Questions regarding precipitous ideals and the ω_1-Chang Model

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December 24, 2010

0.1 Question. Does Bounded Martin's Maximum imply that NS_{ω_1} is precipitous?

Since BMM is preserved by forcings which do not add a subset of ω_1, a negative answer to Question 0.1 would be given by a positive answer to the following question.

0.2 Question. If Bounded Martin's Maximum holds, is it possible to destroy the precipitousness of NS_{ω_1} without adding a subset of ω_1?

Bounded Martin's Maximum may not be the most relevant hypothesis for Question 0.2, but some hypothesis is needed, as the answer is certainly no if NS_{ω_1} is $\aleph_1$-dense, or $\aleph_1$-dense in densely many places. So one could modify Question 0.2 as follows.

0.3 Question. If there exists $A \in NS^+_{\omega_1}$ such that for no stationary $B \subseteq A$ is $NS_{\omega_1} \restriction B$ $\aleph_1$-dense, is it possible to destroy the precipitousness of NS_{ω_1} without adding a subset of ω_1?

Woodin has shown that it is possible to force over the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ to destroy the saturation of NS_{ω_1} without adding an ω_1-sequence of ordinals. This argument can be strengthened to destroy presaturation as well. Thus BMM implies neither of these saturation properties. One can also shown via an iterated forcing argument that BMM does not imply saturation, and I believe that this argument can be modified to show that BMM does not imply presaturation. The following question remains open, however.

0.4 Question. Does Martin's Maximum imply that it is possible to destroy the saturation of NS_{ω_1} without adding a subset of ω_1?

Consider the following game $\mathcal{G}$ of length $\omega_1 + 1$. In each round α, Player I chooses a set $A_\alpha \in NS^+_{\omega_1}$, and Player II chooses a set $X_\alpha \subseteq NS^+_{\omega_1}$ of size $\aleph_1$. Player II must play to ensure that for all $\alpha < \beta$, $A_\beta \setminus A_\alpha \in NS_{\omega_1}$, and, for each $B \in X_\alpha$, $A_\alpha \cap B \in NS^+_{\omega_1}$ implies $A_\beta \cap B \in NS^+_{\omega_1}$. Player II wins the run of the game if in any round he cannot play. Say that NS_{ω_1} is *game complete* if Player II does not have a winning strategy in this game. In the $\mathbb{P}_{max}$ extension, NS_{ω_1}

is game complete, and game completeness implies that it is possible to force that NS_{ω_1} is not saturated, without adding a subset of ω_1. Game completeness also implies precipitousness.

0.5 Question. Does Martin's Maximum imply that NS_{ω_1} is game complete?

Given a cardinal κ, the κ-Chang Model (κ-CM) is $L(Ord^\kappa)$, L of all κ-sequences of ordinals. Precipitousness of NS_{ω_1} is computed in ω_1-CM.

0.6 Question. Is it (always) possible to change the theory of ω_1-CM without adding a subset of ω_1?

One could ask the same question about arbitrary κ.

0.7 Question. Given a cardinal κ, is it (always) possible to change the theory of κ-CM without adding a subset of κ?

Again, Questions 0.6 and 0.7 can be varied by context. One interesting case is when Bounded Martin's Maximum holds.

For fun, one can ask the following, and similar questions for arbitrary κ.

0.8 Question. It is consistent with all large cardinals that ω_1-CM is correct about ω_3?

Precipitousness of NS_{ω_1} is a local property, in that it is decided in $H((2^{\aleph_1})^+)$.

0.9 Question. Is precipitousness of NS_{ω_1} a local property in ω_1-CM?

0.10 Question. Is game completeness of NS_{ω_1} computed in ω_1-CM?

If NS_{ω_1} is saturated, then there exists for each ordinal α a canonical function for α, that is, a function from ω_1 to the ordinals which is forced to represent α in all V-generic ultrapowers for NS_{ω_1}. The existence of canonical functions for each ordinal implies precipitousness.

0.11 Question. Does Bounded Martin's Maximum imply that there is a canonical function for each ordinal?

Again, a negative answer to Question 0.11 would be given by a positive answer to Question 0.12.

0.12 Question. If Bounded Martin's Maximum holds, is it possible to force the nonexistence of a canonical function for some ordinal without adding a subset of ω_1?

0.13 Question. Is the existence of a canonical function for each ordinal a local property?

0.14 Question. Is the existence of a canonical function for each ordinal a local property in ω_1-CM?

Finally, we ask a variation of a question asked by Justin Moore. Recall that Shelah has shown in ZFC that there exists a club guessing sequence at ω_2.

0.15 Question. Is is consistent with all large cardinals that there exists a sequence in ω_1-CM which is club guessing at ω_2 in V?

Tetsuya Ishiu and I have noticed that if $\mathrm{MA}_{\aleph_1}$ holds then it is possible to add a club subset of ω_2 not guessed by any ground model sequence, without adding an ω_1-sequence of ordinals.