Increasing $\delta_1^2$ and Namba-style forcing

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Abstract
We isolate a forcing which increases the value of $\delta_1^2$ while preserving $\omega_1$ under the assumption that there is a precipitous ideal on $\omega_1$ and a measurable cardinal.

1 Introduction

The problem of comparison between ordinals defined in descriptive set theory such as $\delta_n^1$, $n \in \omega$ and cardinals such as $\aleph_n$, $n \in \omega$ has haunted set theorists for decades. In this paper, we want to make a humble comment on the comparison between $\delta_1^2$ and $\omega_2$.

Hugh Woodin showed [6] that if the nonstationary ideal on $\omega_1$ is saturated and there is a measurable cardinal then $\delta_2^1 = \aleph_2$. Thus the iterations for making the nonstationary ideal saturated must add new reals, and they must increase $\delta_2^1$. It is a little bit of a mystery how this happens, since the new reals must be born at limit stages of the iteration and no one has been able to construct a forcing increasing the ordinal $\delta_2^1$ explicitly. The paper [7] shed some light on this problem; it produced a single step Namba type forcing which can increase $\delta_2^1$ in the right circumstances. In this paper we clean up and optimize the construction and prove:

**Theorem 1.1.** Suppose that there is a normal precipitous ideal on $\omega_1$ and a measurable cardinal $\kappa$. For every ordinal $\lambda \in \kappa$ there is an $\aleph_1$ preserving poset forcing $\delta_2^1 > \lambda$.

An important disclaimer: this result cannot be immediately used to iterate and obtain a model where $\delta_2^1 = \aleph_2$ from optimal large cardinal hypotheses.

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The forcing obtained increases $\delta_2$ once, to a value less than $\omega_2$. If the reader wishes to iterate the construction in order to obtain a model where $\delta_2 = \omega_2$, he will encounter the difficult problem of forcing a precipitous ideal on $\omega_1$ by an $\aleph_1$-preserving poset. Forcing $\delta_2 = \aleph_2$ may be possible with some other type of accumulation of partial orders obtained in this paper.

The notation in this paper is standard and follows [2]. After the paper was written we learned that a related construction was discovered by Jensen [4]: a Namba-type forcing in the model $L[U]$ with one measurable cardinal introducing a mouse which iterates to any length given beforehand.

2 Generic ultrapowers, iterations, and $\delta_2^1$

In order to prepare the ground for the forcing construction, we need to restate several basic definitions and claims regarding the generic ultrapowers and their iterations.

Definition 2.1. [3] Suppose that $J$ is a $\sigma$-ideal on $\omega_1$. If $G \subset \mathcal{P}(\omega_1) \setminus J$ is a generic filter, then we consider the generic ultrapower $j : V \to N$ modulo the filter $G$, in which only the ground model functions are used. If the model $N$ is wellfounded, it is identified with its transitive collapse, and the ideal $J$ is called precipitous.

The following definitions and facts have been isolated in [6].

Definition 2.2. [6] Suppose that $M$ is a countable transitive model, and $M \models \"J$ is a precipitous ideal\". An iteration of length $\beta \leq \omega_1$ of the model $M$ is a sequence $M_\alpha : \alpha \leq \beta$ of models together with commuting system of elementary embeddings; successor stages are obtained through a generic ultrapower, and limit stages through a direct limit. A model is iterable if all of its iterands are wellfounded.

Definition 2.3. [1] Suppose $J$ is a precipitous ideal on $\omega_1$. An elementary submodel $M$ of a large structure with $j \in M$ is selfgeneric if for every maximal antichain $A \subset \mathcal{P}(\omega_1) \setminus J$ in the model $M$ there is a set $B \in A \cap M$ such that $M \cap \omega_1 \in B$. In other words, the filter $\{B \in M \cap \mathcal{P}(\omega_1) \setminus J : M \cap \omega_1 \in B\}$ is an $M$-generic filter.

Note that if $M$ is a selfgeneric submodel, $N$ is the Skolem hull of $M \cup \{M \cap \omega_1\}$, and $j : M \to N$ is the elementary embedding between the transitive collapses induced by $id : M \to N$, then $j$ is a generic ultrapower of the model $M$ by the generic filter identified in the above definition. The key observation is that selfgeneric models are fairly frequent:

Proposition 2.4. Suppose that $J$ is a precipitous ideal on $\omega_1$ and $\mu > 2^{\aleph_1}$ is a regular cardinal. The set of countable selfgeneric elementary submodels of $H_\mu$ is stationary in $[H_\mu]^{\aleph_0}$.  

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Proof. Suppose that $f : H^\omega_\mu \to H_\mu$ is a function; we must find a selfgeneric submodel of $H_\mu$ closed under it. Let $G \subseteq \mathcal{P}(\omega_1) \setminus J$ be a generic filter and $j : V \to N$ be the associated generic ultrapower embedding. Note that $j''H^\omega_\mu$ is a selfgeneric submodel of $j(H_\mu)$ closed under the function $j(f)$; it is not in general an element of the model $N$. Consider the tree $T$ of all finite attempts to build a selfgeneric submodel of $j(H_\mu)$ closed under the function $j(f)$. Then $T \in N$ and the previous sentence shows that the tree $T$ is illfounded in $V[G]$. Since the model $N$ is transitive, it must be the case that the tree $T$ is illfounded in $N$ too, and so $M \models \exists T$ is a countable selfgeneric elementary submodel of $j(H_\mu)$ closed under the function $j(f)$. An elementarity argument then yields a countable selfgeneric elementary submodel of the structure $H_\mu$ closed under the function $f$ in the ground model as desired.

Our approach to increasing $\delta^2_3$ is in spirit the same as that of Woodin. We start with a ground model $V$ with a precipitous ideal $J$ on $\omega_1$, a measurable cardinal $\kappa$, and an ordinal $\lambda \in \kappa$. Choose a regular cardinal $\mu$ between $\lambda$ and $\kappa$. In the generic extension $V[G]$, it will be the case that $\omega^V_1 = \omega^V_1[G]$ and $\kappa$ is still measurable and moreover there is a countable elementary submodel $M \prec H^V_\mu$ such that

- $M$ is selfgeneric
- $\bar{M}$ is iterable
- $\lambda$ is a subset of one of the iterands of $\bar{M}$.

In fact, it will be the case that writing $M_\alpha, \alpha \in \omega_1$ for the models obtained by transfinite inductive procedure $M_0 = M$, $M_{\alpha+1} =$ Skolem hull of $M_\alpha \cup \{M_\alpha \cap \omega_1\}$, and $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for limit ordinals $\alpha$, and writing $M_\alpha$ for the respective transitive collapses, the models $M_\alpha$ are all selfgeneric, the models $\bar{M}_\alpha, \alpha \leq \omega_1$ constitute an iteration of the model $M$, and $\lambda \in \bigcup_\alpha M_\alpha$. By Lemma 4.7 of [6], $\delta^2_3$ must be larger than the cumulative hierarchy rank of the model $M_{\omega_1}$, which by the third item is at least $\lambda$. Note that the model $M$ cannot be an element of the ground model.

It may seem that adding a model $M$ such that all the models $M_\alpha, \alpha \in \omega_1$ are selfgeneric is an overly ambitious project. The forcing will in fact add a countable set $\{f_\alpha : n \in \omega\} \subseteq H^\omega_\mu$ such that every countable elementary submodel containing it as a subset is necessarily selfgeneric. It will also add a countable set $\{g_\alpha : n \in \omega\} \subseteq H^\omega_\mu$ of functions from $\omega_1^{<\omega}$ to $\omega$ such that $\lambda = \bigcup_\alpha \mathrm{rng}(g_\alpha)$. This will be achieved by a variation of the classical Namba construction by an $\aleph_1$-preserving forcing of size $< \kappa$. In the generic extension, use the measurability of $\kappa$ to find an elementary submodel $N$ of a large structure containing $J, \mu, \kappa$ as well as the functions $f_n, g_n, n \in \omega$ such that the ordertype of $N \cap \kappa$ is $\omega_1$, and consider the transitive collapse $\bar{N}$ of the model $N \cap V$. It is iterable by Lemma 4.5 of [6]. This means that even the transitive collapse $\bar{M}$ of the model $M = N \cap H^V_\mu$ is iterable, since it is a rank-initial segment of $\bar{N}$.
and every iteration of $M$ extends to an iteration of $N$. Thus the model $M$ is as desired, and this will complete the proof.

3 A class of Namba-like forcings

Definition 3.1. Suppose that $X$ is a set and $I$ is a collection of subsets of $X$ closed under subsets, $X /\notin I$. The forcing $Q_I$ consists of all nonempty trees $T \subset X^{<\omega}$ such that every node $t \in T$ has an extension $s \in T$ such that $\{x \in X : s \upharpoonright \omega x \in T\} \notin I$. The ordering is that of inclusion.

It is not difficult to see that the forcing $Q_I$ adds a countable sequence of elements of the underlying set $X$. The only property of the generic sequence we will use is that it is not a subset of any ground model set in the collection $I$. The usual Namba forcing is subsumed in the above definition: just put $X = \aleph_2$ and $I =$ all subsets of $\omega_2$ of size $\aleph_1$. A small variation of the argument in [5] will show that whenever $I$ is an $\omega_2$-complete ideal then the forcing $Q_I$ preserves $\aleph_1$ and if in addition CH holds then no new reals are added. We want to increase the ordinal $\delta^2_1$, so we must add new reals, and so we must consider weaker closure properties of the collection $I$. The following definition is critical.

Definition 3.2. Suppose that $J$ is an ideal on a set $Y$, $X$ is a set, and $I$ is a collection of subsets of $X$. We say that $I$ is closed under $J$-integration if for every $J$-positive set $B \subset Y$ and every set $D \subset B \times X$ whose vertical sections are in $I$ the set $\int_B D dJ = \{x \in X : \{y \in B : \langle y, x \rangle \notin D\} \in J\} \subset X$ is also in the collection $I$.

We will use this definition in the context of a precipitous ideal $J$ on $\omega_1$. In this case, the closure under $J$ integration allows of an attractive reformulation:

Proposition 3.3. Suppose that $J$ is a precipitous ideal on $\omega_1$ and $I$ is a collection of subsets of some set $X$ closed under inclusion. Then $I$ is closed under $J$-integration if and only if $P(\omega_1) \setminus J$ forces that writing $j : V \rightarrow M$ for the generic ultrapower, the closure of $I$ under $J$ integration is equivalent to the statement that for every set $A \subset X$ not in $I$, the set $j''A$ is not covered by any element of $j(I)$.

Proof. For the left-to-right implication, assume that $I$ is closed under $J$ integration. Suppose that some condition forces that $\hat{C} \in j(I)$ is a set; strengthening this condition of necessary we can find a set $B \in P(\omega_1) \setminus J$ and a function $f : B \rightarrow I$ such that $B \vdash \hat{C} = j(f)(\omega_1)$. Let $D \subset B \times X$ be defined by $\langle \alpha, x \rangle \in D \iff x \in f(\alpha)$ and observe that $\int_B D dJ \in I$. Thus, if $A \notin I$ is a set, it contains an element $x \notin \int_B D dJ$, then the set $B' = \{\alpha \in B : x \notin f(\alpha)\} \subset B$ is $J$-positive and as a $P(\omega_1) \setminus J$ condition it forces $j(x) \notin \hat{C}$ and $j(A) \subsetneq \hat{C}$. The opposite implication is similar.

The reader should note the similarity between the above definition and the Fubini properties of ideals on Polish spaces as defined in [8].
The basic property of the class of forcings we have just introduced is the following.

**Proposition 3.4.** Suppose that $J$ is a precipitous ideal on $\omega_1$, $X$ is a set, and $I$ is a collection of subsets of the set $X$ closed under $J$ integration. Then the forcing $Q_I$ preserves $\aleph_1$.

**Proof.** Suppose that $T \models \dot{f} : \check{\omega} \rightarrow \check{\omega}_1$ is a function. A usual fusion argument provides for a tree $S \subset T$ in the poset $Q_I$ such that for every node $t \in S$ on the $n$-th splitting level the condition $S \upharpoonright t$ decides the value of the ordinal $\dot{f}(\check{n})$ to be some definite ordinal $g(t) \in \omega_1$. Here, $S \upharpoonright t$ is the tree of all nodes of the tree $S$ inclusion-compatible with $t$. To prove the theorem, it is necessary to find a tree $U \subset S$ and an ordinal $\alpha \in \omega_1$ such that the range $g''U$ is a subset of $\alpha$.

For every ordinal $\alpha \in \omega_1$ consider a game $G_\alpha$ between Players I and II in which the two players alternate for infinitely many rounds indexed by $n \in \omega$, Player I playing nodes $t_n \in T$ on the $n$-th splitting level of the tree $T$ and Player II answering with a set $A_n \in I$. Player I is required to play so that $t_0 \subset t_1 \subset \ldots$ and the first element on the sequence $t_{n+1} \setminus t_n$ is not in the set $A_n$. He wins if the ordinals $g(t_n), n \in \omega$ are all smaller than $\alpha$.

It is clear that these games are closed for Player I and therefore determined. Note that if Player I has a winning strategy $\sigma$ in the game $G_\alpha$ for some ordinal $\alpha \in \omega_1$, then the collection of all nodes which can arise as the answers of strategy $\sigma$ to some play by Player II forms a tree $U$ in $Q_I$ and $g''U \subset \alpha$. Thus the following claim will complete the proof of the theorem.

**Claim 3.5.** There is an ordinal $\alpha \in \omega_1$ such that Player I has a winning strategy in the game $G_\alpha$.

Assume for contradiction that Player II has a winning strategy $\sigma_\alpha$ for every ordinal $\alpha \in \omega_1$. Let $M \prec H_\kappa$ be a selfgeneric countable elementary submodel of some large structure containing the sequence of these strategies as well as $X, I, J$. We will find a legal counterplay against the strategy $\sigma_\beta$ in which Player I uses only moves from the model $M$. It is clear that in such a counterplay, the ordinals $g(t_n), n \in \omega$ stay below $\beta$. Therefore Player I will win this play, and that will be the desired contradiction.

The construction of the counterplay proceeds by induction. Build nodes $t_n, n \in \omega$ of the tree $S$ as well as subsets $B_n, n \in \omega$ of $\omega_1$ so that

1. $B_0 \supset B_1 \supset \ldots$ are all $J$-positive sets in the model $M$ such that $\beta \in B_n$ for every number $n$.

2. $t_0 \subset t_1 \subset \ldots \subset t_n$ are all in the model $M$ and they form a legal finite counterplay against all strategies $\sigma_\alpha, \alpha \in B_n$, in particular, against the strategy $\sigma_\beta$.

Suppose that the node $t_n \in S \cap M$ and the set $B_n$ have been found. Consider the set $D = \{(\alpha, x) : \alpha \in B_n, x \in \sigma_\alpha(t_n)\} \subset B \times X$. Its vertical sections are sets in the collection $I$, and by the assumptions so are the integrals $\int_C D \, dJ$. 

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for all \( J \)-positive sets \( C \subset B_n \). Since the node \( t_n \in S \) has more than \( I \) many immediate successors, it follows that the set \( A = \{ C \subset B_n : C \notin J \text{ and } \exists x \in X \forall \alpha \in C t_\alpha \in S \land x \notin \sigma_\alpha(t_\alpha) \} \) is dense in \( \mathcal{P}(\omega_1) \setminus J \) below the set \( B_n \). This set is also in the model \( M \) and by the selfgenericity there is a point \( x \in X \cap M \) such that \( t_\alpha \in S \) and the set \( S_{n+1} = \{ \alpha \in B_n : x \notin \sigma_\alpha(t_\alpha) \} \) is in the set \( A \cap M \) and contains the ordinal \( \beta \). The node \( t_{n+1} \supset t_n \) is then just any node at \( n + 1 \)-st splitting level extending \( t_\alpha \). Clearly, \( t_{n+1} \in M \) by the elementarity of the model \( M \). This concludes the inductive construction and the proof. \( \square \)

As the last remark in this section, the class of sets \( I \) closed under \( J \)-integration is itself closed on various operations, and this leads to simple operations on the partial orders of the form \( Q_\kappa \). We will use the following operation. If \( X_0, X_1 \) are disjoint sets and \( I_0 \subset \mathcal{P}(X_0) \) and \( I_1 \subset \mathcal{P}(X_1) \) are sets closed under subsets and \( J \)-integration, then also the set \( A \subset \mathcal{P}(X_0 \cup X_1) \) defined by \( A \in K \) if either \( A \cap X_0 \in I_0 \) or \( A \cap X_1 \in I_1 \) is closed under subsets and \( J \)-integration. It is easy to see that the forcing \( Q_\kappa \) adds an \( \omega \) sequence of elements of \( X_0 \cup X_1 \) which cofinally often visits both sets and its intersection with \( X_0 \) or \( X_1 \) is not a subset of any ground model set in \( I_0 \) or \( I_1 \) respectively.

4 Wrapping up

Fix a normal precipitous ideal \( J \) on \( \omega_1 \), a measurable cardinal \( \kappa \), and an ordinal \( \lambda < \kappa \). Theorem 1.1 is now proved through identification of several interesting collections of sets closed under \( J \)-integration. This does not refer to the precipitousness of the \( \sigma \)-ideal \( J \) anymore.

**Definition 4.1.** \( X_0 \) is the set of all functions from \( \omega_1^{<\omega} \) to \( \lambda \). \( I_0 \subset \mathcal{P}(X_0) \) is the closure of the set of its generators under subset and \( J \)-integration, where the generators of \( I_0 \) are the sets \( A_\alpha = \{ g \in X_0 : \alpha \notin \text{rng}(g) \} \). \( \alpha \in \lambda \).

The obvious intention behind the definition is that if \( \{ g_n : n \in \omega \} \subset X_0 \) is a set of functions which is not covered by any element of the set \( I_0 \) then \( \bigcup \text{rng}(g_n) = \lambda \). With the previous section in mind, we must prove that \( X_0 \notin I_0 \). Unraveling the definitions, it is clear that it is just necessary to prove that whenever \( n \) is a natural number, \( S \subset \omega_1^n \) is a \( J^n \)-positive set, and \( D \subset S \times X_0 \) is a set whose vertical sections are \( I_0 \)-generators, then the integral \( \int_S D \ dJ^n \) is not equal to \( X_0 \). Here \( J^n \) is the usual \( n \)-fold Fubini power of the ideal \( J \). Let \( g : \omega_1^n \to \lambda \) be a function such that for every \( n \)-tuple \( \bar{\beta} \in S \), the vertical section \( D_{\bar{\beta}} \) is just the generator \( A_{g(\bar{\beta})} \). Then clearly \( g \notin \bigcup_{\bar{\beta} \in S} D_{\bar{\beta}} \), in particular \( g \notin \int_S D \ dJ^n \) and \( \int_S D \ dJ^n \neq X_0 \).

**Definition 4.2.** \( X_1 \) is the set of all functions with domain \( \omega_1^{<\omega} \times \mathfrak{A} \) and range a subset of \( \omega_1 \times \mathcal{P}(\omega_1) \). Here \( \mathfrak{A} \) is the set of all maximal antichains in the forcing \( \mathcal{P}(\omega_1) \setminus J \). The set \( I_1 \) is the closure of the set of its generators under subset and \( J \)-integration, where the generators of \( I_1 \) are the sets of the form \( A_{\alpha, Z} = \{ f \in X_1 : \)
for every finite sequence $\vec{\beta} \in \alpha^{<\omega}$, $f(\vec{\beta}, Z)(0) \in \alpha$ and $f(\vec{\beta}, Z)(1)$ is not a set in $Z$ containing $\alpha$, where $\alpha \in \omega_1$ and $Z \in \mathfrak{A}$ are arbitrary.

The obvious intention behind this definition is that whenever $\{f_n : n \in \omega\}$ is a countable subset of $X_1$ which is not covered by any element of the set $I_1$ then every countable elementary submodel $M \prec H_\mu$ containing all these functions must be self-generic: whenever $Z \in M$ is a maximal antichain in $\mathcal{P}(\omega_1) \setminus J$, writing $\alpha = M \cap \omega_1$, there must be a number $n$ such that $f_n \notin A_{\alpha, Z}$. Perusing the definition of the set $A_{\alpha, Z}$ and noting that $M$ is closed under the function $f_n$, we conclude that it must be the case that for some finite sequence $\vec{\beta} \in \alpha^{<\omega}$ the value $f_n(\vec{\beta}, Z) \in M$ must be a set in $Z$ containing the ordinal $\alpha$. Since the maximal antichain $Z$ was arbitrary, this shows that $M$ is self-generic as required.

We must prove that $X_1 \notin I_1$. This is a rather elementary matter, nevertheless it is somewhat more complicated than the 0 subscript case. Unraveling the definitions, it is clear that it is just necessary to prove that whenever $n$ is a natural number, $S \subset \omega_1^n$ is a $J^n$-positive set, and $D \subset S \times X_0$ is a set whose vertical sections are $I_1$-generators, then the integral $\int_S D \, dJ^n$ is not equal to $X_1$. Here $J^n$ is the usual $n$-fold Fubini power of the ideal $J$. Fix then $n \in \omega$, a $J^n$-positive set $S \subset \omega_1^n$, and the set $D \subset S \times X_1$; we must find a function $f \in X_1$ and a $J_n$-positive set $U \subset S$ such that $\forall \vec{\beta} \in U \langle \vec{\beta}, f \rangle \notin D$. For every sequence $\vec{\beta} \in S$ choose a countable ordinal $\alpha(\vec{\beta})$ and a maximal antichain $Z(\vec{\beta}) \subset \mathcal{P}(\omega_1) \setminus J$ such that $D_{\vec{\beta}} = A_{\alpha(\vec{\beta}), Z(\vec{\beta})}$. Use standard normality arguments to find numbers $m, k \leq n$ and a $J^n$-positive set $T \subset S$ consisting of increasing sequences such that

- for a sequence $\vec{\beta} \in T$, the value of $\alpha(\vec{\beta})$ depends only on $\vec{\beta} \upharpoonright m$ and $\alpha(\vec{\beta}) \geq \vec{\beta}(m - 1)$
- the value of $Z(\vec{\beta})$ depends only on $\vec{\beta} \upharpoonright k$ and the partial map $\pi$ with domain $\omega_1^k$, defined by $Z(\vec{\beta}) = \pi(\vec{\beta} \upharpoonright k)$ whenever $\vec{\beta} \in T$, is countable-to-one.

There are now several cases.

- There is a $J^n$-positive set $U \subset T$ such that $\alpha(\vec{\beta}) > \vec{\beta}(m - 1)$. Here, consider the function $f \in X_1$ such that $f(\vec{\beta} \upharpoonright m, Z) = \alpha(\vec{\beta})$ for every sequence $\vec{\beta} \in U$ and every maximal antichain $Z$. Clearly, $f \notin \bigcup_{\vec{\beta} \in U} D_{\vec{\beta}}$ as required: for every sequence $\vec{\beta} \in U$, it is the case that $\alpha(\vec{\beta}) = f(\vec{\beta} \upharpoonright m, Z(\vec{\beta}))(0)$ and so the ordinal $\alpha(\vec{\beta})$ does not have the required closure property with respect to $f$.
- The first case fails and $k \geq m$. Here, define the map $f \in X_1$ by $f(0, Z)(0) = \sup\{\vec{\beta}(k - 1) : \vec{\beta} \in T \text{ and } Z = Z(\vec{\beta})\} + 1$ for every maximal antichain $Z$. The set $U = \{y \in T : \alpha(\vec{\beta}) = \vec{\beta}(m - 1)\}$ and the map $f$ are as required: again, for every sequence $\beta \in U$ the ordinal $\alpha(\vec{\beta}) \leq \vec{\beta}(k - 1) < f(0, Z(\vec{\beta}))(0)$ does not have the required closure properties.
The first case fails and $k < m$. Define the function $f \in X_1$ in the following way. For every sequence $\vec{\gamma} \in \omega_1^{m-1}$, if the set $W_{\vec{\gamma}} = \{ \alpha \in \omega_1 : \exists \vec{\beta} \in T \vec{\gamma} \alpha \subset \vec{\beta} \text{ and } \alpha = \alpha(\vec{\beta}) \}$ is J-positive, let $f(\vec{\gamma}, \pi(\vec{\gamma} \upharpoonright k))$ to be some element of the maximal antichain $\pi(\vec{\gamma} \upharpoonright k)$ with J-positive intersection with $W_{\vec{\gamma}}$. The set $U = \{ \vec{\beta} \in T : \alpha(\vec{\beta}) = \vec{\beta}(m) \text{ and } \vec{\beta}(m) \in f(\vec{\beta} \upharpoonright (m-1), \pi(\vec{\beta} \upharpoonright k)) \}$ is then $J^\omega$ positive and $f \notin \bigcup_{\vec{\beta} \in U} D_{\vec{\beta}}$ as required: the ordinal $\alpha(\vec{\beta})$ belongs to the set $f(\vec{\beta} \upharpoonright k, Z(\vec{\beta})) \in Z(\vec{\beta})$.

Thus $X_1 \notin I_1$.

To conclude the proof of Theorem 1.1, just form a collection $K \subset P(X_0 \cup X_1)$ as in the end of the previous section and force with the poset $Q_K$. Since $K$ is closed under $J$-integration, the forcing preserves $\aleph_1$. It also adds sets $\{ f_n : n \in \omega \} \subset X_1$ and $\{ g_n : n \in \omega \} \subset X_0$ with the required properties, showing that in the generic extension, $\delta_1^2 > \lambda$.

References


