Discontinuous homomorphisms, selectors and automorphisms of the complex field *

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Abstract

We show, assuming a weak form of the Axiom of Choice, that the existence of a discontinuous homomorphism between separable Banach spaces induces a selector for the Vitali equivalence relation $\mathbb{R}/\mathbb{Q}$. In conjunction with a result of Di Prisco and Todorcevic, this shows that a nonprincipal ultrafilter on the integers is not sufficient to construct a discontinuous automorphism of the complex field, confirming a conjecture of Simon Thomas.

Assuming the Zermelo-Fraenkel axioms for set theory (ZF), the Axiom of Choice (AC) implies that every vector space has a basis (in fact the two statements are equivalent over ZF [1]). The existence of a basis for the vector space $\mathbb{R}$ over the field of scalars $\mathbb{Q}$ in turn implies, in ZF, the existence of a selector for the Vitali equivalence relation $\mathbb{R}/\mathbb{Q}$ (the equivalence relation on $\mathbb{R}$ defined by the formula $x - y \in \mathbb{Q}$) and the existence of a discontinuous homomorphism from the group $(\mathbb{R}, +)$ to itself (see [8, 9, 6], for instance). We show, using a weak form of Choice ($\mathsf{CC}_{\mathbb{R}}$, which asserts the existence of Choice function for each countable set of subsets of $\mathbb{R}$) that the existence of a discontinuous homomorphism from $(\mathbb{R}, +)$ to itself implies the existence of a selector for $\mathbb{R}/\mathbb{Q}$. Our result applies to the additive group of any separable Banach space in place of $(\mathbb{R}, +)$.

A *selector* for an equivalence relation E on a set X is a subset of X meeting each E-equivalence class in exactly one point. The classical construction of a nonmeasurable Vitali set begins by using AC to find a selector for $\mathbb{R}/\mathbb{Q}$. Instead of $\mathbb{R}/\mathbb{Q}$ however we will work with the equivalence relation E_0 of mod-finite equivalence for subsets of ω; our introduction of $\mathbb{R}/\mathbb{Q}$ is only for the expository benefit of readers who are less familiar with E_0. The equivalence relations $\mathbb{R}/\mathbb{Q}$

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and $\mathcal{P}(\omega)/E_0$ are both hyperfinite and nonsmooth, so Borel bi-embeddable (see [4]), which implies among other things that the existence of a selector for either of these equivalence relations implies the existence of one for the other.

The result in this paper confirms a conjecture of Simon Thomas saying that the existence of a nonprincipal ultrafilter on the integers is consistent with the nonexistence of a discontinuous automorphism of the complex field. We briefly give some background information connecting our result to his conjecture. Let P be the set of primes, and for each $p \in P$ let $\bar{\mathbb{F}}_p$ be the algebraic closure of the field $\mathbb{F}_p$ of size p. Given a nonprincipal ultrafilter U on P, the U-ultraproduct $\prod_U \bar{\mathbb{F}}_p$ is an algebraically closed field of characteristic 0 and cardinality $2^{\aleph_0}$. It follows that if AC holds (or just if there is a wellordering of $\mathcal{P}(\omega)$) this ultraproduct is isomorphic to the complex field $(\mathbb{C}, +, \cdot)$ (see [2], for instance). Even without AC, this ultraproduct has $2^{\aleph_0}$ many automorphisms induced by the powers of the Frobenius automorphisms of the fields $\bar{\mathbb{F}}_p$ (see [5, 12]).

Di Prisco and Todorcevic proved in [3] that a certain strong Ramsey principle for countable products of finite sets holds in Solovay's model $L(\mathbb{R})$ from [14]. This principle has implications for forcing extensions of $L(\mathbb{R})$ via the partial order $\mathcal{P}(\omega)/\mathrm{Fin}$ (such an extension has the form $L(\mathbb{R})[U]$, where U is a nonprincipal ultrafilter on ω). For instance [3], it implies that there is no E_0-selector in this model. Thomas observed that this Ramsey principle also precludes the existence of an injection from $\prod_U \bar{\mathbb{F}}_p$ into $\mathbb{C}$ in $L(\mathbb{R})[U]$. He then conjectured that there are no discontinuous automorphisms of $(\mathbb{C}, +, \cdot)$ in this model, i.e., that the only automorphisms are the identity function and complex conjugation. Our result confirms this conjecture, as the restriction of such an automorphism to $(\mathbb{C}, +)$ would be a discontinuous homomorphism. We state this formally in Corollaries 0.2 and 0.4 below. We note that $\mathsf{CC}_{\mathbb{R}}$ holds in $L(\mathbb{R})[U]$, as it is an inner model of a model AC with the same set of real numbers.

Let us say that an abelian topological group $(G, +)$ is *suitable* if there is an invariant metric d inducing the topology on G such that

- G is complete with respect to d;
- letting $\mathbf{0}$ be the identity element of G, $d(\mathbf{0}, n \cdot x) = n \cdot d(\mathbf{0}, x)$ holds for all $x \in G$ and $n \in \omega$ (where $n \cdot x$ denotes the result of adding x to itself n times).

The additive group of a Banach space is suitable, under the metric given by the norm. Moreoever, Theorem 1.2 of [13] shows that a group is suitable if and only if it is isomorphic to closed subset of real Banach space under its addition operation. Note that the second condition above implies that a bounded metric cannot witness suitability. When working with a fixed suitable group $(G, +)$ and a witnessing metric d_G, we will write $\mathbf{0}_G$ for the identity element of G, $|x|_G$ for $d_G(\mathbf{0}_G, x)$ and $B_G(x, \epsilon)$ for $\{y \in G : d(x, y) < \epsilon\}$.

Theorem 0.1 (ZF)**.** *Suppose that $(G, +)$ and $(K, +)$ are suitable topological groups, and that $h\colon (G, +) \to (K, +)$ is a homomorphism. If there exists a convergent sequence $\langle x_i : i \in \omega\rangle$ in G such that $\langle h(x_i) : i \in \omega\rangle$ does not converge to $h(\lim_{n\in\omega} x_i)$, then there is a selector for E_0.*

Proof. Let d_G and d_K be metrics witnessing the respective suitability of $(G, +)$ and $(K, +)$, and let h and $\langle x_i : i \in \omega \rangle$ be as in the statement of the theorem. By the invariance of d_G and d_K, it suffices to consider the case where $\langle x_i : i \in \omega \rangle$ converges to $\mathbf{0}_G$. Since $h(\mathbf{0}_G) = \mathbf{0}_K$, we have that $\langle h(x_i) :\in \omega \rangle$ does not converge to $\mathbf{0}_K$, which means that for some $\epsilon > 0$ the set of $i \in \omega$ with $|h(x_i)|_K \geq \epsilon$ is infinite.

We may now find a sequence $\langle y_i : i \in \omega \rangle$ of elements of G such that

1. for each $i \in \omega$ there exist $k \in \omega$ and $n \in \omega \setminus \{0\}$ such that $y_i = n \cdot x_k$;
2. for all $i < j$ in ω, $|y_j|_G < |y_i|_G/3$;
3. for all $i \in \omega$, $|h(y_i)|_K > i + \sum_{j<i} |h(y_j)|_K$.

To see this, let y_0 be any element of $\{x_i : i \in \omega\} \setminus \{\mathbf{0}_G\}$. Given $j \in \omega$ and $\{y_i : i \leq j\}$, let $n \in \omega \setminus \{0\}$ be such that

$$n \cdot \epsilon > (j + 1) + \sum_{i<j+1} |h(y_i)|_K.$$

There exists then a $k \in \omega$ such that $|x_k|_G < |y_i|_G/3n$ for all $i \leq j$ and such that $|h(x_k)|_K \geq \epsilon$. Then $y_{j+1} = n \cdot x_k$ is as desired.

Condition (2) on $\langle y_i : i \in \omega \rangle$ implies that each value $|y_i|_G$ is more than $\sum\{|y_j|_G : j > i\}$. This in turn, along with the completeness of G, implies that $\sum_{i \in A} y_i$ converges for each $A \subseteq \omega$. Let $Y = \{y_i : i \in \omega\}$ and let Y^+ be the set of elements of G which are sums of (finite or infinite) subsets of Y. By condition (2) on Y, each $y \in Y^+$ is equal to $\sum\{y_i : i \in S_y\}$ for a unique subset S_y of ω. Let F be the equivalence relation on Y^+ where $y_0 F y_1$ if and only if S_{y_0} and S_{y_1} have finite symmetric difference (i.e., $S_{y_0} E_0 S_{y_1}$). By condition (3) on $\langle y_i : i < \omega \rangle$, if yFy' and i is the maximum point of disagreement between S_y and $S_{y'}$, then $d_K(h(y), h(y')) > i$. It follows that the h-preimage of each set of the form $B_K(\mathbf{0}_K, M)$ (for $M \in \mathbb{R}^+$) intersects each F-equivalence class in only finitely many points (since if $2M \leq i$, then for every y in this intersection the set $S_y \setminus i$ is the same). It follows from this (and the fact that there is a Borel linear order $<$ on Y^+ induced by the natural lexicographic order on $\mathcal{P}(\omega)$) that there is an F-selector : for each F-equivalence class, let $M \in \mathbb{Z}^+$ be minimal so that the h-preimage of $B_K(\mathbf{0}_K, M)$ intersects the class, and then pick the $<$-least element of this intersection. Since Y^+/F is isomorphic to $\mathcal{P}(\omega)/E_0$ via the map $y \mapsto S_y$, there is then an E_0-selector. □

Theorem 0.1 does not require the Axiom of Choice, but in general it may require some form of Choice to find a sequence $\langle x_i : i < \omega \rangle$ as in the statement of Theorem 0.1, given a discontinuous homomorphism on a suitable group.

Corollary 0.2 ($\mathsf{ZF} + \mathsf{CC}_{\mathbb{R}}$). *If there is a discontinuous homomorphism between suitable groups of cardinality $2^{\aleph_0}$ then there is a selector for E_0.*

Proof. Let $(G, +)$ and $(K, +)$ be suitable groups of cardinality $2^{\aleph_0}$, and let h be a discontinuous homomorphism from $(G, +)$ to $(K, +)$. Let d_G and d_K be metrics on G and K witnessing suitability. Since h is discontinuous, and d_G and d_K are invariant, there exists an $\epsilon > 0$ such that for each $\delta > 0$ there exists an $x \in B_G(\mathbf{0}_G, \delta)$ with $h(x) \notin B_K(\mathbf{0}_K, \epsilon)$. For each $i \in \omega$, let X_i be the set of $x \in B_G(\mathbf{0}_G, 1/(i+1))$ such that $h(x) \notin B_K(\mathbf{0}_K, \epsilon)$. Then each X_i is nonempty, and by $\mathsf{CC}_{\mathbb{R}}$ there is a sequence $\langle x_i : i \in \omega \rangle$ with each x_i in the corresponding X_i. Now we may apply Theorem 0.1. $\square$

Rephrasing in terms of Banach spaces gives the following.

Corollary 0.3 ($\mathsf{ZF} + \mathsf{CC}_{\mathbb{R}}$)**.** *If there is a discontinuous homomorphism between separable Banach spaces then there is a selector for E_0.*

Combined with the results of Di Prisco and Todorcevic cited above, we have the following corollary, which says that the assumption of the existence of a nonprincipal ultrafilter on the integers is not sufficient to define a third automorphism of the complex field. The strongly inaccessible cardinal in the hypothesis (which we conjecture to be unnecessary) comes from the construction of the model $L(\mathbb{R})$ in [14].

Corollary 0.4. *If the theory ZF is consistent with the existence of a strongly inaccessible cardinal, then it is also consistent with the conjunction of the following three statements:*

- $\mathsf{CC}_{\mathbb{R}}$ *holds;*
- *there is a nonprincipal ultrafilter on ω;*
- *there are exactly two automorphisms of the complex field.*

We end with some related questions. The intended context for each question is the theory $\mathsf{ZF} + \mathsf{CC}_{\mathbb{R}}$, although the versions for other forms of AC may be interesting.

1. Does the existence of a discontinuous homomorphism on $(\mathbb{R}, +)$ imply the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$?
2. Does the existence of a discontinuous homomorphism of $(\mathbb{R}, +)$ imply the existence of a discontinuous automorphism of $(\mathbb{C}, +, \cdot)$?
3. If we drop the second condition from the definition of suitability, does Theorem 0.1 still hold? In particular, does it hold for addition modulo 1 on the interval $[0, 1)$?

We thank Paul McKenney for reminding us of Question (1). It is shown in [10], assuming the existence of a proper class of Woodin cardinals, that the existence of an E_0-selector does not imply the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. The forthcoming [11] shows that only a single strongly inaccessible cardinal is necessary for this result, and in fact that, assuming the consistency of a strongly inaccessible cardinal, the existence of an E_0-selector does not imply the existence of a discontinuous homomorphism on $(\mathbb{R}, +)$.

This paper is part of the project started in [10] and continued in [11], which studies fragments of AC holding in generic extensions of Solovay models. Our proof of Theorem 0.1 was discovered by adapting arguments from [10], with additional inspiration from [7].

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