Another c.c.c. forcing that destroys presaturation

Paul B. Larson   Teruyuki Yorioka

January 10, 2010

An ideal I on ω_1 is said to be *strong* if I is precipitous and in any forcing extension by $\mathcal{P}(\omega_1)/I$, $j(\omega_1{}^V) = \omega_2{}^V$ holds, where j is the corresponding embedding [1, Definition 5.6.], and *presaturated* if for any countable collection $\mathcal{D}$ of predense subsets of $\mathcal{P}(\omega_1)/I$ there are densely many $A \in I^+$ such that for each $D \in \mathcal{D}$, A intersects at most $\aleph_1$ many members of D positively. It is a standard fact, and not hard to see, that if I is presaturated, then the forcing $\mathcal{P}(\omega_1)/I$ does not collapse ω_2 [1, §4]. In their study of ideals on ω_1, Baumgarter and Taylor [1, Theorem 5.7.] show that strong ideals are preserved by c.c.c. forcing, and mention that they do not know whether the same holds for presaturated ideals. They also mention that they do not know whether these two notions are equivalent. These questions were answered by Veličković [5, Theorem 4.6.], who showed (from the consistency of ZFC+SPFA) that consistently the nonstationary ideal on ω_1 (NS_{ω_1}) is saturated and there is a c.c.c. forcing which adds a Kurepa tree, and thus destroys the presaturation of NS_{ω_1}. Here we show that the main argument from [4] gives another example.

Theorem 0.1. *If ZF is consistent with the Axiom of Determinacy, then it is consistent that NS_{ω_1} is saturated and there exists a Suslin tree S such that the forcing to specialize S with finite conditions destroys the presaturation of NS_{ω_1}.*

By a theorem of Baumgartner (see [2, Theorem 3]), the forcing to specialize S (i.e., to cover it with countably many antichains) with finite conditions is c.c.c. in this model, as S has no cofinal branches.

Proof of Theorem 0.1. In [4], a model is constructed (assuming ZF + AD) in which NS_{ω_1} is saturated and there is a Suslin tree S such that, when one forces with $\mathcal{P}(\omega_1)/NS_{\omega_1}$ and takes the corresponding generic embedding $j\colon V \to M$, $j(S)$ contains a cofinal branch in $V[G]$ (though of course not in M, where it is a Suslin tree). This means that in this model there are functions $f_\alpha\colon \omega_1 \to S$ ($\alpha < \omega_2$) such that for all $\alpha < \beta < \omega_2$, the set $\{\gamma < \omega_1 \mid f_\alpha(\gamma) <_S f_\beta(\gamma)\}$ contains a club. Let $V[H]$ be an extension of V by this specializing forcing for S, and consider a $V[H]$-generic filter G for $\mathcal{P}(\omega_1)/NS_{\omega_1}$ with corresponding embedding $j\colon V[H] \to M$. Then $\langle j(f_\alpha)(\omega_1{}^V) : \alpha < \omega_2{}^V\rangle$ forms a (possibly non-cofinal) path through $j(S)$, which is special in M. Therefore $\omega_2{}^V$ is countable in $V[H][G]$ (which means that NS_{ω_1} is not presaturated in $V[H]$). □

References

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Department of Mathematics
Miami University
Oxford, Ohio 45056
USA
larsonpb@muohio.edu

Department of Mathematics
Faculty of Science, Shizuoka University
Ohya 836, Shizuoka, 422-8529
JAPAN
styorio@ipc.shizuoka.ac.jp