GUESSING CLUBS IN THE GENERALIZED CLUB FILTER

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Abstract. We present principles for guessing clubs in the generalized club filter on $\mathcal{P}_\kappa\lambda$. These principles are shown to be weaker than classical diamond principles but often serve as sufficient substitutes. One application is a new construction of a λ^+-Suslin-tree using assumptions different from previous constructions. The other application partly solves open problems regarding the cofinality of reflection points for stationary subsets of $[\lambda]^{\aleph_0}$.

1. Introduction

Club guessing principles have been studied intensely in the literature, a major source being [10]. But in all of these references, the guessing sequences anticipate clubs of ordinals. The purpose of this note is to introduce principles that guess clubs in the generalized club filter on $\mathcal{P}_\kappa\lambda$. Throughout the whole paper, the notion of a club always refers to the club filter that is generated by the sets

$$\mathcal{C}_f = \{x \in \mathcal{P}_\kappa\lambda : x \text{ is closed under } f\},$$

where $f : {}^{<\omega}\lambda \longrightarrow \lambda$. Some references refer to this as 'strongly club' as opposed to 'Jech clubs' which are unbounded sets that are closed under chains of length less than κ. We generally prefer to write $[\lambda]^{<\kappa}$ for $\mathcal{P}_\kappa\lambda$. For the future, we will also make the implicit assumption that κ and λ are regular cardinals and usually $\kappa \leq \lambda$.

Section 2 of this article contains the definition of and basic facts about $\curlywedge^*$, the newly introduced principle. We show in Section 3 that the guessing of clubs in the generalized club filter is a fairly weak assumption if the guessing attempts are on ordinals of small cofinality. For example, $2^\lambda = \lambda^+$ would suffice to guarantee a variety of guessing

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principles for clubs in $[\lambda^+]^{<\kappa}$. In Section 4 we give an independence result using iterated forcing to demonstrate that even full GCH does not imply $⅄^*$ if the guessing attempts are made on ordinals of maximal cofinality. The last two sections deal with applications: in Section 5, a λ^+-Suslin-tree is constructed from GCH and a club guessing principle on ordinals of maximal cofinality. This is related to the old problem if GCH always constructs a Suslin-tree on successors of regular cardinals. Finally, in Section 6 we show that guessing on ordinals of cofinality ω can be used to thin out stationary subsets of $[\omega_n]^{\aleph_0}$ and thereby remove all possible reflection points with countable cofinality but preserving stationarity of the original set. Problems regarding the cofinality of reflection points for stationary subsets of $[\lambda]^{\aleph_0}$ have frequently been asked in the literature.

As additional guidelines for general set theory, we recommend the sources [6] and [8]. For more information about issues related to proper forcing and iterations, we suggest [11]. The authors would like to thank Dieter Donder, Yo Matsubara, and Hiroshi Sakai for their helpful comments.

2. The principle

The following definition seems to be in the spirit of guessing clubs in the generalized club filter on $[\lambda]^{<\kappa}$.

1 Definition. Let $E \subseteq \lambda$ be stationary. Then $⅄^*(\kappa, E)$ is the statement that there is a sequence $\langle \mathcal{F}_\delta : \delta \in E \rangle$ such that

(1) $\mathcal{F}_\delta$ is club in $[\delta]^{<\kappa}$ for all δ in E, and
(2) for all clubs $\mathcal{D} \subseteq [\lambda]^{<\kappa}$ there is a club $C \subseteq \lambda$ such that for all $\delta \in C \cap E$ we have that $\mathcal{F}_\delta \leq^* \mathcal{D}$.

Where in (2), $\mathcal{A} \leq^* \mathcal{B}$ means that there is x of size less than κ such that $x \subseteq y \in \mathcal{A}$ implies $y \in \mathcal{B}$ for all y.

We also say that $\langle \mathcal{F}_\delta : \delta \in E \rangle$ is *tail club guessing*. Defining $\mathcal{A} \leq^* \mathcal{B}$ like this seems to be the right notion for saying that "a tail of $\mathcal{A}$ is included in $\mathcal{B}$" in the context of subsets of $[\lambda]^{<\kappa}$. The cardinal κ is supposed to be clear from the context whenever we use this notation. Note that our new statement can be viewed as a $\diamondsuit^*$ spin-off.[1] We will go on to show that $⅄^*(\kappa, E)$ is strictly weaker than $\diamondsuit^*(E)$. The following facts help determining the status of $⅄^*(\kappa, E)$:

[1] If $E \subseteq \lambda$ then $\diamondsuit^*(E)$ means that there is a sequence $\langle \mathcal{S}_\delta : \delta \in E \rangle$ where $|\mathcal{S}_\delta| \leq |\delta|$ and such that for every $S \subseteq \lambda$ there is a club $C \subseteq \lambda$ such that for all $\delta \in C \cap E$ we have that $S \cap \delta \in \mathcal{S}_\delta$. Standard arguments show that $\diamondsuit^*(\lambda)$ holds in the constructible universe if and only if λ is not ineffable [1, p.328].

2 Lemma. *Let $E \subseteq \lambda$ be stationary.*

(1) *$\diamondsuit^*(E)$ implies $\curlywedge^*(\kappa, E)$ for all $\kappa \leq \lambda$.*
(2) *$\curlywedge^*(\kappa, E)$ is preserved by κ-cc forcings.*

Proof. For (1), let $\diamondsuit^*(E)$ guess all functions $f : {}^{<\omega}\lambda \longrightarrow \lambda$ via a sequence $\langle \mathcal{S}_\delta : \delta \in E \rangle$. Then each $\mathcal{S}_\delta$ consists of less than λ-many functions $f^\delta_\xi : {}^{<\omega}\delta \longrightarrow \delta$ ($\xi < \mu$) that are guessing each $f : {}^{<\omega}\lambda \longrightarrow \lambda$ club many times. Now for each $\delta \in E$ find a club $\mathcal{F}_\delta$ in $[\delta]^{<\kappa}$ such that y is closed under f^δ_ξ whenever $\xi \in y \in \mathcal{F}_\delta$. One easily checks that this suffices.

(2) follows easily from the following standard fact:

2.1 Claim. *If $\mathcal{D} \subseteq [\lambda]^{<\kappa}$ is a club in some κ-cc extension, then there is a club $\mathcal{D}_0 \subseteq \mathcal{D}$ in the ground model.*

Proof of Claim 2.1. Let $\dot{\mathcal{D}}$ be a name for a club in $[\lambda]^{<\kappa}$ in the κ-cc extension. Then

$$\mathcal{D}_0 = \{x \in [\lambda]^{<\kappa} : \Vdash x \in \dot{\mathcal{D}}\}$$

is a club in V. □

□

Notice finally that both principles $\curlywedge^*(\kappa, E)$ and $\diamondsuit^*(E)$ increase in logical strength as E gets bigger.

3. Small cofinality

Let us denote the set $\{\gamma \in [\lambda, \lambda^+) : \omega \leq \mathrm{cf}(\gamma) < \kappa\}$ by $S^{<\kappa}_{\lambda^+}$. The next theorem shows that $\curlywedge^*(\kappa, S^{<\kappa}_{\lambda^+})$ is pretty weak in logical strength. When compared to Lemma 2(1), the assumptions needed here are far weaker than the previous $\diamondsuit^*(S^{<\kappa}_{\lambda^+})$.

3 Theorem. *Let $\kappa \leq \lambda$. The following are equivalent*[2]*:*

(i) *$\curlywedge^*(\kappa, S^{<\kappa}_{\lambda^+})$.*
(ii) *There is a club $\mathcal{F} \subseteq [\lambda^+]^{<\kappa}$ such that for every club $\mathcal{D} \subseteq [\lambda^+]^{<\kappa}$ there is a club $C \subseteq \lambda^+$ such that for all $\delta \in C \cap S^{<\kappa}_{\lambda^+}$ we have that $\mathcal{F} \cap [\delta]^{<\kappa} \leq^* \mathcal{D}$.*
(iii) *There are λ^+-many clubs in $[\lambda]^{<\kappa}$ such that the collection of these is cofinal in the $\geq^*$-ordering.*

Considering (ii), it is an interesting fact that the witness for the principle $\curlywedge^*(\kappa, S^{<\kappa}_{\lambda^+})$ can actually be taken to be a single club $\mathcal{F}$ in $[\lambda^+]^{<\kappa}$. We still chose to formulate $\curlywedge^*(\kappa, S^{<\kappa}_{\lambda^+})$ in the way given above because it is more in the style of classical guessing principles.

[2]The global assumption that λ is regular is actually not necessary for this particular theorem.

Proof of Theorem 3. It is straightforward to check that (ii) implies (i), so we show (iii)⇒(ii). To this end, suppose that the sequence $\langle \mathcal{C}_\alpha : \alpha < \lambda^+ \rangle$ is $\geq^*$-cofinal in the clubs on $[\lambda]^{<\kappa}$. For each $\eta \in [\lambda, \lambda^+)$ fix a bijection $g_\eta : \lambda \to \eta$ and let

$$g_{\eta *} : [\lambda]^{<\kappa} \to [\eta]^{<\kappa}$$

be the induced bijection. If $\lambda \leq \eta < \lambda^+$ and $\alpha < \lambda^+$, then we set $\mathcal{C}^\eta_\alpha = g_{\eta *}{}''\mathcal{C}_\alpha$. One checks that $\langle \mathcal{C}^\eta_\alpha : \alpha < \lambda^+ \rangle$ is $\geq^*$-cofinal in the clubs on $[\eta]^{<\kappa}$ for each $\eta \in [\lambda, \lambda^+)$. Now fix a bijection $g : [\lambda, \lambda^+) \times \lambda^+ \to \lambda^+$ and let

$$F_{g(\eta,\alpha)} = \{x \in [\lambda^+]^{<\kappa} : x \cap \eta \in \mathcal{C}^\eta_\alpha\}$$

for each $(\eta, \alpha) \in [\lambda, \lambda^+) \times \lambda^+$. Note that F_γ is club in $[\lambda^+]^{<\kappa}$ for each $\gamma < \lambda^+$. Define

$$\mathcal{F} = \mathop{\triangle}_{\gamma < \lambda^+} F_\gamma = \{x \in [\lambda^+]^{<\kappa} : \forall \gamma \in x \ (x \in F_\gamma)\}.$$

Clearly, $\mathcal{F}$ is a club subset of $[\lambda^+]^{<\kappa}$. We show that this $\mathcal{F}$ works: let $\mathcal{D}$ be any club subset of $[\lambda^+]^{<\kappa}$ generated by $f : {}^{<\omega}\lambda^+ \longrightarrow \lambda^+$. Then set

$$D = \{\eta \in [\lambda, \lambda^+) : \eta \text{ is closed under } f\}.$$

Note that D is a club subset of λ^+. For each $\eta \in D$ there exists an $h(\eta)$ such that $\mathcal{C}^\eta_{h(\eta)} \leq^* \mathcal{D} \cap [\eta]^{<\kappa}$. Let D' be the set of ordinals which are closed under both g and h and set $D'' = S^{<\kappa}_{\lambda^+} \cap D' \cap \lim D$. Note that D'' is a relative club subset of $S^{<\kappa}_{\lambda^+}$. For each $\delta \in D''$ there is an increasing sequence $\langle \delta_i : i < \mathrm{cf}(\delta) \rangle \subseteq D$ converging to δ. For each $i < \omega$ pick an $x_i \in [\delta_i]^{<\kappa}$ which witnesses $\mathcal{C}^{\delta_i}_{h(\delta_i)} \leq^* \mathcal{D} \cap [\delta_i]^{<\kappa}$, i.e. $x_i \subseteq y \in \mathcal{C}^{\delta_i}_{h(\delta_i)}$ implies $y \in \mathcal{D}$. Now let

$$x = \bigcup_{i < \mathrm{cf}(\delta)} \big(x_i \cup \{g(\delta_i, h(\delta_i))\}\big),$$

and note that $x \in [\delta]^{<\kappa}$. Whenever $x \subseteq y \in \mathcal{F} \cap [\delta]^{<\kappa}$ then we have $y \in F_{g(\delta_i, h(\delta_i))}$ and thus $y \cap \delta_i \in \mathcal{C}^{\delta_i}_{h(\delta_i)}$ holds. But since $x_i \subseteq y \cap \delta_i$ we have that $y \cap \delta_i$ is in $\mathcal{D}$ and therefore closed under f. Then clearly, y is closed under f. This shows that $\mathcal{F} \cap [\delta]^{<\kappa} \leq^* \mathcal{D}$.

Regarding (i)⇒(iii), let $\langle \mathcal{F}_\delta : \delta \in E \rangle$ be a $\curlywedge^*(\kappa, S^{<\kappa}_{\lambda^+})$-sequence and, for each $\delta \in S^{<\kappa}_{\lambda^+}$, fix a ladder δ_i $(i < \mathrm{cf}(\delta))$. We may assume without restriction for all $x \in \mathcal{F}_\delta$ that $\{\delta_i : i < \mathrm{cf}(\delta)\} \subseteq x$ and that x is closed under the bijections $g_{\delta_i} : \lambda \longrightarrow \delta_i$ used above. Now define a sequence of clubs in $[\lambda]^{<\kappa}$ by letting

$$\mathcal{C}_\delta = \{x \cap \lambda : x \in \mathcal{F}_\delta\}$$

for all $\delta \in S^{<\kappa}_{\lambda^+}$.

3.1 Claim. $\mathcal{C}_\delta$ $(\delta \in S^{<\kappa}_{\lambda^+})$ *is* $\geq^*$*-cofinal in the clubs on* $[\lambda]^{<\kappa}$.

Proof of Claim 3.1. If $\mathcal{C} \subseteq [\lambda]^{<\kappa}$ is club, define

$$\mathcal{D} = \{y \in [\lambda^+]^{<\kappa} : y \cap \lambda \in \mathcal{C}\}. \tag{3.1}$$

Then find $\delta \in S^{<\kappa}_{\lambda^+}$ such that $\mathcal{F}_\delta \leq^* \mathcal{D}$ witnessed by some $b \in [\delta]^{<\kappa}$, i.e.

$$\forall x \ \ b \subseteq x \in \mathcal{F}_\delta \text{ implies } x \in \mathcal{D}. \tag{3.2}$$

Now set

$$a = \bigcup_{i<\mathrm{cf}(\delta)} g^{-1}_{\delta_i} {}''(b \cap \delta_i). \tag{3.3}$$

We claim that $\mathcal{C}_\delta \leq^* \mathcal{C}$ is witnessed by a. So assume there is an $x \in \mathcal{F}_\delta$ such that $a \subseteq x \cap \lambda$. But x is closed under the relevant bijections g_{δ_i} for all $i < \mathrm{cf}(\delta)$, hence $b \subseteq x$. By (3.2), x is in $\mathcal{D}$ and therefore $x \cap \lambda \in \mathcal{C}$. This suffices. □

By Claim 3.1, we have found a collection of size λ^+ of clubs in $[\lambda]^{<\kappa}$ that is cofinal in the $\geq^*$-ordering. This finishes the proof. □

We remark that the simple cardinal arithmetic $2^\lambda = \lambda^+$ implies that there are λ^+-many clubs in $[\lambda]^{<\kappa}$ such that the collection of all these generates the club filter and is therefore cofinal in the $\geq^*$-ordering. It will be shown in the next section of this article that tail club guessing principles defined on ordinals of higher cofinality are considerably stronger than that.

As to the possible failure of $⅄^*(\kappa, S^{<\kappa}_{\lambda^+})$, if λ is regular one can add λ^{++}-many Cohen-subsets of λ to create a model in which the equivalent statements (i)-(iii) of Lemma 3 are false. We leave the details to the interested reader.

4. Maximal cofinality

We assume GCH throughout this section and remember that κ and λ are always assumed to be regular. Similar to previous notation, we denote the set $\{\gamma < \lambda^+ : \mathrm{cf}(\gamma) = \lambda\}$ by $S^\lambda_{\lambda^+}$. We want to investigate the status of $⅄^*(\kappa, S^\lambda_{\lambda^+})$: it will be shown that $⅄^*(\kappa, S^\lambda_{\lambda^+})$ does not follow from GCH which means that it is much stronger than $⅄^*(\kappa, S^{<\kappa}_{\lambda^+})$ and cannot be characterized analogously to Theorem 3.

Let $\langle \mathcal{F}_\delta : \delta \in S^\lambda_{\lambda^+} \rangle$ be a $⅄^*(\kappa, S^\lambda_{\lambda^+})$-sequence. Then define a forcing $\mathcal{Q}_\mathcal{F}$ in the following way: conditions are functions $f : {}^{<\omega}\gamma \longrightarrow \gamma$, where $\gamma < \lambda^+$ is such that for all $\beta \in S^\lambda_{\lambda^+} \cap (\gamma + 1)$ we have that $\mathcal{F}_\beta \not\leq^* \mathcal{C}_f$, where $\mathcal{C}_f$ is the club generated by f. The ordering on $\mathcal{Q}_\mathcal{F}$ is reverse inclusion.

4 Lemma. $\mathcal{Q}_\mathcal{F}$ *is* λ*-closed and* λ^+*-distributive.*

Proof. The λ-closure should be clear and to show λ^+-distributivity, let $\dot\tau$ be a name for an λ-sequence, g an arbitrary condition in $\mathcal{Q}_\mathcal{F}$, and M an elementary substructure of H_θ for some sufficiently large regular θ such that $|M| = \lambda$ with $\mathcal{F}, g, \dot\tau \in M$, ${}^{<\lambda}M \subseteq M$, and $\delta = M \cap \lambda^+ \in S^\lambda_{\lambda^+}$. Fix a sequence $\langle \delta_i : i < \lambda\rangle$ such that $\delta_i \nearrow \delta$ and let x_i $(i < \lambda)$ be an enumeration of $[\delta]^{<\kappa}$. Now build a descending sequence of conditions $\langle f_i : i < \lambda\rangle$ of $M \cap \mathcal{Q}_\mathcal{F}$ with $f_0 \leq g$ such that for all $i < \lambda$,

(1) ${}^{<\omega}\delta_i \subseteq \mathrm{dom}(f_i)$,
(2) f_i decides the value of $\dot\tau$ at i,
(3) there is $x_i \subseteq y \in \mathcal{F}_\delta$ such that y is not closed under f_i.

This construction can be carried out since $\mathcal{Q}_\mathcal{F}$ is λ-closed. Now let $p = \bigcup_{i<\lambda} f_i$. Note that $\mathrm{dom}(p) = {}^{<\omega}\delta$ by (1). But then p is a condition below g by (3) and decides $\dot\tau$ by (2). □

The argument for Lemma 4 also shows that every condition can be properly extended by increasing its domain. Thus it follows that a generic filter $G \subseteq \mathcal{Q}_\mathcal{F}$ adds a club $\mathcal{D}_G \subseteq [\lambda^+]^{<\kappa}$ whose existence destroys the tail club guessing properties of $\langle \mathcal{F}_\delta : \delta \in S^\lambda_{\lambda^+}\rangle$. The rest of the section depends heavily on iteration lemmas from [13], so we would like to remind the reader of some definitions from that paper.

5 Definition. If $S \subseteq S^\lambda_{\lambda^+}$ is stationary then a substructure $M \prec \mathrm{H}_\theta$ of size λ will be called S-*good* whenever $M \cap \lambda^+ \in S$ and M is closed under countable sequences. A λ-closed forcing notion $\mathcal{Q}$ is called *strongly S-complete* if for all but non-stationarily many S-good structures M we have that every $(M, \mathcal{Q})$-generic sequence of conditions $\langle q_\xi : \xi < \lambda\rangle$ has a lower bound in $\mathcal{Q}$.

We chose not to reproduce Shelah's notions in the most general form. Our presentation here is basically a special case of the machinery in [13]. The same holds for future definitions.

6 Lemma. *Strongly S-complete forcings are λ^+-distributive and preserve stationary subsets of S.* □

7 Remark. *Let $E \subseteq S^\lambda_{\lambda^+}$ be stationary. The poset $\mathcal{Q}_\mathcal{F}(E)$ is like above but with the weakened requirement that only for all $\beta \in E \cap (\gamma + 1)$ we have that $\mathcal{F}_\beta \not\leq^* \mathcal{C}_f$. Then $\mathcal{Q}_\mathcal{F}(E)$ is again λ-closed and λ^+-distributive. Moreover, if we let $S = S^\lambda_{\lambda^+} \setminus E$ then $\mathcal{Q}_\mathcal{F}(E)$ is strongly S-complete.*

Proof. The proof for λ-closure and λ^+-distributivity is as before in Lemma 4. So we only have to show that $\mathcal{Q}_\mathcal{F}(E)$ is strongly S-complete. To this end, let $N \prec \mathrm{H}_\theta$ be an S-good elementary substructure of size λ such that $\delta = N \cap \lambda^+ \in S$ and let $\langle p_i : i < \lambda\rangle$ be a $\mathcal{Q}_\mathcal{F}(E)$-generic sequence over N. We define $p^* = \bigcup_{i<\lambda} p_i$. Then p^* is clearly a condition in $\mathcal{Q}_\mathcal{F}(E)$ since S and E are disjoint and therefore $\delta \notin E$. □

8 Definition. Let θ be a sufficiently large regular cardinal. A continuous increasing sequence $(N_i : i \leq \lambda)$ is called *S-suitable* if for all $i \leq \lambda$:

(1) $N_i \prec \mathrm{H}_\theta$ is of size λ,
(2) $(N_\xi : \xi \leq i) \in N_{i+1}$,

and there is a club $X \subseteq \lambda$ such that for all $i \in X$

(3) N_{i+1} is S-good.

9 Definition. A forcing notion $\mathcal{Q}$ is *really S-complete* if it is

(1) λ-closed,
(2) strongly S-complete, and
(3) whenever $\bar{N} = (N_i : i \leq \lambda)$ is S-suitable witnessed by the club $X \subseteq \lambda$ and $r \in N_0 \cap \mathcal{Q}$ then INC(omplete) does not have a winning strategy in the following game $\mathcal{G}(\bar{N}, X, r)$ of length λ:

COM	ζ_0, p_0	$\ldots$	ζ_i, p_i	$\ldots$
INC	$\bar{q}_0$	$\ldots$	$\bar{q}_i$	$\ldots$

where for all $i < \lambda$
(a) $\zeta_i \in X$ and $\zeta_i < \zeta_j$ for all $i < j$,
(b) $p_i \in N_{\zeta_i+1} \cap \mathcal{Q}$,
(c) $\bar{q}_i = (q_i(\xi) : \xi < \lambda) \subseteq N_{\zeta_i+1} \cap \mathcal{Q}$ is $\mathcal{Q}$-generic over N_{ζ_i+1},
(d) $\bar{q}_i \in N_{\zeta_i+2}$,
(e) $r \geq p_i \geq q_i(0)$,
(f) $q_i(\xi) \geq p_j$ for all $\xi < \lambda$ and $i < j$.

The player COM(plete) wins the play of the game iff the sequence $(p_i : i < \lambda)$ has a lower bound in $\mathcal{Q}$.

10 Remark.

(i) *Recalling the definitions above, we see that the sequences $\bar{q}_i$ played by INC will always have a lower bound in $\mathcal{Q}$. This is because $\mathcal{Q}$ is S-complete, the sequence $\bar{q}_i$ is $\mathcal{Q}$-generic over N_{ζ_i+1}, and N_{ζ_i+1} is S-good.*
(ii) *Standard arguments show that really S-complete forcings preserve all stationary subsets of λ^+.*

The following theorem is one of the crucial iteration lemmas of [13] that will be used in the proof of Theorem 13.

11 Theorem. *Let $\langle \mathbb{P}_i, \tilde{\mathbb{Q}}_i : i < \gamma \rangle$ be a λ-support iteration such that for each $i < \gamma$*

$$\Vdash_{\mathbb{P}_i} \text{“}\tilde{\mathbb{Q}}_i \textit{ is really } S\textit{-complete.”}$$

Then the forcing notion $\mathbb{P}_\gamma$ is really S-complete. □

12 Lemma. *Suppose $E \subseteq S^{\lambda}_{\lambda^+}$ and $S = S^{\lambda}_{\lambda^+} \setminus E$ are stationary. Then $\mathcal{Q}_{\mathcal{F}}(E)$ is really S-complete.*

Proof. We already showed in Remark 7 that $\mathcal{Q}_{\mathcal{F}}(E)$ is λ-closed and strongly S-complete. So we are left with showing (3) of Definition 9. To this end, let $(N_i : i \leq \lambda)$ be an S-suitable sequence witnessed by the club $X \subseteq \lambda$ and let $r \in N_0 \cap \mathcal{Q}$. We actually describe a winning strategy for player COM in the game $\mathcal{G}(\bar{N}, X, r)$.

We have no problem unless $\delta = N_\lambda \cap \lambda^+ \in E$. The following argument is similar to the proof of Lemma 4: player COM creates his strategy by fixing an enumeration x_i $(i < \lambda)$ of $[\delta]^{<\kappa}$. At stage $i < \lambda$, he picks $\zeta_i \in X$ above all ζ_ξ's played so far with the additional requirements that there is $y \in \mathcal{F}_\delta$ such that $x_i \subseteq y$ and

$$y \not\subseteq \bigcup_{\xi < i} N_{\zeta_\xi + 1}. \tag{4.1}$$

Then COM extends all conditions played so far to a p_i such that

$$p_i \Vdash \text{“}y \text{ is not closed under the generic } G\text{”}. \tag{4.2}$$

The requirement (4.1) guarantees that this can be carried out. But now $(p_i : i < \lambda)$ has a lower bound in $\mathcal{Q}$ since we stipulated (4.2) for unboundedly many y's. □

13 Theorem. GCH *does not imply* $\lambda^*(\kappa, S^{\lambda}_{\lambda^+})$ *for any* $\aleph_0 < \kappa \leq \lambda$.

Proof. Start with a model of GCH and fix two sets $E, S \subseteq S^{\lambda}_{\lambda^+}$ stationary such that $E \cap S = \emptyset$. Now define a λ-support iteration of length λ^{++}. In each step $i < \lambda^{++}$, we force with $\tilde{\mathbb{Q}}^i_{\mathcal{F}}(E)$ to deal with a guessing sequence of the form $\langle \mathcal{F}_\delta : \delta \in S^{\lambda}_{\lambda^+} \rangle$ that is given to us by a book-keeping device. Note that this will destroy the guessing properties of $\langle \mathcal{F}_\delta : \delta \in S^{\lambda}_{\lambda^+} \rangle$. Remember also that each $\tilde{\mathbb{Q}}^i_{\mathcal{F}}(E)$ is forced to be really S-complete by Lemma 12. We have thus defined a λ-support iteration of the form

$$\langle \mathbb{P}_i, \tilde{\mathbb{Q}}^i_{\mathcal{F}}(E) : i < \lambda^{++} \rangle,$$

so by Theorem 11 we conclude that $\mathcal{P} = \mathcal{P}_{\lambda^{++}}$ is really S-complete. Consulting Lemma 6 and Remark 10(ii), this is enough to make sure that $\mathcal{P}$ is λ^+-distributive and that all stationary subsets of λ^+ are preserved. Finally, $\mathcal{P}$ has the λ^{++}-chain-condition by the standard line of reasoning using the fact that every iterand has size $\leq \lambda^+$. The λ^{++}-chain-condition ensures, like in standard arguments, that every sequence $\langle \mathcal{F}_\delta : \delta \in S^{\lambda}_{\lambda^+} \rangle$ appears at some initial step of the iteration, so that every potential sequence is finally taken care of. With the properties just mentioned, it is also easy to see that $\mathcal{P}$ preserves GCH. This finishes the proof. □

The authors do not know for a fact if GCH is consistent with the failure of $\curlywedge^*(\kappa, S)$ for every stationary $S \subseteq S^\lambda_{\lambda^+}$ since the methods presented here cannot settle this question. It is conjectured though that the older (and more involved) Shelah-techniques of [14] can be applied to show that the above actually is consistent.

5. Suslin-trees

Let us turn to applications of the club guessing principles presented above. We give an interesting application of $\curlywedge^*$-sequences defined on $S^\lambda_{\lambda^+}$. Again, assume that λ is regular throughout. The following theorem of Shelah from [10, p.126] is used, where $\mathrm{nacc}(A)$ is the set of all non-accumulation points of A, i.e. the set $A \setminus \lim(A)$.

14 Theorem. *There is a sequence $(C_\alpha : \alpha \in S)$ such that*

(1) *$S \subseteq S^\lambda_{\lambda^+}$ is stationary*
(2) *$C_\alpha \subseteq \alpha$ is a club of order-type λ*
(3) $\mathrm{nacc}(C_\alpha) \subseteq S^\lambda_{\lambda^+}$
(4) *if $D \subseteq \lambda^+$ is club then there is $\alpha \in S$ such that*

$$\sup(D \cap \mathrm{nacc}(C_\alpha)) = \alpha.$$

□

The application we present is the construction of a λ^+-Suslin-tree from GCH and $\curlywedge^*(\lambda, S^\lambda_{\lambda^+})$. Note that the consistency of GCH + "no ω_2-Suslin-trees" is still an open question. The following construction originally raised hopes that GCH actually does imply the existence of an ω_2-Suslin-tree. But in light of Theorem 13, this old question is now more open than ever. It should be mentioned that Jensen [7] was the first to construct a λ^+-Suslin-tree in a similar fashion but using stronger square- and guessing-principles in the constructible universe.

15 Theorem. $2^{<\lambda} = \lambda \,+\, 2^\lambda = \lambda^+ \,+\, \curlywedge^*(\lambda, S^\lambda_{\lambda^+})$ *implies the existence of a λ-closed λ^+-Suslin-tree.*

Proof. Fix enumerations $\mathcal{P}(\alpha) = \{W^\alpha_\gamma\}_{\gamma<\lambda^+}$ for all $\alpha < \lambda^+$ and then define $W^\alpha_X = \{W^\alpha_\gamma\}_{\gamma \in X}$. Let $(C_\alpha : \alpha \in S)$ be as in Theorem 14, where we enumerate $C_\alpha = \{\alpha_\xi\}_{\xi<\lambda}$. Furthermore, take $\langle \mathcal{F}_\delta : \delta \in S^\lambda_{\lambda^+}\rangle$ to be $\curlywedge^*(\lambda, S^\lambda_{\lambda^+})$-guessing. Remember that $\mathcal{F}_\delta$ is club in $[\delta]^{<\lambda}$ and $|\delta| = \lambda$, so we may assume without restriction that $\mathcal{F}_\delta$ is of the form $(F^\xi_\delta : \xi < \lambda)$ such that

- $F^\xi_\delta \in [\delta]^{<\lambda}$ for all $\xi < \lambda$,

- the sequence $(F^\xi_\delta : \xi < \lambda)$ is continuously $\subsetneq$-increasing, and
- $\alpha_{\xi+1} \in F^{\xi+1}_\alpha$.

Now construct a binary λ^+-tree T by induction on the levels so that the following holds for every $\alpha < \lambda^+$:

$$\text{For every } x \in T_{<\alpha} \text{ there is } y \in T_\alpha \text{ such that } x <_T y, \tag{5.1}$$

and simultaneously carry along an enumeration of T in the usual way. To start, let $T_0 = \lambda$. Once $T_{\leq\alpha}$ is constructed, let every $x \in T_\alpha$ have exactly two successors at level $(\alpha+1)$. If $\mathrm{cf}(\alpha) < \lambda$ then T_α extends all branches through $T_{<\alpha}$ and if $\alpha \in S^\lambda_{\lambda^+} \setminus S$ then choose any normal extension T_α of size λ. If $\alpha \in S$ then for any $x \in T_{<\alpha}$ we will construct a branch b_x through x and cofinal in $T_{<\alpha}$: first pick $\xi_0 < \lambda$ such that α_{ξ_0} is larger than the height of x and pick $x_{\xi_0} \in T_{\alpha_{\xi_0}}$ above x. Now by induction on $\xi \in [\xi_0, \lambda)$ we will construct $x_\xi \in T_{\alpha_\xi}$ as follows: if x_ξ is constructed, then pick $x_{\xi+1} \in T_{\alpha_{\xi+1}}$ above x_ξ, so that the branch determined by $x_{\xi+1}$ intersects with all $A \in W^{\alpha_{\xi+1}}_{F^{\xi+1}_\alpha}$ that are maximal antichains in $T_{<\alpha_{\xi+1}}$. Note that this is possible because $\alpha_{\xi+1}$ has cofinality λ and we are diagonalizing through less than λ-many antichains. If $\xi < \lambda$ is a limit, then let x_ξ be the limit of $\{x_\eta\}_{\xi_0 \leq \eta < \xi}$ in $T_{<\alpha}$. At the end of the day, let b_x be the α-branch determined by $\{x_\xi\}_{\xi_0 \leq \xi < \lambda}$.

Now set $T_\alpha = \{b_x : x \in T_{<\alpha}\}$ and $T = \bigcup_{\alpha<\lambda^+} T_\alpha$. Note that by an easy inductive argument, (5.1) was achieved at all levels. The following claim will finish the proof.

15.1 Claim. *T has no antichains of size λ^+.*

Proof of Claim 15.1. Assume that $\mathcal{A}$ is a maximal antichain of size λ^+ and let the function $f : \lambda^+ \longrightarrow \lambda^+$ be defined by $f(\alpha) = \gamma$ iff $\mathcal{A} \cap \alpha = W^\alpha_\gamma$. Then define the club

$$D = \{\delta < \lambda^+ : \mathcal{A} \cap \delta \text{ is a maximal antichain in } T_{<\delta} \text{ and } T_{<\delta} = \delta\}.$$

Now we can use both guessing sequences simultaneously to find $\delta \in S$ such that

$$\sup(D \cap \mathrm{nacc}(C_\delta)) = \delta \text{ and} \tag{5.2}$$

$$\text{each set in a tail of } \mathcal{F}_\delta \text{ is closed under } f. \tag{5.3}$$

If $b \in T_\delta$ then by construction there is $x \in T_{<\delta}$ such that $b = b_x$. Applying (5.2) and (5.3), there is $\xi < \lambda$ such that

- $\mathrm{ht}(x) < \delta_{\xi+1}$,
- $\delta_{\xi+1} \in D$, and
- $F^{\xi+1}_\delta$ is closed under f.

This means, again by construction, that $b_x \cap \mathcal{A} \cap \delta_{\xi+1} \neq \emptyset$, since

$$\mathcal{A} \cap \delta_{\xi+1} \in W^{\delta_{\xi+1}}_{F^{\xi+1}_{\delta}}$$

as $\delta_{\xi+1} \in F^{\xi+1}_{\delta}$ and $F^{\xi+1}_{\delta}$ is closed under f. So $b_x \cap \mathcal{A} \neq \emptyset$, which shows that $\mathcal{A}$ is sealed off at δ. □

□

6. Stationary reflection

We would like to shift the attention to the following question which was asked in [2], [3], [4], and other places in the literature. This question has been formulated in various ways, but the basic problem reads:

16 Question. *Are the following equivalent for a regular λ?*

- *every stationary $\mathcal{E} \subseteq [\lambda]^{\aleph_0}$ reflects to a set X of size $\aleph_1$ containing all countable ordinals.*[3]
- *every stationary $\mathcal{E} \subseteq [\lambda]^{\aleph_0}$ reflects to a set X of size $\aleph_1$ containing all countable ordinals with the additional property that* $\mathrm{cf}(\mathrm{otp}\ X) = \omega_1$.

Using the principle $\curlywedge^*(\aleph_1, S^{\omega}_{\lambda})$, we can now shed new light on this question even though the most general case seems to be still open. Remember that S^{ω}_{λ} denotes the collection of all ω-cofinal ordinals below the cardinal λ.

17 Definition. If $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$ and $x \in [\lambda]^{\aleph_0}$ then we define

$$\mathcal{B}(x) = \{y \in \mathcal{B} : x \subseteq y\},$$

the set of all supersets of x in $\mathcal{B}$. Furthermore, the union of all supersets of x in $\mathcal{B}$, i.e.

$$\bigcup \mathcal{B}(x) = \bigcup \{y \in \mathcal{B} : x \subseteq y\},$$

is said to be the *$\mathcal{B}$-coverage of x*.

We need the following sequence of lemmas.

18 Lemma. *If $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$ then we can partition $\mathcal{B}$ into two sets $\mathcal{B}^{(0)}$ and $\mathcal{B}^{(1)}$ such that*

1. *$\mathcal{B}^{(0)}$ has no $\subsetneq$-increasing chains of length ω_1, and*
2. *every $x \in \mathcal{B}^{(1)}$ has uncountable $\mathcal{B}^{(1)}$-coverage.*

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[3] Following [3], we say for short that $\mathcal{E}$ *reflects to X*.

Proof. Given $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$, we iteratively remove all sets with countable coverage, i.e. define

$$\begin{aligned} \mathcal{B}_0 &= \mathcal{B} \\ \mathcal{B}_{\xi+1} &= \{x \in \mathcal{B}_\xi : x \text{ has uncountable } \mathcal{B}_\xi\text{-coverage}\} \\ \mathcal{B}_\xi &= \bigcap_{\zeta<\xi} \mathcal{B}_\zeta \text{ if } \xi \text{ is limit.} \end{aligned}$$

There must be an ordinal ∞ such that $\mathcal{B}_\infty = \mathcal{B}_{\infty+1}$. Then set

$$\mathcal{B}^{(1)} = \mathcal{B}_\infty \text{ and } \mathcal{B}^{(0)} = \mathcal{B} \setminus \mathcal{B}^{(1)}.$$

Clearly, every member of $\mathcal{B}^{(1)}$ has uncountable $\mathcal{B}^{(1)}$-coverage because $\mathcal{B}_\infty = \mathcal{B}_{\infty+1}$. On the other hand it is straightforward to check that, by construction of the $\mathcal{B}_\xi$'s, $\mathcal{B}^{(0)}$ does not contain any $\subsetneq$-increasing chains of length ω_1. □

19 Lemma. *Suppose $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$ for some regular λ and x_α $(\alpha < \gamma)$ is a (possibly incomplete) list of members of $\mathcal{B}$. For each $\alpha < \gamma$, assume that $\mathcal{A}_\alpha$ is a $\subseteq$-cofinal subset of $\mathcal{B}(x_\alpha)$ that does not contain any continuous, increasing $\subsetneq$-chains of length $\omega + 1$. Define a sequence $\mathcal{A}'_\alpha$ $(\alpha < \gamma)$ inductively:*

$$\mathcal{A}'_\alpha = \{y \in \mathcal{A}_\alpha : \forall \xi < \alpha \, \forall x \in \mathcal{A}'_\xi \;\; y \nsubseteq x \text{ and } x \nsubseteq y\}.$$

Then $\mathcal{A}' = \bigcup_{\alpha<\gamma} \mathcal{A}'_\alpha$ is cofinal in $\bigcup_{\alpha<\gamma} \mathcal{B}(x_\alpha)$ and contains no continuous, increasing $\subsetneq$-chain of length $\omega + 1$.

Proof. The proof is by induction on γ. To check cofinality, consider an $x \in \mathcal{B}(x_\alpha)$ for some $\alpha < \gamma$. Since $\mathcal{A}_\alpha$ is $\subseteq$-cofinal in $\mathcal{B}(x_\alpha)$, we may assume that $x \in \mathcal{A}_\alpha$. Without loss of generality, $x \notin \mathcal{A}'_\alpha$ and x is not contained in any member of $\bigcup_{\xi<\alpha} \mathcal{A}'_\xi$. So there must be an $a \in \mathcal{A}'_\xi$ for some $\xi < \alpha$ such that $a \subset x$. But note that by induction hypothesis, $\bigcup_{\zeta<\alpha} \mathcal{A}'_\zeta$ is $\subseteq$-cofinal in $\bigcup_{\zeta<\alpha} \mathcal{A}_\zeta$. Note also that $\mathcal{A}_\xi \cap \mathcal{B}(a)$ is $\subseteq$-cofinal in $\mathcal{B}(a)$. Thus, there must be $y \in \bigcup_{\zeta<\alpha} \mathcal{A}'_\zeta$ such that $x \subseteq y$.

The lack of continuous, increasing $\subsetneq$-chains of length $\omega + 1$ in $\mathcal{A}'$ follows by construction, using the fact that no individual $\mathcal{A}_\alpha$ contains such a sequence. □

20 Lemma. *Assume* CH *and $1 \leq n < \omega$. Let $\mathcal{B} \subseteq [\omega_n]^{\aleph_0}$ be such that every member of $\mathcal{B}$ has uncountable $\mathcal{B}$-coverage. Then there is $\mathcal{A} \subseteq \mathcal{B}$ which is $\subseteq$-cofinal in $\mathcal{B}$ but contains no continuous, increasing $\subsetneq$-sequence of length $\omega + 1$.*

Proof. We go by induction on n. The following claim is crucial.

20.1 Claim. *Assume that every* $y \in \mathcal{B}(x)$ *has* $\mathcal{B}$*-coverage of cardinality* ω_n*. Then there is* $\mathcal{A}(x)$ *which is* $\subseteq$*-cofinal in* $\mathcal{B}(x)$ *such that* $\mathcal{A}(x)$ *does not contain any continuous, increasing* $\subsetneq$*-chains of length* $\omega + 1$*.*

Proof of Claim 20.1. Enumerate $\mathcal{B}(x) = \{x_\alpha : \alpha < \omega_n\}$, this uses CH. For each $\alpha < \omega_n$, choose $y_\alpha \in \mathcal{B}(x_\alpha)$ such that

$$y_\alpha \nsubseteq \bigcup_{\xi<\alpha} y_\xi,$$

or in other words, every y_α contains a brand new ordinal. Then the set $\mathcal{A}(x) = \{y_\alpha : \alpha < \omega_n\}$ contains no continuous, increasing $\subsetneq$-chains of length $\omega + 1$. □

So we have two cases in which we can thin out a set $\mathcal{B}(x)$ to a $\subseteq$-cofinal $\mathcal{A}(x)$ which contains no continuous, increasing $\subsetneq$-chains of length $\omega + 1$.

Case 1: if every superset of x in $\mathcal{B}$ has $\mathcal{B}$-coverage of cardinality ω_n, this is by Claim 20.1.

Case 2: if x has $\mathcal{B}$-coverage of cardinality less than ω_n, this is by induction hypothesis.

It is straightforward to check that the set of $x \in \mathcal{B}$ satisfying Case 1 or Case 2 is $\subseteq$-cofinal in $\mathcal{B}$. So we a have a situation that allows us to apply Lemma 19. This finishes the proof. □

21 Lemma. *Assume* CH *and* $1 \leq n < \omega$*. Then for every* $\mathcal{B} \subseteq [\omega_n]^{\aleph_0}$ *there is an* $\mathcal{A} \subseteq \mathcal{B}$ *such that* $\mathcal{A}$ *is* $\subseteq$*-cofinal in* $\mathcal{B}$ *and* $\mathcal{A}$ *does not reflect to any set of size* $\aleph_1$*.*

Proof. First partition $\mathcal{B}$ into two pieces as in Lemma 18. Then apply Lemma 20 to the piece $\mathcal{B}^{(1)}$ to get an $\mathcal{A}^{(1)} \subseteq \mathcal{B}^{(1)}$. Now $\mathcal{B}^{(0)} \cup \mathcal{A}^{(1)}$ will do the job. □

22 Theorem. *Assume* CH *and* $\curlywedge^*(\aleph_1, S^\omega_{\omega_n})$ *for some* $2 \leq n < \omega$*. Then every stationary* $\mathcal{B} \subseteq [\omega_n]^{\aleph_0}$ *can be refined to a stationary* $\mathcal{A} \subseteq \mathcal{B}$ *with the property that* $\{x \in \mathcal{A} : sup(x) = \gamma\}$ *does not reflect to a set of size* $\aleph_1$ *for all* $\gamma \in S^\omega_{\aleph_n}$*.*

Proof. Let $\mathcal{F} \subseteq [\omega_n]^{\aleph_0}$ be as stated in Theorem 3(ii). We may assume that every element of $\mathcal{F}$ has limit order type with a supremum in $S^\omega_{\aleph_n}$. For each $\eta \in S^\omega_{\aleph_n}$, let $\mathcal{F}^\eta = \{x \in \mathcal{F} \mid \sup(x) = \eta\}$. If $\mathcal{B} \subseteq [\omega_n]^{\aleph_0}$ is stationary, we may also assume that $\mathcal{B} \subseteq \mathcal{F}$. For every $\eta \in S^\omega_{\aleph_n}$, apply Lemma 21 to find a $\subseteq$-cofinal $\mathcal{A}^\eta \subseteq \mathcal{B} \cap \mathcal{F}^\eta$ such that $\mathcal{A}$ does not reflect to any set of size $\aleph_1$. Let

$$\mathcal{A} = \bigcup_{\eta \in S^{\omega}_{\aleph_n}} \mathcal{A}^\eta.$$

It suffices to show that $\mathcal{A}$ is stationary, so pick a club $\mathcal{D} \subseteq [\omega_n]^{\aleph_0}$. We may assume that $\mathcal{D} \subseteq \mathcal{F}$. Let $\mathcal{D}^*$ be the set of $x \in [\omega_n]^{\aleph_0}$ containing increasing ordinals $\langle \mu_i \mid i < \omega \rangle$ such that

- $\sup_{i<\omega} \mu_i = \sup x$,
- $\mathcal{D} \cap [\mu_i]^{\aleph_0}$ is club in $[\mu_i]^{\aleph_0}$ for each $i < \omega$,
- every element of $\mathcal{F} \cap [\mu_i]^{\aleph_0}$ containing $x \cap \mu_i$ is in $\mathcal{D}$.

It is easy to see that $\mathcal{D}^*$ is club. Now pick $x \in \mathcal{B} \cap \mathcal{D}^*$ and let $\sup x = \eta$. Then $x \in \mathcal{B} \cap \mathcal{F}^\eta$, and thus there is $y \in \mathcal{A}^\eta$ containing x. By the definition of $\mathcal{D}^*$ we have $y \cap \mu_i \in \mathcal{D}$ for each $i < \omega$, and therefore $y \in \mathcal{D} \cap \mathcal{A}$. □

23 Corollary. *Assume* CH *and* $2^{\aleph_{n-1}} = \aleph_n$ *for some* $2 \leq n < \omega$. *If every stationary subset of* $[\omega_n]^{\aleph_0}$ *reflects, then every stationary subset of* $[\omega_n]^{\aleph_0}$ *reflects to a set of ordinals of uncountable ordertype.*

Proof. Directly from Theorems 3 and 22. □

If $n = 2$ then we can get by without CH.

24 Corollary. *Assume* $2^{\aleph_1} = \aleph_2$. *If every stationary subset of* $[\omega_2]^{\aleph_0}$ *reflects, then every stationary subset of* $[\omega_2]^{\aleph_0}$ *reflects to a set of ordinals of uncountable ordertype.*

Proof. Note that if $\mathcal{E} \subseteq [\omega_2]^{\aleph_0}$ is stationary, we may assume that

$$E = \{\delta \in S^{\aleph_0}_{\aleph_2} : \mathcal{E} \cap [\delta]^{\aleph_0} \text{ is stationary in } [\delta]^{\aleph_0}\}$$

is stationary. Otherwise, throw away all elements of $\mathcal{E}$ whose supremum is in E to reach the conclusion of the corollary.

So given that E is stationary, we can do without Lemmas 18-21: if $\mathcal{E} \cap [\delta]^{\aleph_0}$ is stationary for some $\delta < \omega_2$, it is straightforward to find an unbounded subset thereof which contains no continuous, increasing $\subsetneq$-sequence of length $\omega + 1$. Now repeat the proof of Theorem 22. □

So the outcome of this section is that the Generalized Continuum Hypothesis is enough to provide a partial answer to Question 16 but we were not able to push the method beyond $\aleph_\omega$. It is possible that similar arguments can be carried out for higher cardinals but there are some problems. For instance, Lemma 20 is false if we replace the cardinal ω_n with ω_ω: assuming that $\aleph_\omega$ is strong limit, Shelah [12] constructs a club subset of $[\omega_\omega]^{\aleph_0}$ with the property that every unbounded subset of it is stationary.

To wrap things up, let us mention a result communicated to us by Donder which says that, in the constructible universe, a much better refinement than Theorem 22 is possible. This is the following fact which has also been established independently by Sakai [9].

25 Theorem. *In L, for all uncountable regular λ, every stationary $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$ can be refined to a stationary $\mathcal{A} \subseteq \mathcal{B}$ such that the* sup-*function on $\mathcal{A}$ is 1-1.*

The interested reader will find that the proof of Theorem 25 is pretty close to the usual condensation argument that $\diamondsuit_\lambda$ holds in L. Note that the conclusion of Theorem 25 is stronger than the conclusion of Theorem 21 but it also requires the stronger assumption of $V = L$. In [9], Matsubara and Sakai use a model of Gitik [5] to show that there can be an inaccessible cardinal λ and a stationary $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$ such that $\mathcal{B}$ cannot be thinned out to a stationary subset on which the sup-function is 1-1. But it seems to be open whether the same can happen if λ is a successor cardinal. In other words, the answer to the following question is not known:

26 Question. *Let $\lambda > \omega_1$ be a successor cardinal. Is it possible to prove in* ZFC *that every stationary $\mathcal{B} \subseteq [\lambda]^{\aleph_0}$ can be thinned out to a stationary $\mathcal{A} \subseteq \mathcal{B}$ on which the* sup-*function is 1-1?*

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