

# *D*-spaces, irreducibility and trees

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## Abstract

We show that the removal of one point from  $2^{\omega_1}$  gives a counterexample to a conjecture of Ishiu on *D*-spaces. We also show that Martin's Axiom implies that there are no Lindelöf non-*D*-spaces that can be written as union of less than continuum many compact subspaces. Finally we show that the property of being a *D*-space is preserved by forcing with trees of height  $\omega$ .

An *open neighborhood assignment* (ONA) on a topological space  $X$  is a function  $N$  which assigns to each point  $x \in X$  an open set  $N(x)$  containing  $x$ . Given an ONA  $N$  on a space  $X$ , and subset  $Y$  of  $X$ , we let  $N[Y]$  denote  $\bigcup\{N(x) \mid x \in Y\}$ . A space  $X$  is a *D-space* [11] if for every ONA  $N$  on  $X$  there is a closed discrete  $C \subseteq X$  such that  $N[C] = X$ . These spaces were introduced by van Dowen in 1979[11], and while they have attracted a lot of attention in recent years[2, 3, 4, 6, 7, 8, 9, 10, 13, 14], many basic questions remain open [12]. Probably the best known is whether every regular Lindelöf space is a *D*-space (see [16]).

In the first section we prove that removing one point from  $2^{\omega_1}$  gives a counterexample to a conjecture of Tetsuya Ishiu, as the resulting space is irreducible but not a *D*-space. In the second section, we prove that, assuming Martin's Axiom, there are no "small" Lindelöf non-*D*-spaces, where "small" means a union of less than continuum many compact subspaces. Finally, in the third section we consider the effects of forcing with trees of height  $\omega$ . For instance, we show that if  $T$  is such a tree and  $X$  is a *D*-space, then  $X$  remains a *D*-space after forcing with  $T$ .

## 1 Irreducibility and the Revised Range Conjecture

Tetsuya Ishiu proposed what he called the *Revised Range Conjecture*, asserting that every topological space  $X$  has a basis  $\mathcal{B}$  such that for any two ONA's  $N_0$ ,

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$N_1$  (on  $X$ ) with the same range  $R \subseteq \mathcal{B}$ , there is a closed discrete set  $C_0$  such that  $N_0[C_0] = X$  if and only if there is a closed discrete set  $C_1$  such that  $N_1[C_1] = X$ . We will see in this section that this conjecture is false.

A topological space  $X$  is said to be *irreducible* [1] if for every open cover  $\mathcal{O}$  of  $X$  there is an open cover  $\mathcal{O}'$  such that each element of  $\mathcal{O}'$  is contained in a member of  $\mathcal{O}$  and contains a point not in any other member of  $\mathcal{O}'$  (such a  $\mathcal{O}'$  is said to be a *minimal open refinement* of  $\mathcal{O}$ ).

**Lemma 1.1.** *Let  $X$  be an irreducible space in which every open set has the same cardinality. If the Revised Range Conjecture holds for  $X$ , then  $X$  is a  $D$ -space.*

*Proof.* Let  $N$  be an ONA on  $X$ . We may assume that the range of  $N$  is contained in a basis  $\mathcal{B}$  witnessing the Revised Range Conjecture for  $X$ . Since  $X$  is irreducible, there exists a minimal open refinement  $\mathcal{O}'$  of the range of  $N$  covering  $X$ . For each  $O \in \mathcal{O}'$  pick a point in  $O$  not in any other member of  $\mathcal{O}'$ , and let  $Y$  be the set of picked points. Then  $Y$  is a closed discrete set, and we can define a partial ONA  $N'$  on  $Y$  by letting  $N'(y)$  be any member of the range of  $N$  containing the member of  $\mathcal{O}'$  containing  $y$ , for each  $y \in Y$ . It suffices now to extend  $N'$  to an ONA on all of  $X$  with the same range as  $N$ . Since  $Y$  is closed discrete, each open set has intersection of size  $|X|$  with the complement of  $Y$ . The range of  $N$  has cardinality  $\kappa \leq |X|$ . Let  $\langle B_\alpha : \alpha < \kappa \rangle$  be a wellordering of the range of  $N$ , and choose points  $\langle x_\alpha : \alpha < \kappa \rangle$  such that each  $x_\alpha \in B_\alpha \setminus (Y \cup \{x_\beta : \beta < \alpha\})$ , and define  $N'(x_\alpha) = B_\alpha$  for each  $\alpha < \kappa$ . For each  $x \in X \setminus (Y \cup \{x_\alpha : \alpha < \kappa\})$ , let  $N'(x) = N(x)$ .  $\square$

In [18] it was shown that the removal of one point from  $2^{\omega_1}$  gives an irreducible space. However, this space is not a  $D$ -space, as shown by the following lemma. Note that every open subset of this space has the same cardinality.

**Lemma 1.2.** *The space  $2^{\omega_1}$  with one point removed contains a closed copy of  $\omega_1$ .*

*Proof.* For simplicity, let the removed point be the constant 0 function. For each  $\alpha < \omega_1$ , let  $x_\alpha$  be  $(\alpha \times \{0\}) \cup ((\omega_1 \setminus \alpha) \times \{1\})$ . The subspace  $\{x_\alpha : \alpha < \omega_1\}$  is closed. Furthermore, if for each  $\beta < \omega_1$ , we let  $O_\beta = \{z \in 2^{\omega_1} \mid z(\beta) = 0\}$  and  $I_\beta = \{z \in 2^{\omega_1} \mid z(\beta) = 1\}$ , then the  $O_\beta$ 's and  $I_\beta$ 's generate the  $\omega_1$ -topology on  $\{x_\alpha : \alpha < \omega_1\}$ .  $\square$

## 2 Lindelöfness and Martin's Axiom

Our second section concerns Lindelöf non- $D$ -spaces and Martin's Axiom (MA). Recall that Martin's Axiom is the statement that if  $P$  is a partial order without uncountable antichains, and  $\mathcal{D}$  is a collection of dense subsets of  $P$  such that  $|\mathcal{D}| < \mathfrak{c}$  (where  $\mathfrak{c}$  denotes the cardinality of the continuum), then there is a filter  $G \subseteq P$  intersecting each element of  $\mathcal{D}$  (see [17], for instance). The *covering number for the meager ideal* ( $cov(\mathcal{M})$ ) is the smallest cardinality of a family of

meager sets of reals whose union is all of  $\mathbb{R}$  (see [5], for instance); restated, it is the smallest cardinality of a collection  $\mathcal{D}$  consisting of dense subsets of the partial order  $(\omega^\omega, \subseteq)$  with the property that no filter intersects every member of  $\mathcal{D}$ . The Baire Category Theorem implies that  $\text{cov}(\mathcal{M}) \geq \aleph_1$ . Martin's Axiom (indeed, its restriction to Cohen forcing) implies that  $\text{cov}(\mathcal{M}) = \mathfrak{c}$ .

We first prove that there are no Lindelöf non- $D$ -spaces of cardinality less than  $\text{cov}(\mathcal{M})$ . One easy consequence is that one can not prove, only assuming ZFC, that there is such space of cardinality  $\aleph_1$ .

We begin with the following.

**Lemma 2.1.** *If  $X$  is a  $T_1$  Lindelöf space and  $N$  is an open neighborhood assignment on  $X$ , then there is a countable  $Y \subseteq X$  such that for every finite  $a \subseteq Y$  and every  $x \in X \setminus N[a]$  there is a  $y \in Y \setminus N[a]$  such that  $x \in N[y]$ .*

*Proof.* We find countable sets  $Y_i \subseteq X$  ( $i < \omega$ ), and let  $\langle a_i : i < \omega \rangle$  be a listing of all the finite subsets of  $\bigcup_{i < \omega} Y_i$ , such that each  $a_i \subseteq \bigcup_{j < i} Y_j$ . Let  $Y_0$  be any countable subset of  $X$  such that  $N[Y_0] = X$ . Given  $a_i$ , consider the open cover of  $X$  given by the restriction of  $N$  to  $a_i \cup (X \setminus N[a_i])$ , and let  $Y_{i+1}$  be the set of  $x$  such that  $N(x)$  is in some fixed countable subcover. Then  $Y = \bigcup_{i < \omega} Y_i$  is as desired.  $\square$

**Theorem 2.2.** *If  $X$  is a  $T_1$  Lindelöf space and  $|X| < \text{cov}(\mathcal{M})$ , then  $X$  is a  $D$ -space.*

*Proof.* Let  $Y = \langle y_i : i < \omega \rangle$  be as in Lemma 2.1, and consider the set  $A$  of  $a \in 2^\omega$  such that for each  $i \in a^{-1}(1)$ ,  $y_i \notin N[\{y_j : j \in i \cap a^{-1}(1)\}]$ . Then  $A$  is a perfect subset of  $2^\omega$ , and for each  $x \in X$  the set of  $a \in A$  with  $x \notin N[\{y_i : i \in a\}]$  is nowhere dense in  $A$ . Since  $|X| < \text{cov}(\mathcal{M})$ , there is an  $a \in A$  such that  $N[a] = X$ .  $\square$

The assumptions of the Theorem 2.2 are implied by MA(Cohen forcing) when  $|X| < \mathfrak{c}$  (see [5]).

**Corollary 2.3** (MA(Cohen forcing)). *If  $X$  is a Lindelöf space such that  $|X| < 2^\omega$ , then  $X$  is a  $D$ -space.*

**Corollary 2.4** (MA(Cohen forcing)). *If  $X$  is a hereditary Lindelöf space such that it is not a  $D$ -space, then  $|X| = 2^\omega$ .*

*Proof.* This is immediate, since if  $Y$  is a hereditary Lindelöf space then  $|Y| \leq 2^\omega$ , by a result of de Groot (see [15]).  $\square$

Modifying the proof of Theorem 2.2, we can obtain a stronger result. First we note the following consequence of Lemma 2.1.

**Lemma 2.5.** *Let  $X$  be a Lindelöf space and  $N$  be an open neighborhood assignment on  $X$ . Then there is a countable  $Y \subseteq X$  such that for every finite  $a \subseteq Y$ , there is  $b \subseteq Y \setminus N[a]$  such that  $X = N[a] \cup N[b]$ .*

Let  $(X, \tau)$  be a topological space. Let  $f : \omega^{<\omega} \longrightarrow X \times \tau$  be a function. If  $s \in \omega^{<\omega}$  and  $f(s) = (x, V)$ , then we denote by  $f_X(s) = x$  and by  $f_\tau(s) = V$ .

The idea for the next lemma is the following: we will construct an  $\omega$ -tree using the  $Y$  given by the previous lemma. The successors of every element of the tree will be all the points of  $Y$  that are “not yet covered” by our construction. At the same time we will assure that every finite subset of  $Y$  that is not yet covered can be added to the tree in finitely many steps.

**Lemma 2.6.** *Let  $(X, \tau)$  be a Lindelöf space and  $N$  be an open neighborhood assignment on  $X$ . Then there is  $f : \omega^{<\omega} \setminus \{\emptyset\} \longrightarrow X \times \tau$  such that:*

- (i) if  $s \in \omega^{<\omega} \setminus \{\emptyset\}$  then  $f_\tau(s) \subseteq N(f_X(s))$ ;
- (ii) if  $r$  is a branch of  $\omega^{<\omega}$ , then  $\{f_X(s) : s \in r\}$  is closed discrete in  $\bigcup\{f_\tau(s) : s \in r\}$ ;
- (iii) if  $C \subseteq X$  is compact, then  $D_C = \{s \in \omega^{<\omega} : C \subseteq \bigcup_{t \leq s} f_\tau(t)\}$  is dense in  $\omega^{<\omega}$ .

*Proof.* Let  $Y$  be as given by Lemma 2.5. We will define  $f : \omega^{<\omega} \setminus \{\emptyset\} \longrightarrow Y \times \tau$  by recursion on the length of  $s$  in such a way that:

- (a) if  $s \in \omega^{<\omega}$  then for every  $n \in \omega$  and every nonzero  $k \leq |s|$ ,  $f_X(s^\frown n) \notin f_\tau(s \upharpoonright k)$ ;
- (b) if  $s \in \omega^{<\omega}$  then  $f_\tau(s) = N(f_X(s)) \setminus F$  where  $F$  is a finite subset of  $Y \setminus \{f_X(s)\}$ ;
- (c) for every  $s \in \omega^{<\omega}$ , if  $y \in Y \setminus \bigcup\{f_\tau(s \upharpoonright k) : 0 < k \leq |s|\}$ , then there is an  $n \in \omega$  such that  $y = f_X(s^\frown n)$ ;
- (d) if  $y = f_X(s^\frown n)$  for some  $s \in \omega^{<\omega}$  and  $n \in \omega$ , then for each finite  $F \subseteq (Y \cap N(y)) \setminus \{y\}$  there is a  $k \in \omega$  such that  $f(s^\frown k) = (y, N(y) \setminus F)$ ;

Note that we can make this construction since  $Y$  is countable and so is  $[Y]^{<\omega}$ .

First we will show that if  $r$  is a branch of  $\omega^{<\omega} \setminus \{\emptyset\}$ , then  $\{f_X(s) : s \in r\}$  has no accumulation points in  $\bigcup\{f_\tau(s) : s \in r\}$ . Let  $x \in \bigcup\{f_\tau(s) : s \in r\}$ . We will show that it is not an accumulation point of  $\{f_X(s) : s \in r\}$ . Let  $s \in r$  such that  $x \in f_\tau(s)$ . Note that  $f_X(t) \notin f_\tau(s)$  for every  $t \in r$ ,  $t > s$ . Then  $x$  is separated from these points and, since there are only finitely many points more in  $r$ , we have that  $x$  is not an accumulation point.

Note that, by Lemma 2.5, we have that, for every  $s \in \omega^{<\omega} \setminus \{\emptyset\}$ ,

$$\bigcup\{f_\tau(s \upharpoonright k) : 0 < k \leq |s|\} \cup \bigcup_{n \in \omega} f_\tau(s^\frown n) = X.$$

For each  $C \subseteq X$ , let  $D_C$  denote the set of  $s \in \omega^{<\omega} \setminus \{\emptyset\}$  such that  $C$  is contained in  $\bigcup\{f_\tau(s \upharpoonright k) : 0 < k \leq |s|\}$ . We will show that when  $C$  is compact,  $D_C$  is dense in  $\omega^{<\omega} \setminus \{\emptyset\}$ . Fix  $s \in \omega^{<\omega}$  and suppose  $s \notin D_C$ . Then  $C \setminus \bigcup\{f_\tau(s \upharpoonright k) : 0 < k \leq |s|\}$

$|s|$  is a compact subset covered by the family  $(f_\tau(s \frown n))_{n \in \omega}$ . Then, there are  $k_1, \dots, k_n \in \omega$  such that

$$C \setminus \bigcup \{f_\tau(s \frown k) : 0 < k \leq |s|\} \subseteq f_\tau(s \frown k_1) \cup \dots \cup f_\tau(s \frown k_n).$$

We can suppose that  $f_X(s \frown k_i) \notin f_\tau(s \frown k_j)$  for all  $1 \leq i < j \leq n$  by property (d). Thus, we can choose  $p_1, \dots, p_n \in \omega$  such that  $f(s \frown p_1) = f(s \frown k_1)$ ,  $f(s \frown p_1 \frown p_2) = f(s \frown k_2)$ , ...,  $f(s \frown p_1 \frown p_2 \frown \dots \frown p_n) = f(s \frown k_n)$ , by property (c). Note that  $C \subseteq \bigcup_{t \leq s \frown p_1 \frown p_2 \frown \dots \frown p_n} f_\tau(t)$ .  $\square$

It was already known that every  $\sigma$ -compact space is a  $D$ -space (see [6], for instance). Lemma 2.6 allows us to improve this result.

**Theorem 2.7.** *Every Lindelöf space which is a union of fewer than  $\text{cov}(\mathcal{M})$  many compact spaces is a  $D$ -space.*

*Proof.* Suppose that  $\kappa$  is a cardinal less than  $\text{cov}(\mathcal{M})$ , and that  $X = \bigcup_{\xi < \kappa} C_\xi$ , where each  $C_\xi$  is compact. Let  $f : \omega^{<\omega} \rightarrow X \times \tau$  be the function given by Lemma 2.6. Note that for every  $\xi < \kappa$  we have that  $D_{C_\xi}$  (as defined in the proof of Lemma 2.6) is dense in  $\omega^{<\omega}$ . Then there is a branch  $r$  of  $\omega^{<\omega}$  such that  $r \cap D_{C_\xi} \neq \emptyset$  for each  $\xi < \kappa$ . Thus  $\bigcup_{s \in r} f_\tau(s) \supseteq \bigcup_{\xi < \kappa} C_\xi = X$ . Since  $\{f_X(s) : s \in r\}$  is closed discrete in  $\bigcup_{s \in r} f_\tau(s)$ , we have that  $\{f_X(s) : s \in r\}$  is closed discrete in  $X$ .  $\square$

### 3 Forcing with trees of height $\omega$

The results of the previous section suggest that some basic facts about  $D$ -spaces may be independent of ZFC. While we do not have such a result, we present in this section two facts about  $D$ -spaces and forcing which may be of some use. These facts concern forcing with trees of height  $\omega$ , and apply the approach of the previous section.

**Theorem 3.1.** *If  $X$  is a  $D$ -space and  $T$  is a tree of height  $\omega$ , then  $X$  remains a  $D$ -space after forcing with  $T$ .*

*Proof.* Let  $\dot{N}$  be a  $T$ -name for an ONA on  $X$ , and let  $\mathcal{B}$  be the set of open subsets of  $X$  which are forced by some condition to be in the range of the realization of  $\dot{N}$ . Let  $\langle p_\alpha : \alpha < \kappa \rangle$  be a wellordering of the elements of  $T$  such that shorter elements are listed before longer ones. We define recursively on  $\alpha$  closed discrete sets  $D_\alpha$  ( $\alpha < \kappa$ ) and functions  $f_\alpha : D_\alpha \rightarrow T$  and  $h_\alpha : D_\alpha \rightarrow \mathcal{B}$  such that, letting  $Y_\alpha$  be the set of  $x$  in any  $D_\beta$  ( $\beta < \alpha$ ) such that  $f_\beta(x) \geq p_\alpha$ :

- for all  $x$  in any  $D_\alpha$ ,  $f_\alpha(x) \leq p_\alpha$  and  $f_\alpha(x) \Vdash \dot{N}(\check{x}) = h_\alpha(x)$ ;
- for all  $\alpha < \kappa$  and for all  $y \in X$ , either there exist  $\beta < \alpha$  and  $x \in D_\beta \cap Y_\alpha$  such that  $y \in h_\beta(x)$  or there exists an  $x \in D_\alpha$  such that  $y \in h_\alpha(x)$ ;
- if  $\beta < \alpha$  and  $x \in D_\beta \cap Y_\alpha$ , then  $h_\beta(x) \cap D_\alpha = \emptyset$ .

(Note that since  $f_\beta(x) \leq p_\beta$  for each  $\beta < \kappa$  and each  $x \in X$ ,  $p_\beta \geq p_\alpha$  whenever  $D_\beta \cap Y_\alpha$  is nonempty; in particular there are only finitely many such  $\beta$ , so  $Y_\alpha$  is closed discrete.)

Supposing that we have constructed  $D_\alpha$ ,  $f_\alpha$  and  $h_\alpha$  for all  $\beta < \alpha$ , let

$$E_\alpha = \bigcup \{h_\beta(x) \mid \beta < \alpha \wedge x \in D_\beta \cap Y_\alpha\}.$$

We define a new ONA  $N_\alpha$  as follows. For each  $x \in E_\alpha$ , let  $N_\alpha(x) = E_\alpha$ . For each  $x \in X \setminus E_\alpha$ , pick a condition  $p(x) \leq p_\alpha$  and an element  $B(x) \in \mathcal{B}$  such that  $p(x) \Vdash \dot{N}(x) = B(x)$ , and let  $N_\alpha(x) = B(x)$ . Then there is a closed discrete set  $D_\alpha^*$  such that  $N_\alpha[D_\alpha^*] = X$ . Let  $D_\alpha = D_\alpha^* \setminus E_\alpha$ . For each  $x \in D_\alpha$ , let  $f_\alpha(x) = p(x)$  and let  $h_\alpha(x) = B(x)$ . This completes the construction.

Let  $g$  be a  $V$ -generic path through  $T$ . For each  $\alpha \in \kappa$ , let  $C_\alpha$  be the set of  $x \in D_\alpha$  such that  $f_\alpha(x) \in g$ . Let

$$C = \bigcup \{C_\alpha \mid \alpha \in \kappa\}.$$

By genericity  $\dot{N}_g[C] = X$ . We will be done once we show that  $C$  is closed discrete.

Pick a point  $y$  in  $X$ . There is a  $x \in C_\beta$  for some  $\beta \in \kappa$  such that  $y \in \dot{N}_g(x)$ . Since  $f_\beta(x) \in g$ ,  $\dot{N}_g(x) = h_\beta(x)$ . Fix  $\gamma < \kappa$  such that  $p_\gamma \in g$ ,  $\gamma > \beta$  and  $p_\gamma \leq f_\beta(x)$ . Since  $D_\alpha \cap h_\beta(x) = \emptyset$  for all  $\alpha > \beta$  with  $p_\alpha \leq f_\beta(x)$ ,  $y$  is not in the closure of

$$\bigcup \{D_\alpha \mid \gamma \leq \alpha < \kappa, p_\alpha \in g\},$$

which contains  $\bigcup \{C_\alpha : \gamma \leq \alpha < \kappa\}$ . On the other hand,

$$C \subseteq \bigcup \{D_\alpha : p_\alpha > p_\gamma\} \cup \bigcup \{C_\alpha \mid \gamma \leq \alpha < \kappa\}.$$

Since  $\bigcup \{D_\alpha : p_\alpha > p_\gamma\}$  is a finite union of closed discrete sets,  $y$  is not in the closure of  $\bigcup \{D_\alpha : p_\alpha > p_\gamma\} \setminus \{y\}$ , either, which shows that  $C$  is closed discrete.  $\square$

A similar argument shows the following result, where we start with a Lindelöf space in the ground model. If  $T$  is a tree and  $S$  is a subset of  $T$ , we say that  $S$  can be refined to an antichain if there is a function  $a: S \rightarrow T$  such that  $a(s) \leq s$  for all  $s \in S$ , and such that the range of  $S$  is an antichain.

**Theorem 3.2.** *If  $X$  is a Lindelöf space and  $T$  is a tree of height  $\omega$  such that every countable subset of  $T$  can be refined to an antichain, then  $X$  is a  $D$ -space after forcing with  $T$ .*

*Proof.* Let  $\dot{N}$  be a  $T$ -name for an ONA on  $X$ , and let  $\mathcal{B}$  be the set of open subsets of  $X$  which are forced by some condition to be in the range of the realization of  $\dot{N}$ . Let  $\langle p_\alpha : \alpha < \kappa \rangle$  be a wellordering of the elements of  $T$  such that shorter elements are listed before longer ones. We define recursively on  $\alpha$  countable sets  $D_\alpha \subseteq X$  ( $\alpha < \kappa$ ) and functions  $f_\alpha: D_\alpha \rightarrow T$  and  $h_\alpha: D_\alpha \rightarrow \mathcal{B}$  such that, letting  $Y_\alpha$  be the set of  $x$  in any  $D_\beta$  ( $\beta < \alpha$ ) such that  $f_\beta(x) \geq p_\alpha$ :

- for all  $x$  in any  $D_\alpha$ ,  $f_\alpha(x) \leq p_\alpha$  and  $f_\alpha(x) \Vdash \dot{N}(\check{x}) = h_\alpha(x)$ ;
- for all  $\alpha < \kappa$  and for all  $y \in X$ , either there exist  $\beta < \alpha$  and  $x \in D_\beta \cap Y_\alpha$  such that  $y \in h_\beta(x)$  or there exists an  $x \in D_\alpha$  such that  $y \in h_\alpha(x)$ ;
- if  $\beta < \alpha$  and  $x \in D_\beta \cap Y_\alpha$ , then  $h_\beta(x) \cap D_\alpha = \emptyset$ ;
- the range of each  $f_\alpha$  is an antichain.

(Note that since  $f_\beta(x) \leq p_\beta$  for each  $\beta < \kappa$  and each  $x \in X$ ,  $p_\beta \geq p_\alpha$  whenever  $D_\beta \cap Y_\alpha$  is nonempty; in particular there are only finitely many such  $\beta$ , so  $Y_\alpha$  is finite.)

Supposing that we have constructed  $D_\alpha$ ,  $f_\alpha$  and  $h_\alpha$  for all  $\beta < \alpha$ , let

$$E_\alpha = \bigcup \{h_\beta(x) \mid \beta < \alpha \wedge x \in D_\beta \cap Y_\alpha\}.$$

We define a new ONA  $N_\alpha$  as follows. For each  $x \in E_\alpha$ , let  $N_\alpha(x) = E_\alpha$ . For each  $x \in X \setminus E_\alpha$ , pick a condition  $p(x) \leq p_\alpha$  and an element  $B(x) \in \mathcal{B}$  such that  $p(x) \Vdash \dot{N}(\check{x}) = B(x)$ , and let  $N_\alpha(x) = B(x)$ . Then there is a countable set  $D_\alpha^*$  such that  $N_\alpha[D_\alpha^*] = X$ . Let  $D_\alpha = D_\alpha^* \setminus E_\alpha$ . For each  $x \in D_\alpha$ , let  $f_\alpha(x)$  be a condition below  $p(x)$  in such a way that the range of  $f_\alpha$  is an antichain, and let  $h_\alpha(x) = B(x)$ . This completes the construction.

Let  $g$  be a  $V$ -generic path through  $T$ . For each  $\alpha \in \kappa$ , let  $C_\alpha$  be the set of  $x \in D_\alpha$  such that  $f_\alpha(x) \in g$ . Let

$$C = \bigcup \{C_\alpha \mid \alpha \in \kappa\}.$$

By genericity  $\dot{N}_g[C] = X$ . We will be done once we show that  $C$  is closed discrete.

Pick a point  $y$  in  $X$ . There is a  $x \in C_\beta$  for some  $\beta \in \kappa$  such that  $y \in \dot{N}_g(x)$ . Since  $f_\beta(x) \in g$ ,  $\dot{N}_g(x) = h_\beta(x)$ . Fix  $\gamma < \kappa$  such that  $p_\gamma \in g$ ,  $\gamma > \beta$  and  $p_\gamma \leq f_\beta(x)$ . Since  $D_\alpha \cap h_\beta(x) = \emptyset$  for all  $\alpha > \beta$  with  $p_\alpha \leq f_\beta(x)$ ,  $y$  is not in the closure of

$$\bigcup \{D_\alpha \mid \gamma \leq \alpha < \kappa, p_\alpha \in g\},$$

which contains  $\bigcup \{C_\alpha : \gamma \leq \alpha < \kappa\}$ . On the other hand, letting

$$Z = \bigcup \{\{z \in D_\alpha \mid f_\alpha(z) \in g\} : \alpha < \gamma\},$$

$$C \subseteq Z \cup \bigcup \{C_\alpha \mid \gamma \leq \alpha < \kappa\}.$$

Since  $Z$  is a finite set,  $y$  is not in the closure of  $Z \setminus \{y\}$ , either, which shows that  $C$  is closed discrete.  $\square$

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