# The canonical function game

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**Abstract**

The canonical function game is a game of length $\omega_1$ introduced by W. Hugh Woodin which falls inside a class of games known as Neeman games. Using large cardinals, we show that it is possible to force that the game is not determined. We also discuss the relationship between this result and $\Sigma^2_2$ absoluteness, cardinality spectra and $\Pi_2$ maximality for $H(\omega_2)$ relative to the Continuum Hypothesis.

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The canonical function game, introduced by W.H. Woodin, is a game of perfect information of length $\omega_1$ between two players, whom we call *Dominating* and *Undominated*. In each round $\alpha$, *Undominated* plays a countable ordinal $u(\alpha)$, and then *Dominating* plays $\sigma_\alpha$, a wellordering of $\alpha$ of ordertype greater than $u(\alpha)$ (if $\alpha \geq \omega$; when $\alpha$ is finite we require only that $\sigma_\alpha$ is a wellordering of $\alpha$; the first $\omega$ moves are irrelevant to the outcome of the game). After all $\omega_1$ rounds have been played, *Dominating* wins the run of the game if and only if there exists a club $C \subset \omega_1$ such that $\sigma_\alpha = \sigma_\beta \cap (\alpha \times \alpha)$ for all $\alpha < \beta$ in $C$.

Given an ordinal $\gamma \in [\omega_1, \omega_2)$, a *canonical function* for $\gamma$ is a function $f\colon \omega_1 \to \omega_1$ for which there exists a bijection $\pi\colon \omega_1 \to \gamma$ such that the set $\{\alpha < \omega_1 \mid f(\alpha) = o.t.(\pi[\alpha])\}$ contains a club subset of $\omega_1$. Any two canonical functions for the same ordinal agree on a club. Futhermore, if $\gamma < \gamma'$ are ordinals in $[\omega_1, \omega_2)$, $f$ is a canonical function for $\gamma$ and $f'$ is a canonical function for $\gamma'$, then $f' > f$ on a club. If $\langle (u(\alpha), \sigma_\alpha) : \alpha < \omega_1 \rangle$ is a run of the canonical function game and $C \subset \omega_1$ is a club witnessing that *Dominating* wins this run of the game, then $\Sigma = \cup\{\sigma_\alpha : \alpha \in C\}$ is a wellordering of $\omega_1$, and the function $f\colon \omega_1 \to \omega_1$ defined by letting $f(\alpha)$ be the ordertype of $\sigma_\alpha$ is a canonical function for the ordertype of $\Sigma$.

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We let *Bounding* denote the statement that every function from $\omega_1$ to $\omega_1$ is dominated by a canonical function on a club. If Bounding fails, then *Undominated* has a simple winning strategy in the canonical function game: he plays so that $u\colon \omega_1 \to \omega_1$ is any function which is not dominated by a canonical function on a a club. Deiser and Donder [1] have shown Bounding to be equiconsistent with a strongly inaccessible limit of measurable cardinals.

In this paper we will show that the canonical function game is consistently undetermined, assuming the consistency of a strongly inaccessible limit of measurable cardinals. Part of the significance of this result is its relation to a class of games known as Neeman games. There are only countably many Neeman games, one for each $n$-ary formula $\phi$ (for some integer $n$) in the expanded language with one unary predicate. Given such a pair $n$, $\phi$, the Neeman game $G_\phi$ is a game of length $\omega_1$ where players $I$ and $II$ collaborate to build a function $a\colon \omega_1 \to \{0,1\}$, with $I$ picking $a(0)$, $II$ picking $a(1)$ and so on, with $I$ picking $a(\gamma)$ for each limit ordinal $\gamma$. After $a$ has been constructed, $I$ wins if and only if there exists a club $C \subset \omega_1$ such that for all $\alpha_1 < \cdots < \alpha_n$ in $C$, $\langle H(\omega_1), a, \in\rangle \models \phi(\alpha_1, \ldots, \alpha_n)$. For a given integer $n$, an $n$-ary Neeman game is the Neeman game corresponding to some $n$-ary formula. The canonical function game can easily be recast as a binary Neeman game, with *Dominating* as $I$ and *Undominated* as $II$ (the fact that the players play in the opposite order in the two games is not important). In contrast to the main result of this paper, Neeman has shown that the existence of an iterable model with indiscernible Woodin cardinals implies that all unary Neeman games are determined [6].

If $B$ is a set of reals, we define the $B$-Neeman game $G_{B,\phi}$, where $\phi$ is an $n$-ary formula in the expanded language with two unary predicates, by saying that $I$ wins if and only if there exists a club $C \subset \omega_1$ such that for all $\alpha_1 < \cdots < \alpha_n$ in $C$, $\langle H(\omega_1), a, B, \in\rangle \models \phi(\alpha_1, \ldots, \alpha_n)$. Woodin has connected the determinacy of Neeman games to the question of $\Sigma^2_2$-absoluteness with the following result.

**Theorem 0.1.** *(Woodin) Suppose that these exists a proper class of supercompact cardinals. Let $\Gamma$ denote the set of all universally Baire sets of reals. The following are equivalent.*

- *For each $B \in \Gamma$, ZFC + $\diamondsuit_G$ implies in $\Omega$-logic that all $B$-Neeman games are determined.*
- *For each $B \in \Gamma$ and for every unary $\Sigma^2_2$ formula $\phi$, ZFC + $\diamondsuit_G$ implies exactly one of $\phi(B)$ and $\neg\phi(B)$ in $\Omega$-logic.*

Here $\diamondsuit_G$ (called *generic Diamond*) is the statement that for each $\Sigma_2$ sentence $\phi$ for $H(\omega_2)$, $\phi$ holds if and only if $Coll(\omega_1, \mathbb{R})$ forces $\phi$; this is a strong form of $\diamondsuit$. We refer the reader to [9] for the definitions of $\Omega$-logic and universally Baire sets of reals, which are not used in this paper (though we note that if $T$ implies $\phi$ in $\Omega$-logic, then $T + \neg\phi$ cannot be forced to hold in a rank initial segment of the universe). Again, the main results in this paper imply that some hypothesis beyond ZFC is required to imply the determinacy of all Neeman games in $\Omega$-logic. Since $\diamondsuit$ implies that Bounding fails, the canonical function game is not a counterexample to $\diamondsuit_G$ implying the determinacy of all Neeman games in $\Omega$-logic, however.

The canonical function game and Theorem 0.1 were presented by Woodin in his talk *Beyond* $\Sigma^2_1$ *absoluteness*, given June 2, 2002 at the Association for Symbolic Logic Annual Meeting at the University of Nevada, Las Vegas (see also [10]). Theorem 1.1 and Corollary 1.4 of this paper answer two questions asked in that talk.

# 1 Indeterminacy of the canonical function game

**Theorem 1.1.** Dominating *does not have a winning strategy in the canonical function game.*

*Proof.* Fix a strategy $\tau$ for *Dominating*. We will construct two plays of the canonical function game such that each is a play by $\tau$ and yet *Dominating* loses at least one of the two plays.

Our two runs of the game will be conducted on boards labelled $a$ and $b$, and we will use *Dominating*$(a)$, *Undominated*$(a)$, $u_a$ and $\sigma^a_\alpha$ to describe one run, and *Dominating*$(b)$, *Undominated*$(b)$, $u_b$ and $\sigma^b_\alpha$ to describe the other.

Let $A$ be a stationary, co-stationary subset of $\omega_1$. For each round $\alpha$, having built both plays up to round $\alpha$, if $\alpha$ is in $A$ then we let $u_a(\alpha) = 0$ and let $\sigma^a_\alpha$ be the move given by $\tau$ for the partial play defined so far on board $a$. Then we let $u_b(\alpha) = o.t.(\sigma^a_\alpha) + 1$, and let $\sigma^b_\alpha$ be the move given by $\tau$ to the partial play given so far on board $b$. If $\alpha$ is not in $A$, then we reverse the roles of $a$ and $b$. That is, we let $u_b(\alpha) = 0$ and let $\sigma^b_\alpha$ be the move given by $\tau$ for the partial play defined so far on board $b$, then we let $u_a(\alpha) = o.t.(\sigma^b_\alpha) + 1$ and we let $\sigma^a_\alpha$ be the move given by $\tau$ to the partial play given so far on board $a$.

The essential point is that, having completely constructed both plays in this manner,

$$\{\alpha < \omega_1 \mid o.t.(\sigma^a_\alpha) > o.t.(\sigma^b_\alpha)\}$$

and

$$\{\alpha < \omega_1 \mid o.t.(\sigma^a_\alpha) < o.t.(\sigma^b_\alpha)\}$$

are both stationary subsets of $\omega_1$. Now, if $C$ and $D$ are club subsets of $\omega_1$ such that

$$\forall \alpha, \beta \in C \ \alpha < \beta \Rightarrow \sigma^a_\alpha = \sigma^a_\beta \restriction \alpha$$

and

$$\forall \alpha, \beta \in D \ \alpha < \beta \Rightarrow \sigma^b_\alpha = \sigma^b_\beta \restriction \alpha,$$

then by taking the intersection of $C$ and $D$ we may assume that $C = D$. Further,

$$\Sigma_a = \bigcup \{\sigma^a_\alpha \mid \alpha \in C\}$$

and

$$\Sigma_b = \bigcup \{\sigma^b_\alpha \mid \alpha \in C\}$$

both define wellorderings of $\omega_1$. Now, if $o.t.(\Sigma_a) < o.t.(\Sigma_b)$, then for club many $\alpha \in C$, $o.t.(\Sigma_a \restriction \alpha) < o.t.(\Sigma_b \restriction \alpha)$. However, this is false, since for each $\alpha \in C$, $\Sigma_a \restriction \alpha = \sigma^a_\alpha$, and $\Sigma_b \restriction \alpha = \sigma^b_\alpha$. The relations $o.t.(\Sigma_a) > o.t.(\Sigma_b)$ and $o.t.(\Sigma_a) = o.t.(\Sigma_b)$ are similarly contradictory. $\square$

Next we will see that is it consistent that *Undominated* fails to have a winning strategy. First we will show that if there exists a measurable cardinal, then for any strategy $\tau$ for *Undominated* there is a semi-proper forcing adding a run of the canonical function game where *Dominating* wins and *Undominated* plays by $\tau$. This implies in particular that Martin's Maximum [3] plus the existence of a measurable cardinal implies that the canonical function game is undetermined. Furthermore, we will see that the indeterminacy of the canonical function game can be forced from a strongly inaccessible limit of measurable cardinals.

Given a strategy $\tau$ for *Undominated* in the canonical function game, let $P_\tau$ be the forcing which adds a run of the game where *Undominated* plays by $\tau$. The conditions in $P_\tau$ are countable partial runs of the game where *Undominated* plays by $\tau$ and *Dominating* was the last to play. The order is extension. Note that $P_\tau$ is countably closed. If $p$ is a condition in $P_\tau$, we let $l(p)$ denote the length of $p$, and we let $\tau(p)$ be the response to $p$ given by $\tau$.

Given a cardinal $\kappa$, let $Q_\kappa$ be the set of pairs $(c, h)$ such that

- $c$ is a closed, bounded subset of $\omega_1$,
- $h$ is an injective function from $max(c)$ to $\kappa$.

Still fixing $\tau$ and $\kappa$, let $PQ^\tau_\kappa$ be the partial order consisting of triples (p,c,h) such that

- $p = \langle (u(\alpha), \sigma_\alpha) : \alpha < l(p) \rangle \in P_\tau$,
- $(c, h) \in Q_\kappa$,
- $l(p) > max(c)$,
- for all $\alpha, \beta \in c$, $\alpha < \beta$ implies that $\sigma_\alpha = \sigma_\beta \restriction \alpha$,
- for all $\alpha, \beta \in max(c)$, if $\alpha \neq \beta$ then $(\alpha, \beta) \in \sigma_{max(c)} \Leftrightarrow h(\alpha) < h(\beta)$.

We say that $(p, c, h) \geq (p', c', h')$ if $p'$ extends $p$, $c'$ end-extends $c$ and $h \subset h'$.

Suppose that $\kappa$ is a cardinal and $\tau$ is a strategy for *Undominated*. Given that $PQ^\tau_\kappa$ preserves $\omega_1$, which we will show in the case when $\kappa$ is measurable, it follows by genericity that $PQ^\tau_\kappa$ adds a run of the canonical function game where *Undominated* plays by $\tau$ and *Dominating* wins.

Let $\mu$ be a normal measure on $\kappa$. Fix a regular cardinal $\theta > 2^\kappa$ and let $X \prec H(\theta)$ be countable with $\mu, \tau \in X$. A condition $p^* \in P_\tau$ is *X-generic* if for all $\alpha \in X \cap \omega_1$, $p^* \restriction \alpha$ is in $X$, and each dense subset of $P_\tau$ in $X$ has some $p^* \restriction \alpha$ ($\alpha \in X \cap \omega_1$) as a member. Likewise, a triple $(p^*, c^*, h^*) \in P_\tau \times Q_\kappa$ is *X-generic* if for each $\alpha \in X \cap \omega_1$,

$$(p^* \restriction (\alpha + 1), c^* \cap (\alpha + 1), h^* \restriction \alpha) \in PQ^\tau_\kappa \cap X,$$

and each dense open subset of $PQ^{\tau}_{\kappa}$ in $X$ contains $(p^*\restriction(\alpha+1), c^*\cap(\alpha+1), h^*\restriction\alpha)$ for some $\alpha \in X \cap \omega_1$. Note that we do not require that $(p^*, c^*, h^*) \in PQ^{\tau}_{\kappa}$ (i.e., we define genericity even for triples which are not conditions in $PQ^{\tau}_{\kappa}$).

Still fixing $X$ and $\mu$, let $A^X_\mu = \bigcap(X \cap \mu)$. It is a standard fact that if $E \subset A^X_\mu$, then

$$X_E = \{f(a) \mid f\colon [\kappa]^{<\omega} \to H(\theta) \wedge f \in X \wedge a \in [E]^{<\omega}\}$$

is an elementary submodel of $H(\theta)$ containing $X$ and end-extending $X$ below $\kappa$ (see, for instance, [4]). Whenever $E$ is countable, we will call any such model $X_E$ a *$\mu$-extension* of $X$.

**Lemma 1.2.** *Let $\mu$ be a normal measure on a cardinal $\kappa$ and let $\tau$ be a strategy for* Undominated *in the canonical function game. Fix a regular cardinal $\theta > 2^\kappa$ and let $X \prec H(\theta)$ be countable with $\mu, \tau \in X$. Let $\delta = X \cap \omega_1$. Let $p^*$ be an $X$-generic condition in $P_\tau$. Then for every $\mu$-extension $Y$ of $X$ and for every pair $(c, h) \in Y \cap Q_\kappa$ such that*

$$(p^*\restriction(max(c) + 1), c, h) \in PQ^{\tau}_{\kappa}$$

*there exists a pair $(c^*, h^*) \in Q_\kappa$ such that $c^*$ end-extends $c$, $h \subset h^*$ and $(p^*, c^*, h^*)$ is $Y$-generic.*

*Furthermore, if o.t.$(Y \cap \kappa) > \tau(p^*)$, then $(c^*, h^*)$ can be chosen so that*

$$((p^*)^\frown(\tau(p^*), \bigcup\{\sigma_\alpha : \alpha \in c^*\}), c^* \cup \{\delta\}, h^*) \in PQ^{\tau}_{\kappa}.$$

*Proof.* Fix $\mu, \kappa, \tau, \theta, X, \delta$ and $p^*$ as given. Let $E \subset A^X_\mu$ be countable and let $Y = X_E$. Fix $(c, h)$ as in the statement of the lemma. We will build $c^*$ and $h^*$ by approximations $c_k, h_k$ in $Y$. Let $c_0 = c$ and $h_0 = h$. Let $D_k$ $(k < \omega)$ enumerate the dense subsets of $PQ^{\tau}_{\kappa}$ in $Y$. Given $c_k$ and $h_k$, we will find $c_{k+1}$ and $h_{k+1}$ in $Y$ extending $c_k$ and $h_k$ such that

$$(p^*\restriction(max(c_{k+1}) + 1), c_{k+1}, h_{k+1}) \in D_k.$$

The key point is that since $Y$ end-extends $X$ below $\kappa$, $p^*$ is $Y$-generic for $P_\tau$. The set of conditions $p$ in $P_\tau$ for which there is some pair $(c', h') \in Q_\kappa$ extending $(c_k, h_k)$ such that $(p, c', h') \in D_k$ (and such that the length of $p$ is equal to $max(c') + 1$) is dense below $p^*\restriction(max(c_k) + 1)$ and is a member of $Y$, so some initial segment of $p^*$ in $Y$ satisfies this condition, enabling the choice of the desired pair $(c_{k+1}, h_{k+1})$.

The last part of the conclusion of the lemma follows from the fact that for each $\gamma \in \kappa$, the set of $(p', c', h') \in PQ^{\tau}_{\kappa}$ with $\gamma \in range(h')$ is dense, which in turn implies that $h^*[\delta] = Y \cap \kappa$. To see that this set is dense, fix $(\bar{p}, \bar{c}, \bar{h}) \in PQ^{\tau}_{\kappa}$ and $\gamma < \kappa$ such that $\gamma \notin range(\bar{h})$. Let $\beta = l(\bar{p}) + \omega$ and extend $\bar{p}$ to a partial play $p' = \langle u(\alpha), \sigma_\alpha : \alpha \leq \beta \rangle$ according to $\tau$ such that the following hold.

- $\sigma_\beta \restriction max(\bar{c}) = \sigma_{\max(\bar{c})}$.
- for all $\alpha < max(\bar{c})$, $(\alpha, max(\bar{c})) \in \sigma_\beta \Leftrightarrow h'(\alpha) < \gamma$.
- $(max(\bar{c}) + 1) \times (\beta \setminus (max(\bar{c}) + 1)) \subset \sigma_\beta$.

Then if $c' = \bar{c} \cup \{\beta\}$ and $h'$ is any suitable extension of $\bar{h} \cup \{(max(\bar{c}), \gamma)\}$ (for example, for each $\alpha \in \beta \setminus (max(\bar{c}) + 1)$ we could let $h'(\alpha)$ be

$$sup(range(\bar{h})) + \gamma + \zeta_\alpha,$$

where $\zeta_\alpha$ is the rank of $\alpha$ in the wellordering $\sigma_\beta$), then $(p', c', h') \leq (\bar{p}, \bar{c}, \bar{h})$ in $PQ^{\tau}_{\kappa}$ and $\gamma \in range(h')$. □

**Theorem 1.3.** *If $\kappa$ is a measurable cardinal, and $\tau$ is a strategy for* Undominated*, then there is a semi-proper forcing adding a run of the game for which* Undominated *plays by $\tau$ and* Dominating *wins.*

*Proof.* The forcing is $PQ^{\tau}_{\kappa}$. We need to see only that this forcing is semi-proper. Let $(p, c, h)$ be a condition in $PQ^{\tau}_{\kappa}$, let $\theta$ be a regular cardinal greater than $2^\kappa$ and let $X$ be a countable elementary submodel of $H(\theta)$ with $\tau, \kappa$ and $(p, c, h)$ in $X$. Let $p^*$ be an $X$-generic condition in $P_\tau$. Let $\mu$ be a normal measure on $\kappa$ in $X$ and let $Y$ be a $\mu$-extension of $X$ such that $o.t.(Y \cap \kappa) > \tau(p^*)$. Then by Lemma 1.2 there is a pair $c^*, h^*$ such that $(p^*, c^*, h^*) \leq (p, c, h)$ and $(p^*, c^*, h^*)$ is a $Y$-generic condition in $PQ^{\tau}_{\kappa}$. □

**Corollary 1.4.** *Martin's Maximum plus the existence of a measurable cardinal implies that* Undominated *does not have a winning strategy in the canonical function game.*

Given a cardinal $\kappa$, let $R_\kappa$ denote the countable support product of all the partial orders $PQ^{\tau}_{\kappa}$ where $\tau$ is a strategy for *Undominated.* The proof of Lemma 1.2 shows that if $\kappa$ is a measurable cardinal then $R_\kappa$ is semi-proper (first take an $X$-generic for the countable support product of all the $P_\tau$'s, then end-extend to a $Y$ such that $o.t.(Y \cap \kappa)$ is greater than all the $\tau(p)$'s, and choose the rest of the generic filter as before; there are several suitable alternate definitions of $R_\kappa$). Now suppose that $\lambda$ is a strongly inaccessible limit of measurable cardinals, and let $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \lambda \rangle$ be an RCS iteration (see [2]) such that each $\mathbb{Q}_\alpha$ is forced to be $\widetilde{R}_\kappa$ for $\kappa$ the least measurable cardinal in the extension by $\mathbb{P}_\alpha$. If $\rho$ is a $\mathbb{P}$-name for a strategy for *Undominated* in the canonical function game, then in the $\mathbb{P}$-extension (in which $\lambda = \omega_2$) there is a $\gamma < \lambda$ such that $G \cap \mathbb{P}_\gamma$ (where $G \subset \mathbb{P}$ is the generic filter) decides $\rho$ on all positions in $V[G \cap \mathbb{P}_\gamma]$. Then $\mathbb{Q}_\gamma$ added a complete run of the game where *Undominated* played by $\rho$ and lost. Putting all of this together, we have the following.

**Theorem 1.5.** *If there exists a strongly inaccessible limit of measurable cardinals then there is a semi-proper forcing making the canonical function game undetermined.*

# 2 ... and the Continuum Hypothesis

Although the forcing $PQ^{\tau}_{\kappa}$ is $(\omega,\infty)$-distributive, we do not know whether the indeterminacy of the canonical function game is consistent with CH. This question raises some interesting issues. The principle below has been known for some time; the name we give for it is new.

**2.1 Definition.** $\clubsuit_c$ (*Club for clubs*) is the statement that there exist $a_\alpha$ ($\alpha$ a countable limit ordinal) such that each $a_\alpha$ is a cofinal subset of $\alpha$ of ordertype $\omega$ and such that for every club subset $C$ of $\omega_1$ there is an $\alpha < \omega_1$ such that $a_\alpha \subset C$.

Call a pair $X, Y$ of countable elementary submodels of $H(\omega_2)$ *good* if either $X \cap \omega_1 \neq Y \cap \omega_1$ or for all club subsets of $\omega_1$ $C \in X$ and $D \in Y$,

$$C \cap D \cap X \cap \omega_1 \neq \emptyset.$$

We let (+) denote the statement that there exists a stationary set $\mathcal{S}$ of countable elementary submodels of $H(\omega_2)$ such that every pair from $\mathcal{S}$ is good.

**Theorem 2.2.** $\clubsuit_c \Rightarrow (+)$.

*Proof.* Let $\langle a_\alpha : \alpha < \omega_1 \text{ limit}\rangle$ witness $\clubsuit_c$, and let $\mathcal{S}$ be the set of countable $X \prec H(\omega_2)$ such that for every club $D \subset \omega_1$ in $X$, $a_{(X\cap\omega_1)} \setminus D$ is bounded in $X \cap \omega_1$. Any pair of members of $\mathcal{S}$ is good. If $\mathcal{S}$ is not stationary, then there exists a continuous, increasing chain $\langle X_\alpha : \alpha < \omega_1\rangle$ of countable elementary submodels of $H(\omega_2)$ not in $\mathcal{S}$. Let $\langle D_\alpha : \alpha < \omega_1\rangle$ enumerate the club subsets of $\omega_1$ in $\cup\{X_\alpha : \alpha < \omega_1\}$, and let $D = \triangle\{D_\alpha : \alpha < \omega_1\rangle$. Let $E \subset \omega$ be the club consisting of all $\beta < \omega_1$ such that $\{D_\alpha : \alpha < \beta\}$ lists the club subsets of $\omega_1$ in $X_\beta$. Now let $\beta$ be such that $a_\beta \subset D \cap E$. Then for each $\alpha < \beta$, $a_{(X\cap\omega_1)} \setminus D_\alpha$ is bounded in $X \cap \omega_1$, so $X_\beta \in \mathcal{S}$, giving a contradiction. □

**Theorem 2.3.** *The statement* (+) *implies that* Undominated *has a winning strategy in the canonical function game.*

*Proof.* Let $\mathcal{S}$ witness (+). The strategy for *Undominated* is, in round $\beta$, if

$$p = \langle (u(\alpha), \sigma_\alpha) : \alpha < \beta\rangle$$

is the play so far and there exist an $X \in \mathcal{S}$ and a complete run of the game $p^* = \langle (u'(\alpha), \sigma'_\alpha) : \alpha < \omega_1\rangle$ in $X$ such that

- $X \cap \omega_1 = \beta$,
- $p = p^* \restriction \beta$,

- *Dominating* wins the run $p^*$,

then choose such a pair $X, p^*$ and let $u(\beta) = o.t.(\sigma'_\beta) + 1$. If there is no such pair $X, p^*$, then let $u(\beta) = 0$. Now suppose that

$$\bar{p} = \langle (u(\alpha), \sigma_\alpha) : \alpha < \omega_1 \rangle$$

is a complete run of the game where *Undominated* has played by this strategy and *Dominating* has won. Then there is a $Y \in \mathcal{S}$ with $\bar{p} \in Y$. Let $\beta = Y \cap \omega_1$. By the rules of the strategy for *Undominated* and the properties of the pair $Y, \bar{p}$, there exist an $X \in \mathcal{S}$ and a complete run of the game $p^* = \langle (u'(\alpha), \sigma'_\alpha) : \alpha < \omega_1 \rangle$ in $X$ such that

- $X \cap \omega_1 = \beta$
- $p^* \restriction \beta = \bar{p} \restriction \beta$
- *Dominating* wins the run $p^*$,
- $u(\beta) = o.t.(\sigma'_\beta) + 1$.

Let $C \in Y$ and $D \in X$ be club subsets of $\omega_1$ witnessing respectively that $\bar{p}$ and $p^*$ are winning plays for *Dominating*. Since $X$ and $Y$ are both in $\mathcal{S}$, $C \cap D \cap \beta$ must be cofinal in $\beta$. Then

$$\sigma_\beta = \cup\{\sigma_\alpha : \alpha \in C \cap D \cap \beta\} = \sigma'_\beta,$$

contradicting the fact that $o.t.(\sigma_\beta) > u(\beta) > o.t.(\sigma'_\beta)$. $\square$

So $\clubsuit_c$ implies that the canonical function game is determined. In [5], it was shown that Bounding is consistent with the Continuum Hypothesis. An important point of the proof of this fact is that the standard forcing to make Bounding hold is $\alpha$-semi-proper, for each countable ordinal $\alpha$, as defined below. Recall that if $P$ is a partial order, $\theta$ is a regular cardinal greater than $2^{|P|}$ and $X$ is a countable elementary submodel of $H(\theta)$ with $P \in X$, then a condition $p \in P$ is $(X, P)$-semi-generic if $p \Vdash \tau \in (\check{X} \cap \omega_1)$ for each $P$-name $\tau$ in $X$ for a countable ordinal.

**2.4 Definition.** Given a countable ordinal $\alpha$, a partial order $P$ is $\alpha$-semi-proper if, whenever $p \in P$, $\theta$ is a regular cardinal greater than $2^{|P|}$, $\leq_\theta$ is a wellordering of $H(\theta)$ and $X_\beta$ ($\beta < \alpha$) are countable elementary submodels of $\langle H(\theta), \leq_\theta, \in \rangle$ with each $\langle X_\gamma : \gamma < \beta \rangle \in X_\beta$ and $p, P \in X_0$, there exists a $p' \leq p$ in $P$ which is $(X_\beta, P)$-semi-generic for each $\beta < \alpha$.

Theorem 2.5 below is a generalization of a standard fact. Along with the observation that the one-step forcing in the iteration to make Bounding hold makes $\clubsuit_c$ hold, it shows that $\clubsuit_c$ holds in all currently known models of Bounding + CH.

**Theorem 2.5.** *The principle $\clubsuit_c$ is preserved by $\omega$-semi-proper forcing.*

*Proof.* Let $P$ be an $\omega$-semi-proper partial order and let $\langle a_\alpha : \alpha < \omega_1 \text{ limit}\rangle$ witness $\clubsuit_c$. Let $p$ be a condition in $P$ and let $\tau$ be a $P$-name for a club subset of $\omega_1$. We will find a $p' \leq p$ and a limit ordinal $\alpha < \omega_1$ such that $p'$ forces that $a_\alpha \subset \tau$. Let $\theta$ be a regular cardinal greater than $2^{|P|}$, let $\leq_\theta$ be a wellordering of $H(\theta)$ and let $\langle X_\alpha : \alpha < \omega_1\rangle$ be a continuous, increasing chain of countable elementary submodels of $\langle H(\theta), \leq_\theta, \in\rangle$ such that $p, P \in X_0$ and each $X_\beta \in X_\alpha$ for all $\beta < \alpha < \omega_1$. Let $D = \{X_\alpha \cap \omega_1 : \alpha < \omega_1 \text{ limit}\}$. Then $D$ is a club subset of $\omega_1$. Let $\alpha < \omega_1$ be such that $a_\alpha \subset D$. Let $\langle \beta_i : i < \omega\rangle$ be an increasing enumeration of $a_\alpha$, and let $\langle \gamma_i : i < \omega\rangle$ be such that each $\beta_i = X_{\gamma_i} \cap \omega_1$. Then $p, P \in X_{\gamma_0}$ and each $\langle X_{\gamma_j} : j < i\rangle \in X_{\gamma_i}$. Therefore, there is a condition $p' \leq p$ in $P$ which is $(X_{\gamma_i}, P)$-semi-generic for each $i < \omega$. This $p'$ then forces that $\{\beta_i : i < \omega\} \subset \tau$. $\square$

This raises two questions.

**2.6 Question.** Does CH imply that *Undominated* has a winning strategy in the canonical function game?

**2.7 Question.** Does Bounding + CH imply $\clubsuit_c$?

A negative answer to Question 2.6 would imply a negative answer to Question 2.7, which in turn would require a new proof of the consistency of Bounding + CH. On the other hand, a positive answer to Question 2.7 would give a positive answer to the following question of Woodin.

**2.8 Question.** ([9]) Do there exist $\Pi_2$ sentences for $H(\omega_2)$ $\phi$ and $\psi$ such that CH + $\phi$ and CH + $\psi$ are both $\Omega$-consistent but CH + $\phi$ + $\psi$ is not?

By contrast, Woodin has shown that all $\Omega$-consistent $\Pi_2$ sentences for $H(\omega_2)$ hold in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$, assuming certain large cardinals [9]. Shelah has shown that CH is consistent with the failure of $\clubsuit_c$ ([8], Chapter XVIII).

# 3 Elementary submodels and absoluteness

Shelah ([8], Chapter XVI) has shown that if there exists a Woodin cardinal, then there is a semi-proper forcing making the nonstationary ideal on $\omega_1$ ($NS_{\omega_1}$) saturated. His argument makes use of the following definitions, the first of which is implicit in [3].

**3.1 Definition.** A set $\mathcal{A} \subset \mathcal{P}(\omega_1) \setminus NS_{\omega_1}$ is *semi-proper* if for any transitive set $M$ closed under sequences of length $2^{\omega_2}$, if $X \prec M$ is countable with $\mathcal{A} \in X$, then there exists a countable $Y \prec M$ such that

- $X \subset Y$,
- $X \cap \omega_1 = Y \cap \omega_1$,
- $Y \cap \omega_1 \in S$ for some $S \in Y \cap \mathcal{A}$.

**3.2 Definition.** Given a set $\mathcal{A} \subset \mathcal{P}(\omega_1) \setminus NS_{\omega_1}$, the *sealing forcing* for $\mathcal{A}$ is the partial order consisting of pairs $(f, c)$ such that $f$ is a function into $\mathcal{A}$ with domain some countable ordinal and $c$ is a closed subset of $dom(f) + 1$ such that for each $\alpha \in c$ there exists a $\beta < \alpha$ with $\alpha \in f(\beta)$, ordered by extension.

If $\mathcal{A} \subset \mathcal{P}(\omega_1) \setminus NS_{\omega_1}$ is semi-proper in the sense of Definition 3.1, then the sealing forcing for $\mathcal{A}$ is semi-proper in the usual sense. Shelah's forcing for making $NS_{\omega_1}$ saturated consists of an iteration of length some Woodin cardinal where at limit stages one forces with the countable support product of all semi-proper sealing forcings as above, and at successor stages with $Coll(\omega_1, 2^{\omega_2})$.

The following definition, taken from [9], is implicit in Shelah's argument.

**3.3 Definition.** Suppose that $\mathcal{A} \subset \mathcal{P}(\omega_1) \setminus NS_{\omega_1}$. Let $T_{\mathcal{A}}$ be the set of countable $X \prec \mathcal{P}(H(\omega_2))$ such that for no countable $Y \prec \mathcal{P}(H(\omega_2))$ does it hold that

- $X \subset Y$,
- $X \cap \omega_1 = Y \cap \omega_1$,
- $Y \cap \omega_1 \in S$ for some $S \in Y \cap \mathcal{A}$.

So if, $\mathcal{A} \subset \mathcal{P}(\omega_1) \setminus NS_{\omega_1}$ is not semi-proper, then $T_{\mathcal{A}}$ is a stationary subset of $\mathcal{P}_{\omega_1}(\mathcal{P}(H(\omega_2)))$. Following [9], if $N \subset M$ are transitive models of ZFC with the same $\omega_1$, say that $M$ is a *good* extension of $N$ if $(T_{\mathcal{A}})^N$ is a stationary set in $M$ for each $\mathcal{A} \subset (\mathcal{P}(\omega_1) \setminus NS_{\omega_1})^N$ which is predense and not semi-proper in $N$.

Now, suppose that $\delta$ is a Woodin cardinal. Let $\mathbb{P}$ be any semi-proper iteration of length $\delta$ where at limit stages we take the countable support product of all semi-proper sealing forcings as above, and at successors to limit stages we pass to a good extension while collapsing $2^{\omega_2}$. Then it follows immediately from Claim XVI 2.8 of [8] or Theorem 2.62 of [9] that $\mathbb{P}$ makes $NS_{\omega_1}$ saturated. (Using this fact, Theorem 2.5 and the fact (shown in [5]) that $\omega$-semi-properness is preserved by Revised Countable Support iterations, it is straightforward to show that the saturation of $NS_{\omega_1}$ is consistent with $\clubsuit_c$.) The key point here is that if $N \subset M$ are transitive models of ZFC such that $M$ is a good extension of $N$ and $(2^{\omega_2})^N$ has cardinality $\aleph_1$ in $M$, then any semi-proper extension of $M$ is a good extension of $N$. So if our iteration uses $Coll(\omega_1, 2^{\omega_2})$ at successors to limit stages and the forcing $R_\kappa$ (for $\kappa$ the least measurable cardinal) defined at the end of Section 1 at all other successor stages, $\mathbb{P}$ forces that $NS_{\omega_1}$ is saturated and the canonical function game is undetermined (since the countable support product of all semi-proper sealing forcings followed by $Coll(\omega_1, 2^{\omega_2})$ doesn't add reals, every strategy for *Undominated* existing after a limit stage of the iteration is still defined on every position two steps later). In fact, the proof of Lemma 1.2 shows that we can use $R_\kappa$ at all successor stages to achieve the same effect (i.e., the $R_\kappa$-extension is also good - the proof of this, relative to the version of Lemma 1.2 for $R_\kappa$, is the same as the proof of Lemma 2.63 in [9]).

For a given real number $x$, let $I_x$ denote the class of indiscernibles for $x$, assuming that $x^\#$ exists. We let $C_x$ denote the class of uncountable cardinals of the inner model $L[x]$. If $\omega_1$ is inaccessible to reals then for each real $x$, $C_x \cap \omega_1$ is a club subset of $\omega_1$. Also, standard arguments show that for each real $x$ and each $\gamma \in C_{x^\#}$, $I_x \cap \gamma$ is definable over $L_\gamma[x^\#]$ and has ordertype $\gamma$, so in particular $C_{x^\#} \subset I_x$.

Woodin [9] has shown that if $NS_{\omega_1}$ is saturated and there exists a measurable cardinal, then every club subset of $\omega_1$ contains $I_x \cap \omega_1$ (and thus $C_{x^\#} \cap \omega_1$) for some real number $x$. It is not hard to see that if

- $\omega_1$ is inaccessible to reals,
- every club subset of $\omega_1$ contains $C_x \cap \omega_1$ for some real $x$,
- there is a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$min(C_x \cap C_y) < min(C_{f(x)} \cap C_{f(y)}),$$

then (+) holds, and in fact there is a club set $\mathcal{C}$ of countable elementary submodels of $H(\omega_2)$ (those closed under $f$) such that each pair from $\mathcal{C}$ is good. While the existence of such a club $\mathcal{C}$ is consistent (there is one in $L$, for instance), the hypotheses of the previous sentence may be contradictory, as far we know. In any case, we have the following theorem.

**Theorem 3.4.** *Suppose that there exists a Woodin cardinal $\delta$ below a measurable cardinal. Then for every function $f\colon \mathbb{R} \to \mathbb{R}$ existing in an inner model whose theory cannot be changed by forcing with a partial order in $V_{\delta+1}$, there exist $x, y \in \mathbb{R}$, such that*

$$min(C_x \cap C_y) = min(C_{f(x)} \cap C_{f(y)}).$$

Woodin has shown that whenever $\delta$ is a limit of Woodin cardinals below a measurable cardinal no forcing construction in $V_\delta$ can change the theory of $L(\mathbb{R})$ (see [4]). Even for the special case of the function $f(x) = x^\#$, we know of no direct proof of Theorem 3.4. Paris [7] has shown that if $a$ and $b$ are reals such that $a \in L[b]$ and $a^\# \notin L[b]$, then there are countable ordinals $\alpha$ and $\beta$ such that every $\alpha$-th $a$-indiscernible above $\beta$ is a $b$-indiscernible. It follows then that if $x, y$ are reals such that $min(C_x \cap C_y) = min(C_{x^\#} \cap C_{y^\#})$, then the set of reals in $L[x] \cap L[y]$ is closed under sharps.

Theorem 3.4 can be easily generalized in a number of ways, none of which we have application for at this time.

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