LOCALLY COMPACT PERFECTLY NORMAL SPACES MAY ALL BE PARACOMPACT

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Abstract. Using results announced by Stevo Todorcevic we establish that if it is consistent that there is a supercompact cardinal then it is consistent that every locally compact perfectly normal space is paracompact. Modulo the large cardinal, this answers a question of S. Watson. We also solve a problem raised by the second author, proving that it is consistent with ZFC that every first countable hereditarily normal countable chain condition space is hereditarily separable. Finally, we show that if it is consistent that there is a supercompact cardinal, it is consistent that every locally compact space with a hereditarily normal square is metrizable.

0. Introduction

Only a few of the implications concerning basic properties in general topology have remained open. One raised by Watson [Wa, Wa2, Wa3] is particularly interesting and is characterized in [Wa3] as his favorite problem:

Is it consistent that every locally compact perfectly normal space is paracompact?

If this implication holds, then locally compact, perfectly normal spaces have a very simple structure; they are simply the topological sum of σ-compact, perfectly normal − hence hereditarily Lindelöf and first countable − spaces. In fact, as we shall see, these pieces may be taken to be hereditarily separable as well.

Continuing the theme of "niceness," let us note that many of the notorious counterexamples of set-theoretic topology are ruled out: every perfectly normal manifold is metrizable, every locally compact normal Moore space is metrizable, there are no Ostaszewski spaces and so forth. Watson [Wa3] remarks, "... *a consistent theorem would be amazing. ... It looks impossible to me.*" The reason for this hyperbole is that, at the time, no known model could embody the required combinatorics.

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In fact, a tantalizing aspect of the problem is that a positive solution follows from the conjunction of two statements known to follow from well-known but mutually inconsistent axioms. Specifically,

1) *$V = L$ implies locally compact perfectly normal spaces are collectionwise Hausdorff.*
2) MA + $\sim$CH *implies locally compact perfectly normal collectionwise Hausdorff spaces are paracompact.*

We shall show that, assuming the existence of large cardinals, there is a model in which both conclusions hold, answering Watson's question. Precisely,

Theorem 1. *If it is consistent there is a supercompact cardinal, it is consistent that every locally compact perfectly normal space is paracompact.*

Variations of the proof of Theorem 1 will also solve two other interesting problems. In *Open Problems in Topology*, the second author [Ta3] asked whether it is consistent that every first countable hereditarily normal countable chain condition space is hereditarily separable. The attraction again was that this followed from consequences of MA + $\sim$CH and V = L. We have:

Theorem 2. *It is consistent with ZFC that every first countable hereditarily normal countable chain condition space is hereditarily separable.*

In [LarTo2] an old problem of Katětov [Ka] was solved by establishing the consistency of *every compact space with a hereditarily normal square is metrizable.* At the cost of a supercompact cardinal, we extend this to locally compact spaces:

Theorem 3. *If it is consistent that there is a supercompact cardinal, it is consistent that every locally compact space with a hereditarily normal square is metrizable.*

1. Notation

Our set-theoretic notation is standard, as in [Ku]. All ω_1-trees are presumed to be normal, in the terminology of [J]. Topological notation is from Engelking [En]. Since we mainly deal with locally compact spaces, it is convenient to assume all spaces are Hausdorff unless otherwise noted. However, note that the various results quoted about normality implying collectionwise Hausdorffness do not in fact require the assumption of Hausdorffness.

If S is a tree and α is an ordinal, we let $S(\alpha)$ denote the αth level of S.

2. Watson's problem and Theorem 2

The context we shall consider is in the same family as that used to prove the consistency of the positive solution to Katětov's problem [LarTo2]. This approach will surely find increasing use in set-theoretic topology since it produces strong "Suslin-type" [KuTa] consequences of MA + $\sim$CH, e.g. all Aronszajn trees are special, subspaces of countably tight compact spaces are hereditarily Lindelöf if and only if they are hereditarily separable, as well as − in the model we produce here − the important consequence of V = L that all normal first countable spaces are collectionwise Hausdorff. These models are all obtained by starting with a model in which there is a *coherent Suslin tree*. This is a Suslin tree $S \subseteq \omega^{<\omega_1}$, closed under finite modifications, such that $\{\alpha \in dom(s) \cap dom(t) : s(\alpha) \neq t(\alpha)\}$ is finite for all $s, t \in S$. The existence of such a tree follows from $\diamondsuit$ [Lar, SZ] and holds after adding one Cohen real [SZ]. Once one has such an S, one then forces the maximal amount of some forcing axiom such as MA_{ω_1} or PFA compatible with the existence of S. Then one forces with S. The details of how to do the penultimate forcing can be found in [F, Lar, Mi, Mi2]. Here we only need to know that these are iterations like those to establish MA_{ω_1} or PFA, but that certain posets are omitted. For various propositions ϕ, the proof that MA_{ω_1} or PFA implies ϕ can be modified to prove that the weaker version of MA_{ω_1} or PFA implies S cannot force ϕ to fail. The ϕ in our case will comprise several propositions that together imply locally compact perfectly normal collectionwise Hausdorff spaces are paracompact. In addition, we either start from L (if we do not require large cardinals so as to obtain as much of PFA as possible) or else a certain Easton model, and observe that the iteration plus the Suslin forcing will not destroy the fact that normal first countable $\aleph_1$-collectionwise Hausdorff spaces are collectionwise Hausdorff. The final step is to show that forcing with S establishes that normal first countable spaces are $\aleph_1$-collectionwise Hausdorff.

We shall first aim to produce the set theory needed to get that locally compact perfectly normal collectionwise Hausdorff spaces are paracompact, and then eliminate the collectionwise Hausdorff hypothesis as previously indicated. We shall need to use some results of Todorcevic [To2], [Fi1]. Consider the axioms:

$\mathbf{MA}_{\omega_1}\mathbf{(S)}$: There exists a coherent Suslin tree S, and if P is a partial order satisfying the countable chain condition which doesn't force an uncountable antichain in S, and D_ξ $(\xi < \omega_1)$ is a sequence of dense open subsets of P, then there is a filter $G \subseteq P$ such that $G \cap D_\xi \neq \emptyset$ for each $\xi < \omega_1$.

PFA(S): There exists a coherent Suslin tree S, and if P is a proper partial order which doesn't force an uncountable antichain in S, and D_ξ $(\xi < \omega_1)$ is a sequence of dense open subsets of P, then there is a filter $G \subseteq P$ such that $G \cap D_\xi \neq \emptyset$ for each $\xi < \omega_1$.

The consistency of $\mathrm{MA}_{\omega_1}(S)$ is established explicitly in [Lar], though very similar constructions had been studied earlier (in [Fa], for instance). The consistency of $\mathrm{PFA}(S)$ (minus the coherence requirement, which presents no additional difficulties), was established in [Mi].

Recall that a space X is *countably tight* if whenever $y \in \overline{Y} \subseteq X$, there is a countable $Z \subseteq Y$ such that $y \in \overline{Z}$. Finite powers of compact countably tight spaces are countably tight [M]. Todorcevic proved:

Lemma 4. (PFA(S)) *If $\dot{K}$ is an S-name for a compact countably tight space, then K is S-forced to be sequential.*

Recall that a subspace Y of a space X is *locally countable* if for each $y \in Y$ there is an open U_y about y containing only countably many members of Y. Y is *σ-discrete* if it is the union of countably many discrete subspaces. Todorcevic then proved:

Theorem 5. (PFA(S)) *If $\dot{K}$ is an S-name of a compact space with finite powers sequential, then S forces that every locally countable subset of K of size $\aleph_1$ is σ-discrete.*

The proof will appear in [Fi1]. A weaker version is proved in [To2].

Using this he got:

Theorem 6. (PFA(S)) *If $\dot{K}$ is an S-name for a compact countably tight space, then $\dot{Y}$ is S-forced to be a hereditarily separable subspace of $\dot{K}$ if and only if $\dot{Y}$ is S-forced to be hereditarily Lindelöf.*

To avoid this somewhat unwieldy way of stating such results, we introduce "PFA(S)[S] implies ϕ" as an abbreviation for "ϕ holds whenever we force with S (a coherent Suslin tree) over a model of PFA(S)." We shall use analogous notation without further explanation. We now can state

Theorem 6′. (PFA(S)[S]) *If Y is a subspace of a compact countably tight space, then Y is hereditarily separable if and only if it is hereditarily Lindeöf.*

Let us also note the following fact which had been established earlier [LT2]:

Lemma 7. ($\mathrm{MA}_{\omega_1}(S)$) *First countable hereditarily Lindelöf spaces are hereditarily separable.*

We will use Lemma 4 to get that the one-point compactification of a locally compact perfectly normal space X is a space to which Theorem 5 can be applied. Standard techniques and Theorem 6 will then yield that X is paracompact if it is collectionwise Hausdorff, so let us establish the theorems on that subject that we need.

Let us recall some standard facts about "normality versus collectionwise normality" [Ta1].

Definition. *Let κ be an infinite cardinal. A topological space is κ-**collectionwise Hausdorff** (<κ-**collectionwise Hausdorff**) if each closed discrete subspace D of size $\leq \kappa$ ($< \kappa$) can be **separated**, i.e., there exist disjoint open sets $\{U_d\}_{d\in D}$ such that $d \in U_d$. A space is **collectionwise Hausdorff** if it is κ-collectionwise Hausdorff for every κ.*

Definition [Fl]**.** *Let λ be a regular uncountable cardinal. $\mathcal{A} = \{A_f : f \in {}^\lambda\lambda\}$ is a **stationary system** for λ if each A_f is a stationary subset of λ, and whenever $\alpha \in \lambda$ and $f, g \in {}^\lambda\lambda$, if $f|\alpha = g|\alpha$ then*

$$A_f \cap (\alpha + 1) = A_g \cap (\alpha + 1).$$

*$\diamondsuit$ **for stationary systems** (at λ) is the assertion that for each stationary system $\mathcal{A}$ for λ, there is a sequence $\{f_\alpha\}_{\alpha<\lambda}$ such that $f_\alpha \in {}^\alpha\alpha$ and for each $f \in {}^\lambda\lambda$ there is a stationary $S \subseteq A_f$ such that $\beta \in S$ implies $f|\beta = f_\beta$.*

Fleissner proved:

Lemma 8. *Suppose κ is a regular uncountable cardinal, GCH holds at κ and above, and $\diamondsuit$ for stationary systems holds for all regular $\lambda \geq \kappa$. Then if X is a normal first countable $< \kappa$-collectionwise Hausdorff space, then X is collectionwise Hausdorff.*

He also probably noticed the following results, but the only reference for them we know of is [Ta2].

Lemma 9. *Suppose λ is a regular uncountable cardinal. Adjoin λ^+ Cohen subsets of λ. Then $\diamondsuit$ for stationary systems holds at λ.*

Lemma 10. *Suppose $\diamondsuit$ for stationary systems holds at the regular uncountable cardinal λ. Force with a λ-chain condition partial order of size $\leq \lambda$. Then $\diamondsuit$ for stationary systems still holds at λ.*

Using these lemmas, it is not difficult to get that normal first countable spaces which are $\aleph_1$-collectionwise Hausdorff will be collectionwise Hausdorff in the model obtained by S-forcing over a model of PFA(S), provided we start with an appropriate model over which to do the PFA(S) iteration. In particular, start with a model in which there is a supercompact cardinal κ. To simplify matters, we could establish GCH below κ by a "mild" forcing [K] keeping κ supercompact. We then make κ indestructible under κ-directed-closed forcing [Lav] and then Easton-force to add λ^+ Cohen subsets of λ for every regular cardinal $\lambda \geq \kappa$ [E]. This will establish $\diamondsuit$ for stationary systems for regular $\lambda \geq \kappa$, while keeping κ supercompact. We then force to create a coherent Suslin tree S, then force PFA(S) and lastly force with S. The iteration of these three forcings has the κ-chain condition and is of size κ, so we have established that normal first countable spaces that are <κ-collectionwise

Hausdorff are collectionwise Hausdorff. It is clear that the straightforward iteration to produce PFA(S) — if it works at all — will produce a model in which $\kappa = \aleph_2$, but in fact Farah [Fa] proves PFA(S) implies OCA, while PFA(S) implies MA(σ-centred) because σ-centred forcing doesn't add uncountable chains to Suslin trees [KuTa] (see also [Lar]). It follows (see [Be]) that

Lemma 11. PFA(S) *implies that* $2^{\aleph_0} = \aleph_2$ *and therefore so does* PFA(S)[S].

The second part of Lemma 11 follows from the fact that forcing with a Suslin tree preserves cardinals and does not add reals. This gives the following lemma.

Lemma 12. *Let κ be a supercompact cardinal, and assume that $\diamondsuit$ for stationary systems holds for every regular cardinal $\lambda \geq \kappa$. In the model obtained by first forcing* PFA(S) *by a κ-c.c. forcing of size κ and then forcing with S, normal first countable $\aleph_1$-collectionwise Hausdorff spaces are collectionwise Hausdorff.*

It remains to prove normal first countable spaces are $\aleph_1$-collectionwise Hausdorff in this model. In order to do that, we prove a purely set-theoretic combinatorial lemma:

Lemma 13. *After forcing with a Suslin tree, the following holds. Suppose that $\{N(\alpha, i) : i < \omega, \alpha < \omega_1\}$ are sets such that for all α, i, $N(\alpha, i+1) \subseteq N(\alpha, i)$. Suppose further that:*

For all $A \subseteq \omega_1$, there is an $f : \omega_1 \to \omega$ such that

$$\bigcup\{N(\alpha, f(\alpha)) : \alpha \in A\} \cap \bigcup\{N(\beta, f(\beta)) : \beta \in \omega_1 \setminus A\} = 0.$$

Then there is a $g : \omega_1 \to \omega$ and a closed unbounded $C \subseteq \omega_1$, such that:

$$\text{whenever } \alpha < \beta \text{ and } C \cap (\alpha, \beta] \neq \emptyset, \text{ then } N(\alpha, g(\alpha)) \cap N(\beta, g(\beta)) = \emptyset.$$

It should be clear that Lemma 13 yields $\aleph_1$-collectionwise Hausdorffness in first countable normal spaces: without loss of generality we may assume that the topology is on a member of V; let the $N(\alpha, i)$'s be a descending neighborhood base at α, where we have labeled the points of a discrete closed subspace of the space X with the countable ordinals. Define $c\colon \omega_1 \to \omega_1$ by letting $c(\alpha) = sup(C \cap \alpha)$, and let $\alpha \sim \beta$ if $c(\alpha) = c(\beta)$. The $\sim$-classes are countable and normality implies $\aleph_0$-collectionwise Hausdorffness, so there is a $q : \omega_1 \to \omega$ such that $c(\alpha) = c(\beta)$ implies $N(\alpha, q(\alpha)) \cap N(\beta, q(\beta)) = 0$. Let $r(\alpha) = \max(g(\alpha),\ q(\alpha))$. Then $\{N(\alpha, r(\alpha)\}_{\alpha < \omega_1}$ is the required separation.

Proof of Lemma 13. Let S be a Suslin tree. Let $\{\dot{N}(\alpha, i) : i < \omega,\ \alpha < \omega_1,\ \alpha < \omega_1\}$ be S-names for subsets of X as in the hypothesis. For $s \in S$, let $\ell(s)$ be the length of s. Since S has countable levels and its corresponding forcing poset is ω-distributive, we can construct an increasing function $h : \omega_1 \to \omega_1$ such that:

For all $\alpha < \omega_1$ and all $s \in S$ with $\ell(s) = h(\alpha)$, s decides all statements of the form "$\dot{N}(\beta, j) \cap \dot{N}(\alpha, i) = 0$", for all $i, j < \omega$ and $\beta < \alpha$.

Let $\dot{A}$ be an S-name for a subset of ω_1 such that for no $\alpha < \omega_1$ does any $s \in S$ with $\ell(s) = h(\alpha)$ decide whether $\alpha \in A$. To define such an $\dot{A}$, for each $\alpha < \omega_1$ pick two successors of each $s \in S$ with $\ell(s) = h(\alpha)$ and let one force $\alpha \in \dot{A}$ and let the other force $\alpha \notin \dot{A}$.

Let $\dot{f}$ be an S-name for a function $f : \omega_1 \to \omega_1$ as in the hypothesis of the lemma, with respect to A. Let C be a closed unbounded subset of ω_1 in V such that for each $s \in S$ with $\ell(s) \in C$, s decides $f|\ell(s)$ and $A|\ell(s)$, and such that for all $\alpha < \beta < \omega_1$, if $\beta \in C$ then $h(\alpha) < \beta$. We will define an S-name $\dot{g}$ for a function from ω_1 to ω such that whenever $\alpha < \beta < \omega_1$,

if $(\alpha, \beta] \cap C \neq \emptyset$, then $N(\alpha, g(\alpha)) \cap N(\beta, g(\beta)) = \emptyset$.

Let $c\colon \omega_1 \to \omega_1$ be defined by $c(\alpha) = sup(C \cap \alpha)$. Fix $\beta < \omega_1$. Each $s \in S$ with $\ell(s) = h(\beta)$ decides $f|c(\beta)$ and $A|c(\beta)$ and "$\dot{N}(\alpha, f(\alpha)) \cap \dot{N}(\beta, i) = 0$" for all $i < \omega$, $\alpha < c(\beta)$, but not whether $\beta \in A$. Fix $s \in S$ with $\ell(s) = h(\beta)$. Since s does not decide whether $\beta \in A$, we claim that there is an $i_0 < \omega$ such that:

for all $\alpha < c(\beta)$ such that $s \Vdash \alpha \in \dot{A}$, $s \Vdash \dot{N}(\alpha, \dot{f}(\alpha)) \cap \dot{N}(\beta, i_0) = \emptyset$.

To see this, extend s to $t \in S$ forcing that $\beta \notin A$ and deciding $f(\beta)$. Let i_0 be the value of $f(\beta)$ as decided by t. Then for each $\alpha < c(\beta)$ such that $s \Vdash \alpha \in \dot{A}$, t forces that $N(\alpha, f(\alpha)) \cap N(\beta, i_0) = \emptyset$, but these facts were already decided by s. Similarly, there is an $i_1 < \omega$ such that:

for all $\alpha < c(\beta)$ such that $s \Vdash \alpha \notin \dot{A}$, $s \Vdash \dot{N}(\alpha, \dot{f}(\alpha)) \cap \dot{N}(\beta, \dot{i}_1) = \emptyset$.

Since s decides $A|c(\beta)$, letting $\bar{i} = \max\{i_0, i_1\}$,

for all $\alpha < c(\beta)$, $s \Vdash \dot{N}(\alpha, \dot{f}(\alpha)) \cap \dot{N}(\beta, \bar{i})) = \emptyset$.

We have such an $\bar{i}_s$ for each s in the $c(\beta)$-th level of the tree, so we can construct a name $\dot{g}$ such that:

$$s \Vdash \dot{g}(\beta) = \max\{\bar{i}_s, \dot{f}(\beta)\}$$

for each $s \in S$ wih $\ell(s) = c(\beta)$. Then $\dot{g}$ is as required. $\square$

The compatibility of "locally countable subspaces of size $\aleph_1$ in a compact countably tight space are σ-discrete" with "normal first countable spaces are collectionwise Hausdorff" enables us to strengthen a variety of results of Balogh [B] and other authors, in particular proving Theorem 1, which we shall now establish.

Following Nyikos [N2], we have:

Definition. *A space X is of* **Type** *I if $X = \bigcup_{\alpha<\omega_1} U_\alpha$, where the U_α's are open, $\overline{U}_\beta \subseteq U_\alpha$ whenever $\beta < \alpha$, $U_\alpha = \bigcup_{\beta<\alpha} U_\beta$ for limit α, and each $\overline{U}_\alpha$ is Lindelöf. The* **skeleton** *of (such a decomposition of) a Type I space is the sequence $\{\overline{U}_\alpha - U_\alpha : \alpha < \omega_1\}$. The $\overline{U}_\alpha - U_\alpha$'s are called* **bones**. *We extend Nyikos' metaphor by calling a selection of one point from each bone a* **bone-scan**.

Modulo nonstationary sets, a Type I space has a unique skeleton.

Lemma 14 [N2]**.** *Any two skeletons of a Type I space agree on a closed unbounded set of bones.*

The relevance of Type I spaces to the problem at hand is that:

Lemma 15. *If X is hereditarily collectionwise Hausdorff, locally hereditarily Lindelöf and subspaces of X are hereditarily Lindelöf if and only if they are hereditarily separable, then X is the disjoint union of clopen Type I spaces.*

To see that this result applies in our setting, let X be locally compact and perfectly normal. Then X is first countable and hereditarily normal, so it will be hereditarily collectionwise Hausdorff in our model. X is also locally hereditarily Lindelöf. Let $Y \subseteq X$. Whether Y is hereditarily Lindelöf or hereditarily separable, it has no uncountable discrete subspace. Since $\overline{Y}$ is collectionwise Hausdorff, it also has no uncountable discrete subspace, and the same holds for its one-point compactification. By a well-known result of Arhangel'skiĭ, (see e.g. [H]) it follows that that compactification has countable tightness. But then by Theorem 6, Y is hereditarily separable if and only if it is hereditarily Lindelöf.

Proof of Lemma 15. The space X has a basis of open sets which are hereditarily Lindelöf and hereditarily separable. We first claim that *hereditarily Lindelöf open sets have hereditarily Lindelöf closure.* To see this, let U be hereditarily Lindelöf and hence hereditarily separable. It follows as usual, by collectionwise Hausdoffness, that $\overline{U}$ has no uncountable discrete subspace. It follows that to show $\overline{U}$ is hereditarily separable and hence hereditarily Lindelöf, it suffices to show each subspace Z of it is locally separable. Without loss of generality, let $W \cap Z$ be a neighborhood of z in Z, where W is a basic open neighborhood of z in X. But then W is hereditarily separable, so $W \cap Z$ is separable.

Towards completing the proof of Lemma 15, let $\mathcal{U}_0$ be a maximal disjoint collection of basic open sets in X. Suppose $\mathcal{U}_\beta$, $\beta < \alpha$ have been defined to be unions of countably many disjoint collections of basic open sets. Let

$$F_\alpha = X - \bigcup_{\beta<\alpha} \bigcup \mathcal{U}_\beta.$$

Let $\mathcal{V}_\alpha$ be a collection of basic open sets such that $\{F_\alpha \cap V : V \in \mathcal{V}_\alpha\}$ is a maximal disjoint collection of relatively open subsets of F_α. Fix a dense countable subset D_V in each $F_\alpha \cap V$. Any selection of points, one from each D_V, yields a discrete subset of $\bigcup \mathcal{V}_\alpha$, which may therefore be separated by basic open sets. We may therefore cover $\bigcup\{D_V : V \in \mathcal{V}_\alpha\}$ by a collection $\mathcal{U}_\alpha$ of basic open sets, such that $\mathcal{U}_\alpha$ is the union of countably many collections of disjoint basic open sets. We claim that

$$X = \bigcup_{\alpha<\omega_1} \bigcup \mathcal{U}_\alpha.$$

Towards a contradiction, suppose that this is false. Let W be a basic open neighborhood of x. Since the F_α's are descending and W is hereditarily Lindelöf, there is an $\alpha < \omega_1$ such that $F_\alpha \cap W = F_{\alpha+1} \cap W$. Then $\bigcup \mathcal{U}_{\alpha+1} \cap W$ is empty, so $F_{\alpha+1}$ is dense in $F_{\alpha+1}$. But that's a contradiction, since $x \in \bigcap\limits_{\alpha<\omega_1} F_\alpha$.

Now $\bigcup\limits_{\alpha<\omega_1} \mathcal{U}_\alpha$ is the union of $\aleph_1$ collections of disjoint open sets. Each member of $\bigcup\limits_{\alpha<\omega_1} \mathcal{U}_\alpha$ meets only countably many elements of each such collection. Therefore X is the sum of clopen subspaces, each composed of the union of $\aleph_1$ basic open sets. Let one of those clopen subspaces, say S, be such that

$$S = \bigcup_{\alpha<\omega_1} S_\alpha,$$

where each S_α is basic open. We work within S. Let $T_0 = S_0$. Suppose $\{T_\beta\}_{\beta<\alpha}$ have been defined so that each T_β is open with hereditarily Lindelöf closure. For α limit, let

$$T_\alpha = \bigcup_{\beta<\alpha} S_\beta \ \cup \bigcup_{\beta<\alpha} T_\beta.$$

Then $\overline{T}_\alpha$ is hereditarily Lindelöf. For $\alpha = \beta + 1$, since $S_\beta \ \cup \ T_\beta$ is hereditarily Lindelöf and hence has (hereditarily) Lindelöf closure, pick basic open $\{W_\alpha^n\}_{n<w}$ such that

$$\overline{S_\beta \cup T_\beta} \subseteq \bigcup_{n<\omega} W_\alpha^n.$$

Let

$$T_\alpha = \bigcup_{n<w} W_\alpha^n.$$

Since for any α, we have $\overline{T}_\alpha \subseteq T_{\alpha+1}$, and $\bigcup_{\alpha<\omega_1} T_\alpha = S$, we have shown that S is Type I. $\square$

In [N2] Nyikos erroneously ascribes to Gruenhage [G] the assertion that *every locally compact perfectly normal collectionwise Hausdorff space is the disjoint union of clopen type I spaces.* Ostaszewski's space [O] is in fact a counterexample. What Gruenhage in fact proved was that *every locally compact perfectly normal space which is collectionwise normal with respect to compact sets is the disjoint union of clopen subspaces, each of which is the union of $\aleph_1$ open subspaces, each with compact closure.* The hypothesis of collectionwise normality with respect to compact sets was later weakened by Junnila (unpublished) to collectionwise Hausdorffness. Under MA_{ω_1} — which was the situation of interest in [N2] — indeed the ascribed assertion holds by Lemma 15, since perfectly normal collectionwise Hausdorff spaces are hereditarily collectionwise Hausdorff, and the equivalence between hereditary Lindelöfness and hereditary separability follows from MA_{ω_1} the same way we did it here. However we think the finer analysis of Lemma 15 is interesting. The proof technique can be found in [B].

Since the disjoint union of clopen paracompact spaces is paracompact, given Lemma 15, we may confine ourselves to considering Type I spaces. For the same reason, we note:

Lemma 16 [N2]**.** *If the skeleton of a Type I space has a closed unbounded set of empty bones, the space is paracompact.*

Nyikos further notes:

Lemma 17 [N2]**.** *If X is locally hereditarily Lindelöf Type I, then X is hereditarily collectionwise Hausdorff if and only if every discrete subspace misses the elements of a skeleton closed unboundedly often.*

This is proved by a standard pressing-down argument. Lastly, we will need two facts due to Balogh.

Definition. *$f : X \to Y$ is* **perfect** *if it is continuous, closed, and inverse images of points are compact.*

The same argument that proves that the set of limit ordinals in ω_1 is not a G_δ extends to show that:

Lemma 18 [B]**.** *A perfectly normal space does not include a perfect pre-image of ω_1.*

We also have:

Lemma 19 [B]**.** *If X is locally compact and countably tight, then the one-point compactification of X is countably tight if and only if X does not include a perfect preimage of ω_1.*

Putting everything together, we have the following.

Proof of Theorem 1. Assume that there exists a supercompact cardinal. By the remarks after Lemma 10, there is a forcing extension $V[G]$ in which there exists a coherent Suslin tree S, PFA(S) holds and $\Diamond$ for stationary systems holds for each regular $\lambda \geq \aleph_2$. Let $H \subset S$ be $V[G]$-generic. We wish to see that in $V[G][H]$ every locally compact perfectly normal space is paracompact. By Lemmas 8 and 13, it suffices to show that in $V[G][H]$ every locally compact perfectly normal collectionwise Hausdorff space is paracompact. Work in $V[G][H]$ and fix such a space X. Note that X is locally hereditarily Lindelöf and hereditarily collectionwise Hausdorff. By Lemma 15 and the remarks afterwards, if X is a perfectly normal locally compact collectionwise Hausdorff space, then X is a disjoint union of clopen Type I spaces. By Lemmas 18 and 19, since each of these Type I spaces is perfectly normal, their one-point compactifications are countably tight. Since a disjoint union of clopen paracompact spaces is paracompact, it suffices to show that these Type I subspaces of X are paracompact. Let Y be one of these subspaces. By Lemmas 4 and 5, every locally countable subset of the one-point compactification of Y is the countable union of discrete subspaces and the same holds for Y itself. Since bone-scans are locally countable, Lemmas 16 and 17 complete the proof. $\square$

We are now ready to prove Theorem 2. First, let us note that the procedures applied above in a PFA(S) context also work for $\mathrm{MA}_{\omega_1}(S)$. That is, do the Easton forcing, force to create S, force $\mathrm{MA}_{\omega_1}(S)$, and lastly force with S. Then one obtains a model in which normal first countable spaces are collectionwise Hausdorff. By Lemma 7, we are then in position to establish Theorem 2. To prove it, let X be a hereditarily normal first countable countable chain condition space. Let $Y \subseteq X$. Since X is hereditarily collectionwise Hausdorff, Y cannot have an uncountable discrete subspace. It is standard that Y must therefore have a dense hereditarily Lindelöf subspace. For recursively define x_α such that $x_\alpha \notin \overline{\{x_\beta : \beta < \alpha\}}$, until for some λ, $\{x_\alpha : \alpha < \lambda\}$ is dense. Claim $\{x_\alpha : \alpha < \lambda\}$ is hereditarily Lindelöf. For if not, there would exist $\{x_{\alpha_\gamma} : \gamma < \omega_1\}$ such that $\{x_{\alpha_\gamma} : \gamma < \delta\}$ is open in $\{x_{\alpha_\gamma} : \gamma < \omega_1\}$, for each $\delta < \omega_1$. But then $\{x_{\alpha_\gamma} : \gamma < \omega_1\}$ is discrete. Thus, since $\{x_\alpha : \alpha < \lambda\}$ is dense and hereditarily Lindelöf, by Lemma 7 Y is separable.

In the $\mathrm{MA}_{\omega_1}(S)$ situation, one can actually rely on L rather than on Easton and so the argument can be simplified somewhat.

Theorem 3

There are a number of consequences of the compatibility of "locally countable subspaces of countably tight compact spaces are σ-discrete" with "all normal first countable spaces are collectionwise Hausdorff" that follow relatively straightforwardly from Balogh's work [B], for example:

Theorem 20. *If it is consistent there is a supercompact cardinal, by S. Todorčević it is consistent that every locally compact perfectly normal space of cardinality* $\aleph_1$ *is metrizable.*

Proof. We first note a space such as in the Theorem has a countable neighborhood around each point. This follows from Lemma 11 and the fact that compact first countable spaces have cardinality either $\aleph_0$ or $2^{\aleph_0}$. But countable compact sets are metrizable and paracompact locally metrizable spaces are metrizable, so we are done.

Balogh [B] proved under MA + $\sim$CH that connected, locally compact, locally hereditarily Lindelöf, hereditarily normal collectionwise Hausdorff spaces are paracompact if and only if they do not include a perfect pre-image of ω_1. We drop two of these conditions and get:

Theorem 21. *If it is consistent there is a supercompact cardinal, it is consistent that locally compact, locally hereditarily Lindelöf, hereditarily normal spaces are paracompact if and only if they do not include a perfect pre-image of* ω_1*.*

This answers a question Balogh asked for manifolds in [B]. Theorem 21 will follow immediately from the following lemma, which Balogh [B] proved from MA_{ω_1}, but just using the consequences mentioned.

Lemma 22. *Suppose first countable hereditarily Lindelöf spaces are hereditarily separable and locally countable subspaces of size* $\aleph_1$ *of a compact countably tight space are* σ*-discrete. Then if* X *is locally hereditarily Lindelöf, hereditarily collectionwise Hausdorff, and can be embedded into a countably tight compact space, then* X *is paracompact.*

This is proved by the same argument as for Theorem 1. In the situation of Theorem 21, we know the one-point compactification of X is countably tight, that X is first countable, and hence that X is hereditarily collectionwise Hausdorff. So Lemma 22 applies.

As a corollary, we will get a metrization theorem which answers a question in [BB]:

Definition. *X has a* $\mathbf{G}_\delta$**-diagonal** *if* $\{\langle x, x\rangle : x = x\}$ *is a* G_δ *in* $X \times X$*.*

Lemma 23 [C]**.** *A countably compact space with a* G_δ*-diagonal is metrizable (and hence compact.)*

Theorem 24. *If it is consistent there is a supercompact cardinal, it is consistent that every locally compact hereditarily normal space with a G_δ-diagonal is metrizable.*

Proof. It suffices to show X is paracompact, since it is locally metrizable. It follows from Lemma 23 that X does not include a perfect pre-image of ω_1, as well as that X is first countable and locally hereditarily Lindelöf. But then Lemma 22 applies.

It was shown in [LarTo2] that in the extension produced by forcing with the Suslin tree S over a model of $\mathrm{MA}_{\omega_1}(S)$, every compact space with hereditarily normal square is metrizable. We shall extend this to locally compact spaces by using the variation on this model considered in this paper, thus obtaining Theorem 3.

Again, we suspect that the supercompact cardinal is not necessary, though we think it unlikely that the locally compact result can be obtained from the compact case for the following reason. Katětov [Ka] proved that every compact space with hereditarily normal cube is metrizable. There is no such ZFC result for locally compact spaces − it is routine to show:

Theorem 25. MA_{ω_1} *implies there is a locally compact non-metrizable space X with X^n hereditarily normal for all $n \in \omega$.*

Proof. This is standard. X will be any subset of the real line of size $\aleph_1$ with the following topology. Let D be countable dense in X in the real line subspace topology. Each point of D we make isolated. For each $x \in X - D$, we fix a sequence from D converging to x, and let a neighborhood of x be $\{x\}$ together with a tail of the sequence. Then X is locally compact and non-metrizable, as is X^n, for each $n \in \omega$. X^n has a weaker separable metrizable topology, as a subspace of $\mathbb{R}^n$. Each point of X^n has a neighborhood base consisting of sets which are compact in that weaker topology. By the following lemma, MA_{ω_1} will imply X^n is hereditarily normal.

Lemma 26 ([We], Section 7.1**).**
Assume MA(σ-centred) + $\sim$CH. Suppose ρ and τ are two topologies on a set X such that
(i) $\rho \subseteq \tau$.
(ii) $\langle X, \rho\rangle$ *is Hausdorff and second countable, and*
(iii) *there is a closed neighborhood base for τ consisting of sets compact in $\langle X, \rho\rangle$.*

Then for all $H, K \in [X]^{<2^{\aleph_0}}$ such that $\overline{H} \cap K = H \cap \overline{K} = \emptyset$ in the τ topology, we have disjoint open U_H and U_K in τ including H and K respectively.

Now to prove Theorem 3, we will need a lemma of Katětov [Ka]:

Lemma 27. *If Y is countably compact and Y^2 is hereditarily normal, then Y is perfectly normal.*

Now suppose that X is locally compact and X^2 is hereditarily normal. By [LT2] it follows that X is locally metrizable and hence locally hereditarily Lindelöf. X is homeomorphic to a subspace of X^2, so it too is hereditarily normal. Since X is locally metrizable, to show it is metrizable it suffices to show it is paracompact. This will follow from Lemma 19 if we can show X includes no perfect pre-image of ω_1. Suppose it had such a pre-image Y. Y would be countably compact and, by Katětov's Lemma, perfectly normal. But that's impossible by Lemma 18.

Another consequence of the approach taken in this paper is the following result.

Theorem 28. *If it is consistent that there is a supercompact cardinal, then it is consistent that every hereditarily normal vector bundle is metrizable.*

Nyikos [N] needed only MA_{ω_1} for the hypothesis of Theorem 28, but he required also that the vector bundle be hereditarily collectionwise Hausdorff. More on vector bundles, including their definition, can be found in [Sp].

Proof of Theorem 28. Vector bundles are manifolds, so by Theorem 21, it suffices to show that hereditarily normal ones don't include perfect preimages of ω_1. By Lemma 15, since our vector bundle V will be hereditarily collectionwise Hausdorff and connected, it will be of Type I. But Nyikos [N] proved that vector bundles of Type I cannot include a perfect preimage of ω_1. $\square$

A more interesting question concerns the metrizability of hereditarily normal manifolds of dimension greater than 1. Nyikos has written several papers on the subject, proving for example from the consistency of a supercompact cardinal that such manifolds are metrizable if in addition they are hereditarily collectionwise Hausdorff [N2]. We make the following conjecture:

Conjecture. *If it is consistent that there is a supercompact cardinal, then it is consistent that every hereditarily normal manifold of dimension greater than* 1 *is metrizable.*

Furthermore, we expect that there is a proof of this conjecture using the approach taken in this paper.

Remarks. According to Todorcevic (personal communication) the supercompact can probably be eliminated in the work of his on which Theorem 1 depends.

Almost all of this paper was written in 2002; it was submitted in 2008. The reason for the delay was that the authors had not seen the still unpublished proofs of Lemma 4 and Theorem 5.

We now know how to obtain Theorem 5; at Todorcevic's suggestion, this will appear in [Fi1]. At the 2006 Prague Topological Symposium Todorcevic announced Lemma 4. He sketched the proof at the conference on *Advances in Set-theoretic Topology, in Honor of T. Nogura* in Erice, Italy in 2008.

In the years following the writing of the first version of this paper, further results have succeeded in weakening the "perfect normality" condition of Theorem 1. This work appears in the preprints [LarTa] and [Ta4].

Whitehead Groups

Just as the questiong of when normality implies collectionwise normal led to many advances in set-theoretic topology, the question of when Whitehead groups are free has been similarly influential in set-theoretic algebra. For a short, accessible introduction to the subjet, see [Ek]. For a comprehensive presentation, see [EM]. All terms not defined here can be found in both references. Here we only want to point out:

Theorem 29. *In the $MA_{\omega_1}(S)[S]$ and $PFA(S)[S]$ models discussed here, in which normal first countable spaces are collectionwise Hausdorff, all Whitehead groups are free.*

Proof. We assume the reader is somewhat familiar with the proof that Whitehead groups are free in L. That proof proceeds by induction on the cardinality of the groups. It is true for countable ones, and for singular cardinals it is true if it is true for smaller cardinals, by a *Singular Compactness Theorem.*

At regular cardinals κ, $\diamondsuit(S)$ for S a stationary subset of κ is sufficient to carry on with the induction. $\diamondsuit$ for stationary systems is a stronger principle, so the case of $\kappa = \aleph_1$ is the only one needing consideration.

Shelah showed that there is a non-free Whitehead group of size $\aleph_1$ if and only if there is a ladder sustem on some stationary subset of ω_1 which has the 2-uniformization property. But such a ladder system determines in a natural way a first countable, normal, non-collectionwise Hausdorff space of size $\aleph_1$. $\square$

It remains to be seen whether there are MA_{ω_1} or PFA consequences holding in these models which would, in conjunction with "all Whitehead groups are free", produce results of algebraic interest.

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