§1. Introduction. Determinacy axioms are statements to the effect that certain games are determined, in that each player in the game has an optimal strategy. The commonly accepted axioms for mathematics, the Zermelo-Fraenkel axioms with the Axiom of Choice (ZFC; see [??, ??]), imply the determinacy of many games that people actually play. This applies in particular to many games of perfect information, games in which the players alternate moves which are known to both players, and the outcome of the game depends only on this list of moves, and not on chance or other external factors. Games of perfect information which must end in finitely many moves are determined. This follows from the work of Ernst Zermelo [??], Dénes König [??] and László Kálmár [??], and also from the independent work of John von Neumann and Oskar Morgenstern (in their 1944 book, reprinted as [??]).

As pointed out by Stanislaw Ulam [??], determinacy for games of perfect information of a fixed finite length is essentially a theorem of logic. If we let $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ be variables standing for the moves made by players player I (who plays $x_1, \ldots, x_n$) and player II (who plays $y_1, \ldots, y_n$), and $A$ (consisting of sequences of length $2n$) is the set of runs of the game for which player I wins, the statement

$$\exists x_1 \forall y_1 \ldots \exists x_n \forall y_n (x_1, y_1, \ldots, x_n, y_n) \in A$$

essentially asserts that the first player has a winning strategy in the game, and its negation,

$$\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n (x_1, y_1, \ldots, x_n, y_n) \notin A$$

essentially asserts that the second player has a winning strategy.\(^1\)

We let $\omega$ denote the set of natural numbers $0, 1, 2, \ldots$; for brevity we will often refer to the members of this set as “integers”. Given sets $X$ and $Y$,
denotes the set of functions from $X$ to $Y$. The Baire space is the space $\omega^\omega$, with the product topology. The Baire space is homeomorphic to the space of irrational real numbers (see [??, p. 9], for instance), and we will often refer to its members as “reals” (though in various contexts the Cantor space $\omega^2$, the set of subsets of $\omega$ ($\mathcal{P}(\omega)$) and the set of infinite subsets of $\omega$ ($[\omega]^\omega$) are all referred to as “the reals”).

Given $A \subseteq \omega^\omega$, we let $G_\omega(A)$ denote the game of perfect information of length $\omega$ in which the two players collaborate to define an element $f$ of $\omega^\omega$ (with player I choosing $f(0)$, player II choosing $f(1)$, player I choosing $f(2)$, and so on), with player I winning a run of the game if and only if $f$ is an element of $A$. A game of this type is called an integer game, and the set $A$ is called the payoff set. A strategy in such a game for player player I (player II) is a function $\Sigma$ with domain the set of sequences of integers of even (odd) length such that for each $a \in \text{dom}(\Sigma)$, $\Sigma(a)$ is in $\omega$. A run of the game (partial or complete) is said to be according to a strategy $\Sigma$ for player player I (player II) if every initial segment of the run of odd (nonzero even) length is of the form $a^\frown (\Sigma(a))$ for some sequence $a$. A strategy $\Sigma$ for player player I (player II) is a winning strategy if every complete run of the game according to $\Sigma$ is in (out of) $A$. We say that a set $A \subseteq \omega^\omega$ is determined (or the corresponding game $G_\omega(A)$ is determined) if there exists a winning strategy for one of the players. These notions generalize naturally for games in which players play objects other than integers (for instance, real games, in which they play elements of $\omega^\omega$) or games which run for more than $\omega$ many rounds (in which case player player I typically plays at limit stages).

The study of determinacy axioms concerns games whose determinacy is neither proved nor refuted by the Zermelo-Fraenkel axioms ZF (without the Axiom of Choice). Typically such games are infinite. Axioms stating that infinite games of various types are determined were studied by Stanislaw Mazur, Stefan Banach and Ulam in the late 1920’s and early 1930’s; were reintroduced by David Gale and Frank Stewart [??] in the 1950’s and again by Jan Mycielski and Hugo Steinhaus [??] in the early 1960’s; gained interest with the work of David Blackwell [??] and Robert Solovay in the late 1960’s; and attained increasing importance in the 1970’s and 1980’s, finally coming to a central position in contemporary set theory.

Mycielski and Steinhaus introduced the Axiom of Determinacy (AD), which asserts the determinacy of $G_\omega(A)$ for all $A \subseteq \omega^\omega$. Work of Banach in the 1930’s shows that AD implies that all sets of reals satisfy the property of Baire. In the 1960’s, Mycielski and Stanislaw Świerczkowski proved that AD implies that all sets of reals are Lebesgue measurable, and Mycielski showed that AD implies countable choice for reals. Together, these results
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show that determinacy provides a natural context for certain areas of mathematics, notably analysis, free of the paradoxes induced by the Axiom of Choice.

Unaware of the work of Banach, Gale and Stewart [??] had shown that the AD contradicts ZFC. However, their proof used a wellordering of the reals given by the Axiom of Choice, and therefore did not give a nondetermined game of this type with definable payoff set. Starting with Banach’s work, many simply definable payoff sets were shown to induce determined games, culminating in D. Anthony Martin’s celebrated 1974 result [??] that all games with Borel payoff set are determined. This result came after Martin had used measurable cardinals to prove the determinacy of games whose payoff set is an analytic sets of reals.

The study of determinacy gained interest from two theorems in 1967, the first due to Solovay and the second to Blackwell. Solovay proved that under AD, the first uncountable cardinal \( \omega_1 \) is a measurable cardinal, setting off a study of strong Ramsey properties on the ordinals implied by determinacy axioms. Blackwell used open determinacy (proved by Gale and Stewart) to reprove a classical theorem of Kazimierz Kuratowski. This also led to the application, by John Addison, Martin, Yiannis Moschovakis and others, of stronger determinacy axioms to produce structural properties for definable sets of reals. These axioms included the determinacy of \( \Delta^e_1 \) sets of reals, for \( n \geq 2 \), statements which would not be proved consistent relative to large cardinals until the 1980’s.

The large cardinal hierarchy was developed over the same period, and came to be seen as a method for calibrating consistency strength. In the 1970’s, various special cases of \( \Delta^e_1 \) determinacy were located on this scale, in terms of the large cardinals needed to prove them. Determining the consistency (relative to large cardinals) of forms of determinacy at the level of \( \Delta^e_1 \) and beyond would take the introduction of new large cardinal concepts. Martin (in 1978) and W. Hugh Woodin (in 1984) would prove \( \Pi^e_1 \)-determinacy and \( \text{AD}^{L(R)} \) respectively, using hypotheses near the very top of the large cardinal hierarchy. In a dramatic development, the hypotheses for these results would be significantly reduced through work of Woodin, Martin and John Steel. The initial impetus for this development was a seminal result of Matthew Foreman, Menachem Magidor and Saharon Shelah which showed, assuming the existence of a supercompact cardinal, that there exists a generic elementary embedding with well-founded range and critical point \( \omega_1 \). Combined with work of Woodin, this yielded the Lebesgue measurability of all sets in the inner model \( L(R) \) from this hypothesis. Shelah and Woodin would reduce the hypothesis for this result further, to the assumption that there exist infinitely many Woodin cardinals below a measurable cardinal.
Woodin cardinals would turn out to be the central large cardinal concept for the study of determinacy. Through the study of tree representations for sets of reals, Martin and Steel would show that $\Pi^1_{n+1}$-determinacy follows from the existence of $n$ Woodin cardinals below a measurable cardinal, and that this hypothesis was not sufficient to prove stronger determinacy results for the projective hierarchy. Woodin would then show that the existence of infinitely many Woodin cardinals below a measurable cardinal implies $\text{AD}^L(\mathbb{R})$, and he would locate the exact consistency strengths of $\Delta^1_2$-determinacy and $\text{AD}^L(\mathbb{R})$ at one Woodin cardinal and $\omega$ Woodin cardinals respectively.

In the aftermath of these results, many new directions were developed, and we give only the briefest indication here. Using techniques from inner model theory, tight bounds were given for establishing the exact consistency strength of many determinacy hypotheses. Using similar techniques, it has been shown that almost every natural statement (i.e., not invented specifically to be a counterexample) implies directly those determinacy hypotheses of lesser consistency strength. For instance, by Gödel’s Second Incompleteness Theorem, $\text{ZFC}$ cannot prove that the $\text{AD}$ holds in $L(\mathbb{R})$, as the latter implies the consistency of the former. Empirically, however, every natural extension of $\text{ZFC}$ without this limitation (i.e., not proved consistent by $\text{AD}$) does appear to imply that $\text{AD}$ holds in $L(\mathbb{R})$. This sort of phenomenon is taken by some as evidence that the statement that $\text{AD}$ holds in $L(\mathbb{R})$, and other determinacy axioms, should be counted among the true statements extending $\text{ZFC}$ (see [??], for instance).

The history presented here relies heavily on those given by Jackson [??], Kanamori [??, ??], Moschovakis [??], Neeman [??] and Steel [??]. As the title suggests, this is a selective and abbreviated account of the history of determinacy. We have omitted many interesting topics, including, for instance, Blackwell games [??, ??, ??] and proving determinacy in second-order arithmetic [??, ??, ??].

§2. Early developments. The first published paper in mathematical game theory appears to be Zermelo’s paper [??] on chess. Although he noted that his arguments apply to all games of reason not involving chance, Zermelo worked under two additional chess-specific assumptions. The first was that the game in question has only finitely many states, and the second was that an infinite run of the game was to be considered a draw. Zermelo specified a condition which is equivalent to having a winning strategy in such a game guaranteeing a win within a fixed number of moves, as well as another condition equivalent to having a strategy guaranteeing that one will not lose within a given fixed number of moves. His analysis implicitly
introduced the notions of game tree, subtree of a game tree, and quasi-strategy.\(^2\)

The paper states indirectly, but does not quite prove, or even define, the statement that in any game of perfect information with finitely many possible positions such that infinite runs of the game are draws, either one player has a strategy that guarantees a win, or both players have strategies that guarantee at least a draw. A special case of this fact is determinacy for games of perfect information of a fixed finite length, which is sometimes called Zermelo’s Theorem.

König \(^3\) applied the fundamental fact now known as König’s Lemma to the study of games, among other topics. While König’s formulation was somewhat different, his Lemma is equivalent to the assertion that every infinite finitely branching tree with a single root has an infinite path (a path can be found by iteratively choosing any successor node such that the tree above that node is infinite). Extending Zermelo’s analysis to games in which infinitely many positions are possible while retaining the condition that each player has only finitely many options at each point, König used the statement above to prove that in such a game, if one player has a strategy (from a given point in the game) guaranteeing a win, then he can guarantee victory within a fixed number of moves. The application of König’s Lemma to the study of games was suggested by von Neumann.

Kálmar \(^4\) took the analysis a step further by proving Zermelo’s Theorem for games with infinitely many possible moves in each round. His arguments proceeded by assigning transfinite ordinals to nodes in the game tree, a method which remains an important tool in modern set theory. Kálmar explicitly introduced the notion of a winning strategy for a game, though his strategies were also quasi-strategies as above. In his analysis, Kálmar introduced a number of other important technical notions, including the notion of a subgame (essentially a subtree of the original game tree), and classifying strategies into those which depend only on the current position in the game and those which use the history of the game so far.\(^3\)

Games of perfect information for which the set of infinite runs is divided into winning sets for each player appear in a question by Mazur in the Scottish Book, answered by Banach in an entry dated August 4, 1935 (see \(^5\), p. 113)). Following up on Mazur’s question (still in the Book), Ulam asked about games where two players collaborate to build an infinite sequence of 0’s and 1’s by alternately deciding each member of the sequence, with the winner determined by whether the infinite sequence constructed

\(^2\)As defined above, a strategy for a given player specifies a move in each relevant position; a quasi-strategy merely specifies a set of acceptable moves. The distinction is important when the Axiom of Choice fails, but is less important in the context of Zermelo’s paper.

\(^3\)See \(^4\) for much more on these papers of Zermelo, König and Kálmar.
falls inside some predetermined set \( E \). Essentially raising the issue of determinacy for arbitrary \( G_\omega(E) \), Ulam asked: for which sets \( E \) does the first player (alternately, the second player) have a winning strategy? (Section 2.1 below has more on the Banach-Mazur game.)

Games of perfect information were formally defined in 1944 by von Neumann and Morgenstern \([??]\). Their book also contains a proof that games of perfect information of a fixed finite length are determined (page 123).

Infinite games of perfect information were reintroduced by Gale and Stewart \([??]\), who were unaware of the work of Mazur, Banach and Ulam (Gale, personal communication). They showed that a nondetermined game can be constructed using the Axiom of Choice (more specifically, from a wellordering of the set of real numbers).\(^4\) They also noted that the proof from the Axiom of Choice does not give a definable undetermined game, and raised the issue of whether determinacy might hold for all games with a suitably definable payoff set. Towards this end, they introduced a topological classification of infinite games of perfect information, defining a game (or the set of runs of the game which are winning for the first player) to be **open** if all winning runs for the first player are won at some finite stage (i.e., if, whenever \( (x_0, x_1, x_2, \ldots) \) is a winning run of the game for the first player, there is some \( n \) such that the first player wins all runs of the game extending \( (x_0, \ldots, x_n) \)). Using this framework, they proved a number of fundamental facts, including the determinacy of all games whose payoff set is a Boolean combination of open sets (i.e., in the class generated from the open sets by the operations of finite union, finite intersection and complementation). The determinacy of open games would become the basis for proofs of many of the strongest determinacy hypotheses. Gale and Stewart also asked a number of important questions, including the question of whether all Borel games are determined (to be answered positively by Martin \([??]\) in 1974).\(^5\) Classifying games by the definability of their payoff sets would be an essential tool in the study of determinacy.

### 2.1. Regularity properties.

Early motivation for the study of determinacy was given by its implications for regularity properties for sets of

\(^4\)Given a set \( Y \), we let \( \mathcal{A}_Y \) denote the statement that whenever \( \{X_a : a \in Y\} \) is a collection of nonempty sets, there is a function \( f \) with domain \( Y \) such that \( f(a) \in X_a \) for all \( a \in Y \). Zermelo’s **Axiom of Choice** (\( \mathcal{A}_C \)) \([??]\) is equivalent to the statement that \( \mathcal{A}_Y \) holds for all sets \( Y \). A linear ordering \( \leq \) of a set \( X \) is a **wellordering** if every nonempty subset of \( X \) has a \( \leq \)-least element. The Axiom of Choice is equivalent to the statement that there exist wellorderings of every set.

König’s Lemma is a weak form of the Axiom of Choice and cannot be proved in \( ZF \) (see \([??]\), Exercise IX.2.18).

\(^5\)The **Borel** sets are the members of the smallest class containing the open sets and closed under the operations of complementation and countable union. The collection of Borel sets is generated in \( \omega_1 \) many stages from these two operations. A natural process assigns a measure to each Borel set (see, for instance, \([??])\).
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reals. In particular, determinacy of certain games of perfect information was shown to imply that every set of reals has the property of Baire and the perfect set property, and is Lebesgue measurable. These three facts themselves each contradict the Axiom of Choice. We will refer to Lebesgue measurability, the property of Baire and the perfect set property as the regularity properties, the fact that there are other regularity properties notwithstanding.

Question 43 of the Scottish Book, posed by Mazur, asks about games where two players alternately select the members of a shrinking sequence of intervals of real numbers, with the first player the winner if the intersection of the sequence intersects a set given in advance. Banach posted an answer in 1935, showing that such games are determined if and only if the given set is either meager (in which case the second player wins) or comeager relative to some interval (in which case the first player wins), i.e., if and only if the given set has the Baire property (see [??, pp. 27-30], [??, pp. 373-374]). This game has come to be known as the Banach-Mazur game. Using an enumeration of the rationals, one can code intervals with rational endpoints with integers, getting a game on integers.

Morton Davis [??] studied a game, suggested by Dubins, where the first player plays arbitrarily long finite strings of 0’s and 1’s and the second player plays individual 0’s and 1’s, with the payoff set a subset of the set of infinite binary sequences as before. Davis proved that the first player has a winning strategy in such a game if and only if the payoff set contains a perfect set, and the second player has a winning strategy if and only if the payoff set is finite or countably infinite. The determinacy of all such games then implies that every uncountable set of reals contains a perfect set (asymmetric games of this type can be coded by integer games of perfect information). It follows that under $\mathsf{AD}$ there is no set of reals whose cardinality falls strictly between $\aleph_0$ and $2^{\aleph_0}$.

Mycielski and Świerczkowski [??] showed that the determinacy of certain integer games of perfect information implies that every subset of the real line is Lebesgue measurable. Simpler proofs of this fact were later given by Leo Harrington (see [??, pp. 375-377]) and Martin [??].

By way of contrast, an argument of Vitali [??] shows that under $\mathsf{ZFC}$ there are sets of reals which are not Lebesgue measurable. Banach and

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6 A set of reals $X$ has the property of Baire if $X \triangle O$ is meager for some open set $O$, where the symmetric difference $A \triangle B$ of two sets $A$ and $B$ is the set $(A \setminus B) \cup (B \setminus A)$, where $A \setminus B = \{x \in A : x \not\in B\}$. A set of reals $X$ has the perfect set property if it is countable or contains a perfect set (an uncountable closed set without isolated points). A set of reals $X$ is Lebesgue measurable if there is a Borel set $B$ such that $X \triangle B$ is a subset of a Borel measure 0 set. See [??].

7 i.e., for every set $X$, if there exist injections $f : \omega \to X$ and $g : X \to 2^\omega$, then either $X$ is countable or there exists a bijection between $X$ and $2^\omega$. 

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Tarski ([??], see also [??]), building on work of Hausdorff [??], showed that under ZFC the unit ball can be partitioned into five pieces which can be rearranged to make two copies of the same sphere, again violating Lebesgue measurability as well as physical intuition. As with the undetermined game given by Gale and Stewart, the constructions of Vitali and Banach-Tarski use the Axiom of Choice and do not give definable examples of nonmeasurable sets. Via the Mycielski-Świątekowski theorem, determinacy results would rule out the existence of definable examples, for various notions of definability.

2.2. Definability. As discussed above, ZFC implies that open sets are determined, and implies also that there exists a nondetermined set. The study of determinacy was to merge naturally with the study of sets of reals in terms of their definability (i.e., descriptive set theory), which can be taken as a measure of their complexity. In this section we briefly introduce some important definability classes for sets of reals. Standard references include [??, ??]. While we do mention some important results in this section, much of the section can be skipped on a first reading and used for later reference.

A Polish space is a topological space which is separable and completely metrizable. Common examples include the integers $\omega$, the reals $\mathbb{R}$, the open interval $(0, 1)$, the Baire space $\omega^\omega$, the Cantor space $\omega^2$ and their finite and countable products. Uncountable Polish spaces without isolated points are a natural setting for studying definable sets of reals. For the most part we will concentrate on the Baire space and its finite powers.

Following notation introduced by Addison [??], open subsets of a Polish space are called $\Sigma^0_1$, complements of $\Sigma^0_n$ sets are $\Pi^0_n$, and countable unions of $\Pi^0_n$ sets are $\Sigma^0_{n+1}$. More generally, given a positive $\alpha < \omega_1$, $\Sigma^0_\alpha$ consists of all countable unions of members of $\bigcup_{\beta<\alpha} \Pi^0_\beta$, and $\Pi^0_\alpha$ consists of all complements of members of $\Sigma^0_\alpha$. The Borel sets are the members of $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha$.

A pointclass is a collection of subsets of Polish spaces. Given a pointclass $\Gamma \subseteq \mathcal{P}(\omega^\omega)$, we let Det($\Gamma$) and $\Gamma$-determinacy each denote the statement that $G_\alpha(A)$ is determined for all $A \in \Gamma$. Philip Wolfe [??] proved $\Sigma^0_2$-determinacy in ZFC. Davis [??] followed by proving $\Pi^0_3$-determinacy. Jeffrey Paris [??] would prove $\Sigma^0_4$-determinacy. However, this result was proved after Martin had used a measurable cardinal to prove analytic determinacy (see Section 5.2).

Continuous images of $\Pi^0_i$ sets are said to be $\Sigma^1_i$, complements of $\Sigma^0_n$ sets are $\Pi^1_n$, and continuous images of $\Pi^0_1$ sets are $\Sigma^1_{n+1}$. For each $i \in \{0, 1\}$ and

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8The papers [??] and [??] appear in the same volume of Fundamenta Mathematicae. The front page of the volume gives the date 1958-1959. The individual papers have the dates 1958 and 1959 on them, respectively.
finite sequence $s\in\omega$, the pointclass $\Delta_n^1$ is the intersection of $\Sigma_n^1$ and $\Pi_n^1$. The boldface projective pointclasses are the sets $\Sigma_n^1$, $\Pi_n^1$ and $\Delta_n^1$ for positive $n\in\omega$. These classes were implicit in work of Lebesgue as early as [??]. They were made explicit in independent work by Nikolai Luzin [??, ??, ??] and Wacław Sierpiński [??]. The notion of a boldface pointclass in general (i.e., possibly non-projective) is used in various ways in the literature. We will say that a pointclass $\Gamma$ is boldface (or closed under continuous preimages or continuously closed) if $f^{-1}[A] \in \Gamma$ for all $A \in \Gamma$ and all continuous functions $f$ between Polish spaces (where $A$ is a subset of the codomain). The classes $\Sigma_0^0$, $\Pi_0^0$, $\Delta_0^0$ are also boldface in this sense.

The pointclass $\Sigma_1^1$ is also known as the class of analytic sets, and was given an independent characterization by Mikhail Suslin [??]: a set of reals $A$ is analytic if and only if there exists a family of closed sets $D_n$ (for each finite sequence $s$ consisting of integers) such that $A$ is the set of reals $x$ for which there is an $\omega$-sequence $S$ of integers such that $x \in \bigcap_{n<\omega} D_{s(n)}$. Suslin showed that there exist non-Borel analytic sets, and that the Borel sets are exactly the $\Delta_1^1$ sets.

We let $\exists^0$ and $\exists^1$ denote existential quantification over the integers and reals, respectively, and $\forall^0$ and $\forall^1$ the analogous forms of universal quantification. Given a set $A \subseteq (\omega)^{k+1}$, for some positive integer $k$, $\exists^1 A$ is the set of $(x_1, \ldots, x_k) \in (\omega)^k$ such that for some $x \in (\omega)$, $(x, x_1, \ldots, x_k) \in A$, and $\forall^1 A$ is the set of $(x_1, \ldots, x_k) \in (\omega)^k$ such that for all $x \in (\omega)$, $(x, x_1, \ldots, x_k) \in A$. Given a pointclass $\Gamma$, $\exists^1 \Gamma$ consists of $\exists^1 A$ for all $A \in \Gamma$, and $\forall^1 \Gamma$ consists of $\forall^1 A$ for all $A \in \Gamma$. It follows easily that for each positive integer $n$, $\exists^1 \Pi_n^1 = \Sigma_{n+1}^0$ and $\forall^1 \Sigma_n^1 = \Pi_{n+1}^1$.

Given a pointclass $\Gamma$, $\overline{\Gamma}$ is the set of complements of members of $\Gamma$, and $\Delta_\Gamma$ is the pointclass $\Gamma \cap \overline{\Gamma}$; $\Gamma$ is said to be selfdual if $\Delta_\Gamma = \Gamma$. A set $A \in \Gamma$ is said to be $\Gamma$-complete if every member of $\Gamma$ is a continuous preimage of $A$. If $\Gamma$ is closed under continuous preimages and $\Gamma$-determinacy holds, then $\overline{\Gamma}$-determinacy holds. Each of the regularity properties for a set of reals $A$ are given by the determinacy of games with payoff set simply definable from $A$ (indeed, continuous preimages of $A$), but not necessarily with payoff $A$ itself. It follows that when $\Gamma$ is a boldface pointclass, $\Gamma$-determinacy implies the regularity properties for sets of reals in $\Gamma$.

A simple application of Fubini’s theorem shows that if $\Gamma$ is a boldface pointclass and there exists in $\Gamma$ a wellordering of a set of reals of positive Lebesgue measure, then there is a non-Lebesgue measurable set in $\Gamma$. Skipping ahead for a moment, in the early 1970’s Alexander Kechris and Martin, using a technique of Solovay called unfolding, proved that for

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For $S$ a function with domain $\omega$, and $n \in \omega$, $S[n] = \{S(0), \ldots, S(n-1)\}$. 
each integer \( n \), the \( \Pi_{n+1}^{1} \)-determinacy plus countable choice for sets of reals implies that all \( \Sigma_{n+1}^{1} \) sets of reals are Lebesgue measurable, have the Baire property and have the perfect set property (see [??, pp. 380-381]).

As developed by Stephen Kleene, the effective (or lightface) pointclasses \( \Sigma_{n}^{0}, \Pi_{n}^{0}, \Delta_{n}^{0} \) and \( \Sigma_{n}^{1}, \Pi_{n}^{1}, \Delta_{n}^{1} \) are formed in the same way as their boldface counterparts, starting instead from \( \Sigma_{1}^{1} \), the collection of open sets \( O \) such that the set of indices for basic open sets contained in \( O \) (under a certain natural enumeration of the basic open sets) is recursive (see [??], for instance). Sets in \( \Sigma_{1}^{0} \) are called semirecursive, and sets in \( \Delta_{1}^{0} \) are called recursive. Given \( a \in \omega \), \( \Sigma_{1}^{0}(a) \) is the collection of open sets \( O \) such that the set of indices for basic open sets contained in \( O \) is recursive in \( a \), and the relativized lightface projective pointclasses \( \Sigma_{n}^{0}(a), \Pi_{n}^{0}(a), \Delta_{n}^{0}(a), \Sigma_{n}^{1}(a), \Pi_{n}^{1}(a), \Delta_{n}^{1}(a) \) are built from \( \Sigma_{1}^{n} \) in the manner above. It follows that each boldface pointclass is the union of the corresponding relativized lightface classes (relativizing over each member of \( \omega \)).

Following [??], a pointclass is adequate if it contains all recursive sets and is closed under finite unions and intersections, bounded universal and existential integer quantification (see [??, p. 119]) and preimages by recursive functions. The relativized lightface projective pointclasses are adequate (see [??, pp. 118-120]).

Given a Polish space \( X \), an integer \( k \), a set \( A \subseteq X^{k+1} \) and \( x \in X \), \( A_{x} \) is the set of \((x_{1}, \ldots, x_{k})\) such that \((x, x_{1}, \ldots, x_{k}) \in A \). A set \( A \subseteq X^{k+1} \) in a pointclass \( \Gamma \) is said to be universal for \( \Gamma \) if each subset of \( X^{k} \) in \( \Gamma \) has the form \( A_{x} \) for some \( x \in X \). Pointclasses of the form \( \Sigma_{n}^{0}, \Pi_{n}^{1} \) have universal members. Those of the form \( \Delta_{n}^{0} \) do not. Each member of each boldface pointclass is of the form \( A_{x} \) for \( A \) a member of the corresponding effective class. Conversely, as each member of each lightface projective pointclass listed above is definable, each member of each corresponding boldface pointclass is definable from a real number as a parameter.

A set of reals is said to be \( \Sigma_{1}^{0} \) (\( \Pi_{1}^{0} \)) if is definable by a formula of the form \( \exists X \subseteq \mathbb{R} \varphi (\forall X \subseteq \mathbb{R} \varphi) \), where all quantifiers in \( \varphi \) range over the reals or the integers.

In the Lévy hierarchy [??], a formula \( \varphi \) in the language of set theory is \( \Delta_{0} \) (equivalently \( \Sigma_{0}, \Pi_{0} \)) if all quantifiers appearing in \( \varphi \) are bounded (see [??, Chapter 13]); \( \Sigma_{n+1} \) if it has the form \( \exists x \psi \) for some \( \Pi_{n} \) formula \( \psi \); and \( \Pi_{n+1} \) if it has the form \( \forall x \psi \) for some \( \Sigma_{n} \) formula \( \psi \). A set is \( \Sigma_{n} \)-definable

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10The statement that whenever \( X_{n} \) (\( n \in \omega \)) are nonempty sets of reals, there is a function \( f : \omega \to \mathbb{R} \) such that \( f(n) \in X_{n} \) for each \( n \). Countable choice for sets of reals is a consequence of \( \AD \), as shown by Mycielski [??] (see Section 2.3).

11A function \( f \) from a Polish space \( X \) to a Polish space \( Y \) is said to be recursive if the set of pairs \( x \in X, n \in \omega \) such that \( f(x) \) is in the \( n \)-th basic open neighborhood of \( Y \) is semi-recursive.
A BRIEF HISTORY OF DETERMINACY

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if it can be defined by a $\Sigma^n$ formula (and similarly for $\Pi^n$). We say that a model $M$ is $\Gamma$-correct, for a class of formulas $\Gamma$, if for all $\varphi$ in $\Gamma$ and $x \in M$, $M \models \varphi(x)$ if and only if $V \models \varphi(x)$. If $M$ is a model of ZF, we say that a set in $M$ is $\Sigma^M_n$ if it is definable by a $\Sigma_n$ formula relativized to $M$ (and similarly for other classes of formulas).

Gödel’s inner model $L$ is the smallest transitive model of ZFC containing the ordinals. For any set $A$, Gödel’s constructible universe $L$ generalizes to two inner models $L(A)$ and $L[A]$, developed respectively by András Hajnal [[??],??] and Azriel Lévy [[??],??] (see [[??], Chapter 13] or [[??], p. 34]). Given a set $A$, $L(A)$ is the smallest transitive model of ZF containing the transitive closure of $\{A\}$ and the ordinals, and $L[A]$ is the smallest transitive model of ZF containing the ordinals and closed under the function $X \mapsto A \cap X$. Alternately, $L(A)$ is constructed in the same manner as $L$, but introducing the members of the transitive closure of the set $\{A\}$ at the first level, and $L[A]$ is constructed as $L$, but by adding a predicate for membership in $A$ to the language. When $A$ is contained in $L$, $L(A)$ and $L[A]$ are the same. While $L[A]$ is always a model of AC, $L(A)$ need not be. Indeed, $L(\mathbb{R})$ is a model of AD in the presence of suitably large cardinals, and is thus a natural example of a “smaller universum” as described in the quote from [[??]] in Section 2.3.

Though it can be formulated in other ways, we will view the set $0^\#$ ("zero sharp") as the theory of a certain class of ordinals which are indiscernibles over the inner model $L$. This notion was independently isolated by Solovay [[??]] and by Jack Silver in his 1966 Berkeley Ph.D. thesis (see [[??]]). The existence of $0^\#$ cannot be proved in ZFC, as it serves as a sort of transcendence principle over $L$. For instance, if $0^\#$ exists then every uncountable cardinal of $V$ is a strongly inaccessible cardinal in $L$. 13 For any set $X$ there is an analogous notion of $X^\#$ ("$X$ sharp") serving as a transcendence principle over $L(X)$ (see [[??]]).

2.3. The Axiom of Determinacy. The Axiom of Determinacy, the statement that all length $\omega$ integer games of perfect information are determined, was proposed by Mycielski and Steinhaus [[??]]. In a passage that anticipated a commonly accepted view of determinacy, they wrote

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12A set $x$ is transitive if $z \in x$ whenever $y \in x$ and $z \in y$. The transitive closure of a set $x$ is the smallest transitive set containing $x$.

13A cardinal $\kappa$ is strongly inaccessible if it is uncountable, regular and a strong limit (i.e., $2^\gamma < \kappa$ for all $\gamma < \kappa$). If $\kappa$ is a strongly inaccessible cardinal, then $V_\kappa$ is a model of ZFC. Hence, by Gödel’s Second Incompleteness Theorem, the existence of strongly inaccessible cardinals cannot be proved in ZFC. See [[??]] for the definition of $V_\alpha$, for an ordinal $\alpha$.

14We continue to use the now-standard abbreviation AD for the Axiom of Determinacy; it was called (A) in [[??]].
It is not the purpose of this paper to depreciate the classical mathematics with its fundamental “absolute” intuitions on the universum of sets (to which belongs the axiom of choice), but only to propose another theory which seems very interesting although its consistency is problematic. Our axiom can be considered as a restriction of the classical notion of a set leading to a smaller universum, say of determined sets, which reflect some physical intuitions which are not fulfilled by the classical sets. Our axiom could be considered as an axiom added to the classical set theory claiming the existence of a class of sets satisfying (A) and the classical axioms (without the axiom of choice).

Mycielski and Steinhaus summarized the state of knowledge of determinacy at that time, including the fact that AD implies that all sets of reals are Lebesgue measurable and have the Baire property, and they noted that by results of Kurt Gödel and Addison, there is in Gödel’s constructible universe \( L \) (and thus consistently with ZFC) a \( \Delta^1_2 \) wellordering of the reals, and thus a \( \Delta^1_2 \) set which is not determined.

In his work, Mycielski proved several fundamental facts about determinacy, including the fact that \( AD \) implies countable choice for set of reals (he credits this result to Świarczowski, Dana Scott and himself, independently). Thus, while \( AD \) contradicts the Axiom of Choice, it implies a form of Choice which suffices for many of its most important applications, including the countable additivity of Lebesgue measure. Via countable choice for sets of reals, \( AD \) implies that \( \omega_1 \) is regular. Mycielski also showed that \( AD \) implies that there is no uncountable wellordered sequence of reals. In conjunction with the perfect set property, this implies that under determinacy, \( \omega_1^V \) is a strongly inaccessible cardinal in the inner model \( L \) (and even in \( L[a] \) for any real number \( a \)), a fact which was to be greatly extended by Solovay, Martin and Woodin. Harrington would show that \( \Pi^1_1 \)-determinacy implies that 0# exists, and thus that \( \Pi^1_1 \)-determinacy is not provable in ZFC.

In the same paper, Mycielski showed that ZF implies the existence of an undetermined game of perfect information of length \( \omega_1 \) where the players play countable ordinals instead of integers. An interesting aspect of the proof is that it does not give a specific undetermined game. As a slight variant on Mycielski’s argument, consider the game in which the first player plays a countable ordinal \( \alpha \) (and then makes no other moves for the rest

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15The ordinal \( \omega_1 \) is the first uncountable ordinal. A cardinal \( \kappa \) is regular if, for every \( \gamma < \kappa \), every function \( f: \gamma \rightarrow \kappa \) has range bounded in \( \kappa \). Under ZFC, every successor cardinal is regular. Solomon Feferman and Lévy (see also [17, pp. 153-154]) showed that the singularity of \( \omega_1 \) is consistent with ZF. Moti Gitik showed that it is consistent with ZF (relative to large cardinals) that \( \omega \) is the largest regular cardinal.
of the game) and the second player plays a sequence of integers coding $\alpha$, under some fixed coding of hereditarily countable sets by reals.\(^{16}\) Since the first player cannot have a winning strategy in this game, determinacy for the game implies the existence of an injection from $\omega_1$ into $\mathbb{R}$, which contradicts AD but is certainly by itself consistent with $\mathsf{ZF}$, as it follows from $\mathsf{ZFC}$. Later results of Woodin would show that, assuming the consistency of certain large cardinal hypotheses, $\mathsf{ZFC}$ is consistent with the statement that every integer game of length $\omega_1$ with payoff set definable from real and ordinal parameters is determined (see Section 6.3, and [??, p. 298]).

Mycielski noted that under AD there are no nonprincipal ultrafilters\(^{17}\) on $\omega$ (this follows from Lebesgue measurability for all sets of reals plus a result of Sierpiński [??] showing that nonprincipal ultrafilters on $\omega$ give rise to nonmeasurable sets of reals), which implies that every ultrafilter (on any set) is countably complete (i.e., closed under countable intersections). Finally, in a footnote on the first page of the paper, Mycielski reiterated a point made in the passage quoted above from his paper with Steinhaus, suggesting that an inner model containing the reals could satisfy AD. In a followup paper, Mycielski [??] presented a number of additional results, including the fact that there is a game in which the players play real numbers whose determinacy implies uniformization (see Section 3.2) for subsets of the plane, another weak form of the Axiom of Choice.

In 1964, a year after Paul Cohen’s invention of forcing, Solovay [??] proved that if there exists a strongly inaccessible cardinal, then in a forcing extension there exists an inner model containing the reals in which every set of reals satisfies the regularity properties from Section 2.1. Shelah [??] later showed that a strongly inaccessible cardinal is necessary, in the sense that the Lebesgue measurability of all sets of reals (and even the perfect set property for $\Pi^1_1$ sets) implies that $\omega_1$ is strongly inaccessible in all models of the form $\mathbf{L}[a]$, for $a \subseteq \omega$. In the introduction to his paper, Solovay conjectured (correctly, as it turned out) that large cardinals would imply that AD holds in $\mathbf{L}(\mathbb{R})$.

The year 1967 saw two major results in the study of determinacy, one by Blackwell [??] and the other by Solovay. Reversing chronological order by a few months, we discuss Blackwell’s result and its consequences in the next section, and Solovay’s in Section 4.

\(^{16}\)The **hereditarily countable** sets are those sets whose transitive closures are countable. Such sets are naturally coded by sets of integers.

\(^{17}\)An **ultrafilter** on a nonempty set $X$ is a collection $U$ of nonempty subsets of $X$ which is closed under supersets and finite intersections, and which has the property that for every $A \subseteq X$, exactly one of $A$ and $X \setminus A$ is in $U$. An ultrafilter is **nonprincipal** if it contains no finite sets. The existence of nonprincipal ultrafilters on $\omega$ follows from $\mathsf{ZFC}$, but (as this result shows) requires the Axiom of Choice.
§3. Reduction and scales. Blackwell [??] used open determinacy to reprove a theorem of Kuratowski [?] stating that the intersection of any two analytic sets $A$, $B$ in a Polish space $\mathcal{F}$ is also the intersection of two analytic sets $A'$ and $B'$ such that $A \subseteq A'$, $B \subseteq B'$ and $A' \cup B' = \mathcal{F}$. Briefly, the argument is as follows. Since $A$ and $B$ are analytic, there exist continuous surjections $f: \omega \to A$ and $g: \omega \to B$. For each finite sequence $\langle n_0, \ldots, n_k \rangle$, let $\Omega(\langle n_0, \ldots, n_k \rangle)$ be the set of $x \in \omega$ with $\langle n_0, \ldots, n_k \rangle$ as an initial segment; let $R(\langle n_0, \ldots, n_k \rangle)$ be the closure (in $\mathcal{F}$) of the $f$-image of $\Omega(\langle n_0, \ldots, n_k \rangle)$; and let $S(\langle n_0, \ldots, n_k \rangle)$ be the closure of the $g$-image of $\Omega(\langle n_0, \ldots, n_k \rangle)$. Then for each $z \in \mathcal{F}$, let $G(z)$ be the game where players player I and player II build $x$ and $y$ in $\omega$, with I winning if for some integer $k$, $z \in R(x(k)) \setminus S(y(k))$, player II winning if for some integer $k$, $z \in S(y(k)) \setminus R(x(k+1))$, and the run of the game being a draw if neither of these happens. Roughly, each player is creating a real ($x$ or $y$) to feed into his function, and trying to maintain for as long as possible that the corresponding output can be made arbitrarily close to the target real $z$; the loser is the first player to fail to maintain this condition. Let $A'$ be the set of $z$ for which player I has a strategy guaranteeing at least a draw, and let $B'$ be the set of $z$ for which player player II has such a strategy. Then the determinacy of open games implies that $\omega = A' \cup B'$, and $A \subseteq A'$, $B \subseteq B'$ and $A' \cap B' = A \cap B$ follow from the fact that $A$ is the range of $f$ and $B$ is the range of $g$. The sets $A'$ and $B'$ are analytic, as $A'$ is a projection of the set of pairs $(\varphi, z)$ such that $\varphi$ is (a code for) a strategy for I in $G(z)$ guaranteeing at least a draw, which is Borel, and similarly for $B'$.

3.1. Reduction, separation, norms and prewellorderings. In his [??], Kuratowski defined the reduction theorem (now called the reduction property) for a pointclass $\Gamma$ to be the statement that for any $A, B$ in $\Gamma$ there exist disjoint $A', B'$ in $\Gamma$ with $A' \subseteq A$, $B' \subseteq B$ and $A' \cup B' = A \cup B$. He showed in this paper that $\Pi^1_1$ and $\Sigma^1_2$ have the reduction property; Addison [??] showed this for $\Pi^1_1(a)$ and $\Sigma^1_2(a)$, for each real number $a$. Blackwell’s argument proves the reduction property for $\Pi^1_1$, working with the corresponding $\Sigma^1_1$ complements.

Kuratowski also defined the first separation theorem (now called the separation property) for a pointclass $\Gamma$ to be the statement that for any disjoint $A, B$ in $\Gamma$ there exists $C$ in $\Delta \Gamma$ with $A \subseteq C$ and $B \cap C = \emptyset$. This property had been studied by Sierpiński [??] and Luzin [??] for initial segments of the Borel hierarchy. Kuratowski also noted that the reduction property for a pointclass $\Gamma$ implies the separation property for $\Gamma$. Luzin [??, pp. 51–55] proved that the pointclass $\Sigma^1_1$ satisfies the separation

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18Blackwell describes the discovery of his proof in [??, p. 26].
19A projection of a set $A \subseteq (\omega)^k$ (for some integer $k \geq 2$) is a set of the form $\{(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k-1}) \mid \exists i(x_0, \ldots, x_{k-1}) \in A\}$, for some $i < k$. 
property, by showing that disjoint $\Sigma^1_2$ sets are contained in disjoint Borel sets. Petr Novikov [??] showed that $\Pi^1_2$ satisfies the separation property and $\Sigma^1_2$ does not. Novikov [??] (in the case of $\Sigma^1_2$ sets) and Addison [??] showed that if $\Gamma$ satisfies the reduction property and has a so-called doubly universal member, and $\Delta^1_\Gamma$ has no universal member, then $\Gamma$ does not have the separation property, so $\check{\Gamma}$ does not have the reduction property. Addison [??, ??] showed that if all real numbers are constructible, then the reduction property holds for $\Sigma^1_k$, for all $k \geq 2$.

Inspired by Blackwell’s argument, Addison and Martin independently proved that $\Delta^1_\Gamma$-determinacy implies that $\Pi^1_3$ has the reduction property. Since the pointclass $\Sigma^1_3$ has a doubly universal member, this shows that $\Delta^1_\Gamma$-determinacy implies the existence of a nonconstructible real. This fact also follows from Gödel’s result (discussed in [??]) that the Lebesgue measurability of all $\Delta^1_2$ sets implies the existence of a nonconstructible real. Determinacy would soon be shown to imply stronger structural properties for the projective pointclasses.

The key technical idea behind the (pre-determinacy) results listed above on separation and reduction for the first two levels of the projective hierarchy was the notion of sieve (in French, crible). This construction first appeared in a paper of Lebesgue [??], in which he proved the existence of Lebesgue-measurable sets which are not Borel. In Lebesgue’s presentation, a sieve is an association of a closed subset $F_r$ of the unit interval $[0,1]$ to each rational number $r$ in this interval. The sieve then represents the set of $x \in [0,1]$ such that $\{r \mid x \in F_r\}$ is wellordered, under the usual ordering of the rationals. Using this approach, Luzin and Sierpiński [??, ??] showed that $\Sigma^1_3$ sets and $\Pi^1_3$ sets are unions of $\aleph_1$ many Borel sets.

Much of the classical work of Luzin, Sierpiński, Kuratowski and Novikov mentioned here was redeveloped in the lightface context by Kleene [??, ??, ??, ??], who was unaware of their previous work. The two theories were unified primarily by Addison (for example, [??]). While Blackwell’s argument generalizes throughout the projective hierarchy, Moschovakis [??, ??, ??, ??, ??], see also [??, pp. 202-206]) developed via the effective theory a generalization of the Luzin-Sierpiński approach (decomposing a set of reals into a wellordered sequence of simpler sets) which could be similarly propagated. Moschovakis’s goal was to find a uniform approach to the theory of $\Pi^1_1$ and $\Sigma^1_1$; he was unaware of either Kuratowski’s work or determinacy (personal communication). He extracted the following notions, for a given pointclass $\Gamma$: a $\Gamma$-norm for a set $A$ is a function $\rho: A \rightarrow \mathbb{O}n$ for

20 Members $U, V$ of a pointclass $\Gamma$ are doubly universal for $\Gamma$ if for each pair $A, B$ of members of $\Gamma$ there exist an $x \in \omega$ such that $U_x = A$ and $V_x = B$. The non-selfdual projective pointclasses ($e.g., \Sigma^1_1(a), \Pi^1_1(a), \Sigma^1_2(a), \Pi^1_2(a), \ldots$) all have doubly universal members.
which there exist relations $R^+ \in \Gamma$ and $R^- \in \bar{\Gamma}$ such that for any $y \in A$,
\[ x \in A \land \rho(x) \leq \rho(y) \leftrightarrow R^+(x,y) \leftrightarrow R^-(x,y); \]
a pointclass $\Gamma$ is said to have the **prewellordering property** if every $A \in \Gamma$ has a $\Gamma$-norm. The prewellordering property was first explicitly formulated by Moschovakis in 1964; the definition just given is a reformulation due to Kechris. Kuratowski [??] and Addison [??] had shown that a variant of the property implies the reduction property; the same holds for the prewellordering property as defined by Moschovakis. Moschovakis applied Novikov’s arguments to show that if $\Gamma$ is a projective pointclass such that $\forall \Delta \Gamma \subseteq \Gamma$, and $\Gamma$ has the prewellordering property, then so does the pointclass $\exists \Delta \Gamma$. Martin and Moschovakis independently completed the picture in 1968, proving what is now known as the First Periodicity Theorem.

**Theorem 3.1 (First Periodicity Theorem).** Let $\Gamma$ be an adequate pointclass and suppose that $\Delta_\Gamma$-determinacy holds. Then for all $A \in \Gamma$, if $A$ admits a $\Gamma$-norm, then $\forall \Delta A$ admits a $\forall \exists \exists \Gamma$-norm.

**Corollary 3.2 ([??, ??]).** Let $\Gamma$ be an adequate pointclass closed under existential quantification over reals, and suppose that $\Delta_\Gamma$-determinacy holds. If $\Gamma$ satisfies the prewellordering property, then so does $\forall \Delta \Gamma$.

**Projective Determinacy (PD)** is the statement that all projective sets of reals are determined. By the First Periodicity Theorem, under Projective Determinacy the following pointclasses have the prewellordering property, for any real $a$:

\[ \Pi^1_1(a), \Sigma^1_2(a), \Pi^1_2(a), \Sigma^1_3(a), \Pi^1_3(a), \Sigma^1_4(a), \ldots \]

By contrast (see [??, pp. 409–410]), in $L$ the pointclasses with the prewellordering property are

\[ \Pi^1_1(a), \Sigma^1_2(a), \Sigma^1_3(a), \Sigma^1_4(a), \Sigma^1_5(a), \Sigma^1_6(a), \ldots \]

**3.2. Scales.** As noted above, the Axiom of Determinacy contradicts the Axiom of Choice, but it is consistent with, and even implies, certain weak forms of Choice. If $X$ and $Y$ are nonempty sets and $A$ is a subset of the product $X \times Y$, a function $f$ **uniformizes** $A$ if the domain of $f$ is the set of $x \in X$ such that there exists a $y \in Y$ with $(x,y) \in A$, and such that for each $x$ in the domain of $f$, $(x,f(x)) \in A$. A consequence of the Axiom of

\[^{21}\text{A prewellordering is a binary relation which is wellfounded, transitive and total. A function } \rho \text{ from a set } X \text{ to the ordinals induces a prewellordering } \preceq \text{ on } X \text{ by setting } a \preceq b \text{ if and only if } \rho(a) \leq \rho(b). \text{ Conversely, a prewellordering } \preceq \text{ on a set } X \text{ induces a function } \rho \text{ from } X \text{ to the ordinals, where for each } a \in X, \text{ the } \preceq \text{-rank of } a \text{ is the least ordinal } \alpha \text{ such that } \rho(b) < \alpha \text{ for all } b \in X \text{ such that } b \preceq a \text{ and } a \not\preceq b. \text{ The range of } \rho \text{ is called the length of } \preceq.}\]
Choice, **uniformization** is the statement that for every $A \subseteq \mathbb{R} \times \mathbb{R}$ there is a function $f$ which uniformizes $A$.

Uniformization is not implied by $\text{AD}$, as it fails in $L(\mathbb{R})$ whenever there are no uncountable wellordered sets of reals ([?]); see Section 3.3.

Uniformization was implicitly introduced by Jacques Hadamard [??], when he pointed out that the Axiom of Choice should imply the existence of functions on the reals which disagree everywhere with every algebraic function over the integers. Luzin [??] explicitly introduced the notion of uniformization and showed that such functions exist. He also announced several results on uniformization, including the fact that all Borel sets (but not all $\Sigma^1_1$ sets) can be uniformized by $\Pi^1_1$ functions. The result on Borel sets was proved independently by Sierpiński. Novikov [??] showed that every $\Sigma^1_1$ set of pairs has a $\Sigma^1_1$ uniformization.

A pointclass $\Gamma$ is said to have the **uniformization property** if every set of pairs in $\Gamma$ is uniformized by a function in $\Gamma$. Motokiti Kondo [??] showed that the pointclasses $\Pi^1_1$ and $\Sigma^1_2$ have the uniformization property. The effective version of this result (i.e., for $\Pi^1_1$ and $\Sigma^1_2$) was proved by Addison. In some sense this is as far as one can go in $\text{ZFC}$: Lévy [??] would show that consistently there exist $\Pi^1_2$ sets that cannot be uniformized by any projective function. Remarkably, Luzin [??] has predicted that the question of whether the projective sets are Lebesgue measurable and satisfy the perfect set property would never be solved.

After studying Kondo’s proof, Moschovakis in 1971 isolated a property for sets of reals which induces uniformizations. Given a set $A$ and an ordinal $\gamma$, a **scale** (or a $\gamma$-scale) on $A$ into $\gamma$ is a sequence of functions $\rho_n: A \to \gamma$ ($n \in \omega$) such that whenever

- $\{x_\iota : \iota \in \omega\} \subseteq A$ and $\lim_{\iota \to \omega} x_\iota = x$, and
- the sequence $\langle \rho_n(x_\iota) : \iota \in \omega\rangle$ is eventually constant for each $n \in \omega$,

then $x \in A$ and, for every $n \in \omega$, $\rho_n(x)$ is less than or equal to the eventual value of $\langle \rho_n(x_\iota) : \iota \in \omega\rangle$. The scale is a $\Gamma$-**scale** if there exist $R^+ \in \Gamma$ and $R^- \in \Gamma$ such that for all $y \in A$ and all $n \in \omega$,

$$x \in A \land \rho_n(x) \leq \rho_n(y) \leftrightarrow R^+(n, x, y) \leftrightarrow R^-(n, x, y).$$

A pointclass $\Gamma$ has the **scale property** if every $A$ in $\Gamma$ has a $\Gamma$-scale. Moschovakis [??] proved the following three theorems about the scale property.

**Theorem 3.3.** If $\Gamma$ is an adequate pointclass, $A \in \Gamma$, and $A$ admits a $\Gamma$-scale, then $\exists^1 A$ admits a $\exists^1 \exists^1 \Gamma$-scale.

**Theorem 3.4** (Second Periodicity Theorem). Suppose that $\Gamma$ is an adequate pointclass such that $\Delta_\Gamma$-determinacy holds. Then for all $A \in \Gamma$, if $A$ admits a $\Gamma$-scale, then $\forall^1 A$ admits a $\forall^1 \exists^1 \Gamma$-scale.
Suppose that $\Gamma$ is an adequate pointclass which is closed under integer quantification. Suppose that $\Gamma$ has the scale property, and that $\Delta_\Gamma$-determinacy holds. Then $\Gamma$ has the uniformization property.

Kondo’s proof of uniformization for $\Pi^1_1$ shows that $\Pi^1_1(a)$ has the scale property for every real $a$ (see [??, p. 419]). It follows that under $\Delta_\Gamma$-determinacy, $\Pi^1_{2n+1}$ and $\Sigma^1_{2n+2}$ have the scale property, and every $\Delta^1_{2n+1}$ relation on the reals can be uniformized by a $\Pi^1_{2n+1}$ relation (and similarly for $\Sigma^1_{2n+2}$). Furthermore, under Projective Determinacy, for any real $a$, the projective pointclasses with the scale property are the same as those with the prewellordering property: $\Pi^1_1(a)$, $\Sigma^1_2(a)$, $\Pi^1_3(a)$, $\Pi^1_4(a)$, $\Sigma^1_6(a)$, etc.

A tree on a set $X$ is a collection of finite sequences from $X$ closed under initial segments. Given sets $X$ and $Z$, a positive integer $k$ and a tree $T$ on $X^k \times Z$, the projection of $T$, $p[T]$, is the set of $x \in (X^\omega)^k$ such that for some $z \in Z^\omega$, $(x|n, z|n) \in T$ for all $n \in \omega$ (strictly speaking, this definition involves the identification of finite sequences of $k$-tuples with $k$-tuples of finite sequences). If one substitutes the Baire space $\omega^\omega$ for $\mathbb{R}$, Suslin’s construction for analytic sets (see Section 2.2) essentially presents them as projections of trees on $\omega \times \omega$, modulo the representation of closed intervals. Many descriptive set theorists, starting perhaps with Luzin and Sierpiński [??], used trees to represent sets of reals, except that they converted these trees to linear orders via what is now known as the Kleene-Brouwer ordering (after [??] and [??]). The explicit use of projections of trees as we have presented them here is due to Richard Mansfield [??]. As pointed out in [??], given an ordinal $\gamma$, a $\gamma$-scale for a subset $A$ of the Baire space naturally gives rise to a tree on $\omega \times \gamma$ such that $p[T] = A$. Given a set $Z$, a subset of the Baire space is said to be $Z$-Suslin if it is the projection of a tree on $\omega \times Z$. Suslin’s representation of analytic sets shows that a set is analytic if and only if it is $\omega$-Suslin. Some authors use “Suslin” to mean “analytic”. We will follow a different usage, however, and say that a subset of the Baire space is Suslin if it is $\gamma$-Suslin for some ordinal $\gamma$.

Given a tree $T$ on $\omega \times Z$ and a wellordering of $Z$, a member of $p[T]$ can be found by following the so-called leftmost infinite branch through $T$ (similar to the proof of König’s Lemma, one picks a path through the tree by taking the least next step which is the initial segment of an infinite path through the tree). In a similar manner, a tree on $(\omega \times \omega) \times \gamma$, for some ordinal $\gamma$, induces a uniformization of the projection of the tree.

### 3.3. The game quantifier

Given a Polish space $X$ and a set $B \subseteq X \times ^\omega$, we let $\mathcal{D}B$ denote the set of $x \in X$ such that $I$ has a winning strategy in $G_\omega(B_x)$. If $\Gamma$ is a pointclass, $\mathcal{D}\Gamma$ is the class $\{\mathcal{D}B \mid B \in \Gamma\}$. The following facts appear in [??, pp. 245-246].
Theorem 3.6. If $\Gamma$ is an adequate pointclass then the following hold.

- $G \Gamma$ is adequate and closed under $\exists^0$ and $\forall^0$.
- $\exists^1 \Gamma \subseteq G \Gamma$ and $\forall^1 \Gamma \subseteq G \Gamma$.
- If $\text{Det}(\Gamma)$ holds, then $G \Gamma \subseteq \forall^1 \exists^1 \Gamma$.

The First Periodicity Theorem can be stated more generally as the fact that if an adequate pointclass $\Gamma$ has the prewellordering property, then so does $G \Gamma$, and the Second Periodicity Theorem can be similarly stated as saying that if an adequate pointclass $\Gamma$ has the scale property, then so does $\sigma \Gamma$ (see [??], pp. 246,267). The propagation of these properties through the projective pointclasses then follows from Theorem 3.6, given that they hold for $\Pi^1_2$ (and its variants).

Modifying the notion of $\Gamma$-scale by dropping the requirement that $\rho_n(x)$ is less than or equal to the eventual value of $\langle \rho_i(x_i) : i \in \omega \rangle$, one gets the notion of $\Gamma$-semiscale. Moschovakis’s Third Periodicity Theorem [??] concerns the definability of winning strategies and is stated using the game quantifier and the notion of semiscale.

Theorem 3.7 (Third Periodicity Theorem). Suppose that $\Gamma$ is an adequate pointclass, and that $\text{Det}(\Gamma)$ holds. Fix $A \subseteq \omega \omega$ in $\Gamma$, and suppose that $A$ admits a $\Gamma$-semiscale and that $I$ has a winning strategy in the game $G_\omega(A)$. Then $I$ has a winning strategy coded by a subset of $\omega$ in $G \Gamma$.

One consequence the Third Periodicity Theorem in conjunction with Theorem 3.6 is the following [??]: for any $n \in \omega$, if $\Sigma^1_{2n}$-determinacy holds, $A \subseteq \omega^\omega$ is $\Sigma^1_{2n}(a)$ for some real $a$ and $I$ has a winning strategy in the game with payoff $A$, then $I$ has a winning strategy coded by a subset of $\omega$ in $\Delta^1_{2n+1}(a)$.

Let $\exists^1$ denote the game quantifier for real games, games of length $\omega$ where the players alternate playing real numbers. Then $\exists^1 \Sigma^1_0$ defines the inductive sets of reals. Moschovakis [??] showed that the inductive sets have the scale property. Moschovakis [??] showed that, assuming the determinacy of all games with payoff in the class built from the inductive sets by the operations of projection and complementation, coinductive sets have scales in this class. Building on this work, Martin and Steel [??] showed that the pointclass $\Sigma^1_2$ has the scale property in $L(R)$. Kechris and Solovay had shown that if there is no wellordering of the reals in $L(R)$, then there exists in $L(R)$ a set of reals that cannot be uniformized, the set of pairs $(x,y)$ such that $y$ is not ordinal definable from $x$ (i.e., definable from $x$ and some ordinals). This set is $\Pi^1_2$ in $L(R)$.

Formally, this definition requires a definable association of $\omega$-sequences of reals to individual reals. Alternately, a set of reals is inductive if it is in $\Sigma^1_{2n}(R)$, where $J$ refers to Ronald Jensen’s constructibility hierarchy and $\kappa_R$ is the least $\kappa$ such that $J_\kappa(R)$ is a model of Kripke-Platek set theory.
The Solovay Basis Theorem says that if $P(A)$ is a $\Sigma^2_2$ relation on subsets of $\omega^\omega$ and there exists a witness to $P(A)$ in $L(R)$, then there is a $\Delta^2_1$ witness. This reflection result, along with the Martin-Steel theorem on scales in $L(R)$, compensates in many circumstances for the fact that not every set of reals has a scale in $L(R)$.

Steel [??] applied Jensen’s fine structure theory [??] to the study of scales in $L(R)$, refining and unifying a great deal of work on scales and Suslin cardinals. Extending [??], he showed that for each positive ordinal $\alpha$, determinacy for all sets of reals in $J_{\alpha}(R)$ implies that the pointclass $\Sigma^1_{J_{\alpha}(R)}$ has the scale property.

Martin [??] showed how to propagate the scale property using the game quantifier for integer games of fixed countable length (this subsumes propagation by the quantifier $G^1_1$), and Steel [??, ??] did the same for certain games of length $\omega_1$.

3.4. Partially playful universes. The periodicity theorems showed that determinacy axioms imply structural properties for sets of reals beyond the classical regularity properties. It remained to show that these hypotheses were necessary. Towards this end, Moschovakis (see [??]) identified for each integer $n$ (under the assumption of $\Delta^1_{1+n}$-determinacy, where $k$ is the greatest even integer less than $n$) the smallest transitive $\Sigma^1_{n+1}$-correct model of $ZF +$ Dependent Choices ($DC$) which contains all the ordinals (Joseph Shoenfield [??] had shown that $L$ is $\Sigma^1_2$-correct).

This model satisfies $AC$ and $\Delta^1_{1+n}$-determinacy and has a $\Sigma^1_{n+1}$ wellordering of the reals. In this model, $\Pi^1_i$ has the scale property for all odd $i \leq n$, and $\Sigma^1_i$ has the scale property for all other positive integers $i$.

Kechris and Moschovakis [??] introduced the models $L[T_{2n+1}]$, where $T_{2n+1}$ denotes the tree for a $\Pi^1_{2n+1}$-scale for a complete $\Pi^1_{2n+1}$ set. Moschovakis showed that $L[T_1] = L$, and conjectured that $L[T_{2n+1}]$ is independent of the choice of complete set and scale when for all $n$. This conjecture was proved by Howard Becker and Kechris in [??].

Solovay [??] showed that if $L \cap R$ is countable, then it is the largest countable $\Sigma^1_2$ set of reals (i.e., a countable $\Sigma^1_2$ set which contains all other such sets). Kechris and Moschovakis [??] showed that for each positive integer $n$, if $\text{Det}(\Delta^1_{2n})$ holds then there exists a largest countable $\Sigma^1_{2n+2}$ set. The largest countable $\Sigma^1_{2n}$ set came to be called $C_{2n}$. Kechris [??] showed that under Projective Determinacy there is for each integer $n$ a largest

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23The Axiom of Dependent Choices ($DC$) is the statement that if $R$ is a binary relation on a nonempty set $X$, and if for each $x \in X$ there is a $y \in X$ such that $x R y$, then there exists an infinite sequence $(x_i : i < \omega)$ such that $x_i R x_{i+1}$ for all $i \in \omega$. This statement is a weakening of the Axiom of Choice, sufficient to prove König’s Lemma, the regularity of $\omega_1$ and the wellfoundedness of ultrapowers by countably complete ultrafilters. See [??].
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countable $\Pi^1_{2n+1}$ set, which he also called $C_{2n+1}$. The case $n = 0$ follows from $ZF + DC$ and was shown independently by David Guaspari, Kechris and Gerald Sacks [??]. Kechris also showed that under Projective Determinacy there are no largest countable $\Sigma^1_{2n+1}$ or $\Pi^1_{2n}$ sets. It follows that under Projective Determinacy the lightface projective pointclasses with a largest countable set are the same as those in the zig-zag pattern above for the prewellordering property and the scale property. Harrington and Kechris [??] showed (under the assumption that $AD$ holds in $L(\mathbb{R})$) that the reals of each $L[T_{2n+1}]$ are exactly $C_{2n+2}$, for all integers $n$ (the case $n = 1$ was due to Kechris and Martin).

Kechris showed (assuming Projective Determinacy) that each model $L[C_{2n}]$ satisfies $\text{Det}(\Delta^1_{2n-1})$ but not $\text{Det}(\Sigma^1_{2n-1})$, and has a $\Delta^1_{2n}$ wellordering of its reals. Martin would show that $\text{Det}(\Delta^1_{2n})$ implies $\text{Det}(\Sigma^1_{2n})$ for each positive integer $n$.

3.5. Wadge degrees. In 1968, William Wadge considered the following game, given two sets of reals $A$ and $B$: player I builds a real $x$, player II builds a real $y$, and player II wins if $x \in A \leftrightarrow y \in B$. Determinacy for this class of games is known as Wadge determinacy. Given two sets of reals $A$, $B$, we say that $A \leq_W B$ (A has Wadge rank less than or equal to $B$, or is Wadge reducible to $B$) if there is a continuous function $f$ such that for all reals $x, x \in A$ if and only if $f(x) \in B$ (i.e., such that $A = f^{-1}[B]$). Wadge determinacy implies that for any two sets of reals $A$, $B$, either $A \leq_W B$ (in the case that player II has a winning strategy) or $\omega^2 \setminus B \leq_W A$ (in the case that player I does), from which it follows that for any two pointclasses closed under continuous preimages, either the two classes are dual (i.e., a pair of the form $\Gamma$, $\bar{\Gamma}$) or one is contained in the other. Wadge showed that $\leq_W$ is wellfounded on the Borel sets, and Martin, using an idea of Leonard Monk, extended this to all sets of reals under $AD + DC$ (see [??]).

Wadge determinacy and the wellfoundedness of the Wadge hierarchy divide $\mathcal{P}(\omega^2)$ into equivalence classes by Wadge reducibility and order these classes into a wellfounded hierarchy, where each level consists either of one selfdual equivalence class, or two non-selfdual classes, one consisting of all the complements of the members of the other. Wadge determinacy also implies that every non-selfdual adequate pointclass has a universal set (see [??], p. 162).

The discovery of Wadge determinacy led to further progress on separation and reduction. Robert Van Wesep [??] proved that under $AD$, if $\Gamma$ is a non-selfdual pointclass which is closed under continuous preimages, then $\Gamma$ and $\bar{\Gamma}$ cannot both have the separation property. Kechris, Solovay and Steel [??] showed that under $AD + DC$, if $\Gamma \subseteq L(\mathbb{R})$ is nonselfdual boldface pointclass and $\Gamma$ is closed under countable intersections and unions and either $\exists^1$ or $\forall^1$,
but not complements, then either $\Gamma$ or $\hat{\Gamma}$ has the prewellordering property. In 1981, Steel [??] showed that under AD, if $\Gamma$ is a nonselfdual pointclass closed under continuous preimages, then either $\Gamma$ or $\hat{\Gamma}$ has the separation property, and if one assumes in addition that $\Delta_\Gamma$ is closed under finite unions, then either $\Gamma$ or $\hat{\Gamma}$ has the reduction property.

§4. Partition properties and the projective ordinals. A cardinal $\kappa$ is **measurable** if there is a nonprincipal $\kappa$-complete ultrafilter on $\kappa$, where $\kappa$-**completeness** means closure under intersections of fewer than $\kappa$ many elements. In ZFC measurable cardinals are strongly inaccessible. In 1967, Solovay (see [??, p. 633] or [??, p. 348]) showed that AD implies that the club filter on $\omega_1$ is an ultrafilter, which implies that $\omega_1$ is a measurable cardinal. Ulam had shown that under ZFC there are stationary, co-stationary subsets of $\omega_1$; Solovay’s result shows the opposite under AD. Solovay also showed that under AD every subset of $\omega_1$ is constructible from a real (i.e., exists in $L[a]$ for some real number $a$). Since the measurability of $\omega_1$ implies that the sharp of each real exists, this gives another proof that the club filter on $\omega_1$ is an ultrafilter, since for any real $a$, if $a^\#$ exists, then every subset of $\omega_1$ in $L[a]$ either contains or is disjoint from a tail of the $a$-indiscernibles below $\omega_1$, which is a club set.

A **Turing degree** is a nonempty subset of $\mathcal{P}(\omega)$ closed under equicomputability. A **cone** of Turing degrees is the set of all degrees above (or computing) a given degree. Martin [??] showed that under AD the cone measure on Turing degrees is an ultrafilter, i.e., that every set of Turing degrees either contains or is disjoint from a cone. This important fact has a relatively short and simple proof: the two players collaborate to build a real, with the winner decided by whether the Turing degree of the real falls inside the payoff set; the cone above the degree of any real coding a winning strategy must contain or be disjoint from the payoff set. Martin used this result to find a simpler proof of the measurability of $\omega_1$. Solovay followed by showing that $\omega_2$ is measurable as well. **Turing determinacy** is the restriction of AD to payoff sets closed under Turing equivalence. This form of determinacy is easily seen to suffice for Martin’s result. In the early 1980’s, Woodin would show that, in $L(\mathbb{R})$, AD and Turing determinacy are equivalent.

Given an ordered set $X$ and an ordinal $\beta$, $[X]^{\beta}$ denotes the set of subsets of $X$ of ordertype $\beta$. Given ordinals, $\alpha$, $\beta$, $\delta$ and $\gamma$, the expression $\alpha \rightarrow (\beta)^\gamma_\delta$ denotes the statement that for every function $f : [\alpha]^\gamma \rightarrow \delta$, there

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24 A subset of an ordinal is **closed unbounded** (or **club**) if it is unbounded and closed in the order topology on the ordinals, and **stationary** if it intersects every club set. The **club filter** on an ordinal $\gamma$ consists of all subsets of $\gamma$ containing a club set.

25 See [??, ??] for more on the Turing degrees, including a more precise statement of their definition.
exists an $X \in [\alpha]^\beta$ such that $f$ is constant on $[X]^\gamma$. Frank Ramsey [??] proved that $\omega \to (\omega)^2_2$ holds for each positive $n \in \omega$ (this fact is known as **Ramsey’s Theorem**). For infinitary partitions, Paul Erdős and András Hajnal [??] showed (in ZFC) that for any infinite cardinal $\kappa$ there is a function $f: [\kappa]^\omega \to \kappa$ such that for every $X \in [\kappa]^\kappa$, the range of $f|X$ is all of $\kappa$.

In 1968, Adrian Mathias [??, ??] showed that $\omega \to (\omega)^2_1$ holds in Solovay’s model from [??], in which all sets of reals satisfy the regularity properties. A set $Y \subseteq [\omega]^\omega$ is said to be **Ramsey** if there exists an $X \in [\omega]^\omega$ such that either $[X]^\omega \subseteq Y$ or $[X]^\omega \cap Y = \emptyset$. The statement $\omega \to (\omega)^2_2$ is equivalent to the statement that every subset of $[\omega]^\omega$ is Ramsey. Prikry [??] showed that under $AD$ (determinacy for games of perfect information of length $\omega$ for which the players play real numbers) every subset of $[\omega]^\omega$ is Ramsey. It follows from the main theorem of [??] that $AD + V=L(\mathbb{R})$ implies that every such set is Ramsey. Whether $AD$ alone suffices is still an open question.

In late 1968, Martin (see [??], p. 392) showed that $AD$ implies $\omega_1 \to (\omega_1)^2_2$ (this implies for instance that the club filter on $\omega_1$ is an ultrafilter). Kenneth Kunen then showed that $AD$ implies that $\omega_1$ satisfies the weak partition property, where a cardinal $\kappa$ satisfies the weak partition property if $\kappa \to (\kappa)^2_2$ holds for every $\alpha < \kappa$. Martin followed by showing that $\omega_1 \to (\omega_1)^2_2$, again under $AD$. The proof actually shows $\omega_1 \to (\omega_1)^2_1$ and $\omega_1 \to (\omega_1)^1_1$ for every countable ordinal $\alpha$. Martin and Paris (in an unpublished note [??], see [??]) showed that under $AD + DC$, $\omega_2$ has the weak partition property.

Before continuing with this line of results, we briefly discuss the Coding Lemma and the projective ordinals.

### 4.1. $\Theta$, the Coding Lemma and the projective ordinals

Following convention, we let $\Theta$ denote the least ordinal that is not a surjective image of $\mathbb{R}$. Under ZFC, $\Theta = \mathfrak{c}^+$, but under $AD$, $\Theta$ is a limit cardinal, as noted by Harvey Friedman (see [??], p. 398). This fact follows from a theorem known as the **Coding Lemma**, due to Moschovakis [??], extending earlier work of Friedman and Solovay.

Given a subset $P$ of some Polish space, let $\Sigma^1_1(P)$ denote the pointclass of sets which are $\Sigma^1_1$-definable using $P$ and individual reals as parameters.

**Theorem 4.1 (Coding Lemma).** Assume $ZF + AD$. Let $\preceq$ be a prewell-ordering of a set of reals $X$. Let $\xi$ be the length of $\preceq$ and let $A$ be a subset of $\xi$. Then there exists a $Y \subseteq X$ in $\Sigma^1_1(\preceq)$ such that $A$ is the set of $\preceq$-ranks of elements of $Y$.

As an immediate consequence, under $AD$, if $\xi < \Theta$, then there is a surjection from $\mathbb{R}$ onto $P(\xi)$ (furthermore, if $\alpha < \Theta^+$ for some wellfounded
model \( M \) of ZF containing the reals, then such a surjection can be found in \( M \). The proof of the Coding Lemma uses a version of Kleene’s Recursion Theorem (first proved in [??] for partial recursive functions on the integers), which can be stated as saying that given a suitable coding under which each real \( x \) codes a continuous partial function \( \hat{x} \) (our notation) on the reals, for each two-variable continuous partial function \( g \) on the reals there is a real \( x \) such that \( \hat{x}(w) = g(x, w) \) for all reals \( w \).

If \( \Gamma \) is a pointclass, \( \delta_\Gamma \) denotes the supremum of the lengths of the prewellorderings of the reals in \( \Delta_\Gamma \). The notation \( \delta^n_\Sigma \) is used to denote \( \delta^n_\Sigma \) (which is the same as \( \delta^n_\Pi \)). The \textbf{projective ordinals} are the ordinals \( \delta^n_\Sigma \), for \( n \in \omega \setminus \{0\} \). It follows from the results of [??] that \( \Sigma^1 \) prewellorderings of the reals have countable length, and therefore that the ordinal \( \delta^1_\Sigma \) is equal to \( \omega_1 \). Moschovakis [??] showed (under AD, using the Coding Lemma) that for each \( n \in \omega \), \( \delta^n_{\Sigma+1} \) is a cardinal, and that \( \delta^1_{\Sigma+1} \) is regular and (using just PD) strictly less than \( \delta^2_{\Sigma+1} \). Martin showed (without AD) that \( \delta^1_2 \leq \omega_2 \) (see [??]); together these results show that under AD, \( \delta^1_2 = \omega_2 \).

Kunen and Martin (see [??]) independently established from ZF + DC that every wellfounded \( \kappa \)-Suslin prewellordering has length less than \( \kappa^+ \) (this fact is sometimes called the \textbf{Kunen-Martin Theorem}). Moschovakis ([??]; see [??], 4C.14) showed (from PD) that any \( \Pi^1_{\Sigma+1} \)-norm on a complete \( \Pi^1_{\Sigma+1} \) set has length \( \delta^1_{\Sigma+1} \) (this result also uses Kleene’s Recursion Theorem). By the scale property for \( \Pi^1_{\Sigma+1} \) sets (under the assumption of DC + \( \Delta^1_{\Sigma+1} \)-determinacy, given \( n \in \omega \) [??]), every \( \Pi^1_{\Sigma+1} \) set (and thus every \( \Sigma^1_{\Sigma+2} \) set) is \( \delta^1_{\Sigma+1} \)-Suslin, and, since \( \delta^1_{\Sigma+1} \) is regular, every \( \Sigma^1_{\Sigma+1} \) set is \( \lambda \)-Suslin for some \( \lambda < \delta^1_{\Sigma+1} \). It follows that under the same hypothesis, \( \delta^1_{\Sigma+2} \leq (\delta^1_{\Sigma+1})^+ \), and under AD that \( \delta^1_{\Sigma+2} = (\delta^1_{\Sigma+1})^+ \) for each \( n \in \omega \).

Kechris [??] proved (assuming AD) that \( \delta^1_{\Sigma+1} \) is a successor cardinal (its predecessor is called \( \lambda_{\Sigma+1} \)). It follows from his arguments, and those of the previous paragraph, that the pointclasses \( \Sigma^1_{\Sigma+2} \) and \( \Sigma^1_{\Sigma+1} \) exactly the \( \delta^1_{\Sigma+1} \)-Suslin and \( \lambda_{\Sigma+1} \)-Suslin sets respectively.

Given an ordinal \( \lambda \), the \( \lambda \)-\textbf{Borel} sets of reals are those in the smallest class containing the open sets and closed under complements and well-ordered unions of length less than \( \lambda \). Martin showed that if \( \kappa \) is a cardinal of uncountable cofinality, then all \( \kappa \)-Suslin sets are \( \kappa^+ \)-Borel. He also showed (using AD + DC, the Coding Lemma and Wadge determinacy) that the \( \delta^1_{\Sigma+1} \)-Borel sets are \( \Delta^1_{\Sigma+1} \), for each \( n \in \omega \) (the reverse inclusion follows from the results of Moschovakis [??] mentioned above). Using this fact, Kechris proved (again, under AD) that \( \lambda_{\Sigma+1} \) has cofinality \( \omega \). It follows (under AD) that \( \delta^1_{\Sigma+1} < \delta^1_{\Sigma+1} \) for each \( n \in \omega \), so that under AD the sequence
\( \delta_{n+1}^1 : n \in \omega \) is a strictly increasing sequence of successor cardinals. Kunen [??] showed that \( \delta_n^1 \) is regular for each positive \( n \in \omega \).

Solovay noted that under \( \text{AD} \), \( \Theta \) is the \( \Theta \)-th cardinal, and that under the further assumption of \( \forall \text{ZF} \Rightarrow \forall \text{L} = \text{L}(\mathbb{R}) \), \( \Theta \) is regular (see [??], p. 398). He showed [??] that under \( \text{DC} \), \( \Theta \) has uncountable cofinality, and also that \( \forall \text{ZF} + \text{AD}_\mathbb{R} + \text{cf}(\Theta) > \omega \) proves the consistency of \( \forall \text{ZF} + \text{AD}_\mathbb{R} \), so that by Gödel’s Second Incompleteness Theorem, if \( \forall \text{ZF} + \text{AD}_\mathbb{R} \) is consistent, then so is \( \forall \text{ZF} + \text{AD}_\mathbb{R} + \text{cf}(\Theta) = \omega \).\textsuperscript{26} Kechris [??], using the proof of the Third Periodicity Theorem and work of Martin, Moschovakis and Steel on scales [??], showed that \( \forall \text{DC} \) follows from \( \forall \text{AD} + \forall \text{L} = \text{L}(\mathbb{R}) \). Woodin (see [??]) strengthened Solovay’s result that \( \forall \text{DC} \) does not follow from \( \forall \text{AD} \) by showing that, assuming \( \forall \text{AD} + \forall \text{L} = \text{L}(\mathbb{R}) \) there is an inner model of a forcing extension satisfying \( \forall \text{ZF} + \forall \text{AD} + \forall \text{AC}_\omega \) (\( \forall \text{DC} \) directly implies \( \forall \text{AC}_\omega \)). Whether \( \forall \text{AD} \) implies \( \forall \text{DC}(\omega) \) (\( \forall \text{DC} \) for relations on \( \omega \)) is still open.

4.2. Partition properties and ultrafilters. Kunen in an unpublished note [??] proved that \( \delta_{2n}^1 \rightarrow (\delta_{2n+1}^1)^2 \) for all positive \( n \in \omega \) and \( \lambda < \omega_1 \), under \( \forall \text{AD} \). He also showed [??] (under the same hypothesis) that \( \delta_{2n}^1 \rightarrow (\delta_{2n+1}^1)^2 \) is false. Martin, in another unpublished note from 1971, showed that \( \delta_{2n+1}^1 \rightarrow (\delta_{2n+1}^1)^2 \) for all positive \( n \in \omega \) and \( \lambda < \omega_1 \), under \( \forall \text{AD} \).

While Erdős and Hajnal [??] had shown how to derive partition properties from measurable cardinals, Eugene Kleinberg proved the following result in the other direction, which shows (via \( \lambda = \omega \)) that \( \delta_n^1 \) is measurable for each positive \( n \in \omega \).\textsuperscript{27}

**Theorem 4.2** ([??]). If \( \lambda < \kappa \), \( \lambda \) is regular, and \( \kappa \rightarrow (\kappa)^{\lambda+1} \) holds, then \( C_\kappa^\lambda \) is a normal ultrafilter over \( \kappa \).

In 1970, Kunen proved, using Martin’s result on the cone measure on the Turing degrees, that under \( \forall \text{AD} \), any \( \omega_1 \)-complete filter on an ordinal \( \lambda < \Theta \) can be extended to an \( \omega_1 \)-complete ultrafilter, and that every ultrafilter on an ordinal less than \( \Theta \) is definable from ordinal parameters (see [??], pp. 399-400). Solovay [??] proved that under \( \forall \text{AD}_\mathbb{R} \), there is a normal ultrafilter on \( \mathcal{P}_{\aleph_1}(\mathbb{R}) \): for each \( A \subseteq \mathcal{P}_{\aleph_1}(\mathbb{R}) \), consider the game where player I and player II collaborate to build a sequence \( \langle s_i : i < \omega \rangle \) consisting of finite sets of reals, and player I wins if and only if \( \bigcup \{ s_i : i \in \omega \} \in A \).\textsuperscript{28} This implies

\textsuperscript{26}The end of Section 6.2 continues this line of results.

\textsuperscript{27}We let \( C_\kappa^\lambda \) denote the filter generated by the set of \( \lambda \)-closed unbounded subsets of \( \kappa \). A filter is **normal** if every regressive function on a set in the filter is constant on a set in the filter.

\textsuperscript{28}Given a cardinal \( \kappa \) and a set \( X \), \( \mathcal{P}_\kappa X \) denotes the collection of subsets of \( X \) of cardinality less than \( \kappa \). An ultrafilter \( U \) on \( \mathcal{P}_\kappa X \) is **normal** if for each \( Y \subseteq U \), if \( f \) is a regressive function on \( Y \) (i.e., if \( \text{dom}(f) = Y \) and \( f(A) \in A \) for all nonempty \( A \subseteq Y \)) then \( f \) is constant on a set in \( U \).
(again, under $\text{AD}_\mathbb{R}$) that for each ordinal $\gamma < \Theta$ there is a normal ultrafilter on $\wp_\omega \gamma$ (i.e., that $\omega_1$ is $\gamma$-supercompact). It is not known whether $\text{AD}$ suffices for this result, though Harrington and Kechris [??] showed that if $\text{AD}$ holds and $\gamma$ is less than a Suslin cardinal, then there is a normal ultrafilter on $\wp_\omega \gamma$. Extending work of Becker [??] (who proved it in the case that $\gamma$ is a Suslin cardinal), Woodin [??] showed that there is just one such ultrafilter for each $\gamma < \Theta$, if either $\text{AD}_\mathbb{R}$ holds or $\text{AD}$ holds and $\gamma$ is below a Suslin cardinal. The end of Section 6.4 mentions more recent progress on these topics.

A cardinal $\kappa$ is said to have the strong partition property if $\kappa \rightarrow (\kappa)^\mu_\mu$ holds for every $\mu < \kappa$. As mentioned above, Martin showed that under $\text{AD}$, $\omega_1$ has the strong partition property. In late 1977, Kechris adapted Martin’s argument to show that under $\text{AD}$ there exists a cardinal $\kappa$ with the strong partition property such that the set of $\lambda < \kappa$ with the strong partition property is stationary below $\kappa$ (see [??, p. 432]). Pushing this further, Kechris, Kleinberg, Moschovakis and Woodin [??] showed (using a uniform version of the Coding Lemma) that $\text{AD}$ implies that unboundedly many cardinals below $\Theta$ have the strong partition property and are stationary limits of cardinals with the strong partition property. They also showed that whenever $\lambda$ is an ordinal below a cardinal with the strong partition property, all $\lambda$-Suslin sets are determined. Using work of Steel [??] and Martin [??], Kechris and Woodin [??] showed that in $\text{L}(\mathbb{R})$, $\text{AD}$ is equivalent to the assertion that $\Theta$ is a limit of cardinals with the strong partition property, and also to the statement that all Suslin sets are determined. James Henle, Mathias and Woodin [??] later showed that the first equivalence does not follow from $\text{ZF} + \text{DC}$, since the existence of a nonprincipal ultrafilter on $\omega$ is consistent with $\Theta$ being a limit of cardinals with the strong partition property.

A key step in the proof of the Kechris-Woodin theorem was a transfer theorem extending results of Harrington and Martin (discussed in Section 5.3). Harrington and Martin had shown from $\text{ZF} + \text{DC}$ that, for each real $a$, $\Pi^1_1(a)$-determinacy is equivalent to determinacy for the larger class $\bigcup_{\beta < \omega_1} \beta$-$\Pi^1_1(a)$. Kechris and Woodin showed, from the same hypothesis, that for all positive integers $k$, $\Delta^1_{2k}$-determinacy is equivalent to $\Sigma^{(2k-1)}$-$\Pi^1_1$-determinacy, where $\Sigma^{(2k-1)}$ indicates an application of $2k - 1$ many instances of the game quantifier $\exists$. By Theorem 3.6, this means that $\Delta^1_{2k}$-determinacy implies $\Pi^1_{2k}$-determinacy. Martin had proved the lightface version in 1973 (see [??]). Later results of Woodin

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29 An ordinal (necessarily a cardinal) $\kappa$ is said to be Suslin if there is a set of reals which is $\kappa$-Suslin but not $\lambda$-Suslin for any $\lambda < \kappa$. 


and Itay Neeman [??] would show that $\Pi_{n+1}^1$-determinacy is equivalent to $\bigcup_{n<\omega} \beta$-$\Pi_{n+1}^1$-determinacy for all $n \in \omega$.

4.3. Cardinals, uniform indiscernibles and the projective ordinals. A cardinal $\kappa$ is Ramsey if for every function $f: [\kappa]^{<\omega} \rightarrow \{0, 1\}$ (where $[\kappa]^{<\omega}$ denotes the finite subsets of $\kappa$) there exists $A \in [\kappa]^\kappa$ such that for each $n \in \omega$, $f[A]^n$ is constant. Measurable cardinals are Ramsey, and if there exists a Ramsey cardinal then the sharp of each real number exists. Assuming the existence of a Ramsey cardinal, Martin and Solovay [??] showed that nonempty $\Sigma^1_3$ subsets of the plane have $\Delta^1_4$ uniformizations.

As mentioned above, Lévy [??] had shown that ZFC does not suffice for this result. Martin and Solovay used an analysis of sharps for reals, and modeled their argument after the proof of the Kondo-Addison theorem. Mansfield [??] extended the Martin-Solovay analysis to show (using a measurable cardinal) that nonempty $\Pi^1_2$ sets are uniformized by $\Pi^1_3$ functions.

Given a positive ordinal $\alpha$, $u_\alpha$ denotes the $\alpha$th uniform indiscernible, the $\alpha$th ordinal which is a Silver indiscernible for each real number. As bijections between $\omega$ and countable ordinals can be coded by reals, the first uniform indiscernible, $u_1$, is $\omega_1$. It follows from the basic analysis of sharps that all uncountable cardinals are uniform indiscernibles, so $u_2 \leq \omega_2$. By applying the Kunen-Martin theorem inside models of the form $L[a]$, for $a$ a real number, and applying the basic analysis of sharps, Martin showed that $\delta^1_3 = u_2$ if the sharp of every real exists (see [??]). Recall that by the results of Section 4.1, $\delta^1_3 = \omega_2$, under AD.

Martin showed from ZF plus the assumption that the sharp of each real exists that every $\Sigma^1_1$ set is $u_\omega$-Suslin, and from AD that $u_\omega = \omega_\omega$ (see [??, pp.203-204]). By the Kunen-Martin Theorem, then, AD implies that $\delta^1_3 \leq \omega_{\omega+1}$. Kunen had shown that if the sharp of every real exists, then $u_{\xi+1}$ has the same cofinality as $u_2$, for every positive ordinal $\xi$ (see [??]). Since $u_\omega = \omega_\omega$, it follows that each $\omega_n$ ($n \geq 2$) is of the form $u_{k+1}$ for some positive integer $k$, and thus that each such $\omega_n$ has cofinality $\omega_2$. It follows that under AD + DC, $\delta^1_3 = \omega_{\omega+1}$, since $\delta^1_3$ is a regular cardinal, and therefore that $\delta^1_3 = \omega_{\omega+2}$. Kunen and Solovay would then show that $u_n = \omega_n$ for all $n$ satisfying $1 \leq n \leq \omega$.

In 1971, Kunen reduced the computation of $\delta^1_3$ to the analysis of certain ultrapowers of $\delta^1_3$ (see [??]; as part of his analysis, Kunen showed that $\delta^1_3$ has the weak partition property, see [??]). The completion of this project was to take another decade. In the early 1980’s, Martin proved new results analyzing these ultrapowers, and Steve Jackson, using joint work with Martin, computed $\delta^1_3$. The following theorem [??, ??] completes the calculation of the $\delta^1_n$’s.
Theorem 4.3 (Jackson). Assume AD. Then for \( n \geq 1 \), \( \delta_{2n+1}^1 \) has the strong partition property and is equal to \( \omega \omega \omega \omega + 1 \), where \( w(1) = \omega \) and \( w(m+1) = \omega \omega (m) \) in the sense of ordinal exponentiation.

Jackson’s proof of this theorem was over 100 pages long. Elements of his argument (as presented in [??]) include the Kunen-Martin theorem, Kunen’s \( \Delta^1_3 \) coding for subsets of \( \omega \omega \omega \), Martin’s theorem that \( \Delta^1_{2n+1} \) is closed under intersections and unions of sequences of sets indexed by ordinals less than \( \delta^1_1 \), and so-called homogeneous trees, a notion which traces back to [??] and a result of Martin discussed in the next section.

§5. Determinacy and large cardinals. As discussed above, a strongly inaccessible cardinal is an uncountable regular cardinal which is closed under cardinal exponentiation. If \( \kappa \) is strongly inaccessible, then \( V_\kappa \) is a model of ZFC, so that the existence of strongly inaccessible cardinals is not a consequence of ZFC. While there is no technical definition of large cardinal, a typical large cardinal notion (in the context of the Axiom of Choice) specifies a type of strongly inaccessible cardinal. Examples of this type include Ramsey cardinals, measurable cardinals, Woodin cardinals and supercompact cardinals. The large cardinal hierarchy orders large cardinals by consistency strength. That is, large cardinal notion \( A \) is below large cardinal notion \( B \) in the hierarchy if the existence of cardinals of type \( B \) implies the consistency of cardinals of type \( A \). It is a striking empirical fact that the large cardinal hierarchy is linear, modulo open questions (the examples just given were listed in increasing order, for instance). Even more striking is the fact that many set-theoretic statements having no ostensible relationship to large cardinals are equiconsistent with some large cardinal notion.\(^{30}\)

By results of Mycielski (discussed in Section 2.3), AD implies that \( \omega_1 \) is strongly inaccessible in \( L \), which means that AD cannot be proved in ZFC. Moreover, Solovay’s result that AD implies the measurability of \( \omega_1 \) implies that under AD, \( \omega_1 \) (as computed in the full universe) is a measurable cardinal in certain inner models of AC, such as \( \text{HOD} \).\(^{31}\) As we shall see in this section, the relationship between large cardinals and determinacy runs in both directions: various forms of determinacy imply the existence of models of ZFC containing large cardinals, and the existence of large cardinals can be used to prove the determinacy of certain definable sets of reals.

5.1. Measurable cardinals. Solovay [??] showed in 1965 that if there exists a measurable cardinal then every uncountable \( \Sigma^1_1 \) set of reals contains

\(^{30}[??]\) is the standard reference for the large cardinal hierarchy.

\(^{31}\)The inner model \( \text{HOD} \) (a model of ZFC) consists of all sets \( x \) such that every member of the transitive closure of \( \{ x \} \) is ordinal-definable (see [??], Chapter 13)).
a perfect set. This result was proved independently by Mansfield (see [??]). Martin [??] showed that in fact analytic determinacy follows from the existence of a Ramsey cardinal.

Roughly, the idea behind Martin’s proof is that if $A$ is the projection of a tree $T$ on $\omega \times \omega$ and $\chi$ is a Ramsey cardinal, one can modify the original game for $A$ to require the second player to play, in addition to his usual moves, a function $G^*: \omega^{<\omega} \to \chi$ witnessing (via the wellfoundedness of the ordinal $\chi$) that the fragment of $T$ corresponding to the real produced by the two players in their moves from the original game has no infinite branches, and thus that this real is not in the projection of $T$. This modified game is closed, and thus determined, by Gale-Stewart. If the second player has a winning strategy in the modified game, then he has a winning strategy in the original game by ignoring his extra moves. In general there is no reason that a winning strategy for the first player in the modified game will induce a winning strategy for the original game. However, if $\chi$ is a Ramsey cardinal, then there is uncountable $X \subseteq \chi$ such that, as long as the range of $G^*$ is contained in $X$, the first player’s strategy does not depend on the extra moves for the second player. Using this fact, the first player can convert his winning strategy in the modified game into a winning strategy in the original game. The notion of a determined (often closed) auxiliary game and a method for transferring strategies from the auxiliary game to the original game is the basis of many determinacy proofs.

Martin later proved the following refinement.

**Theorem 5.1.** If the sharp of every real exists, then $\Pi^1_1$-determinacy holds.

In the 1970’s Kunen and Martin independently developed the notion of a **homogeneous** tree, following a line of ideas deriving from Martin’s proof of $\Pi^1_1$-determinacy (see [??]). Given a set $Z$ and a cardinal $\kappa$, a tree on $\omega \times Z$ is said to be $\kappa$-**homogeneous** if for each $\sigma \in \omega^{<\omega}$ there is a $\kappa$-complete ultrafilter $\mu_\sigma$ on $Z^{[\sigma]}$ such that

- for each $\sigma \in \omega^{<\omega}$, $\{z : (\sigma, z) \in T\} \in \mu_\sigma$;
- $p[T]$ is the set of $x \in \omega^\omega$ such that the sequence $\langle \mu_x | i : i \in \omega \rangle$ is countably complete.\(^{32}\)

A tree is said to be **homogeneous** if it is $\aleph_1$-homogeneous. A set of reals is said to be **homogeneously Suslin** if it is the projection of a homogeneous tree. There are related notions of **weakly homogeneous** tree and **weakly homogeneously Suslin** set of reals, involving a more involved relationship with a set of ultrafilters. Though it was not the original definition, let us just say that a tree on a set of the form $\omega \times (\omega \times Z)$

\(^{32}\)i.e., for each sequence $\langle A_i : i \in \omega \rangle$ such that each $A_i \in \mu_x | i$ there exists a $t \in Z^\omega$ such that $t|i \in A_i$ for each $i$.\)
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is weakly homogeneous if and only if the corresponding tree on \((\omega \times \omega) \times Z\) is homogeneous, and note that a set of reals is weakly homogeneously Suslin if and only if it is the projection of a homogeneously Suslin set of pairs.

Martin’s proof then shows the following.

**Theorem 5.2 (Martin).** Homogeneously Suslin sets are determined.

The unfolding argument mentioned in Section 2.2 then shows that weakly homogeneously Suslin sets satisfy the regularity properties.

In retrospect, Martin’s proof of analytic determinacy can be broken into two parts, the fact that homogeneously Suslin sets are determined, and the fact that if there is a Ramsey cardinal then \(\Pi^1_1\) sets are homogeneously Suslin.

The results of [??] can similarly be reinterpreted. If \(\Pi^1_1\) sets are homogeneously Suslin, then \(\Sigma^1_2\) sets are weakly homogeneously Suslin. The Martin-Solovay construction can be seen as a method for taking a \(\gamma\)-weakly homogeneous tree \(T\) (for some cardinal \(\gamma\)) and producing a tree \(S\) on \(\omega \times \gamma'\), for some ordinal \(\gamma'\), projecting to the complement of the projection of \(T\). From this follows that all \(\Pi^1_1\) sets, and thus all \(\Sigma^1_1\) sets, are projections of trees on the product of \(\omega\) with some ordinal. More sophisticated arguments can be carried out from the existence of sharps, using the fact that sharps give ultrafilters over certain inner models.

### 5.2. Borel determinacy.

In 1968, Friedman [??] showed that the Replacement axiom is necessary to prove Borel determinacy, even for sets invariant under Turing degrees (he also showed that analytic determinacy cannot hold in a forcing extension of \(L\)). As refined by Martin, his results show (for each \(\alpha < \omega_1\)) that ZFC - Power Set - Replacement + “the \(\alpha\)th iteration of the power set of \(\omega\) exists” does not prove the determinacy of all \(\Sigma^{\omega_0}_{\omega_1+\alpha+3}\) sets.

James Baumgartner mixed the method of Martin’s \(\Pi^1_1\)-determinacy proof with Davis’s \(\Sigma^0_3\)-determinacy proof to give a new proof of \(\Sigma^0_3\)-determinacy in ZFC. Using a similar approach, Martin proved \(\text{Det}(\Sigma^0_3)\) from the existence of a weakly compact cardinal,33 and then Paris [??] proved it in ZFC. Paris noted at the end of his paper that his argument could be carried out without the power set axiom, assuming instead only that the ordinal \(\omega_1\) exists.

Andreas Blass [??] and Mycielski (1967, unpublished) independently proved that \(\text{AD}_\omega\) is equivalent to determinacy for integer games of length \(\omega^2\). The key idea in Blass’s proof was to reduce determinacy in the given game to determinacy in another, auxiliary, game in such a way that one

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33A cardinal \(\kappa\) is **weakly compact** if \(\kappa \rightarrow (\kappa)^2\). Weakly compact cardinals are below the existence of \(0^\#\) and above strongly inaccessible cardinals in the consistency strength hierarchy (see [??, pp. 76,472]).
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player’s moves in the auxiliary game correspond to fragments of his strategy in the original game. Martin [?] used this basic idea to prove Borel determinacy in 1974 (the auxiliary game was in fact an open game). In his [?], Martin gave a short, inductive, proof of Borel determinacy, and introduced the notion of unrolling a set of reals—roughly, finding an association of the set to a clopen set in a larger domain with a map sending strategies in one game to strategies in the other. In his [?], Martin extended this method to games of length $\omega$ played on any (possibly uncountable) set, with Borel payoff (in the corresponding sense). Neeman [?, ?] would unravel $\Pi_1^e$ sets from the assumption of a measurable cardinal $\kappa$ of Mitchell rank $\kappa^{++}$ (proved to be an optimal hypothesis by Steel [?]; see [?], pp. 357-360 for the definition of Mitchell rank). Complementing Friedman’s theorem, Martin proved that for each $\alpha < \omega_1$, the determinacy of each Boolean combination of $\Sigma^e_\alpha + 2$ sets follows from $\text{ZF} - \text{Power Set} - \text{Replacement} + \text{E}_1$ Replacement + “the $\alpha$th iteration of the power set of $\omega$ exists”.

5.3. The difference hierarchy. Given a countable ordinal $\alpha$ and a real $a$, a set of reals $X$ is said to be $\alpha\cdot \Pi_1^1(a)$ if there is wellordering of $\omega$ of length $\alpha$ recursive in $a$ with corresponding rank function $R: \omega \to \alpha$ and a $\Pi_1^1(a)$ subset $A$ of $\omega \times \omega$ such that

- for all $n, m \in \omega$, if $R(n) < R(m)$ then
  
  \( \{x : (m, x) \in A\} \subseteq \{x : (n, x) \in A\} \);

- $X$ is the set of reals $x$ for which the least $\xi$ such that either $\xi = \alpha$ or $\xi < \alpha$ and $(R^{-1}(\xi), x) \notin A$ is odd.

This notation has its roots in [?]. When $a$ is itself recursive one writes $\alpha\cdot \Pi_1^1(a)$. The union of the sets $\alpha\cdot \Pi_1^1(a)$ for all reals $a$ is denoted $\alpha\cdot \Pi_1^1$. The union of the sets $\alpha\cdot \Pi_1^1$ for all $\alpha < \omega_1$ is denoted $\text{Diff}(\Pi_1^1)$. Note that $\text{Diff}(\Pi_1^1)$ is a proper subclass of $\Delta_1^1$.

Friedman [?] extended Theorem 5.1 to show that $\text{Det}(3\cdot \Pi_1^1)$ follows from the existence of the sharp of every real. Martin in 1975 then extended this result to show that the existence of $0^\#$ is equivalent to $\text{Det}(\bigcup_{\beta < \omega_2} \beta\cdot \Pi_1^1)$ (see [?]). Harrington [?] then proved the converse to Theorem 5.1 by showing that $\text{Det}(\Pi_1^1(a))$ implies the existence of $a^\#$, for each real $a$.

For the purposes of the next theorem, say that a model has $\alpha$ measurable cardinals and indiscernibles if there exists a set of ordertype $\alpha$ consisting of measurable cardinals of the model, and there exist uncountably many ordinal indiscernibles of the model above the supremum of these measurable cardinals. Martin proved the following theorem after Harrington’s result.

**Theorem 5.3.** For any real $a$ and any ordinal $\alpha$ recursive in $a$, the following are equivalent.

- $\text{Det}(\bigcup_{\beta < \omega_2} (\omega^2 \cdot \alpha + \beta)\cdot \Pi_1^1(a))$.
• Det(\(\omega^2 \cdot \alpha + 1\)-\(\Pi^1_1\)(a)).

• There is an inner model of ZFC containing a and having \(\alpha\) many measurable cardinals and indiscernibles.

Still, a large-cardinal consistency proof of Det(\(\Delta^1_2\)), the hypothesis used by Addison and Martin in their extension of Blackwell’s argument, remained beyond reach. John Green showed that Det(\(\Delta^1_2\)) implies the existence of an inner model with a measurable cardinal of Mitchell rank 1.

5.4. Larger cardinals. In Section 4 we defined a measurable cardinal to be a cardinal \(\kappa\) such that there exists a nonprincipal \(\kappa\)-complete ultrafilter on \(\kappa\). Equivalently, under the Axiom of Choice, \(\kappa\) is measurable if and only if there is a nontrivial elementary embedding \(j\) from the full universe \(V\) into some inner model \(M\) whose critical point is \(\kappa\), i.e., such that \(\kappa\) is the least ordinal not mapped to itself by \(j\). Many large cardinal notions can be expressed both in terms of ultrafilters and in terms of embeddings, though in the Choiceless context (without the corresponding form of Loś’s Theorem, see [??, p. 159]) it is the definition in terms of ultrafilters which is relevant. For instance, a cardinal \(\kappa\) is supercompact if for each \(\lambda > \kappa\) there exists a normal fine ultrafilter on \(\mathcal{P}_\kappa \lambda\). Under the Axiom of Choice, \(\kappa\) is supercompact if and only if for every \(\lambda > \kappa\) there is an elementary embedding \(j\) from \(V\) into an inner model \(M\) such that the critical point of \(j\) is \(\kappa\) and \(M\) is closed under sequences of length \(\lambda\). Every supercompact cardinal is a limit of measurable cardinals. An even larger large cardinal notion is the huge cardinal, where an uncountable cardinal \(\kappa\) is huge if for some cardinal \(\lambda > \kappa\) there is a \(\kappa\)-complete normal fine ultrafilter on \([\lambda]^\kappa\) (where “normal” and “fine” are defined in analogy with the supercompact case, see [??, p. 331]). Under AC, \(\kappa\) is huge if and only if there is an elementary embedding \(j: V \rightarrow M\) with critical point \(\kappa\) such that \(M\) is closed under sequences of length \(j(\kappa)\). The existence of huge cardinals does not imply the existence of supercompact cardinals, but it does imply their consistency.

Kunen [??] put a limit on the large cardinality hierarchy, showing in ZFC that there is no nontrivial elementary embedding from \(V\) into itself. A corollary of the proof is that for any elementary embedding \(j\) of \(V\) into any inner model \(M\), if \(\delta\) is the least ordinal above the critical point of \(j\) sent to itself by \(j\), then \(V_{\delta+1} \not\subseteq M\). In 1978, Martin [??] proved \(\Pi^1_2\)-determinacy from the hypothesis I2, which states that for some ordinal \(\delta\) there is a nontrivial elementary embedding of \(V\) into an inner model \(M\) with critical point less than \(\delta\) such that \(V_{\delta} \subseteq M\) and \(j(\delta) = \delta\).

\[^{34}\text{Given a cardinal } \kappa \text{ and a set } X, \text{ a collection } U \text{ of subsets of } \mathcal{P}_\kappa X \text{ is } \text{fine } \text{if it contains the collection of supersets of each element of } \mathcal{P}_\kappa X.\]
In 1979, Woodin proved that for each $n \in \omega$, $\Pi^1_{n+1}$ follows (in ZF) from
the existence of an $n$-fold strong rank-to-rank embedding.\textsuperscript{35} For $n = 1$, this is essentially
the theorem of Martin just mentioned. For $n > 1$, these
axioms are incompatible with the Axiom of Choice, by Kunen’s theorem, though they are not known to be inconsistent with ZF.

In 1984, Woodin proved $\text{AD}^\text{L}(\mathbb{R})$ from $\text{I}_0$, the statement that for some ordinal $\delta$ there is a nontrivial elementary embedding from $L(V_{\delta+1})$ into itself with critical point below $\delta$, thus verifying Solovay’s conjecture that $\text{AD}^\text{L}(\mathbb{R})$ would follow from large cardinals. $\text{I}_0$ is one of the strongest large cardinal hypotheses not known to be inconsistent. The inner model program at the time had produced models for many measurable cardinals, hypotheses far short of $\text{I}_2$, and so there was little hope of showing that $\text{I}_2$ and $\text{I}_0$ were necessary for these results.

New large cardinal concepts would prove to be the missing ingredient. Given an ideal $I$ on a set $X$, forcing with the Boolean algebra given by
the power set of $X$ modulo $I$ gives a $V$-ultrafilter on the power set of $X$.\textsuperscript{36} The ideal $I$ is said to be \textbf{precipitous} if the ultrapower of $V$ by this generic ultrafilter is wellfounded in all generic extensions. If the underlying set $X$ is a cardinal $\kappa$, the ideal $I$ is said to be \textbf{saturated} if the Boolean algebra $\mathcal{P}(\kappa)/I$ has no antichains of cardinality $\kappa^+$.\textsuperscript{37} If $\kappa$ is a regular cardinal, saturation of $I$ implies precipitousness. Huge cardinals were invented by Kunen \textsuperscript{[??]}, who used them to produce a saturated ideal on $\omega_1$.

In early 1984, Matthew Foreman, Menachem Magidor and Shelah \textsuperscript{[??]} showed that if there exists a supercompact cardinal—a hypothesis much weaker than $\text{I}_0$ or $\text{I}_2$—then there is an $\omega_1$-preserving forcing making the nonstationary ideal on $\omega_1 (\text{NS}_{\omega_1})$ saturated.

Foreman (see \textsuperscript{[??]}) and Magidor \textsuperscript{[??]} had earlier made a connection between generic elementary embeddings\textsuperscript{38} and regularity properties for reals. Magidor \textsuperscript{[??]} in particular had shown that the Lebesgue measurability

\textsuperscript{35}For positive $n \in \omega$, an \textbf{$n$-fold strong rank-to-rank embedding} is a sequence of elementary embeddings $j_1, \ldots, j_n$ such that for some cardinal $\lambda$,

\begin{itemize}
  \item $j_i: V_{\lambda+1} \to V_{\lambda+1}$ whenever $1 \leq i \leq n$,
  \item $\kappa_\omega(j_i) < \kappa_\omega(j_{i+1})$ for all $i < n$,
\end{itemize}

where $\kappa_\omega(j)$ denotes the first fixed point of an elementary embedding $j$ above the critical point.

\textsuperscript{36}An \textbf{ideal} is a collection of sets closed under subsets and finite unions. Given a
model $M$ and a set $X$ in $M$, an \textbf{$M$-ultrafilter} is a subset of $\mathcal{P}(X) \cap M$ closed under supersets and finite intersections such that for every $A \subseteq X$ in $M$, exactly one of $A$ and $X \setminus A$ is in $U$. Note that $U$ does not need to be an element of $M$.

\textsuperscript{37}An \textbf{antichain} in a partial order (or a Boolean algebra) is a set of pairwise incompatible elements. In the case of a Boolean algebra of the form $\mathcal{P}(\kappa)/I$, an antichain is a collection of subsets of $\kappa$ not in $I$ which pairwise have intersection in $I$.

\textsuperscript{38}A \textbf{generic elementary embedding} is an elementary embedding of the universe $V$ into some class model $M$ which is definable in a forcing extension of $V$. 
of $\Sigma^1_3$ sets followed from the existence of a generic elementary embedding with critical point $\omega_1$ and wellfounded image model (the existence of such an embedding follows from the Foreman-Magidor-Shelah result mentioned above). Woodin noted that these arguments plus earlier work of his (see [??]) could be used to extend this to Lebesgue measurability for all projective sets. Woodin also noted that arguments from [??] could be used to prove the Lebesgue measurability of all sets of reals in $L(\mathbb{R})$, if one could force to produce a saturated ideal on $\omega_1$ without adding reals. Shelah then noted that techniques from [??] could be modified to do just that. It followed then that the existence of a supercompact cardinal implies that all sets of reals in $L(\mathbb{R})$ are Lebesgue measurable.

Woodin and Shelah then addressed the problem of weakening the hypotheses needed for the Lebesgue measurability of all projective sets of reals. Woodin noted that a superstrong cardinal sufficed. Shelah then isolated a weaker notion now known as a **Shelah cardinal**, and showed that the existence of $n + 1$ Shelah cardinals implies that $\Sigma^1_{n+2}$ sets are Lebesgue measurable.

**Definition 5.4.** A cardinal $\kappa$ is a **Shelah cardinal** if for every $f: \kappa \to \kappa$ there is an elementary embedding $j: V \to N$ with critical point $\kappa$ such that $V_{j(f)(\kappa)} \subseteq N$.

Woodin noted that by modifying Shelah’s definition one obtained a weaker, still sufficient, hypothesis, now known as a Woodin cardinal.

**Definition 5.5.** A cardinal $\delta$ is a **Woodin cardinal** if for each function $f: \delta \to \delta$ there exists an elementary embedding $j: V \to M$ with critical point $\kappa < \delta$ closed under $f$ such that $V_{j(f)(\kappa)} \subseteq M$.

Woodin proved that the existence of $n$ Woodin cardinals below a measurable cardinal implies the Lebesgue measurability of $\Sigma^1_{n+2}$ sets, the same amount of measurability that would follow from $\Pi^1_{n+1}$-determinacy. All of this work was done within a few weeks of the Foreman-Magidor-Shelah result on the saturation of $NS_{\omega_1}$. In [??] the hypothesis for the statement that all sets of reals in $L(\mathbb{R})$ are Lebesgue measurable and have the property of Baire was reduced to the existence of ordertype $\omega + 1$ many Woodin cardinals. The hypothesis was to be reduced even further.

Woodin extracted from the Foreman-Magidor-Shelah results a one-step forcing for producing generic elementary embeddings with critical point $\omega_1$, and developed it into a general method, now known as the **stationary tower**. Using this he showed (by the fall of 1984, see his [??]) that if there exists a supercompact cardinal (or a strongly compact cardinal), then every

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39 We follow the account in [??].
set of reals in $L(R)$ is weakly homogeneously Suslin. (Steel and Woodin would show in 1990 that this conclusion in turn implies $AD^{L(R)}$.)

Steel had been working on the problem of finding inner models for supercompact cardinals. Inspired by the results of Foreman, Magidor, Shelah and Woodin, he began to work on producing models for Woodin cardinals, and had some partial results by the spring of 1985, producing inner models with certain weak variants of Woodin cardinals. These models were generated by sequences of extenders, directed systems of ultrafilters which collectively generate elementary embeddings whose images contain more of $V$ than possible for embeddings generated by a single ultrafilter. Special cases of extenders had appeared in Jensen’s proof of the Covering Lemma. The general notion (which first appeared in [??]) is Jensen’s simplification of the notion of hypermeasure, which was introduced by Mitchell [??]. Steel and Martin saw that the problem of building models with Woodin cardinals was linked to the problem of proving determinacy, and they set their sights on this problem in the late spring of 1985.

One key combinatorial problem related to elementary embeddings is whether infinite iterations of these embeddings produce wellfounded models. Kunen [??] had shown that the answer was positive for iterations derived from a single ultrafilter. With extenders the situation was more complicated, as the iterations did not need to be linear but could produce trees of models with no rule for finding a path through the tree leading to a wellfounded model (indeed, this nonlinearity was essential, since otherwise the models would have simply definable wellorderings of their reals). The simplest such tree, a so-called alternating chain, is countably infinite and consists of two infinite branches. Martin and Steel saw that the issue of wellfoundedness for the direct limits along the two branches was linked. This observation led to the following theorem, proved in August of 1985.

**Theorem 5.6 (Martin-Steel [??]).** Suppose that $\lambda$ is a Woodin cardinal and $A$ is a $\lambda^+$-weakly homogeneously Suslin set of reals. Then for any $\gamma < \lambda$, $\omega_\omega \setminus A$ is $\gamma$-homogeneously Suslin.

It follows from this and the fact that analytic sets are homogeneously Suslin in the presence of a measurable cardinal that if there exist $n$ Woodin cardinals below a measurable cardinal, then $\Pi^1_{n+1}$ sets are determined, and that Projective Determinacy follows from the existence of infinitely many Woodin cardinals.

Combined with Woodin’s application of the stationary tower mentioned above, the Martin-Steel theorem implied that $AD^{L(R)}$ follows from the existence of a supercompact cardinal. By the end of 1985, Woodin had improved the hypothesis to the existence of infinitely many Woodin cardinals below a measurable cardinal (see [??]).
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Theorem 5.7 (Woodin). If there exist infinitely many Woodin cardinals below a measurable cardinal, then AD holds in \( L(\mathbb{R}) \).

In the spring of 1986, Martin and Steel \cite{martin_steel} produced extender models (i.e., models of the form \( L[E] \), with \( E \) a sequence of extenders) with \( n \) Woodin cardinals and \( \Delta^1_{n+2} \) wellorderings of the reals. Such a model necessarily has a \( \Sigma^1_{n+2} \) set which is not Lebesgue measurable, and fails to satisfy \( \Pi^1_{n+1} \)-determinacy.

Skipping ahead for a moment, let \( (\ast)_n \) be the statement that for each real \( x \) there exists an iterable model \( M \) containing \( x \) and \( n \) Woodin cardinals plus the sharp of \( V^M_{\delta} \), for \( \delta \) the largest of these Woodin cardinals. For odd \( n \), the equivalence of \( \Pi^1_{n+1} \)-determinacy and \( (\ast)_n \) was proved by Woodin in 1989. That \( (\ast)_n \) implies \( \Pi^1_{n+1} \)-determinacy for all \( n \) was proved by Neeman \cite{neeman} in 1994. Roughly, Neeman’s methods work by considering a modified game in which one player builds an iteration tree and makes moves in the image of the original game by the embeddings given by the tree. In 1995, Woodin proved that \( \Pi^1_{n+1} \)-determinacy implies \( (\ast)_n \) for even \( n > 0 \).

Woodin followed his Theorem 5.7 by determining the exact consistency strength of \( AD \). The forward direction of Theorem 5.8 below (proved in \cite{woodin}) shows from \( ZF + AD \) that there exist infinitely many Woodin cardinals in an inner model of a forcing extension (\( HOD \) of the forcing extension with respect to certain parameters) of \( V \). The proof built on a sequence of results, starting with Solovay’s theorem that \( AD \) implies that \( \omega_1 \) is a measurable cardinal, which, as mentioned above, also shows that \( \omega_1 \) (as defined in \( V \)) is measurable in the inner model \( HOD \). Becker (see \cite{becker}) had shown that, under \( AD \), \( \omega^V_1 \) is the least measurable in \( HOD \). Becker, Martin, Moschovakis and Steel then showed that under \( AD + V = L(\mathbb{R}) \), \( \delta^V_1 \) is \( \beta \)-strong in \( HOD \), where \( \beta \) is the least measurable cardinal greater than \( \delta^V_1 \) in \( HOD \).\footnote{The cardinal \( \delta^V_1 \) is the supremum of the lengths of the \( \Delta^2 \) wellorderings of the reals; under \( AD + V = L(\mathbb{R}) \) it is also the largest Suslin cardinal. A cardinal \( \kappa \) is \( \beta \)-strong if there is an elementary embedding \( j: V \to M \) with critical point \( \kappa \) such that \( V_\beta \subseteq M \), and \( <\delta \)-strong if it is \( \beta \)-strong for all \( \beta < \delta \).} In the 1980’s, Woodin showed under the same hypothesis that \( \delta^V_1 \) is \( \beta \)-strong in \( HOD \) for every \( \beta < \Theta \) (and that \( \delta^V_1 \) is the least ordinal with this property), and that \( \Theta \) is Woodin in \( HOD \).

Theorem 5.8 (Woodin). The following are equiconsistent.

- \( ZF + AD \)
- There exist infinitely many Woodin cardinals.

The following theorem illustrates the reverse direction of the equiconsistency (see \cite{woodin}). It can be seen as a special case of the Derived Model Theorem, discussed in Section 6.2. The partial order \( \text{Col}(\omega, <\delta) \) consists of all finite partial functions \( p \) from \( \omega \times \delta \) to \( \delta \), with the requirement that
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\( p(n, \alpha) \in \alpha \) for all \((n, \alpha)\) in the domain of \( p \). The order is inclusion. If \( \delta \) is a regular cardinal, then \( \delta \) is the \( \omega_1 \) of any forcing extension by \( \text{Col}(\omega, < \delta) \).

**Theorem 5.9** (Woodin). Suppose that \( \lambda \) is a limit of Woodin cardinals, and \( G \subseteq \text{Col}(\omega, < \lambda) \) is \( \mathbb{V} \)-generic filter. Let \( \mathbb{R}^* = \bigcup \{ \mathbb{R}^{\mathbb{V}[G[a]]} : \alpha < \lambda \} \). Then \( \text{AD} \) holds in \( \mathbb{L}(\mathbb{R}^*) \).

The results of Section 5.3 illustrate the difficulties in proving the determinacy of \( \Pi^1_2 \) sets. Woodin resolved this problem in 1989. The forward direction of Theorem 5.10 is proved in [??]. The proof was inspired in part by a result of Kechris and Solovay [??], saying that in models of the form \( \mathbb{L}[a] \) for \( a \subseteq \omega, \Delta^1_2 \)-determinacy implies the determinacy of all ordinal definable sets of reals. Standard arguments show that if \( \Delta^1_2 \) determinacy holds, then it holds in \( \mathbb{L}[x] \) for some real \( x \). Woodin showed that if \( \mathbb{V} \) is \( \mathbb{L}[x] \) for some real \( x \), and \( \Delta^1_2 \)-determinacy holds, then \( \omega_2^{\mathbb{L}[x]} \) is a Woodin cardinal in \( \text{HOD} \). Recall (from the end of Section 4.2) that \( \Delta^1_2 \)-determinacy and \( \Pi^1_2 \)-determinacy are equivalent, by a result of Martin.

**Theorem 5.10** (Woodin). The following are equiconsistent.

- \( \text{ZFC + Det}(\Delta^1_2) \).
- \( \text{ZFC + There exists a Woodin cardinal} \).

The following theorem illustrates the reverse direction. Its proof can be found in [??, p. 1926]. The partial order \( \text{Col}(\omega, \delta) \) is the natural one for making \( \delta \) countable: it consists of all finite partial functions from \( \omega \) to \( \delta \), ordered by inclusion.

**Theorem 5.11** (Woodin). If \( \delta \) is a Woodin cardinal and \( G \subseteq \text{Col}(\omega, \delta) \) is a \( \mathbb{V} \)-generic filter, then \( \Delta^1_2 \)-determinacy holds in \( \mathbb{V}[G] \).

§6. Later developments. In this final section we briefly review some of the developments that followed the results of the previous section. As discussed in the introduction, the set of topics presented here is by no means complete. The first subsection briefly introduces a regularity property for sets of reals which is induced by forcing-absoluteness. The second and third discuss forms of determinacy ostensibly stronger than \( \text{AD} \), in models larger than \( \mathbb{L}(\mathbb{R}) \). The next subsection discusses applications of determinacy to the realm of \( \text{AC} \), via producing models of \( \text{AC} \) by forcing over models of determinacy. In the last two we present some results which derive forms of determinacy from their ostensibly weak consequences, or from statements having no obvious relationship to determinacy. Many of the results of the last two subsections are applications of the study of canonical inner models for large cardinals.
6.1. Universally Baire sets. As discussed above in Sections 5.1 and 5.4, homogeneously Suslin and weakly homogeneously Suslin sets of reals played an important role in applications of large cardinals to regularity properties for sets of reals, as early as the 1969 results of Martin and Solovay. Qi Feng, Magidor and Woodin [??] introduced a related tree representation property for sets of reals. Given a cardinal $\kappa$, a set $A \subseteq \omega^\omega$ is $\kappa$-\textbf{universally Baire} if there exist trees $S$, $T$ such that $p[S] = A$ and $S$ and $T$ project to complements in every forcing extension by a partial order of cardinality less than or equal to $\kappa$.

Woodin (see [??, ??]) showed that if $\delta$ is a Woodin cardinal, then $\delta$-universally Baire sets of reals are $<\delta$-weakly homogeneously Suslin. It follows from the arguments of [??] that if $A \subseteq \omega^\omega$ is $\kappa^+$-weakly homogeneously Suslin, then it is $\kappa$-universally Baire. Combining these facts with Theorem 5.6 gives the following.

\textbf{Theorem 6.1.} If $\delta$ is a limit of Woodin cardinals, then the following are equivalent, for all sets of reals $A$.

- $A$ is $<\delta$ homogeneously Suslin.
- $A$ is $<\delta$ weakly homogeneously Suslin.
- $A$ is $<\delta$-universally Baire.

Feng, Magidor and Woodin showed that if $\delta_0 < \delta_1$ are Woodin cardinals, then every $\delta_1$-universally Baire set is determined (this follows from Theorem 5.6 and the result of Woodin mentioned before the previous paragraph). Neeman later improved this, showing that if $\delta$ is a Woodin cardinal, then all $\delta$-universally Baire sets are determined. In addition to the following theorem, Feng, Magidor and Woodin showed that $\text{Det}(\Pi^1_1)$ is equivalent to the statement that every $\Sigma^1_2$ set of reals is universally Baire.

\textbf{Theorem 6.2} (Feng, Magidor and Woodin [??]). Assume $\text{AD}^{\mathcal{L}(\mathbb{R})}$. Then the following are equivalent.

- $\text{AD}^{\mathcal{L}(\mathbb{R})}$ holds in every forcing extension.
- Every set of reals in $\mathcal{L}(\mathbb{R})$ is universally Baire.

Woodin’s \textit{Tree Production Lemma} is a powerful means for showing that sets of reals are universally Baire (see [??]). Woodin’s proof of Theorem 5.7 proceeded by applying the lemma to the set $\mathbb{R}^\#$. Informally, the lemma can be interpreted as saying that a set of reals $A$ is $\delta$-universally Baire if for every real $r$ generic for a partial order in $\mathcal{V}_\delta$, either $r$ is in the image of $A$ for every $Q_{<\delta}$-embedding\footnote{The partial order $Q_{<\delta}$ is one form of Woodin’s stationary tower, mentioned after Definition 5.5.} for which $r$ is in the image model, or $r$ is in the image of $A$ for no such embedding.

\footnote{The set $A$ is $<\kappa$-\textbf{universally Baire} if it is $\gamma$-universally Baire for all $\gamma < \kappa$, and \textbf{universally Baire} if it is universally Baire for all $\kappa$.}
Theorem 6.3 (Tree Production Lemma). Suppose that $\delta$ is a Woodin cardinal. Let $\varphi$ and $\psi$ be binary formulas, and let $x$ and $y$ be arbitrary sets, and assume that the empty condition in the stationary tower $Q_{<\delta}$ forces that for each real $r$,

$$M \models \psi(r,j(y)) \iff V[r] \models \varphi(r,x),$$

where $j: V \to M$ is the induced elementary embedding. Then $\{r : \psi(r,y)\}$ is $<\delta$-universally Baire.

6.2. $\text{AD}^+$ and $\text{AD}_R$. Moschovakis [??] proved that under $\text{AD}$, if $\lambda$ is less than $\Theta$, $A$ is a set of functions from $\omega$ to $\lambda$ and $A$ is Suslin and co-Suslin, then the game $G\omega(A)$ is determined, where here the players play elements of $\lambda$. Woodin formulated the following axiom, which, assuming $\text{AD}$, holds in every inner model containing the reals whose sets of reals are all Suslin (in $V$). A set of reals $A$ is said to be $\infty$-Borel if there exist a set of ordinals $S$ and binary formula $\varphi$ such that $A = \{x \in \mathbb{R} : L[x,S] \models \varphi(x,S)\}$. For example, a Suslin representation for a set of reals witnesses that the set is $\infty$-Borel.

Definition 6.4. $\text{AD}^+$ is the conjunction of the following statements.

- $\text{DC}(\omega\omega)$
- Every set of reals is $\infty$-Borel.
- If $\lambda < \Theta$ and $\pi: \lambda^\omega \to \omega^\omega$ is a continuous function, then $\pi^{-1}[A]$ is determined for every $A \subseteq \omega^\omega$.

It is an open question whether $\text{AD}$ implies $\text{AD}^+$, though it is known that $\text{AD}^+$ holds in all models of $\text{AD}$ of the form $L(A,\mathbb{R})$, where $A$ is a set of reals (some of the details of the argument showing this appear in [??]). It is not known whether $\text{AD}_R$ implies $\text{AD}^+$, though $\text{AD}^+$ does follow from $\text{AD}_R + \text{DC}$.

The following consequences of $\text{AD}^+$ were announced in [??].

Theorem 6.5 ($\text{ZF} + \text{DC}(\omega\omega)$). If $\text{AD}^+$ holds and $V = L(\mathcal{P}(\mathbb{R}))$, then

- the pointclass $\Sigma^2_1$ has the scale property,
- every $\Sigma^2_1$ set of reals is the projection of a tree in $\text{HOD}$,
- every true $\Sigma_1$-sentence is witnessed by a $\Delta^2_1$ set of reals.

Woodin’s Derived Model Theorem, proved around 1986, gives a means of producing models of $\text{AD}^+$. The model $L(\mathbb{R}^*,\text{Hom}^*)$ in the following theorem is said to be a derived model (over the ground model). A tree $T$ is said to be $<\lambda$-absolutely complemented if there is a tree $S$ such that $p[T] = \mathbb{R} \setminus p[S]$ in all forcing extensions by partial orders of cardinality less than $\lambda$.

Given an ordinal $\lambda$, $G \subseteq \text{Col}(\omega, <\lambda)$ and $\alpha < \lambda$, we let $G|\alpha$ denote $G \cap \text{Col}(\omega, <\alpha)$. The model $V(\mathbb{R}^*)$ in the following theorem can be defined as either $\bigcup_{\alpha \in \text{Ord}} L(V,\mathbb{R}^*)$ or $\text{HOD}^{V[G]}_{\mathbb{R}^*}$. Given a pointclass $\Gamma$, $M_\Gamma$-
denotes the collection of transitive sets \( x \) such that \( \langle x, \in \rangle \) is isomorphic to \( \langle R/E, F/E \rangle \), for some \( E, F \in \Gamma \) such that \( E \) is an equivalence relation on \( R \) and \( F \) is an \( E \)-invariant binary relation on \( R \). Models of the form \( L(\Gamma, R^*) \) below are called derived models. See [??] for an earlier version of the theorem.

**Theorem 6.6 (Derived Model Theorem; Woodin).** Let \( \lambda \) be a limit of Woodin cardinals. Let \( G \subseteq \text{Col}(\omega, <\lambda) \) be a \( V \)-generic filter. Let

- \( R^* \) be \( \bigcup_{\alpha<\lambda} R^*[G(\alpha)] \);
- \( \text{Hom}^* \) be the collection of sets of the form \( p[T] \cap R^* \), for \( T \) a \(<\lambda\)-absolutely complemented tree in \( V[G(\alpha)] \) for some \( \alpha < \lambda \);
- \( \Gamma \) be the collection of sets of reals \( A \) in \( V[G] \) such that \( L(A, R^*) \models AD^+ \).

Then

1. \( L(\Gamma, R^*) \models AD^+ \).
2. \( \text{Hom}^* \) is the collection of Suslin, co-Suslin sets of reals in \( L(\Gamma, R^*) \).
3. \( M_\Gamma \prec \Sigma_1 L(\Gamma, R^*) \).

Woodin also showed that item (3) above is equivalent to \( AD^+ \), assuming \( AD^+ V = L(P(R)) \). The Derived Model Theorem has a converse, also due to Woodin, which says that all models of \( AD^+ \) arise in this fashion.

**Theorem 6.7 (Woodin).** Let \( M \) be a model of \( AD^+ \), and let \( \Gamma \) be the collection of sets of reals which are Suslin, co-Suslin in \( M \). Then in a forcing extension of \( M \) there is an inner model \( N \) such that \( L(\Gamma, R^*) \) is a derived model over \( N \).

In unpublished work, Woodin has shown that over \( AD \), \( AD_R \) is equivalent to some of its ostensibly weak consequences (see [??]). The implication from (2) to (1) in the following theorem is due independently to Martin. The implication from (1) to (2) relies heavily on work of Becker [??]. Recall that Mycielski (see Section 2.3) showed that (1) implies (3); the implication from (2) to (3) is mentioned in Section 3.2.

**Theorem 6.8 (Woodin).** Assume \( ZF + DC \). Then the following are equivalent.

1. \( AD_R \)
2. \( AD \) + Every set of reals is Suslin
3. \( AD \) + Uniformization

Woodin would also produce models of \( AD_R \) from large cardinals.

**Theorem 6.9 (Woodin).** Suppose that there exists a cardinal \( \delta \) of cofinality \( \omega \) which is a limit of Woodin cardinals and \(<\delta\)-strong cardinals. Then there is a forcing extension in which there is an inner model containing the reals and satisfying \( AD_R \).
Steel, using earlier work of Woodin, completed the equiconsistency with the following theorem.

**Theorem 6.10 (Steel).** If $\text{AD}_R$ holds, then in a forcing extension there is a proper class model of ZFC in which there exists a cardinal $\delta$ of cofinality $\omega$ which is a limit of Woodin cardinals and $<\delta$-strong cardinals.

Recall from Section 4.1 that $\Theta$ is defined to be the least ordinal which is not a surjective image of the reals. Consideration of ordinal definable surjections gives the **Solovay sequence**, $\langle \vartheta_\alpha : \alpha \leq \Omega \rangle$. This sequence is defined by letting $\vartheta_0$ be the least ordinal which is not the surjective image of an ordinal definable function on the reals, and, for each $\alpha < \Omega$, letting $\vartheta_{\alpha+1}$ be the least ordinal which is not a surjective image of $\mathcal{P}(\vartheta_\alpha)$ via an ordinal definable function. Taking limits at limit stages and continuing until $\vartheta_\Omega = \Theta$ completes the definition. The consistency strength of $\text{AD}^+ + \vartheta_\alpha = \Theta$ increases with $\alpha$.

In $L(\mathbb{R})$, $\vartheta_0 = \Theta$. Woodin proved that, assuming $\text{AD}^+$, $\text{AD}_R$ is equivalent to the assertion that the Solovay sequence has limit length. Woodin also showed, under the same assumption, $\vartheta_\alpha$ is a Woodin cardinal in $\text{HOD}$, for all nonlimit $\alpha \leq \Omega$.

In unpublished work, Woodin showed that if it is consistent that there exists a Woodin limit of Woodin cardinals, then it is consistent that there exist sets of reals $A$ and $B$ such that the models $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$ each satisfy $\text{AD}$ but $L(A, B, \mathbb{R})$ does not. Woodin also showed that in this case $L(\Gamma, \mathbb{R}) \models \text{AD}_R + \text{DC}$, where $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R}) \cap L(B, \mathbb{R})$. Grigor Sargsyan showed that if there exist models $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$ as above then there is a proper class model of $\text{AD}_R$ containing the reals in which $\Theta$ is regular.

**6.3. Long games.** As mentioned above, Blass [??] and Mycielski showed that determinacy for games of length $\omega^2$ is equivalent to $\text{AD}_R$. For each $n \in \omega$, determinacy for games of length $\omega + n$ is equivalent to $\text{AD}$ (think of the game as being divided in two parts, where in the first part (of length $\omega$) the players try to obtain a position from which they have a winning strategy in the second; the winning strategy in the second part can be coded by an integer, and thus uniformly chosen).

Martin and Woodin independently showed that $\text{AD}_R$ is equivalent to determinacy for games of length $\alpha$ for each countable $\alpha \geq \omega^2$. Determinacy for games of length $\omega^2$ easily gives uniformization. It follows from this and Theorem 6.8 that $\text{AD}_R$ is equivalent to determinacy for games of length $\alpha$ for each countable $\alpha \geq \omega \cdot 2$.

While $\text{AD}$ does not imply uniformization, the Second Periodicity Theorem (Theorem 3.4) shows that $\text{PD}$ implies the uniformization of projective sets. It follows that $\text{PD}$ is equivalent to $\text{PD}$ for games of length less than $\omega^2$. As noted by Neeman [??], the techniques from the Blass-Mycielski
result above can be used to prove the determinacy of games of length $\omega^2$ with analytic payoff from $\text{AD}^{L(R)}$ plus the existence of $\mathbb{R}^\#$.

Steel [??] considered continuously coded games, games where each stage of the game is associated with an integer, and the game ends when an associated integer is repeated. Such a game must end after countably many rounds, but runs of the game can have any countable length. Steel proved that $\text{ZF} + \text{AD} + \text{DC} + \text{"every set of reals has a scale" + "}\omega_1\text{ is }\mathcal{P}(\mathbb{R})\text{-supercompact}"$ implies the determinacy of all continuously coded games.

None of the results mentioned so far in this section involves proving determinacy directly from large cardinals. Instead they show that some form of determinacy for short games with complicated payoff implies determinacy for longer games with simpler payoff. Proving long game determinacy from large cardinals was pioneered, and extensively developed, by Neeman, who established a number of results on games of variable countable length, and even length $\omega_1$ (see [??], [??], [??]). Neeman’s techniques built on the proof of PD from Woodin cardinals by Martin and Steel, using iteration trees. In many cases, his proofs proceed from essentially optimal hypotheses. The proofs of many of these results reduced the determinacy of long games to the iterability of models containing large cardinals.

For example, given $C \subseteq \mathbb{R}^{<\omega_1}$, let $G_{\text{local}}(L, C)$ be the game where players player I and player II alternate playing natural numbers so as to define elements $z_\xi$ of the Baire space. The game ends at the first $\gamma$ such that $\gamma$ is uncountable in $L[z_\xi : \xi < \gamma]$, with player I winning if the sequence $\langle z_\xi : \xi < \gamma \rangle$ is in $C$. It follows from mild large cardinal assumptions (for instance, the existence of the sharp of every subset of $\omega_1$) that $\gamma$ must be countable.

Given a pointclass $\Gamma$, a set $C$ consisting of countable sequences of reals is said to be $\Gamma$ in the codes if the set of reals coding members of $C$ (under a suitably definable coding) is in $\Gamma$.

Theorem 6.11 (Neeman). Suppose that there exists a measurable cardinal above a Woodin limit of Woodin cardinals. Then the games $G_{\text{local}}(L, C)$ are determined for all $C$ which are $\mathcal{D}_{\omega_1}(<\omega^2-\Pi_1^1)$ in the codes.

The preceding theorem is obtained by combining the results of [??] and [??]. The proof proceeds by constructing an iterable class model $M$ with a cardinal $\vartheta$ such that $\vartheta$ is a Woodin limit of Woodin cardinals in $M$ and countable in $\mathbb{V}$ [??]. Using inner model theory, Neeman then transformed the iteration strategy of $M$ into a winning strategy in $G_{\text{local}}(L, C)$.

Adapting Kechris and Solovay’s proof that $\Delta^1_1$-determinacy implies the existence of a real $x$ such that $L[x]$ satisfies the determinacy of all ordinal definable sets of reals (discussed before Theorem 5.10), Woodin proved that the amount of determinacy in the conclusion of Theorem 6.11 implies
that there exists a set $A \subseteq \omega_1$ such that in $L[A]$, all games on integers of length $\omega_1$ with payoff definable from reals and ordinals are determined (see Exercise 7F.15 of [??]). Larson and Shelah [??] showed that it is possible to force that some integer game of length $\omega_1$ with definable payoff is undetermined.

We give one more result of Neeman, proving the determinacy of certain games of length $\omega_1$. In Theorem 6.12 below, $\mathcal{L}^+$ is the language of set theory with one additional unary predicate. Given an integer $k$ and a sequence $S$ of stationary sets indexed by $[\omega_1]^{<k}$, $[S]$ is the collection of increasing $k$-tuples $(\alpha_0, \ldots, \alpha_{k-1})$ from $\omega_1$ such that each initial segment of length $j \leq k$ is in $S(\alpha_0, \ldots, \alpha_j)$. The game $G_{\omega_1,k}(S, \varphi)$ is a game of length $\omega_1$ in which the players collaborate to build a function $f : \omega_1 \to \omega_1$. Then player I wins if there is a club $C$ such that

$$\langle L_{\omega_1}, r \rangle \models \varphi(\alpha_0, \ldots, \alpha_{k-1})$$

for all $(\alpha_0, \ldots, \alpha_{k-1}) \in [S] \cap [C]^k$, and player II wins if there is a club $C$ such that

$$\langle L_{\omega_1}, r \rangle \models \neg \varphi(\alpha_0, \ldots, \alpha_{k-1})$$

for all $(\alpha_0, \ldots, \alpha_{k-1}) \in [S] \cap [C]^k$. Though there can be runs of the game for which neither player wins, determinacy for this game in the sense of Theorem 6.12 refers to the existence of a strategy for one player or the other that guarantees victory.

The model $0^W$ is the minimal iterable fine structural inner model $M$ which has a top extender predicate whose critical point is Woodin in $M$. The existence of such a model is not known to follow from large cardinals.

The last part of the conclusion of Theorem 6.12 extends a result of Martin, who showed that for any recursive enumeration $\langle B_i : i < \omega \rangle$ of the $<\omega^2$-$\Pi^1_1$ sets, the set of $i$ such that player I has a winning strategy in $G_{\omega_1}(B_i)$ is recursively isomorphic to $0^*$. Theorem 6.12 (Neeman [??]). Suppose that $0^W$ exists. Let $k < \omega$. Let $\mathcal{S}$ be a sequence of mutually disjoint stationary sets indexed by $[\omega_1]^{<k}$. Let $\varphi$ be a $\mathcal{L}^+$ formula with $k$ free variables. Then the game $G_{\omega_1,k}(\mathcal{S}, \varphi)$ is determined. Furthermore, the winner of each such game depends only on $\varphi$ and not on $\mathcal{S}$, and the set of $\varphi$ for which the first player has a winning strategy is recursively equivalent to the canonical real coding $0^W$.

If one allows the members of $\mathcal{S}$ all to be $\omega_1$, then there are undetermined games of this type, as observed by Greg Hjorth (see [??]). If one allows the members of $\mathcal{S}$ all to be $\omega_1$ and changes the winning condition for player II to be simply the negation of the winning condition for player I then one can force from a strongly inaccessible limit of measurable cardinals that some game of this type is not determined [??].
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Given a set $A \subseteq \omega \omega$, $G_{\text{open} \omega \omega}(A)$ is the game of length $\omega_1$ in which player I and player II collaborate to build a function from $\omega_1$ to $\omega$, with player I winning if some proper initial segment of the play is in $A$. The determinacy result in Theorem 6.12 includes the determinacy of all games $G_{\text{open} \omega \omega}(A)$ for sets $A$ which are $\Pi^1_1$ in the codes. Combining Neeman’s proof of Theorem 6.12 with his own theory of hybrid strategy mice, Woodin proved that if there exist proper class many Woodin limits of Woodin cardinals then $\text{AD}^+$ holds in the Chang Model, the smallest inner model of ZF containing the ordinals and closed under countable sequences.

6.4. Forcing over models of determinacy. Steel and Van Wesep [??] showed that by forcing over a model of $\text{AD}_R + \text{“} \Theta \text{ is regular} \text{”}$ (the hypothesis they used was actually weaker) one can produce a model of ZFC in which $\text{NS}_{\omega_1}$ is saturated and $\delta^1_2 = \omega_2$. This was the first consistency proof of either of these two statements with ZFC. Martin had conjectured that “$\forall n \in \omega \delta^1_n = \aleph_n$” is consistent with ZFC, and this verified the conjecture for the case $n = 2$. Woodin [??] subsequently reduced the hypothesis to $\text{AD}$.

Shelah [??] later showed that it was possible to force the saturation of $\text{NS}_{\omega_1}$ from a Woodin cardinal. Woodin [??] proved that the saturation of $\text{NS}_{\omega_1}$ plus the existence of a measurable cardinal implies that $\delta^1_2 = \omega_2$. Woodin then turned his proof into a general method for producing models of ZFC by forcing over models of determinacy. The most general form of this method, a partial order called $\mathbb{P}_{\text{max}}$, consists roughly of a directed system containing all countable models of ZFC in which $\text{NS}_{\omega_1}$ is saturated and $\delta^1_2 = \omega_2$. There are many variants of $\mathbb{P}_{\text{max}}$. One of these variants, called $\mathbb{Q}_{\text{max}}$, produces a model in which $\text{NS}_{\omega_1}$ is $\aleph_1$-dense (i.e., $\mathcal{P}(\omega_1)/\text{NS}_{\omega_1}$ has a dense subset of cardinality $\aleph_1$; this implies saturation), from the assumption that $\text{AD}$ holds in $L(R)$. No other method is known for producing a model of ZFC in which $\text{NS}_{\omega_1}$ is $\aleph_1$-dense.

Steel [??] showed that under $\text{AD}$, $\text{HOD}^{L(R)}$ is an extender model below $\Theta$. Woodin then showed that the entire model $\text{HOD}^{L(R)}$ is a model of the form $L[\vec{E}, \Sigma]$, where $\vec{E}$ is a sequence of extenders and $\Sigma$ is an iteration strategy corresponding to this sequence. Using this approach, Steel showed that for every regular $\kappa < \Theta$, the $\omega$-club filter over $\kappa$ is an ultrafilter in $L(R)$. Woodin used this to show that $\omega_1$ is $<\Theta$-supercompact in $L(R)$. Previously it was known only that $\omega_1$ is $\lambda$-supercompact for $\lambda$ below the supremum of the Suslin cardinals (see the paragraph after Theorem 4.2).

Woodin also used the inner models approach to show that, in $L(R)$, $\omega_1$ is huge to $\kappa$ for each measurable $\kappa < \Theta$, improving results of Becker.
Neeman [??] used this approach to prove, for each $\lambda < \Theta$, the uniqueness of the normal ultrafilter on $\mathcal{P}(\mathbb{R})$ witnessing the $\lambda$-supercompactness of $\omega_1$. Previously this too was known only for $\lambda < \delta_2^1$ (this is also discussed in the paragraph after Theorem 4.2). Neeman [??] and Woodin independently used this approach to show that, assuming $\text{AD} + V = L(\mathbb{R})$, one could force without adding reals to obtain $\text{ZFC} + \delta_n^1 = \omega_2$, for any $n \geq 3$. It is still unknown whether $\delta_m^1$ can equal $\omega_n$ for any $m \geq n \geq 2$ (under $\text{ZFC}$).

6.5. Determinacy from its consequences. Woodin [??] conjectured that Projective Determinacy follows from the statement that all projective sets are Lebesgue measurable, have the Baire property and can be uniformized by projective functions (all consequences of $\text{PD}$). This conjecture was refuted by Steel in 1997. If one requires the uniformization property for the scaled projective pointclasses, then the conjecture is still open. Woodin did prove the following version of the conjecture in the late 1990’s, using work of Steel in inner model theory. Recall that $\text{AD}$ implies the three statements below (see Sections 2.1 and 3.3).

**Theorem 6.13 (Woodin).** Assuming $\text{ZF} + \text{DC} + V = L(\mathbb{R})$, the Axiom of Determinacy follows from the conjunction of the following three statements.

- Every set of reals is Lebesgue measurable.
- Every set of reals has the property of Baire.
- Every $\Sigma^1_1$ subset of $2^{<\omega}$ can be uniformized.

Woodin had proved another equivalence in the early 1980’s.

**Theorem 6.14 (Woodin).** Assume $\text{ZF} + \text{DC} + V = L(\mathbb{R})$. Then the following are equivalent.

- $\text{AD}$
- Turing determinacy.

It is apparently an open question whether $\text{AD}$ follows from $\text{ZF} + \text{DC} + V = L(\mathbb{R})$ plus either of (a) for every $\alpha < \Theta$ there is a surjection of $^{<\omega} \omega$ onto $\mathcal{P}(\alpha)$; (b) $\Theta$ is inaccessible.

Determinacy would turn out to be necessary for some of its earliest applications. For instance, Steel [??] showed that $\Sigma^1_2$-separation plus the existence of sharps for all reals implies $\Delta^1_2$-determinacy. Applying related work of Steel, Hjorth [??] showed that $\Pi^1_2$-determinacy follows from Wadge determinacy for $\Pi^1_2$ sets. Earlier, Harrington had shown that, for each real $x$, $\Pi^1_1(x)$-Wadge determinacy implies that $x^#$ exists. It is open whether Wadge determinacy for the projective sets implies $\text{PD}$.

6.6. Determinacy from other statements. Determinacy axioms such as $\text{PD}$ and $\text{AD}^L(\mathbb{R})$ imply the consistency of $\text{ZFC}$ (plus certain large cardinal statements) and so cannot be proved in $\text{ZFC}$. Empirically, however, these
statements appear to follow from every natural statement of sufficient consistency strength. This includes a number of statements ostensibly having little relation to determinacy. In this section we give a few examples of this phenomenon. Most of these arguments use inner model theory, and our presentation relies heavily on [??].

The following theorem shows, among other things, that in the presence of large cardinals, even mere forcing-absoluteness for the theory of \(L(\mathbb{R})\) implies \(AD_{L(\mathbb{R})}\). The theorem is due to Steel and Woodin independently (see [??]).

**Theorem 6.15.** Suppose that \(\kappa\) is a measurable cardinal. Then the following are equivalent.

- For all partial orders \(P \in V_\kappa\), the theory of \(L(\mathbb{R})\) is not changed by forcing with \(P\).
- For all partial orders \(P \in V_\kappa\), \(AD\) holds in \(L(\mathbb{R})\) after forcing with \(P\).
- For all partial orders \(P \in V_\kappa\), all sets of reals in \(L(\mathbb{R})\) are Lebesgue measurable after forcing with \(P\).
- For all partial orders \(P \in V_\kappa\), there is no \(\omega_1\)-sequence of reals in \(L(\mathbb{R})\) after forcing with \(P\).

A sequence \(C = \langle C_\alpha : \alpha < \lambda \rangle\) (for some ordinal \(\lambda\)) is said to be **coherent** if each \(C_\beta\) is a club subset of \(\beta\), and \(C_\alpha = \alpha \cap C_\beta\) whenever \(\alpha\) is a limit point of \(C_\beta\). A **thread** of such a coherent sequence \(C\) is a club set \(D \subseteq \lambda\) such that \(C_\alpha = \alpha \cap D\) for all limit points \(\alpha\) of \(D\). The principle \(\square(\lambda)\) says that there is a coherent sequence of length \(\lambda\) with no thread. The principle \(\square_\kappa\) says that there is a coherent sequence \(C\) of length \(\kappa^+\) such that the ordertype of \(C_\alpha\) is at most \(\kappa\), for each limit \(\alpha < \lambda\) (in which case there cannot be a thread). These principles were isolated in the 1960’s by Jensen [??], who showed that \(\square_\kappa\) holds in \(L\) for all infinite cardinals \(\kappa\) (see [??], p. 141).

Todorcevic [??] showed that the Proper Forcing Axiom (PFA) implies that \(\square(\kappa)\) fails for all cardinals \(\kappa\) of cofinality at least \(\omega_2\), from which it follows that \(\square_\kappa\) fails for all uncountable cardinals. The failure of these square principles implies the failure of covering theorems for certain inner models, from which one can derive inner models with large cardinals. Using this general approach, Ernest Schimmerling [??] proved that PFA implies \(\Delta^1_2\)-determinacy. Woodin extended this proof to show that PFA implies \(\text{PD}\).

In 1990, Woodin also showed that PFA plus the existence of a strongly inaccessible cardinal implies \(AD_{L(\mathbb{R})}\). His proof introduced a technique known as the core model induction, an application of descriptive set theory and inner model theory. Roughly, the idea is to inductively work through the Wadge degrees to build canonical inner models which are correct for each Wadge class. The induction works through the gap structure highlighted
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This general approach had previously been used by Kechris and Woodin (see the end of Section 4.2).

Alessandro Andretta, Neeman and Steel showed that PFA plus the existence of a measurable cardinal implies the existence of a model of AD containing all the reals and ordinals. Steel showed that if \( \Box_\kappa \) fails for a singular strong limit cardinal \( \kappa \), then AD holds in \( L(\mathbb{R}) \). Building on Steel’s work, Sargsyan produced a model of \( AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”} \) from the same hypothesis.

The following theorem is due to Steel. Schimmerling had previously obtained PD from the same assumption.

**Theorem 6.16.** If \( \kappa \geq \max\{\aleph_2, c\} \) and \( \Box(\kappa) \) and \( \Box_\kappa \) fail, then AD holds in \( L(\mathbb{R}) \).

Todorcevic and Boban Veličković showed that PFA implies that \( 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 \). This gives another route towards showing that PFA implies that the AD holds in \( L(\mathbb{R}) \). In May 2011, Andrés Caicedo, Larson, Sargsyan, Ralf Schindler, Steel and Martin Zeman showed that the hypothesis of Theorem 6.16 (with \( \kappa = \aleph_2 \)) can be forced (using \( P_{\max} \)) over a model of AD in which \( \Theta \) and some other member of the Solovay sequence are both regular.

Schimmerling and Zeman used the core model induction to prove the following theorem. They had previously derived Projective Determinacy from the failure of a weaker version of \( \Box_\kappa \) at a weakly compact cardinal; Woodin had then derived AD in \( L(\mathbb{R}) \).

**Theorem 6.17.** If \( \kappa \) is a weakly compact cardinal and \( \Box_\kappa \) fails, then AD holds in \( L(\mathbb{R}) \).

As discussed in Section 6.4, Woodin showed using a variation of \( P_{\max} \) that over a model of AD one can force to produce a model of ZFC in which the nonstationary ideal on \( \omega_1 \) is \( \aleph_1 \)-dense. Using the core model induction, he showed that the \( \aleph_1 \)-density of \( NS_{\omega_1} \) implies AD in \( L(\mathbb{R}) \).

Steel had previously shown, using inner models, that Projective Determinacy follows from CH plus the existence of a homogeneous ideal on \( \omega_1 \) (a weaker assumption that the \( \aleph_1 \)-density of \( NS_{\omega_1} \), which is in fact inconsistent with CH, by a theorem of Shelah). He had also shown that if \( NS_{\omega_1} \) is saturated and there is a measurable cardinal, then \( \Delta^1_2 \)-determinacy holds. The hypothesis of the measurable cardinal was later removed in collaboration with Jensen.

Using the core model induction, Richard Ketchersid showed that if the restriction of \( NS_{\omega_1} \) to some stationary set \( S \subseteq \omega_1 \) is \( \aleph_1 \)-dense, and the restriction of the generic elementary embedding corresponding to forcing with \( P(S)/NS_{\omega_1} \) to each ordinal is an element of the ground model,
then there is a model of $\text{AD}^+ + \psi_0 < \Theta$ containing the reals and the ordinals. Also using this method, Sargsyan would deduce the consistency of $\text{AD}_\omega + \text{"\Theta is regular"}$ from the same hypothesis. This gives an equiconsistency, as Woodin has shown how to force the hypothesis over a model of $\text{AD}_\omega + \text{"\Theta is regular."}$ In yet another application of the core model induction, Steel and Stuart Zoble [??] derived $\text{AD}^{L(\mathbb{R})}$ from a consequence of Martin’s Maximum isolated by Todorcevic, known as the Strong Reflection Principle at $\omega_2$.

We conclude with three more examples. Silver [??] proved that if $\kappa$ is a singular cardinal of uncountable cofinality and $2^\alpha = \alpha^+$ for club many $\alpha < \kappa$, then $2^\kappa = \kappa^+$. Gitik and Schindler (see [??]) showed that if $\kappa$ is a singular cardinal of uncountable cofinality and the set of $\alpha < \kappa$ for which $2^\alpha = \alpha^+$ is stationary and costationary, then PD holds. Schindler (in the same paper) showed that if $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} > \aleph_\omega$, then PD holds. It is not known whether either of these results can be strengthened to obtain $\text{AD}^{L(\mathbb{R})}$.

A cardinal $\kappa$ is said to have the Tree Property if every tree of height $\kappa$ with all levels of cardinality less than $\kappa$ has a cofinal branch (i.e., if there are no $\kappa$-Aronszajn trees). Foreman, Magidor and Schindler [??] showed that if there exist infinitely many cardinals $\delta$ above the continuum such that the tree property holds at $\delta$ and at $\delta^+$, then PD holds. The hypothesis of this statement had been shown consistent relative to the existence of infinitely many supercompact cardinals by James Cummings and Foreman [??]. It is not known whether the conclusion can be strengthened to $\text{AD}^{L(\mathbb{R})}$.

Finally, as mentioned in Section 2.3, Gitik showed that if there is a proper class of strongly compact cardinals, then there is a model of ZF in which all infinite cardinals have cofinality $\omega$. Using the core model induction, Daniel Busche and Schindler [??] showed that this statement implies that PD holds, and that $\text{AD}$ holds in the $L(\mathbb{R})$ of a forcing extension of $\text{HOD}$. 

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