Increasing $\boldsymbol{\delta}^1_2$ and Namba-style forcing*

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Abstract

We isolate a forcing which increases the value of $\boldsymbol{\delta}^1_2$ while preserving ω_1 under the assumption that there is a precipitous ideal on ω_1 and a measurable cardinal.

1 Introduction

The problem of comparison between ordinals defined in descriptive set theory such as $\boldsymbol{\delta}^1_\mathbf{n}, n \in \omega$ and cardinals such as $\aleph_n, n \in \omega$ has haunted set theorists for decades. In this paper, we want to make a humble comment on the comparison between $\boldsymbol{\delta}^1_2$ and ω_2.

Hugh Woodin showed [6] that if the nonstationary ideal on ω_1 is saturated and there is a measurable cardinal then $\boldsymbol{\delta}^1_2 = \aleph_2$. Thus the iterations for making the nonstationary ideal saturated must add new reals, and they must increase $\boldsymbol{\delta}^1_2$. It is a little bit of a mystery how this happens, since the new reals must be born at limit stages of the iteration and no one has been able to construct a forcing increasing the ordinal $\boldsymbol{\delta}^1_2$ explicitly. The paper [7] shed some light on this problem; it produced a single step Namba type forcing which can increase $\boldsymbol{\delta}^1_2$ in the right circumstances. In this paper we clean up and optimize the construction and prove:

Theorem 1.1. *Suppose that there is a normal precipitous ideal on ω_1 and a measurable cardinal κ. For every ordinal $\lambda \in \kappa$ there is an $\aleph_1$ preserving poset forcing $\boldsymbol{\delta}^1_2 > \lambda$.*

An important disclaimer: this result cannot be immediately used to iterate and obtain a model where $\boldsymbol{\delta}^1_2 = \aleph_2$ from optimal large cardinal hypotheses.

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The forcing obtained increases $\boldsymbol{\delta}^1_2$ *once*, to a value less than ω_2. If the reader wishes to iterate the construction in order to obtain a model where $\boldsymbol{\delta}^1_2 = \omega_2$, he will encounter the difficult problem of forcing a precipitous ideal on ω_1 by an $\aleph_1$-preserving poset. Forcing $\boldsymbol{\delta}^1_2 = \aleph_2$ may be possible with some other type of accumulation of partial orders obtained in this paper.

The notation in this paper is standard and follows [2]. After the paper was written we learned that a related construction was discovered by Jensen [4]: a Namba-type forcing in the model $L[U]$ with one measurable cardinal introducing a mouse which iterates to any length given beforehand.

2 Generic ultrapowers, iterations, and $\boldsymbol{\delta}^1_2$

In order to prepare the ground for the forcing construction, we need to restate several basic definitions and claims regarding the generic ultrapowers and their iterations.

Definition 2.1. [3] Suppose that J is a σ-ideal on ω_1. If $G \subset \mathcal{P}(\omega_1) \setminus J$ is a generic filter, then we consider the *generic ultrapower* $j : V \to N$ modulo the filter G, in which only the ground model functions are used. If the model N is wellfounded, it is identified with its transitive collapse, and the ideal J is called *precipitous.*

The following definitions and facts have been isolated in [6].

Definition 2.2. [6] Suppose that M is a countable transitive model, and $M \models$"J is a precipitous ideal". An *iteration* of length $\beta \leq \omega_1$ of the model M is a sequence $M_\alpha : \alpha \leq \beta$ of models together with commuting system of elementary embeddings; successor stages are obtained through a generic ultrapower, and limit stages through a direct limit. A model is *iterable* if all of its iterands are wellfounded.

Definition 2.3. [1] Suppose J is a precipitous ideal on ω_1. An elementary submodel M of a large structure with $j \in M$ is *selfgeneric* if for every maximal antichain $A \subset \mathcal{P}(\omega_1) \setminus J$ in the model M there is a set $B \in A \cap M$ such that $M \cap \omega_1 \in B$. In other words, the filter $\{B \in M \cap \mathcal{P}(\omega_1) \setminus J : M \cap \omega_1 \in B\}$ is an M-generic filter.

Note that if M is a selfgeneric submodel, N is the Skolem hull of $M \cup \{M \cap \omega_1\}$, and $j : \bar{M} \to \bar{N}$ is the elementary embedding between the transitive collapses induced by $id : M \to N$, then j is a generic ultrapower of the model M by the genric filter identified in the above definition. The key observation is that selfgeneric models are fairly frequent:

Proposition 2.4. *Suppose that J is a precipitous ideal on ω_1 and $\mu > 2^{\aleph_1}$ is a regular cardinal. The set of countable selfgeneric elementary submodels of H_μ is stationary in $[H_\mu]^{\aleph_0}$.*

Proof. Suppose that $f : H_\mu^{<\omega} \to H_\mu$ is a function; we must find a selfgeneric submodel of H_μ closed under it. Let $G \subset \mathcal{P}(\omega_1) \setminus J$ be a generic filter and $j : V \to N$ be the associated generic ultrapower embedding into a transitive model. Note that $j''H_\mu^V$ is a selfgeneric submodel of $j(H_\mu)$ closed under the function $j(f)$; it is not in general an element of the model N. Consider the tree T of all finite attempts to build a selfgeneric submodel of $j(H_\mu)$ closed under the function $j(f)$. Then $T \in N$ and the previous sentence shows that the tree T is illfounded in $V[G]$. Since the model N is transitive, it must be the case that the tree T is illfounded in N too, and so $M \models$there is a countable selfgeneric elementary submodel of $j(H_\mu)$ closed under the function $j(f)$. An elementarity argument then yields a countable selfgeneric elementary submodel of the structure H_μ closed under the function f in the ground model as desired. $\square$

Our approach to increasing $\boldsymbol{\delta^1_2}$ is in spirit the same as that of Woodin. We start with a ground model V with a precipitous ideal J on ω_1, a measurable cardinal κ, and an ordinal $\lambda \in \kappa$. Choose a regular cardinal μ between λ and κ. In the generic extension $V[G]$, it will be the case that $\omega_1^V = \omega_1^{V[G]}$ and κ is still measurable and moreover there is a countable elementary submodel $M \prec H_\mu^V$ such that

- M is selfgeneric
- $\bar{M}$ is iterable
- λ is a subset of one of the iterands of $\bar{M}$.

In fact, it will be the case that writing $M_\alpha, \alpha \in \omega_1$ for the models obtained by transfinite inductive procedure $M_0 = M$, $M_{\alpha+1} =$Skolem hull of $M_\alpha \cup \{M_\alpha \cap \omega_1\}$, and $M_\alpha = \bigcup_{\beta \in \alpha}$ for limit ordinals α, and writing $\bar{M}_\alpha$ for the respective transitive collapses, the models M_α are all selfgeneric, the models $\bar{M}_\alpha, \alpha \leq \omega_1$ constitute an iteration of the model $\bar{M}$, and $\lambda \subset \bigcup_\alpha M_\alpha$. By Lemma 4.7 of [6], $\boldsymbol{\delta^1_2}$ must be larger than the cumulative hierarchy rank of the model $\bar{M}_{\omega_1}$, which by the third item is at least λ. Note that the model M cannot be an element of the ground model.

It may seem that adding a model M such that all the models $M_\alpha, \alpha \in \omega_1$ are selfgeneric is an overly ambitious project. The forcing will in fact add a countable set $\{f_n : n \in \omega\} \subset H_\mu^V$ such that every countable elementary submodel containing it as a subset is necessarily selfgeneric. It will also add a countable set $\{g_n : n \in \omega\} \subset H_\mu^V$ of functions from $\omega_1^{<\omega}$ to ω such that $\lambda = \bigcup_n \operatorname{rng}(g_n)$. This will be achieved by a variation of the classical Namba construction by an $\aleph_1$-preserving forcing of size $< \kappa$. In the generic extension, use the measurability of κ to find an elementary submodel N of a large structure containing J, μ, κ as well as the functions $f_n, g_n, n \in \omega$ such that the ordertype of $N \cap \kappa$ is ω_1, and consider the transitive collapse $\bar{N}$ of the model $N \cap V$. It is iterable by Lemma 4.5 of [6]. This means that even the transitive collapse $\bar{M}$ of the model $M = N \cap H_\mu^V$ is iterable, since it is a rank-initial segment of $\bar{N}$

and every iteration of $\bar{M}$ extends to an iteration of $\bar{N}$. Thus the model M is as desired, and this will complete the proof.

3 A class of Namba-like forcings

Definition 3.1. Suppose that X is a set and I is a collection of subsets of X closed under subsets, $X \notin I$. The forcing Q_I consists of all nonempty trees $T \subset X^{<\omega}$ such that every node $t \in T$ has an extension $s \in T$ such that $\{x \in X : s^\frown x \in T\} \notin I$. The ordering is that of inclusion.

It is not difficult to see that the forcing Q_I adds a countable sequence of elements of the underlying set X. The only property of the generic sequence we will use is that it is not a subset of any ground model set in the collection I. The usual Namba forcing is subsumed in the above definition: just put $X = \aleph_2$ and $I =$all subsets of ω_2 of size $\aleph_1$. A small variation of the argument in [5] will show that whenever I is an $< \aleph_2$-complete ideal then the forcing Q_I preserves $\aleph_1$ and if in addition CH holds then no new reals are added. We want to increase the ordinal δ^1_2, so we must add new reals, and so we must consider weaker closure properties of the collection I. The following definition is critical.

Definition 3.2. Suppose that J is an ideal on a set Y, X is a set, and I is a collection of subsets of X. We say that I is *closed under J integration* if for every J-positive set $B \subset Y$ and every set $D \subset B \times X$ whose vertical sections are in I the set $\int_B D \; dJ = \{x \in X : \{y \in B : \langle y, x\rangle \notin D\} \in J\} \subset X$ is also in the collection I.

We will use this definition in the context of a precipitous ideal J on ω_1. In this case, the closure under J integration allows of an attractive reformulation:

Proposition 3.3. *Suppose that J is a precipitous ideal on ω_1 and I is a collection of subsets of some set X closed under inclusion. Then I is closed under J-integration if and only if $\mathcal{P}(\omega_1) \setminus J$ forces that writing $j : V \to M$ for the generic ultrapower, the closure of I under J integration is equivalent to the statement that for every set $A \subset X$ not in I, the set $j''A$ is not covered by any element of $j(I)$.*

Proof. For the left-to-right implication, assume that I is closed under J integration. Suppose that some condition forces that $\dot{C} \in j(I)$ is a set; strengthening this condition of necessary we can find a set $B \in \mathcal{P}(\omega_1) \setminus J$ and a function $f : B \to I$ such that $B \Vdash \dot{C} = j(f)(\check{\omega}_1)$. Let $D \subset B \times X$ be defined by $\langle \alpha, x\rangle \in D \leftrightarrow x \in f(\alpha)$ and observe that $\int_B D \; dJ \in I$. Thus, if $A \notin I$ is a set, it contains an element $x \notin \int_B D \; dJ$, then the set $B' = \{\alpha \in B : x \notin f(\alpha)\} \subset B$ is J-positive and as a $\mathcal{P}(\omega_1) \setminus J$ condition it forces $j(x) \notin \dot{C}$ and $j(\check{A}) \not\subset \dot{C}$.

The opposite implication is similar. □

The reader should note the similarity between the above definition and the Fubini properties of ideals on Polish spaces as defined in [8].

The basic property of the class of forcings we have just introduced is the following.

Proposition 3.4. *Suppose that J is a precipitous ideal on ω_1, X is a set, and I is a collection of subsets of the set X closed under J integration. Then the forcing Q_I preserves $\aleph_1$.*

Proof. Suppose that $T \Vdash \dot{f} : \check{\omega} \to \check{\omega}_1$ is a function. A usual fusion argument provides for a tree $S \subset T$ in the poset Q_I such that for every node $t \in S$ on the n-th splitting level the condition $S \restriction t$ decides the value of the ordinal $\dot{f}(\check{n})$ to be some definite ordinal $g(t) \in \omega_1$. Here, $S \restriction t$ is the tree of all nodes of the tree S inclusion-compatible with t. To prove the theorem, it is necessary to find a tree $U \subset S$ and an ordinal $\alpha \in \omega_1$ such that the range $g''U$ is a subset of α.

For every ordinal $\alpha \in \omega_1$ consider a game G_α between Players I and II in which the two players alternate for infinitely many rounds indexed by $n \in \omega$, Player I playing nodes $t_n \in T$ on the n-th splitting level of the tree T and Player II answering with a set $A_n \in I$. Player I is required to play so that $t_0 \subset t_1 \subset \dots$ and the first element on the sequence $t_{n+1} \setminus t_n$ is not in the set A_n. He wins if the ordinals $g(t_n), n \in \omega$ are all smaller than α.

It is clear that these games are closed for Player I and therefore determined. Note that if Player I has a winning strategy σ in the game G_α for some ordinal $\alpha \in \omega_1$, then the collection of all nodes which can arise as the answers of strategy σ to some play by Player II forms a tree U in Q_I and $g''U \subset \alpha$. Thus the following claim will complete the proof of the theorem.

Claim 3.5. *There is an ordinal $\alpha \in \omega_1$ such that Player I has a winning strategy in the game G_α.*

Assume for contradiction that Player II has a winning strategy σ_α for every ordinal $\alpha \in \omega_1$. Let $M \prec H_\kappa$ be a selfgeneric countable elementary submodel of some large structure containing the sequence of these strategies as well as X, I, J. Let $\beta = M \cap \omega_1$. We will find a legal counterplay against the strategy σ_β in which Player I uses only moves from the model M. It is clear that in such a counterplay, the ordinals $g(t_n), n \in \omega$ stay below β. Therefore Player I will win this play, and that will be the desired contradiction.

The construction of the counterplay proceeds by induction. Build nodes $t_n, n \in \omega$ of the tree S as well as subsets $B_n, n \in \omega$ of ω_1 so that

- $B_0 \supset B_1 \supset \dots$ are all J-positive sets in the model M such that $\beta \in B_n$ for every number n
- $t_0 \subset t_1 \subset \dots \subset t_n$ are all in the model M and they form a legal finite counterplay against all strategies $\sigma_\alpha, \alpha \in B_n$, in particular, against the strategy σ_β.

Suppose that the node $t_n \in S \cap M$ and the set B_n have been found. Consider the set $D = \{\langle \alpha, x \rangle : \alpha \in B_n, x \in \sigma_\alpha(t_n)\} \subset B \times X$. Its vertical sections are sets in the collection I, and by the assumptions so are the integrals $\int_C D \; dJ$

for all J-positive sets $C \subset B_n$. Since the node $t_n \in S$ has more than I many immediate successors, it follows that the set $A = \{C \subset B_n : C \notin J$ and $\exists x \in X\ \forall \alpha \in C\ t_n^\frown x \in S \land x \notin \sigma_\alpha(t_n)\}$ is dense in $\mathcal{P}(\omega_1) \setminus J$ below the set B_n. This set is also in the model M and by the selfgenericity there is a point $x \in X \cap M$ such that $t_n^\frown x \in S$ and the set $S_{n+1} = \{\alpha \in B_n : x \notin \sigma_\alpha(t_n)\}$ is in the set $A \cap M$ and contains the ordinal β. The node $t_{n+1} \supset t_n$ is then just any node at $n+1$-st splitting level extending $t_n^\frown x$. Clearly, $t_{n+1} \in M$ by the elementarity of the model M. This concludes the inductive construction and the proof. □

As the last remark in this section, the class of sets I closed under J-integration is itself closed on various operations, and this leads to simple operations on the partial orders of the form Q_I. We will use the following operation. If X_0, X_1 are disjoint sets and $I_0 \subset \mathcal{P}(X_0)$ and $I_1 \subset \mathcal{P}(X_1)$ are sets closed under subsets and J integration, then also the set $K \subset \mathcal{P}(X_0 \cup X_1)$ defined by $A \in K$ if either $A \cap X_0 \in I_0$ or $A \cap X_1 \in I_1$ is closed under subsets and J-integration. It is easy to see that the forcing Q_K adds an ω sequence of elements of $X_0 \cup X_1$ which cofinally often visits both sets and its intersection with X_0 or X_1 is not a subset of any ground model set in I_0 or I_1 respectively.

4 Wrapping up

Fix a normal precipitous ideal J on ω_1, a measurable cardinal κ, and an ordinal $\lambda < \kappa$. Theorem 1.1 is now proved through identification of several interesting collections of sets closed under J-integration. This does not refer to the precipitousness of the σ-ideal J anymore.

Definition 4.1. X_0 is the set of all functions from $\omega_1^{<\omega}$ to λ. $I_0 \subset \mathcal{P}(X_0)$ is the closure of the set of its generators under subset and J-integration, where the generators of I_0 are the sets $A_\alpha = \{g \in X_0 : \alpha \notin \operatorname{rng}(g)\}$ for $\alpha \in \lambda$.

The obvious intention behind the definition is that if $\{g_n : n \in \omega\} \subset X_0$ is a set of functions which is not covered by any element of the set I_0 then $\bigcup_n \operatorname{rng}(g_n) = \lambda$. With the previous section in mind, we must prove that $X_0 \notin I_0$. Unraveling the definitions, it is clear that it is just necessary to prove that whenever n is a natural number, $S \subset \omega_1^n$ is a J^n-positive set, and $D \subset S \times X_0$ is a set whose vertical sections are I_0-generators, then the integral $\int_S D\ dJ^n$ is not equal to X_0. Here J^n is the usual n-fold Fubini power of the ideal J. Let $g : \omega_1^n \to \lambda$ be a function such that for every n-tuple $\vec{\beta} \in S$, the vertical section $D_{\vec{\beta}}$ is just the generator $A_{g(\vec{\beta})}$. Then clearly $g \notin \bigcup_{\vec{\beta} \in S} D_{\vec{\beta}}$, in particular $g \notin \int_S D\ dJ^n$ and $\int_S D\ dJ^n \neq X_0$.

Definition 4.2. X_1 is the set of all functions with domain $\omega_1^{<\omega} \times \mathfrak{A}$ and range a subset of $\omega_1 \times \mathcal{P}(\omega_1)$. Here $\mathfrak{A}$ is the set of all maximal antichains in the forcing $\mathcal{P}(\omega_1) \setminus J$. The set I_1 is the closure of the set of its generators under subset and J-integration, where the generators of I_1 are the sets of the form $A_{\alpha,Z} = \{f \in X_1 :$

for every finite sequence $\vec{\beta} \in \alpha^{<\omega}$, $f(\vec{\beta}, Z)(0) \in \alpha$ and $f(\vec{\beta}, Z)(1)$ is not a set in Z containing $\alpha\}$, where $\alpha \in \omega_1$ and $Z \in \mathfrak{A}$ are arbitrary.

The obvious intention behind this definition is that whenever $\{f_n : n \in \omega\}$ is a countable subset of X_1 which is not covered by any element of the set I_1 then every countable elementary submodel $M \prec H_\mu$ containing all these functions must be self-generic: whenever $Z \in M$ is a maximal antichain in $\mathcal{P}(\omega_1) \setminus J$, writing $\alpha = M \cap \omega_1$, there must be a number n such that $f_n \notin A_{\alpha,Z}$. Perusing the definition of the set $A_{\alpha,Z}$ and noting that M is closed under the function f_n, we conclude that it must be the case that for some finite sequence $\vec{\beta} \in \alpha^{<\omega}$ the value $f_n(\vec{\beta}, Z) \in M$ must be a set in Z containing the ordinal α. Since the maximal antichain Z was arbitrary, this shows that M is self-generic as required.

We must prove that $X_1 \notin I_1$. This is a rather elementary matter, nevertheless it is somewhat more complicated than the 0 subscript case. Unraveling the definitions, it is clear that it is just necessary to prove that whenever n is a natural number, $S \subset \omega_1^n$ is a J^n-positive set, and $D \subset S \times X_0$ is a set whose vertical sections are I_1-generators, then the integral $\int_S D \, dJ^n$ is not equal to X_1. Here J^n is the usual n-fold Fubini power of the ideal J. Fix then $n \in \omega$, a J^n-positive set $S \subset \omega_1^n$, and the set $D \subset S \times X_1$; we must find a function $f \in X_1$ and a J_n-positive set $U \subset S$ such that $\forall \vec{\beta} \in U \ \langle \vec{\beta}, f \rangle \notin D$. For every sequence $\vec{\beta} \in S$ choose a countable ordinal $\alpha(\vec{\beta})$ and a maximal antichain $Z(\vec{\beta}) \subset \mathcal{P}(\omega_1) \setminus J$ such that $D_{\vec{\beta}} = A_{\alpha(\vec{\beta}), Z(\vec{\beta})}$. Use standard normality arguments to find numbers $m, k \leq n$ and a J^n-positive set $T \subset S$ consisting of increasing sequences such that

- for a sequence $\vec{\beta} \in T$, the value of $\alpha(\vec{\beta})$ depends only on $\vec{\beta} \restriction m$ and $\alpha(\vec{\beta}) \geq \vec{\beta}(m-1)$
- the value of $Z(\vec{\beta})$ depends only on $\vec{\beta} \restriction k$ and the partial map π with domain ω_1^k, defined by $Z(\vec{\beta}) = \pi(\vec{\beta} \restriction k)$ whenever $\vec{\beta} \in T$, is countable-to-one.

There are now several cases.

- There is a J^n-positive set $U \subset T$ such that $\alpha(\vec{\beta}) > \vec{\beta}(m-1)$. Here, consider the function $f \in X_1$ such that $f(\vec{\beta} \restriction m, Z) = \alpha(\vec{\beta})$ for every sequence $\vec{\beta} \in U$ and every maximal antichain Z. Clearly, $f \notin \bigcup_{\vec{\beta} \in U} D_{\vec{\beta}}$ as required: for every sequence $\vec{\beta} \in U$, it is the case that $\alpha(\vec{\beta}) = f(\vec{\beta} \restriction m, Z(\vec{\beta}))(0)$ and so the ordinal $\alpha(\vec{\beta})$ does not have the required closure property with respect to f.
- The first case fails and $k \geq m$. Here, define the map $f \in X_1$ by $f(0, Z)(0) = \sup\{\vec{\beta}(k-1) : \vec{\beta} \in T \text{ and } Z = Z(\vec{\beta})\} + 1$ for every maximal antichain Z. The set $U = \{y \in T : \alpha(\vec{\beta}) = \vec{\beta}(m-1)\}$ and the map f are required: again, for every sequence $\beta \in U$ the ordinal $\alpha(\vec{\beta}) \leq \vec{\beta}(k-1) < f(0, Z(\vec{\beta}))(0)$ does not have the required closure properties.

- The first case fails and $k < m$. Define the function $f \in X_1$ in the following way. For every sequence $\vec{\gamma} \in \omega_1^{m-1}$, if the set $W_{\vec{\gamma}} = \{\alpha \in \omega_1 : \exists \vec{\beta} \in T\ \vec{\gamma}^\frown \alpha \subset \vec{\beta}$ and $\alpha = \alpha(\vec{\beta})\}$ is J-positive, let $f(\vec{\gamma}, \pi(\vec{\gamma} \restriction k))$ to be some element of the maximal antichain $\pi(\vec{\gamma} \restriction k)$ with J-positive intersection with $W_{\vec{\gamma}}$. The set $U = \{\vec{\beta} \in T : \alpha(\vec{\beta}) = \vec{\beta}(m)$ and $\vec{\beta}(m) \in f(\vec{\beta} \restriction (m-1), \pi(\vec{\beta} \restriction k)\}$ is then J^n positive and $f \notin \bigcup_{\vec{\beta} \in U} D_{\vec{\beta}}$ as required: the ordinal $\alpha(\vec{\beta})$ belongs to the set $f(\vec{\beta} \restriction k, Z(\vec{\beta})) \in Z(\vec{\beta})$.

Thus $X_1 \notin I_1$.

To conclude the proof of Theorem 1.1, just form a collection $K \subset \mathcal{P}(X_0 \cup X_1)$ as in the end of the previous section and force with the poset Q_K. Since K is closed under J-integration, the forcing preserves $\aleph_1$. It also adds sets $\{f_n : n \in \omega\} \subset X_1$ and $\{g_n : n \in \omega\} \subset X_0$ with the required properties, showing that in the generic extension, $\delta^1_2 > \lambda$.

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