$\mathbb{P}_{\max}$ VARIATIONS FOR SEPARATING CLUB GUESSING PRINCIPLES

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Abstract. In his book on $\mathbb{P}_{max}$ [6], Woodin presents a collection of partial orders whose extensions satisfy strong club guessing principles on ω_1. In this paper we employ one of the techniques from this book to produce $\mathbb{P}_{max}$ variations which separate various club guessing principles. The principle (+) and its variants are weak guessing principles which were first considered by the second author [3] while studying games of length ω_1. It was shown in [1] that the Continuum Hypothesis does not imply (+) and that (+) does not imply the existence of a club guessing sequence on ω_1. In this paper we give an alternate proof of the second of these results, using Woodin's $\mathbb{P}_{max}$ technology. We also present a variation which produces a model with a ladder system which weakly guesses each club subset of ω_1 club often. We show that this model does not satisfy the Interval Hitting Principle, thus separating these statements. The main technique in this paper, in addition to the standard $\mathbb{P}_{max}$ machinery, is the use of condensation principles to build suitable iterations.

In Chapter 8 of his book on $\mathbb{P}_{max}$ [6], Woodin presents a collection of $\mathbb{P}_{max}$ variations whose extensions satisfy strong club guessing principles on ω_1, along with the statement that the nonstationary ideal on ω_1 (NS_{ω_1}) is saturated (see pages 499-500, for instance). In this paper we employ one of the techniques from that chapter to produce $\mathbb{P}_{max}$ variations which separate various club guessing principles. The arguments and results in this paper are significantly simpler than the ones used there. The separation of club guessing principles is carried out via iterations; no local forcing arguments are used. We present these iteration arguments in full and outline the way in which they are incorporated in the standard $\mathbb{P}_{max}$ machinery.

The principle (+) and its variants are weak guessing principles which were first considered by the second author [3] while studying games of length ω_1. It was shown in [1] that the Continuum Hypothesis does not imply (+) and that (+) does not imply the existence of a club guessing sequence on ω_1. In this paper we give an alternate proof of the second of these results, using Woodin's $\mathbb{P}_{max}$ technology. With the $\mathbb{P}_{max}$ approach it is more natural to produce sequences which weakly guess clubs at club many points, so our model for (+) satisfies a strengthening of (+) for which the guessing happens club often. As always with $\mathbb{P}_{max}$ variations, the continuum has cardinality $\aleph_2$ in our models. This research was done at the same time as [1], though the results in that paper were proved first. As a warm-up we present a variation which produces a model with a ladder system which weakly guesses each club subset of ω_1 club often. We show that this model does not satisfy the Interval Hitting Principle, thus separating these statements.

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Section 10.2 of [4] outlines a general method for producing $\mathbb{P}_{max}$ variations relative to a given Σ_2 sentence for $H(\aleph_2)$. The variations presented here do not literally fall inside this framework, but are similar in spirit. One difference is that the variations presented here use Woodin's stationary tower in place of the Boolean algebra $\mathbb{P}(\omega_1)/NS_{\omega_1}$, as explained in Section 1. Modulo this difference, the variation presented in Section 2 would fall inside the framework of [4], except that the sentence that it preserves, club weak club guessing, is Σ_2 in $H(\aleph_2)$ with a predicate for NS_{ω_1}, which can easily be added to the framework of [4]. The variation presented in Section 3 is still further removed, as the sentence it preserves, $(+)^c_\omega$, is not in general expressible in $H(\aleph_2)$.

The main technique in this paper, in addition to the standard $\mathbb{P}_{max}$ machinery, is the use of condensation principles to build suitable iterations.

1. Condensation

Given a cardinal κ, we let $H(\kappa)$ denote the collection of sets whose transitive closure has cardinality less than κ. Woodin [6, Definition 8.15] defines the *strong condensation principle* for $H(\kappa)$ to be the statement that there is a function $F\colon \kappa \to H(\kappa)$ such that, for all $X \prec \langle H(\kappa), F, \in\rangle$, $F_X = F\restriction(Ord \cap M_X)$, where F_X and M_X are the images of F and M under the transitive collapse of X. We will use a consequence of strong condensation for $H(\kappa)$ (for any $\kappa \geq \aleph_2$), which we will call *weak condensation* for $H(\aleph_2)$. For our purposes, a *club* set of countable subsets of an uncountable set Z is the set of all countably infinite subsets of Z closed under a given finitary function on Z. We note that if $\mathcal{C}$ is such a club, and A is a set in Z, then $\{X \in \mathcal{C} \mid A \in X\}$ is club, and $\{X \cap \omega_1 \mid X \in \mathcal{C}\}$ is a club subset of ω_1 in the usual sense (see [2], for instance).

1.1 Definition. *Weak condensation for* $H(\aleph_2)$ is the statement that there exist ν_α and $\mathcal{N}_\alpha = \{N^\alpha_\beta : \beta < \nu_\alpha\}$ $(\alpha < \omega_1)$ such that for each $\alpha < \omega_1$,

- $\nu_\alpha \in \omega_1$;
- for all $\beta < \nu_\alpha$, N^α_β is the transitive collapse of a countable elementary submodel of $H(\aleph_2)$ with $\omega_1^{N^\alpha_\beta} = \alpha$;
- for all $\beta < \gamma < \nu_\alpha$, $N^\alpha_\beta \subseteq N^\alpha_\gamma$;

and such that for club many countable elementary submodels X of $H(\aleph_2)$, the transitive collapse of X is an element of $\mathcal{N}_{X\cap\omega_1}$.

The club guessing principles which hold in the models in this article are not preserved by the usual forcings to make the nonstationary ideal on ω_1 precipitous, as they guess club often. For this reason, we use Woodin's stationary tower to generate the elementary embeddings which are used to define the order on our partial orders. We will be using the so-called *countable tower* $\mathbb{Q}_{<\delta}$, where δ is presumed to be a Woodin cardinal, see [2], for instance). Given a model M of ZFC and a Woodin cardinal δ of M, an *iteration* of $(M, \mathbb{Q}^M_{<\delta})$ (see [6, Definition 5.19]) consists of a family of models M_α $(\alpha \leq \omega_1)$ and a commuting system of elementary embeddings $j_{\alpha,\beta}$ $(\alpha \leq \beta \leq \omega_1)$ such that

- $M_0 = M$;
- each embedding $j_{\alpha,\beta}$ maps from M_α to M_β;

- each $j_{\alpha,\alpha+1}$ is a generic embedding derived from forcing over M_α with $\mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$, and $M_{\alpha+1}$ is the corresponding generic ultrapower (replaced with its Mostowski collapse if it is wellfounded);
- for each limit ordinal $\beta \leq \omega_1$, M_β is the direct limit of the models M_α $(\alpha < \beta)$ under the maps $j_{\alpha,\gamma}$ $(\alpha \leq \gamma < \beta)$ and each map $j_{\alpha,\beta}$ $(\alpha < \beta)$ is induced by this directed system.

We often use j to indicate the embedding j_{0,ω_1} derived from an iteration, and sometimes refer to the iteration itself as j. If M is a countable transitive model of ZFC and δ is a Woodin cardinal of M, an iteration

$$j\colon (M, \mathbb{Q}^M_{<\delta}) \to (M^*, \mathbb{Q}^{M^*}_{<j(\delta)})$$

is *full* if every member a of $\mathbb{Q}^{M^*}_{<j(\delta)}$ is stationary in M^* (i.e., $a = j_{\alpha,\omega_1}(a_\alpha)$ for some $\alpha < \omega_1$ and some

$$a_\alpha \in \mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)},$$

and the set of $\beta \in [\alpha, \omega_1)$ such that $j_{\alpha,\beta}(a_\alpha)$ is in the generic filter at stage β is a stationary subset of ω_1). If M is a countable transitive model of ZFC and δ is a Woodin cardinal in M, we say that the pair $(M, \mathbb{Q}^M_{<\delta})$ is *iterable* if M^* is wellfounded for every iteration

$$j\colon (M, \mathbb{Q}^M_{<\delta}) \to (M^*, \mathbb{Q}^{M^*}_{<j(\delta)})$$

of $(M, \mathbb{Q}^M_{<\delta})$. If A is a set of reals, we say that $(M, \mathbb{Q}^M_{<\delta})$ is A-*iterable* if it is iterable and if $A \cap M^* = j(A \cap M)$ for every iteration $j\colon (M, \mathbb{Q}^M_{<\delta}) \to (M^*, \mathbb{Q}^{M*}_{<j(\delta)})$ of $(M, \mathbb{Q}^M_{<\delta})$.

The proof of Theorem 8.42 from [6] shows that the conditions in the partial orders defined in this paper exist in suitable generality.

Theorem 1.2 (Woodin)**.** *If AD holds in $L(\mathbb{R})$ and A is a set of reals in $L(\mathbb{R})$ then there exist a countable transitive model M of ZFC and an ordinal $\delta \in M$ such that*

- *δ is a Woodin cardinal in M;*
- *$(M, \mathbb{Q}^M_{<\delta})$ is A-iterable;*
- *$\langle V_{\omega+1} \cap M, A \cap M, \in\rangle \prec \langle V_{\omega+1}, A, \in\rangle$;*
- *strong condensation holds for $H(\kappa)$ in M, where κ is the least strongly inaccessible cardinal of M.*

2. Club weak club guessing

This section is mostly a warm-up for the next section, in which our main result is proved. We prove one separation result here, mainly to illustrate our approach in a simpler setting. It is likely that other separation results can be proved in a similar fashion.

We let *club weak club guessing* denote the statement that there is a sequence $\langle a_\alpha : \alpha < \omega_1\rangle$ such that each a_α is a cofinal subset of α of ordertype at most ω, and such that for every club $C \subseteq \omega_1$, $a_\alpha \cap C$ is infinite for club many α. In this section we present a $\mathbb{P}_{max}$ variation for the existence of a club weak club guessing sequence. We drop the conventional naming system for variations of $\mathbb{P}_{max}$ and call the partial order in this section $\mathbb{P}_0$.

2.1 Definition. The partial order $\mathbb{P}_0$ consists of all pairs $\langle M, A, \delta, X\rangle$ such that

(1) M is a countable transitive model of ZFC;
(2) δ is a Woodin cardinal in M;
(3) $(M, \mathbb{Q}^M_{<\delta})$ is iterable;
(4) A is a club weak club guessing sequence in M;
(5) X is a set in M consisting of iterations $j\colon (N, \mathbb{Q}^N_{<\kappa}) \to (N^*, \mathbb{Q}^{N^*}_{<j(\kappa)})$ which are full in M, for some $\langle N, B, \kappa, Y\rangle \in \mathbb{P}_0$, with $j(B) = A$ and $j(Y) \subseteq X$, such that no two distinct members of X are iterations of the same pair.

The order on $\mathbb{P}_0$ is as follows: $\langle M, A, \delta, X\rangle < \langle N, B, \kappa, Y\rangle$ if there exists a full iteration $j\colon (N, \mathbb{Q}^N_{<\kappa}) \to (N^*, \mathbb{Q}^{N^*}_{<j(\kappa)})$ in X.

The following statement was introduced by Kunen in unpublished work.

2.2 Definition. The Interval Hitting Principle (IHP) is the statement that there exists a set $\{b_\alpha : \alpha < \omega_1\}$ such that each b_α is a cofinal subset of α of ordertype at most ω and such that for every club $C \subseteq \omega_1$ there is a limit ordinal $\alpha < \omega_1$ such that for all but finitely many $\beta \in b_\alpha$, $C \cap [\beta, \min(b_\alpha \setminus (\beta + 1))$ is nonempty.

We will show that, assuming that AD holds in $L(\mathbb{R})$, the $\mathbb{P}_0$-extension of $L(\mathbb{R})$ satisfies club weak club guessing but not IHP. The following lemma is the key step in proving each of these facts. The lemma shows, assuming weak condensation for $H(\aleph_2)$, that it is possible to iterate a countable transitive model in such a way that a given club weak club guessing sequence in the countable model is mapped to such a sequence in V (i.e., in the model constructing the iteration) while simultaneously mapping a witness to IHP to a sequence which fails to witness IHP.

Lemma 2.3. *Suppose that $(M, \mathbb{Q}^M_{<\delta})$ is an iterable pair, and that*

$$A = \{a_\alpha : \alpha < \omega_1^M\}$$

is a club weak club guessing sequence in M. Suppose that

$$B = \{b_\alpha : \alpha < \omega_1^M\}$$

is a set in M such that each b_α is a cofinal subset of α of ordertype at most ω. Suppose that weak condensation holds for $H(\aleph_2)$. Then there is an iteration $j\colon (M, \mathbb{Q}^M_{<\delta}) \to (M^*, \mathbb{Q}^{M^*}_{<j(\delta)})$ such that $j(A)$ witnesses club weak club guessing and the critical sequence of j shows that $j(B)$ does not witness IHP.

Proof. We let the usual iteration construction determine cofinally many members of each generic filter, including the first member. This guarantees the genericity of each filter and fullness of the iteration.

For each $\alpha < \omega_1$, we let a^*_α and b^*_α be the unique members of $j(A)$ and $j(B)$ respectively which are cofinal subsets of α. Each a^*_α and each b^*_α are determined by $j_{0,\gamma}$, where $\gamma < \omega_1$ is minimal such that $j_{0,\gamma}(\omega_1^M) > \alpha$.

Let $\mathcal{N}_\alpha = \{N^\alpha_\beta : \beta < \nu_\alpha\}$ $(\alpha < \omega_1)$ witness weak condensation for $H(\aleph_2)$.

In order to ensure that the critical sequence of j shows that $j(B)$ does not witness IHP, we include in the construction of G_α (for each limit $\alpha < \omega_1$) a stage for each $\beta < \alpha$, where we ensure that the interval between some consecutive pair of elements of $b^*_{\omega_1^{M_\alpha}}$ above β will be disjoint from the critical sequence of j. At the stage for each such β, some set $d \in \mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$ has been chosen to put into G_α. Since each b_η is a cofinal subset of η of ordertype at most ω, there must be some $n \in \omega$ such that

for cofinally many $\gamma < \omega_1^{M_\alpha}$ there exist $d_\gamma \leq d$ such that for all $Y \in d_\gamma$, the n-th element of $b^*_{Y \cap \omega_1^{M_\alpha}}$ exists and is greater than γ. Let n_0 be the least such n. There exist an ordinal of the form $\xi + 1 < \alpha$ and a

$$d' \in \mathbb{Q}^{M_{\xi+1}}_{<j_{0,\xi+1}(\delta)}$$

such that $j_{\xi+1,\alpha}(d') = d$ and $\omega_1^{M_\xi} > \beta$. Then n_0 satisfies the same definition in $M_{\xi+1}$ with d' in place of d, so there is a condition $d'' \leq d'$ in $\mathbb{Q}^{M_{\xi+1}}_{<j_{0,\xi+1}(\delta)}$ such that the n_0-th and (n_0+1)-st elements of $b^*_{Y \cap \omega_1^{M_{\xi+1}}}$ are the same values ζ and ζ' (greater than $\omega_1^{M_\xi}$ and less than $\omega_1^{M_{\xi+1}}$) for all $Y \in d''$. Then $j_{\xi+1,\alpha}(d'') \leq d$ forces that ζ and ζ' are the n_0-th and (n_0+1)-st elements of $b^*_{\omega_1^{M_\alpha}}$, and no member of the critical sequence is in the interval $[\zeta, \zeta')$, as desired.

In order to guarantee that $j(A)$ is a club weak club guessing sequence, the new part of the construction of each G_α includes a stage for each pair (β, C), where $\beta < \omega_1^{M_\alpha}$ and C is a club subset of $\omega_1^{M_\alpha}$ in some member of $\mathcal{N}_{\omega_1^{M_\alpha}}$. At this stage, some set $d \in \mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$ has been chosen to put into G_α. If there is a $d' \leq d$ forcing that $\gamma \in a^*_{\omega_1^{M_\alpha}}$ for some $\gamma \in C \setminus \beta$, then we choose such a d' to put into G_α (and we say that we have met the challenge (β, C)). Otherwise, we do nothing at this stage.

Having completed the construction, suppose that C is a club subset of ω_1. Let $X \prec H(\aleph_2)$ be in the club corresponding to $\mathcal{N}_\alpha$ $(\alpha < \omega_1)$, with j and C in X. Let $\alpha = X \cap \omega_1$, and note that $\omega_1^{M_\alpha} = \alpha$. We will show that we met every challenge $(\beta, C \cap \alpha)$ in the construction of G_α, which implies that $a_\alpha \cap C$ is cofinal in α. The transitive collapse of X, call it N, is in $\mathcal{N}_\alpha$. Since j is full, every member of $\mathbb{Q}^{M_{\omega_1}}_{<j(\delta)}$ is stationary in V, so every member of $\mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$ is stationary in N. Fix $\beta < \alpha$ and let $d \in \mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$ be the condition which was put into G_α just before the stage of the construction of G_α corresponding to $(\beta, C \cap \alpha)$. The set f consisting of those $\gamma < \omega_1^{M_\alpha}$ which are forced by some $b' \leq b$ to be in a^*_α is stationary in M_α. To see this, note that $j_{0,\alpha}(A) = \{a^*_\gamma : \gamma < \alpha\}$ is a club weak club guessing sequence in M_α. Let e be a club subset of α in M_α. Then there is in M_α a club set $e' \subseteq \alpha \setminus (\beta + 1)$ such that $e \cap a^*_\gamma$ is cofinal in γ for all $\gamma \in e'$. Let d_0 be the set of $Y \in d$ such that $Y \cap \alpha \in e'$. Then $d_0 \leq d$ in $\mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$. There is a regressive function g on d_0 such that each $g(Y)$ is an element of $a^*_{Y \cap \alpha} \cap (e \setminus \beta)$. This function is constant on a condition $d' \leq d$ in $\mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$. If γ is this constant value, then d' forces that $\gamma \in a^*_\alpha \cap e$, which shows that $\gamma \in f \cap e$. Since M_α is stationarily correct in N, f is stationary in N, which means that $(f \cap C) \setminus \beta$ is nonempty. Thus we could and did meet the challenge at every stage of the construction of G_α corresponding to a pair $(\beta, C \cap \alpha)$, for some $\beta < \alpha$. It follows that $a^*_\alpha \cap C$ is cofinal in α. □

Assuming $\mathrm{AD}^{L(\mathbb{R})}$, the basic analysis of the $\mathbb{P}_0$ extension requires only Lemma 2.3 in addition to standard $\mathbb{P}_{max}$ arguments. Theorem 1.2 implies that $\mathbb{P}_0$ conditions exist, and Theorem 1.2 and Lemma 2.3 together imply that every $\mathbb{P}_0$ condition has a stronger condition below it (the version of this fact corresponding to $\mathbb{P}_1$ is sketched after the proof of Lemma 3.3).

The ω-closure of $\mathbb{P}_0$ is proved using the adaptation of Lemma 2.3 to limit structures (minus the part of the argument regarding IHP, which is not necessary).

Lemma 2.4. *If $AD^{L(\mathbb{R})}$ holds, then every descending chain of $\mathbb{P}_0$ conditions of length ω has a lower bound in $\mathbb{P}_0$.*

Aside from notational complications, the iteration argument needed to prove Lemma 3.4 is essentially the same as the proof of Lemma 2.3. Briefly, given a descending sequence of conditions $p_i = \langle M_i, A_i, \delta_i, X_i \rangle$, one forms a limit sequence $\langle M_i^*, \delta_i^* : i < \omega \rangle$ by composing the embeddings witnessing the order on these conditions. The images of the sets A_i under this composition are all the same set A^*. One then applies Theorem 1.2 to find a suitable iterable pair $(N, \mathbb{Q}^N_{<\kappa})$ with $\langle M_i^*, \delta_i^* : i < \omega \rangle \in H(\aleph_1)^N$ and, working in N, constructs a full iteration j of $\langle M_i^*, \delta_i^* : i < \omega \rangle$ (in the corresponding sense) for which the image of A^* is a club weak club guessing sequence in N. Again, the construction of j is just like the argument for Lemma 2.3. The condition $\langle N, j(A^*), \kappa, Y \rangle$ is then below each p_i, where Y is the union of the images of the sets X_i under the composition of the relevant embeddings.

Suppose that AD holds in $L(\mathbb{R})$, and let $G \subset \mathbb{P}_0$ be an $L(\mathbb{R})$-generic filter. For each $p = \langle M, A, \delta, X \rangle \in G$, if we consider all the stronger conditions $\langle N, B, \kappa, Y \rangle$ in G, we see that the iterations of $(M, \mathbb{Q}^M_{<\delta})$ in these sets Y must be initial segments of one another, and must in fact all be proper initial segments of one iteration j_p of $(M, \mathbb{Q}^M_{<\delta})$ of length ω_1. We let $\mathcal{P}(\omega_1)_G$ denote the union of all sets of the form $j_p(\mathcal{P}(\omega_1)^M)$ for all $p = \langle M, A, \delta, X \rangle \in G$. The definition of the order on $\mathbb{P}_0$ implies that $j_p(A)$ is the same set for all $p = \langle M, A, \delta, X \rangle \in G$; we call this set A_G.

The following theorem gives the basic analysis of the $\mathbb{P}_0$ extension. Again, the proof of the following theorem involves only the modification of the corresponding proof for $\mathbb{P}_{max}$, using the version of Lemma 2.3 for building a descending ω_1-sequence of conditions. We refer the reader to [6] for the definition of ψ_{AC}. For our purposes, the only relevant fact about ψ_{AC} is that we can deduce the Axiom of Choice from ψ_{AC} in the $\mathbb{P}_0$ extension.

Theorem 2.5. *Assume $AD^{L(\mathbb{R})}$. Then the following hold in the $\mathbb{P}_0$-extension of $L(\mathbb{R})$.*

(1) $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$;
(2) ω_1-DC;
(3) NS_{ω_1} is saturated;
(4) ψ_{AC};
(5) A_G is a club weak club guessing sequence;
(6) $\neg$ IHP.

Briefly, the proofs of the first two parts of Theorem 2.5 involve coding a given $\mathbb{P}_0$-name for a set of reals (below a given condition) with a set of reals B, and then applying Theorem 1.2 to obtain an B-iterable pair $(M, \mathbb{Q}^M_{<\delta})$ (this the the use of B-iterability). Working inside M, one builds a descending ω_1-sequence of conditions which realizes $\aleph_1$-much of this name (from the perspective of M). The standard argument for doing this requires only choosing the first element of each generic filter correctly. The new steps from the proof of Lemma 2.3 can be worked in to preserve the selected club weak club guessing sequences. The last four parts of the theorem use the fact that $\mathcal{P}(\omega_1)_G = \mathcal{P}(\omega_1)$ in the $\mathbb{P}_0$-extension. As with the other parts of the theorem, the standard proofs of the third and fourth parts require only choosing the first element of each generic filter correctly, so the new steps can be worked in. That A_G witnesses club weak club guessing in the extension follows from

$\mathcal{P}(\omega_1)_G = \mathcal{P}(\omega_1)$ and the fact that the selected club weak club guessing sequence A is preserved from one condition to the next. That IHP fails in the extension is due to Lemma 2.3, which in conjunction with Theorem 1.2 says that any IHP-sequence in any condition can be mapped to a sequence which fails to witness IHP in a stronger condition.

The *Optimal Iteration Lemma* [5] for the existence of a club weak club guessing sequence is the version of Lemma 2.3 obtained by replacing the hypothesis of weak condensation with the assumption of a club weak club guessing sequence (and removing the part concerning IHP). We do not know whether this lemma holds. As a result, we do not know if $\mathbb{P}_0$ extensions are Π_2-maximal for $H(\aleph_2)$ relative to the existence of a club weak club guessing sequences, or whether this form of Π_2-maximality is even possible. Similarly, we do not know if $\mathbb{P}_0$ is homogeneous.

3. (+) and its variants

The original principle (+) asserts the existence of a stationary (i.e., intersecting each such club) family $\mathcal{F}$ consisting of countable elementary submodels of $H(\aleph_2)$, with the property that whenever M, N are in F and $M \cap \omega_1 = N \cap \omega_1$, all club subsets of ω_1 in M intersect all club subsets of ω_1 in N cofinally below $M \cap \omega_1$. This was reformulated by Justin Moore as the following statement: there exist $\langle f_\alpha : \alpha < \omega_1 \rangle$ such that each f_α is a (possibly empty) set of club subsets of α which pairwise have intersection cofinal in α, such that for every club subset C of ω_1 there is an α with $C \cap \alpha \in f_\alpha$. Here we consider a strengthening of this principle.

3.1 Definition. The principle $(+)^c_\omega$ asserts the existence of a sequence

$$\langle f_\alpha : \alpha < \omega_1 \rangle$$

such that

- for each $\alpha < \omega_1$,
  - each element of f_α is a club subset of α;
  - f_α is closed under finite intersections;
- for every club $C \subseteq \omega_1$, $\{\alpha < \omega_1 \mid C \cap \alpha \in f_\alpha\}$ contains a club.

Our goal in this section is to produce a $\mathbb{P}_{max}$ variation in whose extension $(+)^c_\omega$ holds yet there is no club guessing sequence on ω_1.

3.2 Definition. The partial order $\mathbb{P}_1$ consists of all pairs $\langle M, F, \delta, X \rangle$ such that

(1) M is a countable transitive model of ZFC + CH;
(2) δ is a Woodin cardinal in M;
(3) $(M, \mathbb{Q}^M_{<\delta})$ is iterable;
(4) $F = \langle f_\alpha : \alpha < \omega_1^M \rangle$ witnesses $(+)^c_\omega$ in M;
(5) X is a set in M, each element of X is an iteration $j\colon (N, \mathbb{Q}^N_{<\kappa}) \to (N^*, \mathbb{Q}^{N^*}_{<j(\kappa)})$ such that
   - j is full in M,
   - there exist H, Y such that $\langle N, H, \kappa, Y \rangle \in \mathbb{P}_1$;
   - $j(Y) \subseteq X$;
   - letting $j(H) = \langle h^*_\alpha : \alpha < \omega_1^N \rangle$, $h^*_\alpha \subseteq f_\alpha$ for all $\alpha < \omega_1^M$;

   and no two distinct members of X are iterations of the same pair.

The order on $\mathbb{P}_1$ is as follows: $\langle M, F, \delta, X\rangle < \langle N, H, \kappa, Y\rangle$ if there exists an iteration $j\colon (N, \mathbb{Q}^N_{<\kappa}) \to (N^*, \mathbb{Q}^{N^*}_{<j(\kappa)})$ in X.

As in the previous section, we use weak condensation to prove our iteration lemma.

Lemma 3.3. *Suppose that*

- $\langle N^\alpha_\beta : \alpha < \omega_1, \beta < \nu_\alpha\rangle$ *witnesses weak condensation for* $H(\aleph_2)$*;*
- $(M, \mathbb{Q}^M_{<\delta})$ *is an iterable pair such that* $M \models CH$*;*
- $\langle f_\alpha : \alpha < \omega_1^M\rangle$ *witnesses* $(+)^c_\omega$ *in* M*;*
- $\langle a_\alpha : \alpha < \omega_1^M\rangle$ *is such that for each limit* $\alpha < \omega_1$*,* a_α *is a cofinal subset of* α *of ordertype at most* ω*.*

Then there exist a full iteration

$$\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \beta < \omega_1, \alpha \le \gamma \le \omega_1\rangle$$

of $(M, \mathbb{Q}^M_{<\delta})$ *and a function* $e\colon \omega_1 \to \omega_1$ *such that, letting*

$$\langle f^*_\alpha : \alpha < \omega_1\rangle = j_{0,\omega_1}(\langle f_\alpha : \alpha < \omega_1^M\rangle)$$

and

$$\langle a^*_\alpha : \alpha < \omega_1\rangle = j_{0,\omega_1}(\langle a_\alpha : \alpha < \omega_1^M\rangle),$$

(1) for every $\alpha < \omega_1$*,*
 (a) $e(\alpha) \le \nu_\alpha$*;*
 (b) the intersection of any member of f^*_α *with any club subset of* α *in* $\bigcup_{\beta<e(\alpha)} N^\alpha_\beta$ *is cofinal in* α*;*
 (c) if α *is a limit ordinal, then* $a^*_{\omega_1^{M_\alpha}} \setminus \{\omega_1^{M_\beta} : \beta < \alpha\}$ *is infinite;*
(2) for each club $C \subseteq \omega_1$*, the set of* α *such that* $C \cap \alpha \in \bigcup_{\beta<e(\alpha)} N^\alpha_\beta$ *contains a club.*

Proof. We construct the iteration in the usual way, with the following modifications. We allow the ordinary construction to determine cofinally many members of each G_β, including the first one, and fill in the intervening steps ourselves. This guarantees the genericity of each filter and the fullness of the iteration.

To ensure conclusion (1c), when α is a limit ordinal we include a stage for each $\xi < \omega_1^{M_\alpha}$, as follows. When we come to this stage, we have some

$$b \in \mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$$

which we have decided to put into G_α. Let $d = \{\omega_1^{M_\xi} : \xi < \alpha\}$. Then $d \setminus e$ is bounded in $\omega_1^{M_\alpha}$ for each club $e \in \mathcal{P}(\omega_1)^{M_\alpha}$. Let

$$x = \{\rho \in (\xi, \omega_1^{M_\alpha}) \mid \{Y \in b \mid \rho \in a^*_{Y \cap \omega_1^{M_\alpha}}\} \in \mathbb{Q}^{M_\alpha}_{j_{0,\alpha}(\delta)}\}.$$

This set is cofinal in $\omega_1^{M_\alpha}$, and its closure cannot be contained in d. Since d is closed, that means that x is not contained in d. Pick an ordinal $\rho \in x \setminus d$, and put

$$\{Y \in b \mid \rho \in a^*_{Y \cap \omega_1^{M_\alpha}}\}$$

in G_β. This condition forces that $\rho \in a^*_{\omega_1^{M_\alpha}}$. Collectively, then, these stages of the construction ensure that $a^*_{\omega_1^{M_\alpha}} \setminus d$ is cofinal in $\omega_1^{M_\alpha}$.

It remains to describe the parts of the iteration which ensure the others parts of the conclusion of the lemma. Note that each f^*_α will be of the form $j_{\beta+1,\omega_1}(f)$, for some f in $M_{\beta+1}$, where β is minimal such that $\omega_1^{M_{\beta+1}} > \alpha$. In fact, f^*_α will be of the form

$$\bigcup_{\gamma\in[\beta+1,\omega_1)} j_{\beta+1,\gamma}(f).$$

We refer to each $j_{\beta+1,\gamma}(f)$ as $f^*_\alpha \cap M_\gamma$. This is a slight (but unambiguous) abuse of notation, as the sets of the form $f^*_\alpha \cap M_\gamma$ will be determined once M_γ is determined, but f^*_α, being in general an uncountable subset of $\mathcal{P}(\alpha)$, will not be defined until the entire construction is completed.

While constructing G_α, we include a stage for each tuple (h, ξ, ζ, c) of the following type:

- $\zeta < \xi \leq \omega_1^{M_\alpha}$;
- h is a function in M_α whose domain b_h is an element of $\mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$ compatible with every such element (i.e., a club of countable sets in $V^{M_\alpha}_{j_{0,\alpha}(\delta)}$);
- in the case $\xi < \omega_1^{M_\alpha}$, the codomain of h is $f^*_\xi \cap M_\alpha$;
- in the case $\xi = \omega_1^{M_\alpha}$, for all $Y \in b_h$, $Y \cap \omega_1^{M_\alpha} \in \omega_1^{M_\alpha}$ and $h(Y) \in f^*_{Y\cap\omega_1} \cap M_\alpha$;
- c is a club subset of ξ in $N_\beta^{\omega_1^{M_\alpha}}$, for some $\beta < \nu_{\omega_1^{M_\alpha}}$.

When we come to the stage for a given (h, ξ, ζ, c), we have some $b \in \mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$ which we have decided to put into G_α. Since b_h is a club, we may assume that $b \leq b_h$. If possible, we find some $\gamma \in (\zeta, \xi) \cap c$ such that

$$\{Y \in b \mid \gamma \in h(Y \cap \bigcup b_h)\}$$

is in $\mathbb{Q}^{M_\alpha}_{<j_{0,\delta}}$, and we put this set in G_α. If there is no such δ, we do nothing at this stage.

Having completed the construction of the iteration, for each $\alpha < \omega_1$, let z_α be the set of $\beta < \nu_\alpha$ such that there exists a countable elementary substructure X of $H(\aleph_2)$ with the iteration $\langle M_\alpha, G_\beta, j_{\alpha,\gamma} : \beta < \omega_1, \alpha \leq \gamma \leq \omega_1\rangle$ as a member, such that the transitive collapse of X is N^α_β. If z_α is empty, let $e(\alpha) = 0$. If z_α has a maximal element β_α, let $e(\alpha) = \beta_\alpha + 1$. Otherwise, let $e(\alpha)$ be the supremum of z_α. In all cases, $e(\alpha) \leq \nu_\alpha$, and conclusions (1a) and (2) are satisfied.

It remains to check that conclusion (1b) is satisfied, i.e., that for each $\xi < \omega_1$, the intersection of any member of f^*_ξ with any member of $\bigcup_{\beta<e(\xi)} N^\xi_\beta$ is cofinal in ξ. We need check only those ξ for which $e(\xi) > 0$. For these ξ, $\omega_1^{M_\xi} = \xi$. Fix such a ξ, and fix a club subset c of ξ in $\bigcup_{\beta<e(\xi)} N^\xi_\beta$. We show by induction on $\alpha \in (\xi, \omega_1)$ that the intersection of any member of $f^*_\xi \cap M_\alpha$ with c is cofinal in ξ. Note that this is preserved automatically at limit stages.

First consider the case $\alpha = \xi + 1$. When we reach the stage for a tuple of the form (h, ξ, ζ, c) in the construction of G_ξ, we have some $b \in \mathbb{Q}^{M_\xi}_{<j_{0,\xi}(\delta)}$ which we have chosen to put into G_ξ. Since b_h and b are compatible, we may assume by shrinking b if necessary that $b \leq b_h$. Consider the set E consisting of those $\gamma \in (\zeta, \omega_1^{M_\xi})$ for which there exists a condition $b' \leq b$ such that $\gamma \in h(Y \cap \bigcup b_h)$ for all $Y \in b'$. Suppose towards a contradiction that E is nonstationary in M_ξ. Then there is a club $D \in \mathcal{P}(\omega_1)^{M_\xi}$ disjoint from E, and, since $\langle f^*_\beta \cap M_\xi : \beta < \xi\rangle$

witnesses $(+)^c_\omega$ in M_ξ, there is a club $D' \in \mathcal{P}(\omega_1 \setminus (\zeta+2))^{M_\xi}$ such that for all $\rho \in D'$, $D \cap \rho \in f^*_\rho \cap M_\xi$. Then there is a condition $b_0 \leq b$ such that for all $Y \in b_0$, $Y \cap \omega_1^{M_\xi} \in D'$, and there is a regressive function on b_0 which picks for each such Y an element of $D \cap h(Y \cap \bigcup b_h)$ greater than ζ. Thinning b_0 to make this regressive function constant gives a contradiction to the claim that D is disjoint from E. Then E is stationary in M_ξ, and also in N^ξ_β for all $\beta < e(\xi)$, since M_ξ is stationarily correct in these models. Since $E \cap c$ is cofinal in ξ, we may choose $\gamma \in E \cap c \cap (\zeta, \xi)$ and $b' \leq b$ such that b' forces the element of $f^*_\xi \cap M_\alpha$ represented by h to intersect c above γ. Applying this argument for every tuple of the form (h, ξ, ζ, c) takes care of the case $\alpha = \xi + 1$.

Now suppose that $\alpha > \xi$ and the induction hypothesis holds for all members of $(\xi, \alpha]$. We show that it holds for $\alpha + 1$. When we reach the stage for a tuple of the form (h, ξ, ζ, c) in the construction of G_α, we have some $b \in \mathbb{Q}^{M_\alpha}_{<j_{0,\alpha}(\delta)}$ which we have chosen to put into G_α. We may assume that $b \leq b_h$ and that $Y \cap \omega_1^{M_\alpha} \in \omega_1^{M_\alpha} \setminus \xi$ for all $Y \in b$. We need to see that there exists a $\gamma \in (\zeta, \xi) \cap c$ such that the set of $Y \in b$ such that $\gamma \in h(Y \cap \bigcup b_h)$ is stationary in M_α. Supposing that there is no γ as desired, then let d be the set of $\gamma \in (\zeta, \xi)$ such that b forces in M_β that γ is not in $[h]_{G_\beta}$. Then $c \setminus (\zeta+1) \subseteq d$. By our induction hypothesis, all elements of $f^*_\xi \cap M_\alpha$ have cofinal intersection with c. So for each $Y \in b$ there is a $\rho(Y) \in d$ such that $\rho(Y) \in h(\alpha)$, and we get a contradiction again by thinning b to make this function constant. This completes the proof. □

The remainder of this section is similar to the end of the previous section. Assuming $\mathrm{AD}^{L(\mathbb{R})}$, the basic analysis of the $\mathbb{P}_1$ extension requires only Lemma 3.3 in addition to standard $\mathbb{P}_{max}$ arguments. Theorem 1.2 implies that $\mathbb{P}_1$ conditions exist, and Theorem 1.2 and Lemma 3.3 together imply that every $\mathbb{P}_1$ condition has a stronger condition below it. That is, if $p = \langle M, F, \delta, X \rangle$ is a $\mathbb{P}_1$ condition, Theorem 1.2 says that there is an iterable pair $(N, \mathbb{Q}^N_{<\kappa})$ such that $p \in H(\aleph_1)^N$ and N satisfies weak condensation for $H(\aleph_2)$. Letting $\langle N^\alpha_\beta : \alpha < \omega_1^N, \beta < \nu_\alpha \rangle$ witnesses weak condensation for $H(\aleph_2)$ in N, apply Lemma 3.3 in N to obtain a full iteration j of $(M, \mathbb{Q}^M_{<\delta})$ and a function $e\colon \omega_1^N \to \omega_1^N$ such that, letting $j(F) = \langle f^*_\alpha : \alpha < \omega_1 \rangle$,

- for every $\alpha < \omega_1$,
  - $e(\alpha) \leq \nu_\alpha$;
  - the intersection of any member of f^*_α with any club subset of α in $\bigcup_{\beta < e(\alpha)} N^\alpha_\beta$ is cofinal in α;
- for each club $C \subseteq \omega_1^N$ in N, the set of α such that $C \cap \alpha \in \bigcup_{\beta < e(\alpha)} N^\alpha_\beta$ contains a club.

Let $Y = j(X) \cup \{j\}$. For each $\alpha < \omega_1^N$, let h^0_α be the collection of club subsets of α in $\bigcup_{\beta < e(\alpha)} N^\alpha_\beta$, and let h_α be the set of all finite intersections of members of $f^*_\alpha \cup h^0_\alpha$. Let $H = \langle h_\alpha : \alpha < \omega_1^N \rangle$. Then $\langle N, H, \kappa, Y \rangle$ is a $\mathbb{P}_1$ condition stronger than p.

The ω-closure of $\mathbb{P}_1$ is proved using the adaptation of Lemma 3.3 to limit structures (minus the part of the argument regarding club guessing, which is not necessary).

Lemma 3.4. *If $AD^{L(\mathbb{R})}$ holds, then every descending chain of $\mathbb{P}_1$ conditions of length ω has a lower bound in $\mathbb{P}_1$.*

Aside from notational complications, the iteration argument needed to prove Lemma 3.4 is essentially the same as the proof of Lemma 3.3. Briefly, given a descending sequence of conditions $p_i = \langle M_i, F_i, \delta_i, X_i \rangle$, one forms a limit sequence $\langle M_i^*, \delta_i^* : i < \omega \rangle$ by composing the embeddings witnessing the order on these conditions. The images of the sequences F_i induce via coordinatewise union a sequence F^* whose members are all closed under finite intersections. One then applies Theorem 1.2 to find a suitable iterable pair $(N, \mathbb{Q}^N_{<\kappa})$ with

$$\langle M_i^*, \delta_i^* : i < \omega \rangle \in H(\aleph_1)^N$$

and, working in N, constructs a full iteration j of $\langle M_i^*, \delta_i^* : i < \omega \rangle$ (in the corresponding sense) for which the image of F^* satisfies the conclusion of Lemma 3.3. Again, the construction of j is just like the argument for Lemma 3.3. Then there is a set H as discussed above (after the proof of Lemma 3.3) such that the condition $\langle N, H, \kappa, Y \rangle$ is below each p_i, where Y is the union of the images of the sets X_i under the composition of the relevant embeddings.

Suppose that AD holds in $L(\mathbb{R})$, and let $G \subset \mathbb{P}_1$ be an $L(\mathbb{R})$-generic filter. For each $p = \langle M, F, \delta, X \rangle \in G$, if we consider all the stronger conditions $\langle N, H, \kappa, Y \rangle$ in G, we see that the iterations of $(M, \mathbb{Q}^M_{<\delta})$ in these sets Y must be initial segments of one another, and must in fact all be proper initial segments of one iteration j_p of $(M, \mathbb{Q}^M_{<\delta})$ of length ω_1. As before, we let $\mathcal{P}(\omega_1)_G$ denote the union of all sets of the form $j_p(\mathcal{P}(\omega_1)^M)$ for all $p = \langle M, A, \delta, X \rangle \in G$. We also get a sequence $F_G = \langle f^G_\alpha : \alpha < \omega_1 \rangle$ such that each f^G_α is the union of all sets of the form f^*_α, where $p = \langle M, F, \delta, X \rangle \in G$, and $j_p(F) = \langle f^*_\alpha : \alpha < \omega_1 \rangle$. The definition of the order on $\mathbb{P}_1$ implies that each f^G_α is a collection of club subsets of α closed under finite intersections.

The following theorem gives the basic analysis of the $\mathbb{P}_1$ extension. Again, the proof of the following theorem involves only the modification of the corresponding proof for $\mathbb{P}_{max}$, using the version of Lemma 3.3 for building a descending ω_1-sequence of conditions. Again, for our purposes, the only relevant fact about ψ_{AC} is that we can deduce the Axiom of Choice from ψ_{AC} in the $\mathbb{P}_1$ extension.

Theorem 3.5. *Assume $AD^{L(\mathbb{R})}$. Then the following hold in the $\mathbb{P}_1$-extension of $L(\mathbb{R})$.*

(1) $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$;
(2) ω_1-DC;
(3) NS_{ω_1} is saturated;
(4) ψ_{AC};
(5) F_G witnesses $(+)^c_\omega$;
(6) there is no club guessing sequence.

Again, the proofs of the first two parts of Theorem 3.5 involve coding a given $\mathbb{P}_1$-name for a set of reals (below a given condition) with a set of reals A, and then applying Theorem 1.2 to obtain an A-iterable pair $(M, \mathbb{Q}^M_{<\delta})$. Working inside M, one builds a descending ω_1-sequence of conditions which realizes $\aleph_1$-much of this name (from the perspective of M). The standard argument for doing this requires only choosing the first element of each generic filter correctly. The new steps from the proof of Lemma 3.3 can be worked in ensure that the image of the selected witness to $(+)^c_\omega$ can be extended to a witness to $(+)^c_\omega$ in the larger model. The last four parts of the theorem use the fact that $\mathcal{P}(\omega_1)_G = \mathcal{P}(\omega_1)$ in the $\mathbb{P}_1$-extension.

As with the other parts of the theorem, the standard proofs of the third and fourth parts require only choosing the first element of each generic filter correctly, so the new steps can be worked in. That F_G witnesses $(+)^c_\omega$ in the extension follows from $\mathcal{P}(\omega_1)_G = \mathcal{P}(\omega_1)$ and the fact that the members of the selected sequence F are extended from one condition to the next. That club guessing fails in the extension is due to Lemma 3.3, which in conjunction with Theorem 1.2 says that any club guessing sequence in any condition can be mapped to a sequence which fails to witness club guessing in a stronger condition.

Again, the *Optimal Iteration Lemma* for $(+)^c_\omega$ is the version of Lemma 3.3 obtained by replacing the hypothesis of weak condensation with $(+)^c_\omega$ itself (and removing the part concerning club guessing sequences). We do not know whether this lemma holds. As a result, we do not know if $\mathbb{P}_1$ extensions are Π_2-maximal for $H(\aleph_2)$ relative to $(+)^c_\omega$, or whether this form of Π_2-maximality is even possible. Similarly, we do not know if $\mathbb{P}_1$ is homogeneous.

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