finite set of ordinals $\bar{\alpha}$ such that $Q = \{q \in X^{<\omega} \times \zeta : \phi(z,\bar{x},\bar{\alpha},A)\}$. Let T be

$$\{(q,(n,\bar{y},\bar{\beta})) \in (X^{<\omega} \times \zeta) \times P_{\gamma,X} : B_{n,\bar{y},\bar{\beta}} = \{C \subseteq \gamma : \phi(z,\bar{x},\bar{\alpha},C)\}\}.$$

Then T is in HOD_X, and Q, which is

$$\{q : \exists\,[n,\bar{y},\bar{\beta}]_{Z,X} \in G_A\,(z,(n,\bar{y},\bar{\beta})) \in T\},$$

is in $\mathrm{HOD}_X[G_A]$. □

The proof of the following theorem, due to Woodin, generalizes the proof of Theorem 10.0.1.[2]

[2]Really? And is this really the main thing to say about the theorem?

Theorem 10.0.2. *Let S be a set of ordinals, and suppose that $V = L(A)$, for some transitive set A. Then* $\mathrm{HOD}_{\{S\}} = L[B]$*, for some set of ordinals B.*

Before proving Theorem 10.0.2, we prove a lemma, and give a proof for the case when $V = L[A]$, for A a set of ordinals.

Lemma 10.0.3. *Let M_1 and M_2 be transitive models of ZFC, with $M_1 \subseteq M_2$. Suppose that $\mathbb{P}$ is a partial order in M_1, $G \subseteq \mathbb{P}$ is an M_2-generic filter and $M_2 \subseteq M_1[G]$. Then $M_1 = M_2$.*

Proof. Since M_1 and M_2 are models of ZFC, it is enough to see that every set of ordinals in M_2 is in M_1. Letting X be a set of ordinals in M_2, we have that $X = \tau_G$, for some $\mathbb{P}$-name τ in M_1. Since G is M_2-generic, this means that some condition in $\mathbb{P}$ decides all of τ, so X is in M_1. □

Now, as a warm-up, suppose that $V = L[A]$, where A is a subset of an ordinal γ. Since Choice holds in $\mathrm{HOD}_{\{S\}}$, there exist in $\mathrm{HOD}_{\{S\}}$ an ordinal η and bijection $\pi\colon \mathbb{V}_{\gamma,\{S\}} \to \eta$. Let $\leq_\gamma$ be the partial order on η induced by π. Let K be the set of pairs

$$(\delta, (n, \bar{x}, \bar{\alpha})) \in \gamma \times P_{\gamma,\{S\}}$$

such that $B_{n,\bar{x},\bar{\alpha}} = \{C \subseteq \gamma : \delta \in C\}$ (i.e., the set K from the proof of Theorem 10.0.1 with $\{S\}$ as X), and let K_γ be the π-image of K, that is,

$$\{(\delta, \pi([(n, \bar{x}, \bar{\alpha})]_{\gamma,\{S\}})) : (\delta, (n, \bar{x}, \bar{\alpha})) \in K\}.$$

By Theorem 10.0.1, A is $\mathbb{V}_{\gamma,\{S\}}$-generic over HOD, via the generic filter G_A. Then $\pi[G_A]$ is $\mathrm{HOD}_{\{S\}}$-generic for $\leq_S$, and A is in $L[\leq_\gamma, K_\gamma][G_A]$. Then

$$V = L[A] = \mathrm{HOD}_{\{S\}}[G_A] = L[\leq_\gamma, K_\gamma][\pi[G_A]],$$

which implies that $\mathrm{HOD}_{\{S\}} = L[\leq_\gamma, K_\gamma]$, by Lemma 10.0.3, with $L[\leq_\gamma, K_\gamma]$ as M_1 and $\mathrm{HOD}_{\{S\}}$ as M_2.

We introduce a new partial order to deal with the general case. Recall that, given an infinite set Z, $\mathrm{Col}^*(\omega, Z)$ is the partial order of finite partial injections from ω to Z (each with domain some $n \in \omega$), ordered by extension. Given a filter $G \subseteq \mathrm{Col}^*(\omega, Z)$, let $a_G = \{(i,j) \in \omega \times \omega : g(i) \in g(j)\}$, where $g = \bigcup G$. Then a_G is a subset of $\omega \times \omega$, and, if Z is transitive, $V[G] = V[a_G]$.[3]

As above, suppose that X and Z are sets (with Z infinite) such that Z is in $\mathrm{OD}_{\mathrm{TC}(X)}$, and let κ be the least cardinal λ such that $X \subseteq V_\lambda$, Z is in $\mathrm{OD}_X^{V_\lambda}$ and every subset of $\mathcal{P}(\mathrm{Col}^*(\omega, Z))$ in $\mathrm{OD}_{\mathrm{TC}(X)}$ is in $\mathrm{OD}^{V_\lambda}_{\mathrm{TC}(X)}$. Let $P^\omega_{Z,X}$ be the set of tuples $(n, m, \bar{x}, \bar{\alpha})$ such that

- $n \in \omega$,
- m is the Gödel number of a binary formula ϕ,

[3]Explain?

- $\bar{x}$ is a finite sequence of elements of $\mathrm{TC}(X)$,
- $\bar{\alpha}$ is a finite sequence of elements of κ and
- the set $B_{n,m,\bar{x},\bar{\alpha}} = \{p \in \mathrm{Col}^*(\omega, Z) : V_\kappa \models \phi(p, \bar{x}, \bar{\alpha})\}$ is nonempty, and all members of $B_{n,m,\bar{x},\bar{\alpha}}$ have domain n.

Given $(n, m, \bar{x}, \bar{\alpha}) \in P^\omega_{Z,X}$ and $k \leq n$, let $B_{n,m,\bar{x},\bar{\alpha}} \restriction k$ denote the set

$$\{p \restriction k : p \in B_{n,m,\bar{x},\bar{\alpha}}\}.$$

This set is evidently also in HOD_X. The set $P^\omega_{Z,X}$ is in HOD_X, as is the reflexive and transitive ordering $\leq^\omega_{Z,X}$ on $P^\omega_{Z,X}$ defined by setting $(n, m, \bar{x}, \bar{\alpha}) \leq^\omega_{Z,X} (k, j, \bar{y}, \bar{\beta})$ if $n \geq k$ and $B_{n,m,\bar{x},\bar{\beta}} \restriction k \subseteq B_{k,j,\bar{y},\bar{\beta}}$. Let $\equiv^\omega_{Z,X}$ be the induced equivalence relation on $P^\omega_{Z,X}$, and let $[n, m, \bar{x}, \bar{\alpha}]^\omega_{Z,X}$ denote the $\equiv^\omega_{Z,X}$-class of a tuple $(n, m, \bar{x}, \bar{\alpha})$. Let $\mathbb{V}^\omega_{Z,X}$ be the partial order whose domain is the set of $\equiv^\omega_{Z,X}$-classes of $P^\omega_{Z,X}$, with the order inherited from $\leq^\omega_{Z,X}$. Each injection $f : \omega \to Z$ defines a filter H_f on $\mathbb{V}^\omega_{Z,X}$, consisting of the $\equiv^\omega_{Z,X}$-classes of those $(n, m, \bar{x}, \bar{\alpha}) \in P^\omega_{Z,X}$ for which $f \restriction n$ is in $B_{n,m,\bar{x},\bar{\alpha}}$.

We will use the following lemma later with $X = \mathrm{TC}(\omega^\omega)$ (without assuming $V = L(\mathbb{R})$).[4]

Lemma 10.0.4. *For any set X, and any infinite set Z in HOD_X, if $G \subseteq \mathrm{Col}^*(\omega, Z)$ is a V-generic filter, and $g = \bigcup G$, then H_g is $\mathbb{V}^\omega_{Z,X}$-generic over HOD_X, and a_g is in $\mathrm{HOD}_X[H_g]$.*

Proof. To see that H_g is $\mathbb{V}^\omega_{Z,X}$-generic over HOD_X, let D be a dense open subset of $\mathbb{V}^\omega_{Z,X}$ in HOD_X, and let p be a condition in $\mathrm{Col}^*(\omega, Z)$. Let D^0 be the set of $(n, m, \bar{x}, \bar{\alpha}) \in P^\omega_{Z,X}$ with $[(n, m, \bar{x}, \bar{\alpha})]^\omega_{Z,X} \in D$. Applying the genericity of G, it suffices to find a condition $p' \leq p$ and a $(n, m, \bar{x}, \bar{\alpha})$ in D^0 such that p' is in $B_{n,m,\bar{x},\bar{\alpha}}$. Let n_p be the domain of p. The set

$$\bigcup\{B_{n,m,\bar{x},\bar{\alpha}} \restriction n_p : (n, m, \bar{x}, \bar{\alpha}) \in D^0, n \geq n_p\},$$

being in HOD_X, must be the set of injections from n_p to Z, since otherwise the complement of this set has the form $B_{n_p,j,\bar{y},\bar{\beta}}$ for some j and $\bar{\beta}$, and we get a contradiction by considering a $(n, m, \bar{x}, \bar{\alpha}) \in D^0$ with $B_{n,m,\bar{x},\bar{\alpha}} \restriction n_p \subseteq B_{n_p,j,\bar{y},\bar{\beta}}$. It follows that there exist $p' \leq p$ and $(n, m, \bar{x}, \bar{\alpha})$ as desired.

As in the proof of Theorem 10.0.1, let K be the set of pairs

$$((i, j), (n, m, \bar{x}, \bar{\alpha})) \in (\omega \times \omega) \times P^\omega_{Z,X}$$

such that $\{i, j\} \subseteq n$ and $p(i) \in p(j)$ for all $p \in B_{n,m,\bar{x},\bar{\alpha}}$. Then again K is in HOD_X, and, as

$$a_g = \{(i, j) \in \omega \times \omega : \exists\, [n, m, \bar{x}, \bar{\alpha}]^\omega_{Z,X} \in H_g\; ((i, j), (n, m, \bar{x}, \bar{\alpha})) \in K\},$$

a_g is in $\mathrm{HOD}_X[H_g]$. □

[4] Where? Also, relativized? And, change to $L(\omega^\omega)$?

Theorem 10.0.2 now follows.

Proof of Theorem 10.0.2. We may assume that $A = V_\alpha$, for some infinite ordinal α (so that A is ordinal definable). In $\mathrm{HOD}_{\{S\}}$, there exist an ordinal γ and bijection $\pi\colon \mathbb{V}^\omega_{A,\{S\}} \to \gamma$. Let $\leq_\gamma$ be the partial order on γ induced by π. Let K be the set of pairs

$$((i,j),(n,m,\bar{x},\bar{\alpha})) \in (\omega\times\omega)\times P^\omega_{A,\{S\}}$$

such that $\{i,j\} \subseteq n$ and $p(i) \in p(j)$ for all $p \in B_{n,m,\bar{x},\bar{\alpha}}$ (i.e., the set K from the proof of Lemma 10.0.4 with A as Z and $\{S\}$ as X), and let K_γ be the π-image of K, that is,

$$\{((i,j),\pi([(n,m,\bar{x},\bar{\alpha})]^\omega_{A,\{S\}})) : ((i,j),(n,m,\bar{x},\bar{\alpha})) \in K\}.$$

Let $G_0 \subseteq \mathrm{Col}^*(\omega, A)$ be a V-generic filter and let $g = \bigcup G_0$. Then $\pi[H_g]$ is a $\mathrm{HOD}_{\{S\}}$-generic filter for $\leq_\gamma$. As in the proof of Lemma 10.0.4, a_g is in the model $L[\leq_\gamma, K_\gamma][\pi[H_g]]$, and therefore so is A. Applying Lemma 10.0.3 with $L[\leq_\gamma, K_\gamma]$ as M_1, $\mathrm{HOD}_{\{S\}}$ as M_2, $\leq_\gamma$ as $\mathbb{P}$ and $\pi[H_g]$ as G, we get that $L[\leq_\gamma, K_\gamma] = \mathrm{HOD}_{\{S\}}$. As the model $L[\leq_\gamma, K_\gamma]$ has the form $L[B]$, for some set of ordinals B, we are done. □

10.1 Codes for projections, and Uniformization

This section uses the results of Chapter 8, using the notation introduced in Sections 8.3 and 8.4. If S is a set of ordinals then $(\equiv_S, \leq_S)$ is an ordered equivalence relation in the sense of Chapter 8. Theorem 1.1.5 shows that AD implies that, for each set of ordinals S, μ_S is an ultrafilter on the corresponding set $\mathcal{D}_S$; if $\mathsf{DC}_\mathbb{R}$ holds then μ_S is countable complete. If there exists any set of ordinals S for which μ_S is a countably complete ultrafilter on $\mathcal{D}_S$, then ω_1^V is measurable (see Remark 1.1.7).

For each set S of ordinals, and each $x \in \omega^\omega$, let Q^S_x denote the definably least poset on the ordinals isomorphic in $\mathrm{HOD}^{L[S,x]}_S$ to the poset $(\mathbb{V}_{\omega,S})^{L[S,x]}$ (via the definability order on $\mathrm{HOD}^{L[S,x]}_S$, say), and let K^S_x (also a set of pairs of ordinals) denote the corresponding version of the set K from the proof of Theorem 10.0.1 (i.e., relabeled as in the proof of Theorem 10.0.2 to refer to Q^S_x). Note that Q^S_x depends only on $[x]_S$. We will use this notation throughout this section, and in other places.[5]

Theorem 10.1.2 below shows that if $\mathsf{DC}_\mathbb{R}$ holds and μ_S is an ultrafilter for each set of ordinals S then the collection of ∞-Borel sets is projectively closed. We start with a lemma whose hypothesis is weaker (and appeal to the fact that the nonexistence of an injection from ω_1^V to ω^ω is equivalent to ω_1^V being strongly inaccessible in every inner model satisfying Choice).

[5]probably

Lemma 10.1.1. *Let S be a set of ordinals, and suppose there is no injection from ω_1^V to ω^ω. Let ϕ be a ternary formula and let B denote the set*

$$\{x \in \omega^\omega : \exists y \in \omega^\omega\, L[S,x,y] \models \phi(S,x,y)\}.$$

Then following are equivalent, for each $x \in \omega^\omega$.

1. *$x \in B$*
2. *For some $w \in \omega^\omega$, for all $z \geq_S w$, $L[S, Q^S_z, K^S_z, x] \models$ "there is some $p \in \mathrm{Col}(\omega, 2^{Q^S_z})$ forcing that $\exists y\, L[S,x,y] \models \phi(S,x,y)$."*
3. *For some $z \in \omega^\omega$, $L[S, Q^S_z, K^S_z, x] \models$ "there is some $p \in \mathrm{Col}(\omega, 2^{Q^S_z})$ forcing that $\exists y\, L[S,x,y] \models \phi(S,x,y)$."*

Proof. That (2) implies (3) is immediate. To see that (3) $\Rightarrow$ (1), fix one such z. The cardinality of Q^S_z in $L[S,z]$ is at most $(2^{2^{\aleph_0}})^{L[S,z]}$, which is below ω_1^V, as ω_1^V is strongly inaccessible in $L[S,z]$. Since ω_1^V is strongly inaccessible in $L[S, Q^S_z, K^S_z, x]$,

$$\mathcal{P}(Q^S_z) \cap L[S, Q^S_z, K^S_z, x]$$

is countable, and $L[S, Q^S_z, K^S_z, x]$-generic filters for Q^S_z exist below each condition. Since forcing with $\mathrm{Col}(\omega, 2^{Q^S_z})$ adds a generic filter for Q^S_z, statement (1) follows.

For (1) $\Rightarrow$ (2), fix $y_0 \in \omega^\omega$ such that $L[S,x,y_0] \models \phi(S,x,y_0)$ and $z \in \omega^\omega$ such that $z \geq_S y_0$ and $z \geq_S x$. Letting $G_z \subseteq Q_z$ be the filter induced by z as in the proof of Theorem 10.0.1, G_z is $\mathrm{HOD}^{L[S,z]}_S$-generic. It follows that G_z is Q^S_z-generic over $L[S, Q^S_z, K^S_z]$ and (using K^S_z) that z is a member of $L[S, Q^S_z, K^S_z][G_z]$. Since x is in $L[S, Q^S_z, K^S_z][G_z]$, $L[S, Q^S_z, K^S_z, x]$ is a generic extension of $L[S, Q^S_z, K_z]$, and $L[S, Q^S_z, K^S_z][G_z]$ is a generic extension of $L[S, Q^S_z, K^S_z, x]$ by a partial order of cardinality at most $(2^{|Q^S_z|})^{L[S, Q^S_z, K^S_z]}$ in $L[S, Q^S_z, K^S_z, x]$. This partial order regularly embeds into $\mathrm{Col}(\omega, 2^{Q^S_z})$ in $L[S, Q^S_z, K^S_z, x]$.[6] This gives (2). $\square$

If DC holds in $L(S,\mathbb{R})$ (as it does if $\mathsf{DC}_{\mathbb{R}}$ holds), then for each set S of ordinals the ultraproduct

$$V^{\mathcal{D}_S}/\mu_S$$

(formed using all S-invariant functions in $L(S,\mathbb{R})$) is wellfounded in $L(S,\mathbb{R})$. We let S^∞, Q^∞_S and K^∞_S be the sets in $\prod_{z \in \omega^\omega} V/\mu_S$ represented by the functions $z \mapsto S$ and $z \mapsto Q^S_z$ and $z \mapsto K^S_z$, respectively.

Theorem 10.1.2. *Assume that $\mathsf{DC}_{\mathbb{R}}$ holds. Let S be a set of ordinals, and suppose that μ_S is an ultrafilter. Let ϕ be a ternary formula, let S be a set of ordinals, and let B be*

$$\{x \in \omega^\omega : \exists y \in \omega^\omega\, L[S,x,y] \models \phi(S,x,y)\}.$$

[6] Explain regularly embeds, and cite McAloon?

Then there exist a set of ordinals T in $\mathrm{OD}_{\{S\}}$ and a binary formula ψ such that

$$B = \{x \in \omega^\omega : L[T,x] \models \psi(T,x)\}.$$

Proof. By $\mathsf{DC}_{\mathbb{R}}$, μ_S is countably complete. Since μ_S is an ultrafilter, ω_1^V is a measurable cardinal, which implies that there is no injection from ω_1^V to ω^ω. We work in $L(S,\mathbb{R})$, which satisfies DC by the assumption that V satisfies $\mathsf{DC}_{\mathbb{R}}$.

Since DC holds in $L(S,\mathbb{R})$, for each $x \in \omega^\omega$ the ultraproduct

$$\prod_{z\in\omega^\omega} L[S, Q^S_z, K^S_z, x]/\mu_S$$

is wellfounded in $L(S,\mathbb{R})$. Furthermore, for each $x \in \omega^\omega$,

$$L[S^\infty, Q^\infty_S, K^\infty_S, x] = \prod_{z\in\omega^\omega} L[S, Q^S_z, K^S_z, x]/\mu_S$$

and, by Lemma 10.1.1, x is in B if and only if $L[S^\infty, Q^\infty_S, K^\infty_S, x]$ satisfies the statement "there is some $p \in \mathrm{Col}(\omega, 2^{Q^\infty_S})$ forcing that

$$\exists y \in \omega^\omega \; L[S^\infty, x, y] \models \phi(S^\infty, x, y)."$$

Then we can let T be set of the ordinals coding the triple $(S^\infty, Q^\infty_S, K^\infty_S)$ under some fixed coding in L of triples of ordinals by ordinals, and let ψ be the corresponding version of the statement just given. $\square$

Theorem 10.1.4 below is a uniformization result derived from Lemma 10.1.1 and Theorem 10.1.2.[7] Given sets B and x, we let $\sigma_{B,x}$ denote $\omega^\omega \cap \mathrm{OD}_{\{B,x\}}$. Recall that for a set $A \subseteq (\omega^\omega)^2$ and $y \in \omega^\omega$, A_y denotes the set $\{z \in \omega^\omega : (y,z) \in A\}$, and that $\exists^{\omega^\omega} A$ denotes the set $\{y \in \omega^\omega : A_y \neq \emptyset\}$. Such a set can be *uniformized* if there is a function $f\colon \exists^{\omega^\omega} A \to \omega^\omega$ such that $(y, f(y)) \in A$ for all $y \in \exists^{\omega^\omega} A$. Recall also that, for $x, y \in \omega^\omega$, $x \oplus y$ denotes the element z of ω^ω such that, for all $n \in \omega$, $z(2n) = x(n)$ and $z(2n+1) = y(n)$.

The proof of Theorem 10.1.4 goes through the following observation.

Lemma 10.1.3. *Suppose that A and B are sets, with $A \subseteq (\omega^\omega)^2$. Suppose that for a Turing cone of $x \in \omega^\omega$, for all $y \in L[x] \cap \exists^{\omega^\omega} A$,*

$$A_y \cap \sigma_{B,x} \neq \emptyset.$$

Then A can be uniformized.

Proof. Let x_0 be a base for a cone witnessing the relevant assumption of the lemma. Given y in $\exists^{\omega^\omega} A$, define $f(y)$ to be the $\mathrm{OD}_{\{B,x_0,y\}}$-least $z \in \sigma_{B,x_0\oplus y}$ with $(y,z) \in A$. $\square$

[7] Somewhere we should define "Turing cone", "the Turing measure is an ultrafilter" and "base of a cone".

Theorem 10.1.4. *Suppose that there is no injection from ω_1 into ω^ω, and that every set of Turing degrees contains or is disjoint from a cone.*[8] *Suppose that $S \subseteq \mathrm{Ord}$ is an ∞-Borel code for a set $A \subseteq \omega^\omega$. Let B be a set.*[9] *Suppose that, for a Turing cone of $x \in \omega^\omega$, $\omega^\omega \cap \sigma_{B,x} \not\subseteq L[S^\infty, Q^\infty_S, K^\infty_S, x]$. Then A can be uniformized.*

Proof. There exists a set $A' \subseteq (\omega^\omega)^2$ such that $\exists^{\omega^\omega} A' = \omega^\omega$, $A'_y = A_y$ for all $y \in \exists^{\omega^\omega} A$ and S is an ∞-Borel code for A'. It suffices then to prove the theorem under the assumption that $\exists^{\omega^\omega} A = \omega^\omega$.

We prove that for a Turing cone of $x \in \omega^\omega$, for all $y \in \omega^\omega \cap L[S, x]$,

$$A_y \cap \sigma_{B,x} \neq \emptyset,$$

and apply Lemma 10.1.3. Let $x_0 \in \omega^\omega$ be such that for all $x \geq_{\mathrm{T}} x_0$,

$$\omega^\omega \cap L[S^\infty, Q^\infty_S, K^\infty_S, x] \subsetneq \sigma^B_x.$$

Applying the assumption that the Turing measure is an ultrafilter, suppose toward a contradiction that for a Turing cone of x there exists a $y \in \omega^\omega \cap L[S, x]$ such that $A_y \cap \sigma^B_x = \emptyset$. Let $x_1 \geq_{\mathrm{T}} x_0$ be a base for a Turing cone witnessing this.

Let ϕ be a ternary formula such that

$$A = \{(y, z) \in (\omega^\omega)^2 : L[S, y, z] \models \phi(S, y, z)\}$$

By Lemma 10.1.1, for every $y \in \omega^\omega$,

$$L[S^\infty, Q^\infty_S, K^\infty_S, y] \models \text{“}V^{\mathrm{Col}(\omega, Q^\infty)} \models \exists z\, L[S, y, z] \models \phi(S, y, z)\text{”}.$$

Let U be an ultrafilter on ω as in Section 8.2, in $L[S^\infty, Q^\infty_S, K^\infty_S, x_1]$, and let $\mathbb{P}$ denote $\mathbb{P}_U^{L[S^\infty, Q^\infty_S, K^\infty_S, x_1]}$. Fix

$$t \in \sigma_{B,x_1} \setminus L[S^\infty, Q^\infty_S, K^\infty_S, x_1].$$

For any $L[S^\infty, Q^\infty_S, K^\infty_S, x_1]$-generic

$$g \subseteq \mathbb{P},$$

and any

$$y \in \omega^\omega \cap L[S^\infty, Q^\infty_S, K^\infty_S, x_1][g],$$

$$L[S^\infty, Q^\infty_S, K^\infty_S, x_1][g] \models \text{“}V^{\mathrm{Col}(\omega, Q^\infty_S)} \models \exists z \in \omega^\omega\, L[S, y, z] \models \phi(S, y, z)\text{”},$$

since $L[S^\infty, Q^\infty_S, K^\infty_S, x_1][g]$ is an outer model of $L[S^\infty, Q^\infty_S, K^\infty_S, y]$.

We have that

$$L[S^\infty, Q^\infty_S, K^\infty_S, x_1] = \prod L[S, Q^S_x, K^S_x, x_1]/\mu_S.$$

Since there is no injection from ω_1 into ω^ω, $L[S^\infty, Q^\infty_S, K^\infty_S, x_1] \cap \mathcal{P}(\mathcal{P}(\omega^\omega))$ is countable, and there is an $x_2 \geq_{\mathrm{T}} x_1$ in ω^ω such that, for all $x \geq_{\mathrm{T}} x_2$,

[8] We should say : The Turing measure is an ultrafilter.
[9] Any set!

1. $L[S^\infty, Q_S^\infty, K_S^\infty, x_1] \cap \mathcal{P}(\mathcal{P}(\omega^\omega)) = L[S, Q_x^S, K_x^S, x_1] \cap \mathcal{P}(\mathcal{P}(\omega^\omega))$,
2. $\mathbb{P} \in L[S, Q_x^S, K_x^S, x_1]$,
3. all $\mathbb{P}$-names in $L[S^\infty, Q_S^\infty, K_S^\infty, x_1]$ for elements of ω^ω are in $L[S, Q_x^S, K_x^S, x_1]$,
4. $1_\mathbb{P}$ forces over $L[S, Q_x^S, K_x^S, x_1]$ that for all $y \in \omega^\omega$,

$$V^{\mathrm{Col}(\omega, Q_x^S)} \models \exists z \in \omega^\omega\, L[S, y, z] \models \phi(S, y, z).$$

As in Section 8.2, there is an $L[S^\infty, Q_S^\infty, K_S^\infty, x_1]$-generic filter g as above such that there is a $w \in \omega^\omega \cap L[g, t]$ which codes an ω-sequence of $\mathbb{P}$-names from $L[S, Q_{x_2}^S, K_{x_2}^S, x_1]$ such that each $\mathbb{P}$-name in $L[S, Q_{x_2}^S, K_{x_2}^S, x_1]$ for a subset of $Q_{x_2}^S$ is forced by $1_\mathbb{P}$ to have the same realization as some member of the sequence.

Consider $x_1 \oplus g$. Suppose that $y \in L[S^\infty, Q_S^\infty, K_S^\infty, x_1 \oplus g] \cap \exists^{\omega^\omega} A$. The following claim finishes the proof of the theorem.

Claim. $A_y \cap \sigma_{B, x_1 \oplus g} \neq \emptyset$.

We prove the claim. Since $t \in \sigma_{B, x_1 \oplus g}$ and $w \in L[g, t]$ there exists a $L[S, Q_{x_2}^S, K_{x_2}^S, x_1][g]$-generic filter $G \subseteq \mathrm{Col}(\omega, Q_{x_2}^S)$ in $\mathrm{OD}_{\{B, x_1, g\}}$. By item (4) from the choice of x_2, there is a

$$z \in \omega^\omega \cap L[S, Q_{x_2}^S, K_{x_2}^S, x_1][g][G]$$

such that $L[S, y, z] \models \phi(S, y, z)$. Since

$$\omega^\omega \cap L[S, Q_{x_2}^S, K_{x_2}^S, x_1][g][G] \subseteq \sigma_{B, x_1 \oplus g},$$

we are done. □

Given sets S, T of ordinals, let us write $<_\mathcal{D}$ to mean that $\omega^\omega \cap L[S, x] \subsetneq L[T, x]$ for a Turing cone of x. Theorem 10.1.4 shows that if the Turing measure is an ultrafilter, every subset of ω^ω is ∞-Borel[10] and $<_\mathcal{D}$ has no greatest element, then Uniformization holds. The converse also holds.

Corollary 10.1.5. *Assume Turing Determinacy + DC, and that every subset of ω^ω is ∞-Borel. Then Uniformization is equivalent to the assertion that $<_\mathcal{D}$ has no greatest element.*[11]

Proof. Let S be a set of ordinals, and let $f : \omega^\omega \to \omega^\omega$ uniformize the set $\{(x, y) \in (\omega^\omega)^2 : y \notin L[S, x]\}$. Let T be an infinity Borel code for $\{(x, i, j) : (i, j) \in f(x)\}$ via formula ϕ. Then for all (x, i, j), $(i, j) \in f(x)$ if and only if $L[T, x] \models \phi(T, x, i, j)$, which shows that $L[T, x]$ is closed under f for all x.[12][13] □

[10] and some other things? $\mathsf{DC}_\mathbb{R}$ at least?

[11] Just $\mathsf{DC}_\mathbb{R}$?

[12] Maybe we have to say something to make this notation official?

[13] Corollary again : Assuming Turing Determinacy plus DC, if A is ∞-Borel and $x \in F(x)$ (?) then $F(x) \subseteq H^\infty[t]\ x \to \omega^\omega \cap L[t, x]$ (?)

10.2 Vopěnka algebras and ∞-Borel sets

Given a set Y consisting of ordinals, the Vopěnka algebra can also be varied to use ∞-Borel codes in $\mathrm{OD}_{\{Y\}}$ to refer to sets. As before, we define one-step and ω-sequence versions. We fix a the least cardinal κ such that for each $n \in \omega$ and each $A \subseteq (\omega^\omega)^n$, if there exist a set of ordinals S in $\mathrm{OD}_{\{Y\}}$ and a formula ϕ such that $A = \{\bar{x} \in (\omega^\omega)^n : L[S, \bar{x}] \models \phi(S, \bar{x})\}$ then there exist a set of ordinals S in $V_\kappa \cap \mathrm{OD}_{\{Y\}}$ and a formula ϕ such that

$$A = \{\bar{x} \in (\omega^\omega)^n : L_\kappa[S, \bar{x}] \models \phi(S, \bar{x})\}.$$

Let $P_{\infty,Y}$ be the set of pairs (m, S) such that

- m is the Gödel number of a binary formula ϕ,
- $S \in V_\kappa \cap \mathrm{OD}_{\{Y\}}$ is a set of ordinals,
- the set $B_{m,S} = \{x \in \omega^\omega : L_\kappa[S, x] \models \phi(S, x)\}$ is nonempty.

Then again we have an induced ordering $\leq_{\infty,Y}$ on $P_{\infty,Y}$ defined by setting

$$(m, S) \leq_{\infty,Y} (j, T)$$

if $B_{m,S} \subseteq B_{j,T}$. Let $\equiv_{\infty,Y}$ be the induced equivalence relation on $P_{\infty,Y}$. Let $\mathbb{V}_{\infty,Y}$ be the partial order whose domain is the set of $\equiv_{\infty,Y}$-classes of $P_{\infty,Y}$, with the order inherited from $\leq_{\infty,Y}$.

For the sequence version, let $P^\omega_{\infty,Y}$ be the set of triples (n, m, S) such that

- $n \in \omega$,
- m is the Gödel number of a binary formula ϕ,
- $S \in V_\kappa \cap \mathrm{OD}_{\{Y\}}$ is a set of ordinals,
- the set $B^\omega_{n,m,S} = \{\bar{x} \in (\omega^\omega)^n : L[S, \bar{x}] \models \phi(S, \bar{x})\}$ is nonempty.

Then again we have an induced ordering $\leq^\omega_{\infty,Y}$ on $P^\omega_{\infty,Y}$ defined by setting

$$(n, m, S) \leq^\omega_{\infty,Y} (k, j, T)$$

if $n \geq k$ and $\{\bar{x}\restriction k : \bar{x} \in B^\omega_{n,m,S}\} \subseteq B^\omega_{k,j,T}$. Let $\equiv^\omega_{\infty,Y}$ be the induced equivalence relation on $P^\omega_{\infty,Y}$. Let $\mathbb{V}^\omega_{\infty,Y}$ be the partial order whose domain is the set of $\equiv^\omega_{\infty,Y}$-classes of $P^\omega_{\infty,Y}$, with the order inherited from $\leq^\omega_{\infty,Y}$.

10.2.1 Remark. By Theorem 10.1.2, for each bounded $S \subseteq \kappa$ in $\mathrm{HOD}_{\{Y\}}$ and each ternary formula ϕ, there exist a bounded $T \subseteq \kappa$ and a binary formula ϕ such that $\{x \in \omega^\omega : \exists y \in \omega^\omega\, L[S, x, y] \models \phi(S, x, y)\} = \{x \in \omega^\omega : L[T, x] \models \psi(T, x)\}$.

10.2.2 Remark. Let (n, m, S) be an element of $P^\omega_{\infty,Y}$, and let ϕ be the formula with Gödel number m. Let n' be an element of $\omega \setminus n$. Then

$$\{\bar{x} \in (\omega^\omega)^{n'} : \bar{x}\restriction n \in B_{n,m,S}\} = \{\bar{x} \in (\omega^\omega)^{n'} : L_\kappa(S, \bar{s}) \models \phi(S, \bar{x}\restriction n)\},$$

so this set has the form $B_{n',k,S}$ for some $k \in \omega$, and $B_{n',k,S} \leq^\omega_{\infty,Y} B_{n,m,S}$. Moreover, there exists in $\mathrm{HOD}_{\{Y\}}$ a function sending each such tuple (n, m, S, n') to a corresponding value k.

10.2.3 Remark. The first paragraph of the proof of Theorem 10.0.1 adapts to prove that for each $x \in \omega^\omega$, the set G_x consisting of those $[(m,S)]_{\infty,\{Y\}} \in P_{\infty,\{Y\}}$ such that $x \in B_{m,S}$ is a $\mathrm{HOD}_{\{Y\}}$-generic filter. This also follows from Lemma 10.2.5 below, using the fact that the first coordinate of a $\mathbb{V}^\omega_{\infty,\{Y\}}$-generic sequence is generic for $\mathbb{V}_{\infty,\{Y\}}$. For each $i \in \omega$, $x(i)$ is the unique $j \in \omega$ such that $[(\emptyset, m_{i,j})]_{\infty,\{Y\}} \in G_x$, where $m_{i,j}$ is the Gödel number of the formula $y(i) = j$. It follows then that $\mathrm{HOD}_{\{Y\}}[x] = \mathrm{HOD}_{\{Y\}}[G_x]$, since G_x is the set of $[(m,S)]_{\infty,Y} \in \mathbb{V}_{\infty,Y}$ such that $L_\kappa[S,x] \models \phi(S,x)$ (where ϕ is the formula with Gödel number m), and this can be computed in $\mathrm{HOD}_{\{Y\}}[x]$. Since every condition in $\mathbb{V}_{\infty,\{Y\}}$ is a member of G_x for some $x \in \omega^\omega$, every $\mathrm{HOD}_{\{Y\}}$-generic filter $G \subseteq \mathbb{V}_{\infty,\{Y\}}$ is equal to G_x, for $x = \{(i,j) : [(\emptyset, m_{i,j})]_{\infty,\{Y\}} \in G\}$.

10.2.4 Remark. Since the first coordinate of a $\mathbb{V}^\omega_{\infty,\{Y\}}$-generic sequence is generic for $\mathbb{V}_{\infty,\{Y\}}$, there is a $\mathbb{V}_{\infty,\{Y\}}$-name $\dot{Q}$ such that $\mathbb{V}^\omega_{\infty,\{Y\}}$ is forcing-equivalent to $\mathbb{V}_{\infty,\{Y\}} * \dot{Q}$, via a map which carries the generic real for $\mathbb{V}_{\infty,\{Y\}}$ to the first coordinate of the generic sequence produced by forcing with $\mathbb{V}^\omega_{\infty,\{Y\}}$.

As with the partial order $\mathbb{V}^\omega_{Z,X}$ above, each injection $f\colon \omega \to \omega^\omega$ defines a filter H_f on $\mathbb{V}^\omega_{\infty,Y}$, consisting of the $\equiv^\omega_{\infty,Y}$-classes of those $(n,m,S) \in P^\omega_{\infty,Y}$ for which $f \restriction n$ is in $B_{n,m,S}$. Lemma 10.2.5 is the version of Lemma 10.0.4 for $\mathbb{V}^\omega_{\infty,Y}$. Theorem 10.1.2 and Remark 10.2.2 are used in the proof of the lemma.

Lemma 10.2.5. *Suppose that* $\mathsf{DC}_\mathbb{R}$ *holds, let* Y *be a set of ordinals and suppose that* μ_S *is an ultrafilter for each set* S *in* $\mathrm{OD}_{\{Y\}}$ *consisting of ordinals. If* $G \subseteq \mathrm{Col}^*(\omega, \omega^\omega)$ *is* V*-generic, and* $g = \bigcup G$*, then* H_g *is* $\mathbb{V}^\omega_{\infty,Y}$*-generic over* $\mathrm{HOD}_{\{Y\}}$*, and* g *is in* $\mathrm{HOD}_{\{Y\}}[H_g]$*.*

Proof. Let $D \subseteq \mathbb{V}^\omega_{\infty,Y}$ be a dense set in $\mathrm{HOD}_{\{Y\}}$, and let p be a condition in $\mathrm{Col}^*(\omega,\omega^\omega)$. Let n_p be the domain of p. By strengthening the conditions in D if necessary, we may assume, applying Remark 10.2.2, that each element of D has the form $[(n,m,S)]^\omega_{\infty,Y}$ for some $(n,m,S) \in P^\omega_{\infty,Y}$ with $n \geq n_p$. Let D_0 be the set of $(n,m,S) \in P^\omega_{\infty,Y}$ with $[(n,m,S)]^\omega_{\infty,Y} \in D$. To show that H_g is $\mathrm{HOD}_{\{Y\}}$-generic, we want to find a condition $p' \leq p$ and a (n,m,S) in D_0 such that p' is in $B_{n,m,S}$. By Remark 10.2.1 there is an $\mathrm{OD}_{\{Y\}}$ function associating each $(n,m,S) \in D_0$ to a tuple (n_p,k,T) such that

- $k \in \omega$,
- $T \in V_\kappa \cap \mathrm{OD}_{\{Y\}}$ and
- $B_{n,m,S} \restriction n_p = B_{n_p,k,T}$.

Since $\mathrm{HOD}_{\{Y\}}$ is a model of Choice, and the collection of ∞-Borel sets is closed under wellordered unions, there exist by the choice of κ a set

$$T_* \in V_\kappa \cap \mathrm{OD}_{\{Y\}}$$

and a binary formula ψ (with Gödel number k_*) such that

$$\{\bar{x} \in (\omega^\omega)^{n_p} : \exists (n,m,S) \in D_0 \; \bar{x} \in B_{n,m,S} \restriction n_p\}$$

is equal to $\{\bar{x} \in (\omega^\omega)^{n_p} : L_\kappa[T_*, \bar{x}] \models \psi(T_*, \bar{x})\}$. Furthermore, B_{n_p,k_*,T_*} must be the set of all injections from n_p to ω^ω, since otherwise the complement of this set has the form $B_{n_p,j,R}$ for some $j \in \omega$ and $R \in V_\kappa \cap \mathrm{OD}_{\{Y\}}$, and we get a contradiction by considering a $(n, m, S) \in D_0$ with

$$\{\bar{x}\restriction n_p : \bar{x} \in B_{n,m,S}\} \subseteq B_{n_p,j,R}.$$

There exists then a triple $(n, m, S) \in D_0$ such that $p \in B_{n,m,S}\restriction n_p$, as desired. This gives the genericity of H_g.

To see that g is in $\mathrm{HOD}_{\{Y\}}[H_g]$, as in the proof of Theorem 10.0.1, let K be the set of pairs

$$((i, j, k), (n, m, S)) \in (\omega \times \omega \times \omega) \times P^\omega_{\infty,Y}$$

such that $i < n$ and $p(i)(j) = k$ for all $p \in B_{n,m,S}$. Then again K is in $\mathrm{HOD}_{\{Y\}}$, and K can be used to define a $\mathbb{V}^\omega_{\infty,Y}$-name $\tau \in \mathrm{HOD}_{\{Y\}}$ such that for every $\mathrm{Col}^*(\omega, \omega^\omega)$-generic function g over V, $\tau_{H_g} = g$. □

The following is the main theorem of this section.[14]

Theorem 10.2.6. *Suppose that $\mathsf{DC}_\mathbb{R}$ holds, let Y be a set of ordinals such that $V{=}L(Y, \mathbb{R})$ and suppose that μ_S is an ultrafilter for each set S in $\mathrm{OD}_{\{Y\}}$ consisting of ordinals. Then for each $\mathrm{OD}_{\{Y\}}$ set $A \subseteq \omega^\omega$ there exist an $\mathrm{OD}_{\{Y\}}$ set S of ordinals and a binary formula ϕ such that*

$$A = \{x \in \omega^\omega : L[S, x] \models \phi(S, x)\}.$$

Proof. Let $\bar{a}$ be a finite set of ordinals, let ψ be a ternary formula, and let A be the set of $x \in \omega^\omega$ for which $\psi(\bar{a}, x, Y)$ holds. By Theorem 10.0.2, we may fix a set B contained in the ordinals such that $\mathrm{HOD}_{\{Y\}} = L[B]$.

By Remark 10.2.4, $\mathbb{V}^\omega_{\infty,\{Y\}}$ is forcing-equivalent to an iteration of the form $\mathbb{V}_{\infty,\{V\}} * \dot{Q}$, for some $\mathbb{V}_{\infty,\{Y\}}$-name $\dot{Q}$. By Remark 10.2.3, for every $x \in \omega^\omega$ there is a $\mathrm{HOD}_{\{Y\}}$-generic filter $G_x \subseteq \mathbb{V}_{\infty,\{Y\}}$ such that $\mathrm{HOD}_{\{Y\}}[G_x] = \mathrm{HOD}_{\{Y\}}[x]$.

Let τ be the $\mathbb{V}^\omega_{\infty,\{Y\}}$-name from the proof of Lemma 10.2.5. Let $\dot{R}$ be a $\mathbb{V}^\omega_{\infty,\{Y\}}$-name in $\mathrm{HOD}_{\{Y\}}$ for the range of the realization of τ. Let $\dot{R}_*$ be the $\mathbb{V}_{\infty,\{Y\}} * \dot{Q}$-name induced by $\dot{R}$ and a map witnessing the forcing-equivalence of $\mathbb{V}_{\infty,\{Y\}} * \dot{Q}$ with $\mathbb{V}^\omega_{\infty,\{Y\}}$ which carries the generic real for $\mathbb{V}_{\infty,\{Y\}}$ to the first coordinate of the generic sequence produced by forcing with $\mathbb{V}^\omega_{\infty,\{Y\}}$, as in Remark 10.2.4. We want to see that for each $x \in \omega^\omega$, all generic $\dot{Q}_{G_x}$-extensions of $\mathrm{HOD}_{\{Y\}}[x]$ agree about whether or not $\phi(\bar{\alpha}, x, Y)$ holds in $L(Y, \dot{R}_*)$. By Lemma 10.2.5, $L(Y, \mathbb{R})$ is one of these extensions. The theorem will then follow, with S a set of ordinals coding B, Y, $\bar{\alpha}$ and $\dot{R}_*$.

If this were not the case there would be $x \in \omega^\omega$ and (n, m, S), (n', m', S') in $P^\omega_{\infty,Y}$ such that

- $x \in B_{n,m,S} \cap B_{n',m',S'}$

[14] It shows that the two versions of the Vopěnka algebra are equivalent!

- $[(n,m,S)]^{\omega}_{\infty,Y} \Vdash$ "$L(\check{Y},\dot{R}) \models \phi(\check{\bar{\alpha}},\tau(0),\check{Y})$";
- $[(n',m',S')]^{\omega}_{\infty,Y} \Vdash$ "$L(\check{Y},\dot{R}) \models \neg\phi(\check{\bar{\alpha}},\tau(0),\check{Y})$";

Let $p\colon n+n' \to \omega^\omega$ and $\pi\colon \omega \to \omega$ be such that

- $p(0) = x$;
- $\pi(0) = 0$;
- $\pi \restriction (n+n')$ is a permutation of $n+n'$;
- $\pi(i) = i$ for all $i \geq n+n'$;
- $p\restriction n \in B_{n,m,S}$;
- the function $p'\colon n' \to \omega^\omega$ defined by setting $p'(i) = p(\pi(i))$ is in $B_{n',m',S'}$.

Let $g\colon \omega \to \omega^\omega$ be the union of a V-generic filter for $\mathrm{Col}^*(\omega,\omega^\omega)$ containing p, and let $g'\colon \omega \to \omega^\omega$ be defined by setting $g'(i) = g(\pi(i))$ for all $i \in \omega$. Then g' is also the union of a V-generic filter for $\mathrm{Col}^*(\omega,\omega^\omega)$, $\dot{R}_{H_g} = \dot{R}_{H_{g'}}$ and $g(0) = g'(0)$, giving a contradiction. □

Part (1) of the following corollary appears as Theorem 1.9 in [2].

Corollary 10.2.7. *Suppose that* $\mathsf{DC}_{\mathbb{R}}$ *holds, let* Y *be a set of ordinals such that* $V{=}L(Y,\mathbb{R})$ *and suppose that* μ_S *is an ultrafilter for each set* $S \subseteq \mathrm{Ord}$ *in* $\mathrm{OD}_{\{Y\}}$*. Then the following hold.*

1. *Every subset of* ω^ω *is* ∞*-Borel.*
2. $\mathrm{HOD}_{\{Y,x\}} = \mathrm{HOD}_{\{Y\}}[x]$ *for all* $x \in \omega^\omega$*.*

Proof. For part (1), every set in $L(Y,\mathbb{R})$ is definable from Y, a finite set of ordinals and a finite subset of ω^ω. It follows then that for each $A \subseteq \omega^\omega$ in $L(Y,\mathbb{R})$, there exist an $\mathrm{OD}_{\{Y\}}$-set $B \subseteq \omega^\omega \times \omega^\omega$ and an $x \in \omega^\omega$ such that $A = \{y \in \omega^\omega : (x,y) \in B\}$. By Theorem 10.2.6, there exist an $\mathrm{OD}_{\{Y\}}$ set S of ordinals and a ternary formula ϕ such that $B = \{(x,y) \in \omega^\omega : L[S,x,y] \models \phi(S,x,y)\}$. It is easy to find a set T of ordinals coding S and x in some simple way, and a binary formula ϕ' such that $A = \{y \in \omega^\omega : L[T,y] \models \phi'(T,y)\}$.

For part (2), fix $x \in \omega^\omega$. Clearly, $\mathrm{HOD}_{\{Y\}}[x]$ is contained in $\mathrm{HOD}_{\{Y,x\}}$. To show the reverse inclusion, let $G_x \subseteq \mathbb{V}_{\omega,Y}$ (the ordinary Vopěnka algebra) be the $\mathrm{HOD}_{\{Y\}}$-generic filter induced by x. By Theorem 10.0.1 it suffices to show that G_x is in $\mathrm{HOD}_{\{Y\}}[x]$.

Let κ_0 be the cardinal κ used in the definition of $P_{\omega,\{Y\}}$ and, for each $(n,\bar{x},\bar{\alpha}) \in P_{\omega,\{Y\}}$ let $B^0_{n,\bar{x},\bar{\alpha}}$ be the set $B_{n,\bar{x},\bar{\alpha}}$ from this definition. Similarly, let κ_1 be the cardinal κ used in the definition of $P_{\infty,\{Y\}}$, and, for each $(m,S) \in P_{\infty,\{Y\}}$ let $B^1_{m,S}$ be the set $B_{m,S}$ from that definition. By Theorem 10.2.6, for each $(n,\bar{x},\bar{\alpha}) \in P_{\omega,\{Y\}}$ there is a pair $(m,S) \in P_{\infty,\{Y\}}$ such that $B^0_{n,\bar{x},\bar{\alpha}} = B^1_{m,S}$. Furthermore, the set of pairs $((n,\bar{x},\bar{\alpha}),(m,S))$ in $P_{\omega,\{Y\}} \times P_{\infty,\{Y\}}$ such that

$B^0_{n,\bar{x},\bar{\alpha}} = B^1_{m,S}$ is in $HOD_{\{Y\}}$. The set of $(m, S) \in P_{\infty,\{Y\}}$ such that $x \in B^1_{m,S}$ is in $\mathrm{HOD}_{\{Y\}}[x]$. It follows that G_x, which is the set of $(n, \bar{x}, \bar{\alpha})$ such that $x \in B^0_{n,\bar{x},\bar{\alpha}}$, is in $\mathrm{HOD}_{\{Y\}}[x]$. □

Chapter 11

Applications

11.1 Producing strong ∞-Borel codes

In this section we use the notion on the equivalence relation $\equiv_S$ from Sections 8.3 and 8.4, where S is a set of ordinals. Recall from Theorem 9.2.3 that subsets of ω^ω with strong ∞-Borel codes are Suslin. The following theorem produces strong ∞-Borel codes. Given a set of ordinals S, we let δ_S^∞ denote $\prod \omega_2^{L[S,x]}/\mu_S$.

Theorem 11.1.1. *Let S be a set of ordinals. Suppose that the following hold.*

- *μ_S is an ultrafilter on $\mathcal{D}_S$.*
- *The ultrapower $\prod \mathrm{Ord}/\mu_S$ is wellfounded, and $\delta_S^\infty < \Theta$.*[1]
- *δ_S^∞-Determinacy.*

Then if S is an ∞-Borel code for a set $A \subseteq \omega^\omega$, then A has a strong ∞-Borel code which is contained in δ_S^∞ and definable from S.[2]

Proof. Since ω-Determinacy (i.e., AD) holds, there is no injection from ω_1 into ω^ω. Then by Theorem 8.0.4 we have that for some $x_0 \in \omega^\omega$ and all $y \in \omega^\omega$ such that $[y]_S \geq_S [x]_S$, GCH holds in $L[S, y]$ below ω_1^V (which is strongly inaccessible in $L[S, y]$ by the nonexistence of an injection from ω_1 into ω^ω). For each such y, it follows that the partial order $\mathbb{V}_{\infty,S}$ (defined at the beginning of Section 10.2) as computed in $L[S, y]$, being isomorphic to the subset relation restricted to a subset of $\mathcal{P}(\omega^\omega)$, has cardinality at most $\aleph_2$ in $L[S, y]$. Furthermore, since antichains in this partial order correspond to pairwise disjoint subsets of $\mathcal{P}(\omega^\omega)$, $\mathbb{V}_{\infty,S}^{L[S,y]}$ is $\aleph_2$-c.c. in $L[S, y]$, and the set of all antichains of $\mathbb{V}_{\infty,S}^{L[S,y]}$ has cardinality $\aleph_2$ in $L[S, y]$. In the proof (2) $\Rightarrow$ (1) of Theorem 9.0.4, the translation from ∞-Borel* codes to ∞-Borel codes is ordinal definable. It follows that for each

[1] Maybe we can weaken this, either by restricting to an inner model, or an initial segment of the ordinals.

[2] Say what the code is

∞-Borel* code in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$ there is an ∞-Borel code in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$ defining the same subset of ω^ω in $L[S,y]$, and furthermore that the set of pairs of codes (of the two types) defining the same subset of ω^ω is also in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$.

In Section 9.1 we defined the relation $\phi \mapsto \phi^*$ on ∞-Borel codes, and noted (for instance in the proof of Theorem 9.1.5) that if κ is a regular cardinal and there does not exist a κ-sequence of disjoint ∞-Borel sets, then each resulting formula ϕ^* must be in $\mathcal{L}_{\kappa,0}$. This means that every set of reals in $L[S,y]$ with an ∞-Borel* code in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$ (i.e., every set of reals corresponding to a condition in $\mathbb{V}^{L[S,y]}_{\infty,S}$) has an ∞-Borel code in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$ which is a bounded subset of $\omega_2^{L[S,y]}$. Furthermore, starting with propositional variables $P_{n,m}$ ($n,m \in \omega$, corresponding to the assertion $z(n)=m$), we can, working by recursion on the complexity of formulas in $\mathcal{L}^{L[S,y]}_{\aleph_2,0} \cap \mathrm{HOD}^{L[S,y]}_{\{S\}}$, associate to each such formula ψ a condition $p_\psi \in \mathbb{V}^{L[S,y]}_{\infty,S}$ in such a way that

- for all $n,m \in \omega$, $p_{P_{n,m}}$ is the condition in $\mathbb{V}^{L[S,y]}_{\infty,S}$ corresponding to the set of $z \in \omega^\omega$ such that $z(n)=m$;
- for $\psi \in \mathcal{L}^{L[S,y]}_{\aleph_2,0} \cap \mathrm{HOD}^{L[S,y]}_{\{S\}}$, $p_{\neg\psi}$ is the complement of p_ψ in $\mathbb{V}^{L[S,y]}_{\infty,S}$;
- for all $\gamma < \omega_2^{L[S,y]}$ and all $\psi \in \mathcal{L}^{L[S,y]}_{\aleph_2,0} \cap \mathrm{HOD}^{L[S,y]}_{\{S\}}$ of form $\bigvee_{\alpha<\gamma} \psi_\alpha$, p_ψ is the supremum in $\mathbb{V}^{L[S,y]}_{\infty,S}$ of $\{p_{\psi_\alpha} : \alpha < \gamma\}$;
- for all $\gamma < \omega_2^{L[S,y]}$ and all $\psi \in \mathcal{L}^{L[S,y]}_{\aleph_2,0} \cap \mathrm{HOD}^{L[S,y]}_{\{S\}}$ of form $\bigwedge_{\alpha<\gamma} \psi_\alpha$, p_ψ is the infimum in $\mathbb{V}^{L[S,y]}_{\infty,S}$ of $\{p_{\psi_\alpha} : \alpha < \gamma\}$.

Since every set of reals in $L[S,y]$ with an ∞-Borel* code in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$ has an ∞-Borel code in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$ which is a bounded subset of $\omega_2^{L[S,y]}$, the range of our association $\psi \mapsto p_\psi$ will be all of $\mathbb{V}^{L[S,y]}_{\infty,S}$. There is an S-invariant function choosing for each $y \in \omega^\omega$ a condition q^y_A in $\mathbb{V}^{L[S,y]}_{\infty,S}$ corresponding in $L[S,y]$ to an ∞-Borel* code for $A \cap L[S,y]$. It follows that we can choose (for each $y \geq_S x_0$, in an S-invariant fashion) a formula $\chi_y \in \mathcal{P}(\omega_2)^{L[S,y]} \cap \mathcal{L}^{L[S,y]}_{\aleph_3,0}$ such that for all $z \in \omega^\omega$ (in V), if $z \models \chi_y$ then z generates a $\mathrm{HOD}^{L[S,y]}_{\{S\}}$-generic filter below q^y_A, so z is in A. For each such y, let T_y be the set of ordinals coding χ_y as in the first paragraph of Chapter 9.

Now we consider the ultrapower $\prod V/\mu_S$, which we have assumed to be wellfounded.[3] Let T be the set of ordinals represented by the function $y \mapsto T_y$. We want to see that T is a strong ∞-Borel code for A. We have that T is a subset of δ^∞_S, which by assumption is less than Θ.

[3]I think we need only $\mathsf{DC}_{\mathbb{R}}$ in this case, since we can assume that we are working in a model of the form $L(S, \omega^\omega, B)$ for some $B \subseteq \omega^\omega$. Check! Recall that by Solovay's theorem we get $\mathsf{DC}_{\mathbb{R}}$ from this assumption.

Since δ^∞_S-Determinacy holds, the game $\mathcal{G}_{\phi_T,\delta}$ from Definition 9.2.1 is determined.[4] We want to see then that player II does not have a winning strategy. Suppose towards a contradiction that Σ is such a strategy. For each $y \in \omega^\omega$ such that $y \geq x_0$, let R_y be the tree of attempts to find a countable $\sigma \subseteq \delta$ resulting from a play of $\mathcal{G}_{\phi_T,\delta}$ according to Σ, and an isomorphism between T_y and $T\restriction\sigma$.[5] If any of these trees is illfounded, then we have a contradiction. If not, then we can find (in an S-invariant fashion) ranking functions ρ_y on each such tree R_y. Let j be the map sending each element of V to the element of $\prod V/\mu_S$ represented by the corresponding constant function. In addition, let R^* and ρ^* be the elements of $\prod V/\mu_S$ represented by the maps $y \mapsto R_y$ and $y \mapsto \rho_y$. Then in the wellfounded model

$$\prod \mathrm{HOD}^{L[S,T,\Sigma,y]}_{\{S,T,\Sigma\}}/\mu_S,$$

ρ^* is a ranking function on the tree of attempts to a find countable $\sigma \subseteq j(\delta)$ resulting from a play of $j(\mathcal{G}_{\phi_T,\delta})$ according to $j(\Sigma)$, and an isomorphism between T and $j(T)\restriction\sigma$. Since $j[\delta]$ is closed under $j(\Sigma)$, and T is isomorphic to $j[T]$, which is $j(T)\restriction j[\delta]$, there is a nonempty subset of R^* without terminal nodes, corresponding to those nodes where the isomorphism is contained in j, and the run of the game is contained in $j[\delta]$. It follows that R^* is illfounded, giving the desired contradiction. $\square$

11.2 ∞-Borel representations from Uniformization

The main theorem of this section is Theorem 11.2.2, which says that if AD and Uniformization hold then all sets of reals are ∞-Borel. Theorem 11.2.1 is a slightly stronger version, in which the assumption of AD is replaced with some of its consequences. Theorem 11.2.1 is Theorem 5.10 of [16], whose presentation we follow closely.

Before beginning the proof of Theorem 11.2.1 we introduce some terminology for treating generic filters over countable structures in terms of descriptive set theory. Given a partial order $\mathbb{P}$, a *nice* $\mathbb{P}$-name for an element of ω^ω is a set τ such that

- each element of τ is a pair $(p,(n,m))$, where $p \in \mathbb{P}$ and $n, m \in \omega$;
- for each $p \in \mathbb{P}$ and each $n \in \omega$, there exist $q \leq p$ and $m \in \omega$ such that $(q,(n,m)) \in \tau$;
- for all $p, p' \in \mathbb{P}$ and all $n, m, m' \in \omega$, if $(p,(n,m))$ and $(p',(n,m'))$ are both in τ, and p and p' are compatible, then $m = m'$.

Given a nice $\mathbb{P}$-name τ and a set $g \subseteq \tau$, we let τ_g be the set of pairs (n,m) for which there exists a $p \in g$ with $(p,(n,m)) \in \tau$. Given a set X, we say that a

[4] This might take some explaining.

[5] More to the point, between the sentences they code.

filter $g \subseteq \mathbb{P}$ is $\mathcal{D}$-*generic* if it intersects each dense subset of $\mathbb{P}$ in X. We say that a set Y consisting of filters on $\mathcal{P}(\mathbb{P})$ is *comeager* if there is a countable set X, such that Y contains every X-generic filter contained in $\mathbb{P}$. We write $p \Vdash^* \tau \in A$ to mean that for a comeager set of filters $g \subseteq \mathbb{P}$, τ_g is in A, and define $p \Vdash^* \tau \notin A$ similarly. We let $D^A_{\mathbb{P},\tau}$ be the set of $p \in \mathbb{P}$ such that $p \Vdash^* \tau \in A$ or $p \Vdash^* \tau \notin A$. If $\mathbb{P}$ is countable, and every subset of ω^ω has the property of Baire, then each set of the form $D^A_{\mathbb{P},\tau}$ is dense in $\mathbb{P}$.

Given $A \subseteq \omega^\omega$, the *term relation* for A is the class $\mathrm{TR}(A)$ consisting of those triples $(\mathbb{P}, p, \tau)$ such that $\mathbb{P}$ is a poset, τ is a nice $\mathbb{P}$-name for an element of ω^ω and $p \Vdash^* \tau \in A$. We write $\dot{t}_{A,\mathbb{P}}$ for the set of (p, τ) for which $(\mathbb{P}, p, \tau) \in \mathrm{TR}(A)$. Note that $\dot{t}_{A,\mathbb{P}}$ is a $\mathbb{P}$-name, as is every subset of it. If $\mathbb{P}$ is a countable partial order and T is a countable set of nice $\mathbb{P}$-names for reals and each set $D^A_{\mathbb{P},\tau}$ $(\tau \in T)$ is dense, then, for comeagerly many filters $g \subseteq \mathbb{P}$ and all $\tau \in T$, $\tau_g \in A$ if and only if, for some $p \in g$, $(p, \tau) \in \dot{t}_{A,\mathbb{P}}$.

For any set M we write $\dot{t}^M_{A,\mathbb{P}}$ for $\dot{t}_{A,\mathbb{P}} \cap M$. A transitive set M is said to be *weakly*-$(A, \mathbb{P})$-*closed* if $\dot{t}^M_{A,\mathbb{P}} \in M$ (applying the definition to the set of names of the form $\check{x}$ $(x \in \omega^\omega \cap M)$ we see that this implies that $A \cap M \in M$). We say that M is *strongly*-$(A, \mathbb{P})$-*closed* if $\dot{t}^M_{A,\mathbb{P}} \in M$ and, for all M-generic filters $g \subseteq \mathbb{P}$, $\dot{t}^M_{A,\mathbb{P},g} = A \cap M[g]$ (here were are referring to existing M-generic filters, not filters existing in a forcing extension of V). We say that M is weakly (or strongly) A-closed if it is weakly (or strongly) $(A, \mathbb{P})$-closed for all partial orders $\mathbb{P} \in M$.

Theorem 11.2.1. *Suppose that* $\mathsf{Uniformization} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$ *holds and that there is a normal fine measure on* $\mathcal{P}_{\aleph_1}(\omega^\omega)$.[6] *Then every subset of* ω^ω *is* ∞-*Borel.*

Proof. Fix $A \subseteq \omega^\omega$. Let B be the set of $(x, y) \in (\omega^\omega)^2$ such that

- x HC-codes a pair $(\mathbb{P}, \tau)$ where $\mathbb{P}$ is a countable poset and τ is a nice $\mathbb{P}$-name for an element of ω^ω;
- y HC-codes a countable set $\mathcal{D}$ consisting of dense open subsets of $\mathbb{P}$, with $D^A_{\mathbb{P},\tau} \in \mathcal{D}$, such that for any $\mathcal{D}$-generic filter $g \subseteq \mathbb{P}$,
  - $\tau_g \in A$ if and only if $\exists p \in g\ p \Vdash^* \tau \in A$, and
  - $\tau_g \notin A$ if and only if $\exists p \in g\ p \Vdash^* \tau \notin A$.

The conclusion of the condition on y is equivalent to : for all $\mathcal{D}$-generic filters $g \subseteq \mathbb{P}$, $\tau_g \in \dot{t}_{A,\mathbb{P},g}$ if and only if $\tau_g \in A$. By the density of the sets $D^A_{\mathbb{P},\tau}$ (mentioned above), the assumption that every set of reals has the property of Baire implies that for each $x \in \omega^\omega$ coding a pair $(\mathbb{P}, \tau)$ as above, there is a $y \in \omega^\omega$ such that $(x, y) \in B$.

Let f be a function uniformizing B, and $F \subseteq \omega^\omega$ code f as follows : $x \in F$ if and only if, letting $x' \in \omega^\omega$ be such that $x'(n) = x(n + 2)$ for all $n \in \omega$,

[6] Do you really need normality?

$x' \in \mathrm{dom}(f)$ and $f(x')(x(0)) = x(1)$. For each $\sigma \in \mathcal{P}_{\aleph_1}(\omega^\omega)$, let N_σ be

$$L_{\omega_1}[\mathrm{TR}(A), \mathrm{TR}(F), \sigma],$$

and let M_σ be

$$\mathrm{HOD}^{N_\sigma}_{\{\mathrm{TR}(A),\mathrm{TR}(F)\}}.$$

Then the models N_σ and M_σ are weakly A-closed and weakly F-closed, and since ω_1 is measurable in V (see the last sentence of Remark 1.1.7 they are models of ZFC.

Claim. *For each $\sigma \in \mathcal{P}_{\aleph_1}(\omega^\omega)$, M_σ is strongly A-closed.*

To prove the claim, fix $\sigma \in \mathcal{P}_{\aleph_1}(\omega^\omega)$ and a partial order $\mathbb{P} \in M_\sigma$. Let C be the set of nice $\mathbb{P}$-names in M_σ for elements of ω^ω. Let $\mathbb{Q}$ be $\mathrm{Col}(\omega, \mathbb{P})$, and for each $\tau \in C$, let $\dot{z}_\tau$ be a nice $\mathbb{Q}$-name for an element of ω^ω coding the pair $(\mathbb{P}, \tau)$.

Since M_σ is weakly F-closed, $\dot{t}^{M_\sigma}_{F,\mathbb{Q}}$ is in M_σ. Let $\mathcal{E}$ be a countable set of dense open subsets of $\mathbb{Q}$, containing each dense open subset of $\mathbb{Q}$ from M_σ and having the property that for each $\mathcal{E}$-generic $h \subseteq \mathbb{Q}$ and all nice names $\rho \in M_\sigma$ for elements of ω^ω, $\rho_h \in F$ if and only if, for some $p \in h$, $(p, \rho) \in \dot{t}^{M_\sigma}_{F,\mathbb{Q}}$. Applying this to the collection of nice $\mathbb{Q}$-names ρ for which it is forced by $1_{\mathbb{Q}}$ that (for some $\tau \in C$) $\rho(n+2) = \dot{z}_\tau(n)$ for all $n \in \omega$, we get that for any $\mathcal{E}$-generic filter $h \subseteq \mathbb{Q}$, and for each $\tau \in C$, $\dot{z}_{\tau,h} \in \omega^\omega$ and $f(\dot{z}_{\tau,h})$ is in $M_\sigma[h]$. Fix such a h, and let $\mathcal{D}_h$ be the collection of dense open subsets of $\mathbb{P}$ coded by the members of $\{f(\dot{z}_{\tau,h}) : \tau \in C\}$.

Let $g \subseteq \mathbb{P}$ be an $M_\sigma[h]$-generic filter. Then g is $\mathcal{D}_h$-generic. It follows then that $\dot{t}^M_{A,\mathbb{P},g} = A \cap M_\sigma[g]$. Since $M_\sigma[h][g] = M_\sigma[g][h]$ (i.e., (g, h) is generic for $\mathbb{P} * \mathbb{Q}$), it follows that $\dot{t}^M_{A,\mathbb{P},g} = A \cap M_\sigma[g]$ holds for any M_σ-generic filter $g \subseteq \mathbb{P}$, as desired. This ends the proof of the claim.

We complete the proof working in the inner model $L(\mathrm{TR}(A), \mathrm{TR}(F), \mu, \omega^\omega)$. Since Uniformization holds, $\mathsf{DC}_{\mathbb{R}}$ holds. It follows that $L(\mathrm{TR}(A), \mathrm{TR}(F), \mu, \omega^\omega)$ satisfies DC, and, since μ is countably complete, $\prod_{\mathcal{P}_{\aleph_1}(\omega^\omega)} M_\sigma/\mu$ (which we will call M_∞) is wellfounded when computed in $L(\mathrm{TR}(A), \mathrm{TR}(F), \mu, \omega^\omega)$. For each $\sigma \in \mathcal{P}_{\aleph_1}(\omega^\omega)$, let $\mathbb{P}_\sigma$ be $(\mathbb{V}_{\infty,\emptyset})^{M_\sigma}$, as defined in Section 10.2. Using the ordinal-definability order, we can pick in each N_σ a set of ordinals S_σ coding the set of conditions in $(\mathbb{V}_{\infty,\emptyset})^{M_\sigma}$ which force that the generic real will be in the realization of $\dot{t}^M_{A,\mathbb{P}_\sigma}$ (for instance, either by listing the relevant pairs from $P_{\infty,\emptyset}$, or converting to a sentence in $\mathcal{L}_{\infty,0}$ using Theorem 9.0.4). Since each real in N_σ is M_σ-generic for $(\mathbb{V}_{\infty,\emptyset})^{M_\sigma}$, we have that, in N_σ, S_σ is an ∞-Borel code (or an ∞-Borel* code, depending on which method of coding we choose) for $A \cap N_\sigma$. Let S_∞ be the μ-ultraproduct of the sets S_σ. Since μ is fine, and since A is the μ-ultraproduct of the sets $A \cap \sigma$, we have that S_∞ is an ∞-Borel code (or ∞-Borel*-code) for A. □

Putting together Theorems 1.1.5 and 11.2.1 with part (3) of Remark 1.0.2, we have the following.

Theorem 11.2.2. *If* AD + Uniformization *holds then every subset of* ω^ω *is* ∞-*Borel.*

11.3 Closure of the Suslin cardinals

Recall from Chapter 6 that a cardinal κ is *Suslin* if there is a set $A \subseteq \omega^\omega$ which is κ-Suslin but not γ-Suslin for any $\gamma < \kappa$. In this section we prove Theorem 11.3.1 below, on the closure of the Suslin cardinals below Θ. Again, we follow the presentation in [16]. [7]

In this section we prove the following theorem.

Theorem 11.3.1 (Steel-Woodin)**.** *Suppose that*

1. $\mathsf{DC}_{\mathbb{R}}$ *holds,*
2. $\kappa < \Theta$ *is a limit of Suslin cardinals,*
3. κ-Determinacy *holds and that*
4. $\langle S_\alpha : \alpha < \kappa \rangle$ *is a sequence of bounded subsets of* κ *which are* ∞-*Borel codes for pairwise disjoint subsets of* 2^ω.

Then κ *is a Suslin cardinal.*

It follows from AD alone (or, more directly part (1) of Corollary 3.0.2 and the fact that AD implies $\mathsf{CC}_{\mathbb{R}}$; see part (1) of Remark 1.0.2) that if $\kappa < \Theta$ is a limit of an ω-sequence of Suslin cardinals, then κ is a Suslin cardinal. We assume then for the rest of this section that $\mathrm{cof}(\kappa)$ is uncountable, in which case (since we are assuming AD) the $<\kappa$-Borel sets and the $<\kappa$-Suslin sets coincide, by Lemma 9.0.9.

The map $\phi \mapsto \phi^*$ from Section 9.1 shows that a sequence $\langle S_\alpha : \alpha < \kappa \rangle$ as in hypothesis (4) of Theorem 11.3.1 exists if there is an ∞-Borel set which is not $<\kappa$-Borel. If $\kappa < \Theta$, then this in turn follows from either the assumption that every subset of 2^ω is ∞-Borel, or the assumption that there is a Suslin cardinal above κ, along with Lemma 9.0.9.

Let μ_S and $[y]_S$ (for $y \in \omega^\omega$ and S a set of ordinals) be as defined in Section 8.4. Since assuming AD, μ_S is an ultrafilter on the set of S-degrees, for each such S, by Corollary 1.1.6. In this section we will use only the case $S = \emptyset$. While this is not necessary for the present argument, we note that the two following lemmas imply the corresponding version where $\emptyset$ is replaced with an arbitrary set of ordinals.

Lemma 11.3.2. *Assume that* AD *holds. If* κ *is a limit of Suslin cardinals of uncountable cofinality, then for all bounded* $T \subseteq \kappa$, $\prod \omega_1^T / \mu_\emptyset < \kappa$.

[7] Various thoughts, idle and otherwise. There is something about the equivalence, given AD, of AD^+ and something about the Suslin cardinals. Is Θ-Determinacy impossible? What is the relationship between the Wadge rank of a set of reals, and the least cardinal for which it is Suslin, if any?

Proof. By Lemma 9.0.9, every $<\kappa$-Borel set (e.g., $\leq_\emptyset \times <^c_T$ from the Theorem 8.4.1) is in $\mathcal{S}_{<\kappa}$. By Remark 6.0.8, $\mathcal{S}_{<\kappa}$ is a projective algebra. By the Kunen-Martin Theorem (Theorem 6.2.1) the supremum of the lengths of the prewellorderings in $\mathcal{S}_{<\kappa}$ is at most κ (and exactly κ by Theorem 6.0.12). The lemma then follows from Theorem 8.4.1. □

Lemma 11.3.3. *Assume that* AD *holds. If κ is a limit of Suslin cardinals of uncountable cofinality, then for each bounded $S \subseteq \kappa$, $\prod \omega_2^{L[S,y]}/\mu_\emptyset < \kappa$.*

Proof. The set of pairs $(x, y) \in \omega^\omega$ such that $x \in L[S, y]$ is $<\kappa$-Borel, so its complement is as well. By Lemma 9.0.9, the complement of S is $<\kappa$-Suslin. The leftmost branch construction for uniformizing Suslin sets then shows that there is a bounded $T \subseteq \kappa$ such that $\omega^\omega \cap L[S, y]$ is a proper subset of $\omega^\omega \cap L[T, y]$ for all $y \in \omega^\omega$ (one could also use Corollary 10.1.5 for this step). It follows from Theorem 8.2.1 then that the set of $[y]_\emptyset$ for which $\omega_2^{L[S,y]} < \omega_1^{L[T,y]}$ is in $\mu_\emptyset$. The lemma then follows from Lemma 11.3.2. □

The proof of Theorem 11.3.1 is completed by appealing to Theorems 9.2.4 and 11.1.1.

Proof of Theorem 11.3.1. We assume that κ has uncountable cofinality, as the countable cofinality case was dealt with just after the statement of the theorem. For each $\alpha < \kappa$, let A_α be the subset of 2^ω coded by S_α. Using the coding at the beginning of Chapter 8, we may find $S_{\alpha,\beta}$ ($\alpha \leq \beta < \kappa$), bounded subsets of κ, such that each $S_{\alpha,\beta}$ is an ∞-Borel code for the set $A_\alpha \times A_\beta$. Lemma 11.3.3 and Theorem 11.1.1 give a sequence $\langle S^*_{\alpha,\beta} : \alpha \leq \beta < \kappa\rangle$, where each $S^*_{\alpha,\beta}$ is a bounded subset of κ and a strong ∞-Borel code for the corresponding set $A_\alpha \times A_\beta$. Let $\leq = \bigcup_{\alpha\leq\beta<\kappa} A_\alpha \times A_\beta$ and observe that this is a prewellordering of length κ. Theorem 9.2.4 then shows that $\leq$ is κ-Suslin. By the Kunen-Martin Theorem (Theorem 6.2.1), $\leq$ cannot be $<\kappa$-Suslin. □

The reverse direction of the following theorem is beyond the scope of this book. See Section 7 of [16] for a proof.

Theorem 11.3.4. *Suppose that* AD + $\mathsf{DC}_\mathbb{R}$ *holds. Then* AD^+ *holds if and only if the set of Suslin cardinals is a closed subset of* Θ.

11.4 Suslin sets and the Solovay sequence

Recall from Theorem 6.4.3 that if Wadge Determinacy+Uniformization+$\aleph_1 \not\leq 2^{\aleph_0}$ holds, then the length of the Solovay sequence is a limit ordinal.

The reverse implication follows from $<\Theta$-Determinacy.

Theorem 11.4.1. *Assuming* $<\Theta$-Determinacy*, if the length of the Solovay sequence is a limit ordinal then every subset of ω^ω is Suslin.*

Proof. This follows from Theorem 8.4.1 and 11.1.1.[8] □

[8] Say more.

11.5 When AD^+ holds and $\mathsf{AD}_{\mathbb{R}}$ fails

By Theorems 11.3.1 and 12.0.1, the context of the following theorem is exactly the case where AD^+ holds and $\mathsf{AD}_{\mathbb{R}}$ fails.

Theorem 11.5.1. *Assume that AD^+ holds and that there is a largest Suslin cardinal. Then there is a set of ordinals S such that $\mathcal{P}(\omega^\omega)$ is contained in $L(S,\mathbb{R})$.*

We begin the proof. Let κ be the largest Suslin cardinal. Recall that $\mathcal{S}_\kappa$ denotes smallest boldface pointclass containing the κ-Suslin subsets of ω^ω. By Theorem 6.0.7, $\mathcal{S}_\kappa$ is not closed under complements. Let T be a tree on $\omega\times\kappa$ such that $\omega^\omega\setminus \mathrm{p}[T]$ is not κ-Suslin.

Claim. *For every set of ordinals S, for a Turing cone of $x\in\omega^\omega$,*

$$\omega^\omega\cap L[S,x]\subseteq L[T,x].$$

Proof. If not, then by Theorem 8.2.1, for a Turing cone of x, $\omega_2^{L[T,x]}$ is countable in $L[S,x]$. Theorem 8.4.1 then implies that $\prod \omega_2^{L[T,x]}/\mu_S$ is less than Θ. Since T is an ∞-Borel code for $\omega^\omega\setminus \mathrm{p}[T]$, Theorem 11.1.1 implies that $\omega^\omega\setminus\mathrm{p}[T]$ is Suslin, giving a contradiction. □

Since we are assuming that every subset of ω^ω is ∞-Borel, the claim and Remark 8.2.5 together give that for every $A\subseteq\omega^\omega$, $A\cap L[T,x]$ is in $L[T,x]$ for a T-cone of x.

As in Section 8.5, for any set S consisting of ordinals, we let $\mathbb{P}_S$ be the partial order of S-positive subsets of ω^ω and let $\mathbb{S}_S$ be the partial order of S-pointed perfect trees, each ordered by $\subseteq$. By Theorem 8.3.1, a V-generic filter G for $\mathbb{P}_S$ induces a generic real x_G for $\mathbb{S}_S$ and vice versa.

Let T^* be the ultrapower of T by the measure μ_T. By Remark 8.5.4, $\mathbb{S}_T$ and $\mathbb{S}_{T^*}$ are the same partial order. We will show that $\mathcal{P}(\omega^\omega)\subseteq L(T^*,\omega^\omega)$. To show this, it suffices (by Solovay's argument mentioned in Remark 8.2.5) to see that $\mathcal{P}(\omega^\omega)^V\subseteq L(T^*,\omega^\omega)[x_G]$ whenever x_G is V-generic for $\mathbb{S}_{T^*}$. We fix then a V-generic filter $G\subseteq\mathbb{P}_T$ and the corresponding real x_G which is V-generic for both $\mathbb{S}_T$ and $\mathbb{S}_{T^*}$.

By Corollary 8.5.3, $\prod_{x\in\omega^\omega} L[T,x]/G$ is isomorphic to $L[T^*,x_G]$. For each $x\in\omega^\omega$, let $A_x = A\cap L[T,x]$. Let $A^*=\prod A_x/G$. Then A^* is in $L[T^*,x_G]$ and $A=A^*\cap(\omega^\omega)^V$. It follows that $A\in L(T^*,\omega^\omega)[x_G]$. This completes the proof.

Chapter 12

$\mathsf{AD}_{\mathbb{R}}$

In this chapter we will prove the following theorem, which is due to Martin and Woodin independently.

Theorem 12.0.1 (Martin, Woodin)**.** *If* AD *holds and every subset of* ω^ω *is Suslin, then* $\mathsf{AD}_{\mathbb{R}}$ *holds.*

We will give two versions of one of the key steps in our proof of this theorem. One is a theorem of Martin (Theorem 12.0.2 below) which we will not prove. The other is a lemma which we will prove in Section 12.1, which can be used in place of Martin's theorem. Our proof of this lemma will not be entirely self-contained, either. We prove a key lemma in Section 12.2 and finish the proof of Theorem 12.0.1 in Section 12.3.

We will be using material on towers of measures. We briefly review some of this material here. More thorough treatments can be found in [20, 18, 26], among other places. Given a set X and an $i \in \omega$, an ultrafilter μ on X^{i+1} *projects* to the ultrafilter μ' on X^i whose members are the set of $A \subseteq X^i$ for which $\{s ^\frown \langle x \rangle : s \in A, x \in X\}$ is in μ (we sometimes refer to ultrafilters as *measures*). Each function on X^i has a natural reinterpretation as function on X^{i+1} (ignoring the last coordinate of the input) and this reinterpretation induces a factor map $j_{\mu',\mu}$ from a μ'-ultrapower to a μ-ultrapower. A *tower* of ultrafilters on X is a sequence $\langle \mu_i :\in \omega \rangle$ such that each μ_i is an ultrafilter on X^i, and each μ_{i+1} projects to the corresponding μ_i. There is a natural direct limit embedding associated to a tower, and the tower is said to be *wellfounded* if the corresponding image model is wellfounded.

Given a tree T on a product $X \times Y$, and an $s \in X^{<\omega}$, $T(s)$ denotes the set $\{t \in Y^{|s|} : (s,t) \in T\}$. A tree T on $\omega \times Y$ (for some set Y) is *homogeneous* if there exists a collection of measures $\{\mu_s : s \in \omega^{<\omega}\}$ such that,

- for each $s \in \omega^{<\omega}$, $T(s) \in \mu_s$, and
- for each $x \in \omega^\omega$, $x \in \mathrm{p}[T]$ if and only if $\langle \mu_{x\restriction n} : n \in \omega \rangle$ is a wellfounded tower.

A tree T on $\omega \times Y$ (for some set Y) is *weakly homogeneous* if there exists a countable set σ such that, for each $x \in \omega^\omega$, $x \in \mathrm{p}[T]$ if and only if there exists a wellfounded tower $\langle \mu_n : n \in \omega \rangle$ such that, for each $n \in \omega$, $\mu_n \in \sigma$ and $T(x \restriction n) \in \mu_n$. We refer the reader to [20, 18, 26] for more on homogeneous trees and weakly homogeneous trees. A subset of ω^ω is *homogeneously Suslin* if it is the projection of a homogeneous tree, and *weakly homgeneously Suslin* if it is the projection of a weakly homogeneous tree (equivalently, if it is the projection, in a different sense,[1] of a homogeneously Suslin subset of $\omega^\omega \times \omega^{\omega 2}$).

The following theorem of Martin appears as Theorem 1.1 of [22].

Theorem 12.0.2 (Martin)**.** *Assuming* $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$*, for any* $A \subseteq \omega^\omega$*,* A *is homogeneously Suslin if and only if* A *and its complement are Suslin.*[3]

12.0.3 Remark. The construction of the Martin-Solovay tree (in [19]; see also Section 1.3 of [18]) implies that the complement of a weakly homogeneously Suslin set is Suslin. It is the other direction of Theorem 12.0.2 which is used in the proof of Theorem 12.0.1.

12.1 Weakly homogeneous trees

We start by proving a theorem of Kunen on definability of measures. Given a set X, we let $\mathrm{meas}(X)$ denote the set of countable complete ultrafilters on X. Recall (from part (3) of Remark 1.0.2) that AD implies that every set of reals has the property of Baire, which implies that every nonprincipal ultrafilter (on any set) is countably complete. The following theorem shows that, assuming AD, every nonprincipal ultrafilter on an ordinal below Θ is ordinal definable. Our proof is taken from [26].

We use the ordered equivalence relation $(\equiv_{\mathrm{M}}, \leq_{\mathrm{M}})$ from Chapter 8 (although any ordered equivalence relation as thick as $(\equiv_{\mathrm{M}}, \leq_{\mathrm{M}})$, such as the Turing degrees or the constructibility degrees) would work just as well). By Theorem 1.1.5, AD implies that the $\equiv_{\mathrm{M}}$-cone measure μ_{M} is an ultrafilter on the set $\mathcal{C}_{\mathrm{M}}$ of $\equiv_{\mathrm{M}}$ equivalence classes.

Theorem 12.1.1 (Kunen)**.** *If* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *holds and* $\kappa < \Theta$*, then every element of* $\mathrm{meas}(\kappa)$ *is ordinal definable.*

Proof. Fix $\kappa < \Theta$, $\mu \in \mathrm{meas}(\kappa)$ and $A \subseteq \omega^\omega$ of Wadge rank κ. By the Moschovakis Coding Lemma[4], there is a surjection $F \colon \omega^\omega \to \mathcal{P}(\kappa)$ in $L(A, \mathbb{R})$. Let γ be the supremum of κ and the Wadge rank of $F^{-1}[\mu]$. The model $L(B, \mathbb{R})$ is the same for all $B \subseteq \omega^\omega$ of Wadge rank γ, which means that this model is ordinal definable. It suffices then to fix such a B and prove that μ is ordinal definable in $L(B, \mathbb{R})$. Note that F and μ are elements of $L(B, \mathbb{R})$.

Since $\mathsf{DC}_{\mathbb{R}}$ holds, DC holds in $L(B, \mathbb{R})$, which means that $(\mathrm{Ord}^{\mathcal{C}_{\mathrm{M}}}/\mu_M)^{L(B,\mathbb{R})}$ is wellfounded. Let

[1] used somewhere in this book?

[2] reference?

[3] Look at [22]. Maybe we just need, and are proving, a weak version of this?

[4] and something else

- F_μ be the function on ω^ω defined by setting $F_\mu(x)$ to be $F(x)$ if $F(x) \in \mu$, and $\kappa \setminus F(x)$ otherwise;
- f_μ be the function on $\mathcal{C}_{\mathrm{M}}$ defined by setting $f_\mu(d)$ to be the least member of $\bigcap\{F_\mu(x) : x \leq_{\mathrm{M}} y\}$, for any $y \in d$ (this does not depend on the choice of y).
- γ_μ be the ordinal represented by f_μ in the ultrapower $(\mathrm{Ord}^{\mathcal{C}_{\mathrm{M}}}/\mu_{\mathrm{M}})^{L(B,\mathbb{R})}$.

Then μ is is definable from γ_μ in $L(B,\mathbb{R})$, as it is the set of $C \subseteq \kappa$ such that, for any function g representing γ_μ in

$$(\mathrm{Ord}^{\mathcal{C}_{\mathrm{M}}}/\mu_M)^{L(B,\mathbb{R})},$$

$\{d \in \mathcal{C}_{\mathrm{M}} : g(d) \in C\} \in \mu_{\mathrm{M}}$. □

12.1.2 Remark. Let κ be an ordinal below Θ, and let $F\colon \omega^\omega \to \mathcal{P}(\kappa)$ be a surjection. If the Wadge ranks of the members of $\{F^{-1}[\mu] : \mu \in \mathrm{meas}(\kappa)\}$ are not cofinal in Θ, then there is a surjection from ω^ω onto $\mathrm{meas}(\kappa)$. By Theorem 12.1.1, each set $F^{-1}[\mu]$ is ordinal definable. By Theorem 6.1.4, if the Wadge ranks of the members of $\{F^{-1}[\mu] : \mu \in \mathrm{meas}(\kappa)\}$ are cofinal in Θ, then there is a counterexample to Uniformization.

The following theorem is our alternative to Theorem 12.0.2. [5] [6]

Lemma 12.1.3. *If* AD *holds, κ is an ordinal below Θ and there is a surjection from ω^ω to $\mathrm{meas}(\kappa^{<\omega})$, then every tree on $\omega \times \kappa$ is weakly homogeneous.*

Proof. Let $F\colon \omega^\omega \to \mathrm{meas}(\lambda^{<\omega})$ be a surjection, and define

$$F_{\mathrm{M}}\colon \mathcal{C}_{\mathrm{M}} \to \mathcal{P}_{\aleph_1}(\mathrm{meas}(\kappa^{<\omega}))$$

by setting $F_{\mathrm{M}}(d)$ to be $F[d]$.[7] Let U be the set of $C \subseteq \mathcal{P}_{\aleph_1}(\mathrm{meas}(\kappa^{<\omega}))$ such that $F_{\mathrm{M}}^{-1}[C] \in \mu_{\mathrm{M}}$. Then U is a fine, countably complete measure on $\mathcal{P}_{\aleph_1}(\mathrm{meas}(\kappa^{<\omega}))$.

Fix a tree T on $\omega \times \kappa$. As usual, given $x \in \omega^\omega$, we let T_x be the set $\{s \in \kappa^{<\omega} : (x\restriction|s|, s) \in T\}$. We will show that the set of $\sigma \in \mathcal{P}_{\aleph_1}(\mathrm{meas}(\kappa^{<\omega}))$ witnessing the weak homogeneity of T is in U.

To do this, we define for each such σ a game G_σ. This game will be closed for player I, and player I will have a winning strategy if σ does not witness the weak homogeneity of T. For each $i \in \omega$, in round i of the game $\mathcal{G}_\sigma$, player I plays $k_i \in \omega$, $\alpha_i \in \kappa$ and $\beta_i \in \mathrm{Ord}$, and player II plays a measure $\mu_i \in \sigma$ concentrating

[5] We're probably not using any of this. Now suppose that we can map ω^ω onto the measures on λ. By Kunen, all measures are ordinal definable. Let B be a prewellordering of length λ, inducing a surjection ρ from $\mathbb{R}$ into $\mathcal{P}(\lambda)$. Then every measure on λ induces an OD_A set of reals. Since $\Theta^{\mathrm{HOD}_A} < \Theta$, we can map the reals onto the measures.

[6] Or this. Under AD all measures on ordinals are countably complete and ordinal-definable. By the coding of measures theorem of Kechris (Kunen??) assuming AD and that there is a Suslin cardinal above κ, there are fewer than Θ many measures on κ (or $\kappa^{<\omega}$ for that matter.)

[7] This proof is copied from Trevor's MO post

on κ^i. Player II is required to play so that for each $i \in \omega$, μ_{i+1} projects to μ_i. Player I is required to play so that for each $i \in \omega$, $(\langle k_0, \ldots, k_i \rangle, \langle \alpha_0, \ldots, \alpha_i \rangle)$ is in T and $j_{\mu_i,\mu_{i+1}}(\beta_i) > \beta_{i+1}$, where $j_{\mu_i,\mu_{i+1}}$ is the factor map corresponding to the projection of μ_{i+1} to μ_i. The first player to break these rules loses; if there is no such player then player I wins.

I	x_0, α_0, β_0		x_1, α_1, β_1		x_2, α_2, β_2		$\ldots$
II		μ_0		μ_1		$\ldots$	

The game G_σ

If σ does not witness that T is weakly homogeneous then there is an $x \in \mathrm{p}[T]$ for which there is no wellfounded tower $\langle \mu_i : i < \omega \rangle$ consisting of measures from σ, with $\{s \in \kappa^i : (x \restriction i, s) \in T\} \in \mu_i$ for each i. Lemma 1.3.8 of [18] shows that in this case there is a continuous witness to the illfoundedness of towers of measures in σ concentrating on T_x. This gives a winning strategy for player I in G_σ, letting $x = \langle k_i : i < \omega \rangle$, $\langle \alpha_i : i < \omega \rangle$ be any fixed witness to $x \in \mathrm{p}[T]$, and the values β_i come from this continuous witness.

It suffices now to show that for U-many σ, I does not have a winning strategy for G_σ. Assume toward a contradiction that for U-almost every σ, player I does has a winning strategy in G_σ. Since each game G_σ is closed, and since the set of possible moves is for player I wellorderable (this is also true for player II, since all measures on $\kappa^{<\omega}$ are ordinal-definable, but not needed for the argument that follows), we can let, for each σ for which I has a winning strategy in in G_σ, $H(\sigma)$ be the winning strategy for I where I plays the least move leading to a subgame where she still has a winning strategy.

Now:

- let k_0 be the element of ω played by $H(\sigma)$ in round 0 for U-many σ;
- let μ_0 be the set of $A \subseteq \kappa$ such that, for U-many σ, the ordinal α_0^σ played by $H(\sigma)$ in round 0 is in A (then μ_0 is a countably complete measure on κ);
- let k_1 be the element of ω played by $H(\sigma)$ in round 1 in response to μ_0, for U-many σ;
- let μ_1 be the set of $A \subseteq \kappa^2$ such that, for U-many σ, letting $\langle \alpha_0^\sigma, \alpha_1^\sigma \rangle$ be as played by $H(\sigma)$ in rounds 0 and 1 in response to μ_0, $\langle \alpha_0^\sigma, \alpha_1^\sigma \rangle \in A$ (then μ_1 is a countably complete measure on κ^2 projecting to μ_0).

Continuing in this way, we get $x = \langle k_i : i \in \omega \rangle \in \omega^\omega$ and a tower of measures $\vec{\mu}$. Each μ_i concentrates on T_x because $(\alpha_0^\sigma, \ldots, \alpha_i^\sigma) \in T_x$. It is a wellfounded tower because if $A_i \in \mu_i$ for all $i < \omega$ then by the countable completeness of U there is a σ such that $(\alpha_0^\sigma, \ldots, \alpha_i^\sigma) \in A_i$ for all $i \in \omega$. However, by the countable completeness of U there is a σ such that $\vec{\mu}$ is a legal play by player

II against player I's winning strategy $H(\sigma)$, so player I's moves β_i induced by playing $H(\sigma)$ against $\vec{\mu}$ continuously witness the illfoundedness of $\vec{\mu}$, giving a contradiction. □

12.1.4 Remark. If all sets of reals are weakly homogeneously Suslin, then all set of reals are homogeneously Suslin. Otherwise, the homogeneously Suslin sets would form a proper initial segment of the Wadge hierarchy, and every subset of ω^ω would be the continuous image of a member of this initial segment. This would induce a surjection from ω^ω onto $\mathcal{P}(\omega^\omega)$.

12.1.5 Remark. If κ is the largest Suslin cardinal, there is no surjection from ω^ω onto the measures on κ (so the induced sets are Wadge-cofinal), since otherwise the Martin-Solovay construction (see Remark 12.0.3) would produce Suslin representations for the complements of κ-Suslin sets, contradicting Theorem 6.0.7.

12.2 Normal measures

In this section we show (assuming $\mathsf{DC}_\mathbb{R}$) that if λ is a Suslin cardinal and λ-Determinacy holds, then there is a normal fine ultrafilter on $\mathcal{P}_{\aleph_1}(\lambda)$.

12.2.1 Definition. We say that an ultrafilter μ on $\mathcal{P}(\mathcal{P}_{\aleph_1}(\lambda))$ is

- *fine* if for all $\alpha \in \lambda$, $\{\sigma \in \mathcal{P}_{\aleph_1}(\lambda) : \alpha \in \sigma\}$ is in μ;
- *normal* if whenever $f\colon \mathcal{P}_{\aleph_1}(\lambda) \to \mathcal{P}_{\aleph_1}(\lambda)$ is such that $p(\sigma)$ is a nonempty subset of σ for each $\sigma \in \mathcal{P}_{\aleph_1}(\lambda)$, there exists an $\alpha \in \lambda$ such that

  $$\{\sigma \in \mathcal{P}_{\aleph_1}(\lambda) : \alpha \in f(\sigma)\}$$

  is in μ.

Recall (from Definition 6.0.4) that a Suslin cardinal is an ordinal λ (necessarily a cardinal) such that some subset of ω^ω is λ-Suslin but not γ-Suslin for any $\gamma < \lambda$.

Theorem 12.2.2. *If λ is a Suslin cardinal, and λ-*Determinacy *+* $\mathsf{DC}_\mathbb{R}$ *holds, then there is a normal fine measure on $\mathcal{P}_{\aleph_1}(\lambda)$.*[8]

Proof. Fix a tree T as given by Lemma 6.0.11. For each $\sigma \subseteq \lambda$, let $T{\restriction}\sigma$ be the set of nodes (s,t) of T for which the range of t is contained in σ. We say that σ is *full* if, for each node (s,t) of $T{\restriction}\sigma$, there exists an $x \in p[T{\restriction}\sigma]$ whose leftmost branch in T contains (s,t).

Given an injection $g\colon \omega \to \lambda$, we let T_g be the tree on $\omega \times \omega$ consisting of those nodes $(s,t) \in \omega^{<\omega} \times \omega^{<\omega}$ such that $(s, g \circ t)$ is in T. Let us say that a function $g\colon \omega \to \lambda$ is *full* if it is injective and its range is is full. We call the function $(s,t) \mapsto (s, g \circ t)$ embedding T_g into T the *g-induced map.*

[8] This is due to Woodin? Compare the result here with Kechris-Harrington.

Claim. *If g and g' are distinct full functions from ω to λ, then $T_g \neq T_{g'}$.*

Proof of Claim. We want to see that, given a full function $g\colon \omega \to \lambda$, the g-induced map can be recovered from T_g without using g. This suffices, as the g-induced map is uniquely determined by g (by the second condition in the conclusion of Lemma 6.0.11; this is the reason for that condition).

We work recursively through T_g (using some wellordering of T_g in ordertype ω which lists each node before its successors), building a $\subseteq$-increasing sequence of finite partial length-preserving embeddings e_k ($k \in \omega$) of T_g into T (and possibly halting if our construction breaks down, although we will eventually see that it doesn't). We let e_0 be the map which sends the empty node to itself. Now suppose that we have built e_k (for some $k \in \omega$), and that (s, t) is the least node of T_g (in our wellordering) outside the domain of e_k. We may suppose that (s, t) is of length $n + 1$, for some $n \in \omega$, and that $(s{\restriction}n, t{\restriction}n)$ is in the domain of e_k. Let E_k be the set of $\gamma < \lambda$ for which there exists a length-preserving embedding of T_g into T extending e_k and mapping (s, t) to $(s, (e_k \circ (t{\restriction}n))^\frown\langle\gamma\rangle)$. If E_k is empty, then we stop the construction. Otherwise, letting γ_* be the least element of E_k, we let

$$e_{k+1} = e_k \cup \{(s, t), (s, (e_k \circ (t{\restriction}n))^\frown\langle\gamma_*\rangle)\}.$$

This completes the construction.

We now verify inductively that our maps e_k agree with the g-induced map. For e_0 this is clear. Supposing that is is true for a given $k \in \omega$, and letting (s, t) be the least node of T_g outside domain of e_k, we need to see that $g(t(n))$ is the least element of E_k (where (s, t) has length $n + 1$). Our induction hypothesis implies that $g(t(n))$ is in E_k, which means that e_{k+1} was defined. Since g is full, there is an $x \in \omega^\omega$ whose T-leftmost branch goes through $(s, g \circ t)$. This shows that $g(t(n))$ is the least element of E_k, as desired, since a length-preserving embedding of T_g into T witnessing the contrary would produce a path though $[T]$ contradicting our assumption that the leftmost branch of x in T contains $(s, g \circ t)$. The completes the proof of the claim. ☐

We define a function F whose domain is the set of trees on $\omega \times \omega$:

- for trees of the form T_g, for g a full function from ω to λ, $F(T_g)$ is the range of g;
- for all other trees on $\omega \times \omega$, F takes the value $\emptyset$.

By the claim, this function is well defined. We now apply λ-$\mathsf{Determinacy}$.

Given a set $A \subseteq \mathcal{P}_{\aleph_1}(\lambda)$, let G_A be the game where

- players I and II build a $\subseteq$-increasing sequence $\langle p_n : n \in \omega\rangle$ of finite partial injections from ω to λ, with n contained in the domain of p_n for each $n \in \omega$;
- letting $g = \bigcup_{n\in\omega} p_n$, player II wins if g is full and $F(T_g)$ is in A.

This game is determined by λ-Determinacy.

Let μ be the set of $A \subseteq \mathcal{P}_{\aleph_1}(\lambda)$ for which II has a winning strategy. We wish to see that μ is a normal fine ultrafilter. To check that μ is fine, fix an $\alpha < \lambda$ and a strategy Σ for player I in the game G_A, where $A = \{x \in \mathcal{P}_{\aleph_1}(\lambda) : \alpha \in x\}$. Using $\mathsf{DC}_{\mathbb{R}}$ and the assumption that every node of T is part of the leftmost branch of some element of $p[T]$, we can find a winning play for II against Σ.

To see that μ is an ultrafilter, fix $A \subseteq \mathcal{P}_{\aleph_1}(\lambda)$. We show first that at least one of A and $\mathcal{P}_{\aleph_1}(\lambda) \setminus A$ is in μ. Fix strategies Σ_A and $\Sigma_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ for player I in G_A and $G_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$, respectively. Using $\mathsf{DC}_{\mathbb{R}}$ we can find a countable full set $\sigma \subseteq \lambda$ which is closed under both Σ_A and $\Sigma_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ (in the sense that a response by either strategy to a sequence of moves whose ranges are contained in σ will a function with range contained in σ). Then, playing as II, we can produce runs of G_A and $G_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ where I plays with Σ_A and $\Sigma_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ respectively, and the range of the resulting function g is σ in each case. This shows that Σ_A and $\Sigma_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ cannot both be winning strategies for I. The same argument, with I and II reversed, shows that A and $\mathcal{P}_{\aleph_1}(\lambda) \setminus A$ cannot both be in μ.

To see that μ is normal, fix a function f which picks a nonempty subset of each $\sigma \in \mathcal{P}_{\aleph_1}(\lambda)$. For each $\alpha \in \lambda$, let A_α be the set of $\sigma \in \mathcal{P}_{\aleph_1}(\lambda)$ for which α is in $f(\sigma)$. We want to see that that player II has a winning strategy in some A_α. Let G be the game in which player II goes first, picking $\alpha < \lambda$, after which the players play G_{A_α}, with I going first as usual. We want to see that player II has a winning strategy in this game. If she doesn't, then I does, by λ-Determinacy. Player I's winning strategy in the game G induces a sequence $\langle \Sigma_\alpha : \alpha < \lambda \rangle$, where each Σ_α is a winning strategy for I in the corresponding game G_{A_α}. Using $\mathsf{DC}_{\mathbb{R}}$ again we can find a countable full set $\sigma \subseteq \lambda$ which is closed under Σ_α for each α in σ. Then we can find runs of the games G_{A_γ} $(\gamma \in \sigma)$, where in each case I plays according to Σ_γ and the range of the resulting function g is σ. This gives a contradiction. $\square$

We note the following theorem, whose proof is essentially the same as the proof of the claim in the proof of Theorem 12.2.2.[9]

Theorem 12.2.3. *Let M be a transitive model of ZF, let λ be an ordinal of M and let T be a tree on $\omega \times \lambda$ such that for each node (s, f) of T there exists an $x \in p[T] \cap M$ such that (s, f) is part of the leftmost branch of x through T. Let $j \colon M \to N$ be an elementary embedding, with N transitive. Then $j[T]$ is in $L[T, j(T)]$.*

Theorem 12.2.4. *Let λ be an ordinal, and let T be a tree on $\omega \times \lambda$. Suppose that M is a transitive set such that $T \in M$ and such that for each node (s, f) of T there exists an $x \in p[T] \cap M$ such that (s, f) is part of the leftmost branch of x through T. Let N be a transitive set and let $j \colon M \to N$ be an elementary embedding. Then $j[T]$ is in $L[T, j(T)]$.*

[9]Choose a version, or remove the theorem.

Proof. The map $j{\restriction}T$ embeds T into $j(T)$. We want to find this map in $L[T, j(T)]$. We use the fact that if

- P is a wellfounded model of ZF (in particular, a forcing extension of $L[T, j(T)]$),
- S and R are trees in P,
- S is countable in P,
- g is, in P, a length-preserving partial embedding of S into R, and
- there is a length-preserving embedding of S into R extending g (in V),

then there is, in P, a length-preserving embedding of S into R extending g.

We work recursively through T, using some wellordering $\langle (s_\alpha, t_\alpha) : \alpha < \delta \rangle$ of T which lists each node before its successors, building a continuous $\subseteq$-increasing sequence of partial length-preserving embeddings e_α ($\alpha < \delta$) of T into $j(T)$ (and possibly halting if our construction breaks down, although we will eventually see that it doesn't). Each e_α will have $\{(s_\beta, t_\beta) : \beta < \alpha\}$ as its domain. Then e_0 is the empty function. We let e_1 be the map which sends the empty node to itself. Now suppose that we have built e_α, for some $\alpha \in (1, \delta)$. Then (s_α, t_α) is of length $n+1$, for some $n \in \omega$, and $(s_\alpha{\restriction}n, t_\alpha{\restriction}n)$ is in the domain of e_α. Let E_α be the set of $\gamma < \lambda$ for which, in any generic extension of $L[T, j(T)]$ in which T is countable, there exists a length-preserving embedding of T into $j(T)$ extending e_α and mapping (s_α, t_α) to $(s_\alpha, (e_\alpha(t_\alpha{\restriction}n))^\frown\langle\gamma\rangle)$. If E_α is empty, then we stop the construction. Otherwise, letting γ_* be the least element of E_α, we let

$$e_{\alpha+1} = e_\alpha \cup \{(s_\alpha, t_\alpha), (s_\alpha, (e_\alpha(t_\alpha{\restriction}n))^\frown\langle\gamma_*\rangle)\}.$$

This completes the construction.

We now verify inductively that our maps e_α agree with $j{\restriction}T$. For e_0 and e_1 this is clear. The limit case of the induction also follows immediately. Supposing that it is true for a given $\alpha < \delta$, we need to see that $j(t_\alpha(n))$ is the least element of E_α (where (s_α, t_α) has length $n+1$). Our induction hypothesis implies that $j(t_\alpha(n))$ is in E_α, which means that e_{k+1} was defined. Fix an $x \in \omega^\omega \cap M$ whose T-leftmost branch goes through (s_α, t_α). Then the leftmost branch for x in $j(T)$ (as computed in any wellfounded model of ZF) goes through $(s_\alpha, j(t_\alpha))$. This shows that $j(t_\alpha(n))$ is the least element of E_α, as desired, since a length-preserving embedding of T into $j(T)$ witnessing the contrary would produce a path though $[T]$ contradicting the fact that the leftmost branch for x in $j(T)$ (as computed in any wellfounded model of ZF) goes through $(s_\alpha, j(t_\alpha))$. □

12.3 Proving $\mathsf{AD}_{\mathbb{R}}$

In this section we complete our proofs of Theorem 12.0.1. Our proofs will show that quasi-$\mathsf{AD}_{\mathbb{R}}$ holds. Then we will be done by Remark 6.1.3. Recall that if every subset of ω^ω is Suslin, then Uniformization holds, and that Uniformization implies $\mathsf{DC}_{\mathbb{R}}$. Fix a bijection $\pi\colon \omega \times \omega \to \omega$, and let π_* be the induced bijection from $(\omega^\omega)^\omega \to \omega^\omega$. We fix a real game G, and let A be the π_* image of its payoff set. By Theorem 12.0.2, we may fix homogeneous trees S and T (on $\omega \times \kappa$, for some cardinal κ) such that $p[S] = A$ and $p[T] = \omega^\omega \setminus A$.

The key remaining step is given in Lemma 12.3.1 below. Note that in the statement of the lemma the trees S and T are homogeneous in V, but not necessarily in M.

Lemma 12.3.1. *Assume that* AD *holds. Let κ be an ordinal. Let G be a real game with payoff set $A \subseteq (\omega^\omega)^\omega$, and let S and T be homogeneous trees on $\omega \times \kappa$ such that $p[S] = \pi_*[A]$ and $p[T] = \omega^\omega \setminus \pi_*[A]$. Let M be an inner model of* ZF *containing S and T, such that $\omega^\omega \cap M$ is countable. Then, in M, the real game with payoff set $\pi_*^{-1}[\mathrm{p}[S]]$ is quasi-determined.*

Proof. Let G_M be the real game with payoff set $\pi_*^{-1}[p[S]]$, as computed in M. We define games G_I^* and G_{II}^*, both in M. In G_I^*, players I and II alternate playing $x_i \in \omega^\omega \cap M$. In each turn, I also plays $\alpha_{i/2} \in \kappa$. Player I wins if and only if $(\pi_*(\langle x_i : i < \omega\rangle), \langle \alpha_i : i < \omega\rangle)$ is in $[S]$.

I	x_0, α_0		x_2, α_1		x_4, α_2		$\ldots$
II		x_1		x_3		$\ldots$	

The game G_I^*.

In the game G_{II}^*, players I and II alternate playing $x_i \in \omega^\omega \cap M$. In each turn, II also plays $\alpha_{(i-1)/2} \in \kappa$. Player II wins if and only if

$$(\pi_*(\langle x_i : i < \omega\rangle), \langle \alpha_i : i < \omega\rangle)$$

is in $[T]$.

I	x_0		$x_2 x$		x_4		$\ldots$
II		x_1, α_0		x_3, α_1		$\ldots$	

The game G_{II}^*.

The first of these games is closed, and the second is open, so they are both quasi-determined. Moreover, for each game there is a quasi-strategy in M which is a winning quasi-strategy in V, since M and V compute the same ranking function when applying the proof of open determinacy to these games. Since $\omega^\omega \cap M$ is countable, such a winning quasi-strategy can be converted in V into a winning strategy.

We want to see that, in M, either I has a winning quasi-strategy in G^*_I or II has a winning quasi-strategy in G^*_{II}, since the corresponding player could use his strategy in the game G_M. It suffices then to derive a contradiction from the assumption that, in V, player II has a winning strategy Σ_{II} in G^*_I and player I has a winning strategy Σ_I in G^*_{II}. This contradiction follows from the fact that if player II has a winning strategy in G^*_I, then player II has a winning strategy in G_M and, similarly that if if player I has a winning strategy in G^*_{II}, then player II has a winning strategy in G_M. These facts are essentially the same, and essentially the same as Martin's theorem that homogeneously Suslin sets are determined.[10] We give a proof for the case where player I has a winning strategy in G^*_{II}.

Let Σ_I be such a strategy. Let $\{\nu_s : s \in \omega^{<\omega}\}$ be a set of measures witnessing that T is homogeneously Suslin. Then for each $s \in \omega^{<\omega}$, ν_s is a countably complete ultrafilter on $\kappa^{|s|}$, and $\{t \in \kappa^{|s|} : (s,t) \in T\}$ is in ν_s. Define a strategy Σ_* for player I in G_M as follows. Let $\Sigma_*(\langle\rangle) = \Sigma_I(\langle\rangle)$. Now fix a sequence $\bar{x} = \langle x_0, \ldots, x_n\rangle \in (\omega^\omega \cap M)^{<\omega}$ with n odd, and let $s_{\bar{x}} \in \omega^{(n+1)/2}$ be the common initial segment of length $(n+1)/2$ of the π_*-values of the elements of $(\omega^\omega)^\omega$ extending $\bar{x}$. Let $\Sigma_*(\bar{x})$ be the unique $y \in \omega^\omega \cap M$ such that the set

$$R^y_{\bar{x}} = \{t \in \kappa^{(n+1)/2} : \Sigma_I(\langle x_0, (x_1, t(0)), \ldots, (x_n, t((n-1)/2))\rangle = y\}$$

is in $\nu_{s_{\bar{x}}}$. Now suppose that $\langle x_i : i < \omega\rangle$ is a run of G_M where I has played according to Σ_* and lost. Since the measures ν_s witness the homogeneity of T, there is an $f \in \kappa^\omega$ such that $f{\restriction}((n+1)/2)$ is in $R^{x_{n+1}}_{\langle x_i : i<n\rangle}$ for each odd $n \in \omega$. Then $\langle x_0, (x_1, f(0)), x_2, (x_3, f(1)), \ldots\rangle$ is a run of G^*_{II} where I plays according to Σ_I and loses, giving a contradiction. □

Proof of Theorem 12.0.1. Fix $A \subseteq (\omega^\omega)^\omega$ and let S and T be homogeneous trees projecting to $\pi_*[A]$ and its complement respectively. By Lemma 10.0.4, we may fix a partial order P on $\Theta^{\mathrm{HOD}_{\{S,T\}}}$ and a set $K \subseteq \Theta^{\mathrm{HOD}_{\{S,T\}}}$, both in $\mathrm{HOD}_{\{S,T\}}$, such that, in any forcing extension of V by $\mathrm{Col}^*(\omega, \mathbb{R})$, there is a $\mathrm{HOD}_{\{S,T\}}$-generic filter $G \subseteq P$ such that $\mathbb{R}^V$ is contained in $L[S,T,P,K][G]$. Let λ be a Suslin cardinal greater than $\Theta^{\mathrm{HOD}_{\{S,T\}}}$ (which exists by Remark 6.0.9 if all subsets of ω^ω are Suslin) and let μ_λ be a normal fine measure on $\mathcal{P}_{\aleph_1}(\lambda)$, as given by Theorem 12.2.2. For each countable $\sigma \subseteq \lambda$, let P_σ and K_σ be the restrictions of P to $\sigma \cap \kappa$. By Lemma 12.3.1, for any such σ, in any inner model of any forcing extension of $L[S,T,P_\sigma,K_\sigma]$ by P_σ, the real game with payoff set $\pi_*^{-1}[\mathrm{p}[S]]$ is determined.

Since μ_λ is a normal fine measure, the ultraproduct

$$\prod_{\sigma \in \mathcal{P}_{\aleph_1}(\lambda)} L[S,T,P_\sigma,K_\sigma]/\mu_\lambda$$

is a wellfounded model $L[S',T',P,K]$, and by elementarity, in any inner model of any forcing extension of $L[S',T',P,K]$ by P, the real game with payoff set

[10] citation

$\pi_*^{-1}[p[S']]$ is determined. However, $p[S'] = p[S] = \pi_*[A]$, and $L[S', T', P, K](\mathbb{R})$ is such an inner model, which shows that the real game with payoff set $\pi_*^{-1}[\mathrm{p}[S]]$ is determined. □

12.3.2 Remark. Our proofs of Theorem 12.0.1 do not appear to easily induce local versions. Each proof uses Theorem 12.2.2, which in turn uses λ-Determinacy. Theorem 7.0.3 (along with Theorem 5.0.2) gives that AD implies the restriction of λ-Determinacy to games with Suslin, co-Suslin payoff sets. Our proofs of Theorem 12.0.1 seem to require then that every subsets of ω^ω is Suslin.

Chapter 13

Questions

1. Does $\mathsf{ZF} + \mathsf{AD}$ imply that the Wadge hierarchy is wellfounded?
2. Does $\mathsf{ZF} + \mathsf{AD}$ imply $\mathsf{DC}_{\mathbb{R}}$?
3. Does $\mathsf{ZF} + \mathsf{AD}$ imply that all subsets of ω^ω are ∞-Borel?
4. Does $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ imply that all subsets of ω^ω are ∞-Borel?
5. Does $\mathsf{ZF} + \mathsf{AD}$ imply Ordinal Determinacy?
6. Does $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ imply Ordinal Determinacy?
7. Does $\mathsf{ZF} + \mathsf{AD}$ imply WF_{T}?
8. Does $\mathsf{ZF} + \mathsf{AD}$ imply that $\chi_{\mathrm{B}} = \kappa_{\mathrm{B}}$?
9. Does $\mathsf{ZF} + \mathsf{AD}$ imply that $\kappa_{\mathrm{B}} = \lambda_{\mathrm{B}}$?
10. Does Turing Determinacy imply AD?
11. Does $\mathsf{ZF} + \mathsf{AD}$ imply $\mathsf{AD}_{\mathbb{R}}$ for Suslin-co-Suslin sets?
12. Does $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ imply $<\Theta$-Determinacy?

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