THREE DAYS OF Ω-LOGIC

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The Zermelo-Fraenkel axioms for set theory with the Axiom of Choice (ZFC) form the most commonly accepted foundations for mathematical practice, yet it is well-known that many mathematical statements are neither proved nor refuted from these axioms. One example is given by Gödel's Second Incompleteness Theorem, which says that no consistent recursive axiom system which is strong enough to describe arithmetic on the integers can prove its own consistency. This presents a problem for those interested in finding an axiomatic basis for mathematics, as the consistency of any such theory would seem to be as well justified as the theory itself. There are many views on how to resolve this situation. While some logicians propose weaker foundations for mathematics, some study the question of how to properly extend ZFC. Large cardinal axioms and their many consequences for small sets have been studied as a particularly attractive means to this goal. Large cardinals resolve the projective theory of the real line and banish the various measure-theoretic paradoxes derivable from the Axiom to Choice from the realm of the definable. Moreover, the large cardinal hierarchy itself appears to serve as universal measuring stick for consistency strength, in that the consistency strength of any natural statement of mathematics (over ZFC) can be located on this hierarchy (see [7] for the definitive reference on large cardinals).

While there is no technical definition of "large cardinal," Woodin's Ω-logic is an attempt to give a formal definition for the set of consequences of large cardinals for rank initial segments of the universe. Two definitions are proposed. The first, a form of generic invariance, is called the semantic relation: a theory T is said to imply a statement φ in Ω-logic if φ holds in every rank initial segment satisfying T in every set forcing extension (we deal only with set forcing in this article). Another more elaborate notion, involving a correctness property with respect to universally Baire sets of reals, plays the role of proofs. Woodin's Ω-conjecture is the statement that these two definitions are

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in fact equivalent. That the proof relation implies the semantic relation is already known (see Theorem 6.4).

Cantor's Continuum Hypothesis (CH), the statement that the cardinality of the set of real numbers is the least uncountable cardinal, is an important test case for those hoping to find correct extensions of ZFC. By a theorem of Levy and Solovay [12], large cardinals cannot settle the Continuum Hypothesis, as it is always possible to force to change the truth value of CH. Indeed, CH may be least complex natural statement of mathematics not resolved by large cardinals, as large cardinals decide the theory of the inner model $L(\mathbb{R})$ (and larger models) in Ω-logic. In his study of Ω-logic, Woodin has proposed a new approach to resolving CH. Assuming certain unresolved questions about the definability of Ω-logic, any statement implying generic invariance for the theory of $H(\aleph_2)$ implies that CH is false. Moreover, the definability of Ω-logic calls into question the idea that generic invariance should be a necessary condition for extensions of ZFC. We briefly survey these arguments in Subsections 8.4 and 8.5.

This paper is a slightly polished version of a lecture series on Ω-logic given at Nagoya University in November of 2009. As such it is a quick tour through the basic concepts of Ω-logic, giving proofs or sketches of proofs when appropriate. We have not attempted to address some of the more technical issues surrounding Ω-logic, such as current attempts to pin down its complexity or to prove the Ω-conjecture. There is a significant overlap with other presentations of Ω-logic, such as [1, 2, 21, 22, 23, 24, 25, 26]. None of the results presented here is due to the author. In Sections 1, 2 and 5-9, all results are due to Woodin unless otherwise noted.

Finally, I am very happy to have the chance to contribute to this special issue in honor of Yuzuru Kakuda, who sponsored me for a very enjoyable and productive year as a JSPS postdoctoral fellow at Kobe University in 1999-2000. I would also like to thank the organizers and attendees of this lecture series for their support and attention.

1. The stationary tower

The stationary tower is a partial order developed by Woodin [20], following on a theorem of Foreman, Magidor and Shelah [5] that it is possible, starting from a supercompact cardinal, to force to make the nonstationary ideal on ω_1 precipitous without collapsing ω_1.

Definition 1.1. A set a is *club* if $\bigcup a$ is nonempty and there is a function $F\colon (\bigcup a)^{<\omega} \to \bigcup a$ such that a is the set of $Y \subseteq \bigcup a$ closed under F. A set a is *stationary* if $\bigcup a$ is nonempty and a intersects every club subset of $\mathcal{P}(\bigcup a)$, i.e., if for every function $F\colon (\bigcup a)^{<\omega} \to \bigcup a$ there is a $Y \in a$ such that $F[[Y]^{<\omega}] \subseteq Y$.

Note that this agrees with the usual notion of stationarity for an unbounded subset of a regular cardinal κ (for clubs, the situation is slightly different : given a club $C \subseteq \kappa$, the limit points of C are the closure points of the function $\alpha \mapsto \min(C \setminus (\alpha+1))$). On the other hand, any set is stationary if it has a (nonempty) $\subseteq$-largest member.

Example 1.2. Given any nonempty set X, the set of $Y \subseteq X$ such that $(Y, \in)$ is an elementary submodel of $(X, \in)$ contains a club. For any infinite cardinal $\kappa \leq |X|$, the set of subsets of X of cardinality κ is stationary.

The intersection of countably many clubs is club, and the intersection of a club set with a stationary set is stationary. We let $[X]^{\aleph_0}$ denote the set of countable subsets of a given set X.

Fact 1.3. *Suppose that $X_0 \subseteq X_1$ are nonempty sets.*

- *If $a \subseteq \mathcal{P}(X_0)$ is stationary, then so is $\{Y \subseteq X_1 \mid Y \cap X_0 \in a\}$.*
- *If $a \subseteq [X_0]^{\aleph_0}$ is stationary, then so is $\{Y \in [X_1]^{\aleph_0} \mid Y \cap X_0 \in a\}$.*
- *If $a \subseteq \mathcal{P}(X_1)$ is stationary, then so is $\{Y \cap X_0 \mid Y \in a\}$.*

Proof. The third is the easiest, since we can extend any $F\colon X_0^{<\omega} \to X_0$ to a function $F'\colon X_1^{<\omega} \to X_1$, and if $Y \in a$ is closed under F', then $Y \cap X_0$ is closed under F.

For the other two, given a function $F\colon X_1^{<\omega} \to X_1$ there is a function $F'\colon X_1^{<\omega} \to X_1$ with the property that $F'[Y^{<\omega}]$ is closed under F and contains Y, for all $Y \subset X_1$ (F' essentially codes all the F-terms). Then if $Y \in a$ is closed under $F' \restriction X_0^{<\omega}$ (modified to take some fixed value in X_0 when F' takes values outside of X_0) , then $F'[Y^{<\omega}]$ is closed under F and $F'[Y^{<\omega}] \cap X_0 = Y$. $\square$

Definition 1.4. Given a (strongly inaccessible) cardinal κ, the stationary tower $\mathbb{P}_{<\kappa}$ is the partial order whose domain is the set of stationary sets in V_κ, where $a \leq b$ if and only if $Y \cap \bigcup b \in b$, for all $Y \in a$. The (countable) stationary tower $\mathbb{Q}_{<\kappa}$ is the same order, restricted to the set of stationary sets in V_κ which consist of countable sets.

Forcing with either $\mathbb{P}_{<\kappa}$ or $\mathbb{Q}_{<\kappa}$ induces an elementary embedding, using functions $f\colon a \to V$, for stationary sets a in the generic filter. Given a relation R in $\{\in, =\}$, and functions $f\colon a \to V$ and $g\colon b \to V$, with a and b in the generic filter G we set fRg in the generic ultrapower if and only if

$$\{Y \subset \bigcup a \cup \bigcup b \mid f(Y \cap \bigcup a) R g(Y \cap \bigcup b)\} \in G.$$

We let $[f]_G$ denote the element of the generic ultrapower represented by f.

Fact 1.5. *If a is stationary and $f(Y) \in Y$ for all $Y \in a$, then f is constant on a stationary set.*

Proof. Otherwise, for each $z \in \bigcup a$, choose $g_z \colon (\bigcup a)^{<\omega} \to \bigcup a$ witnessing that $f^{-1}[\{z\}]$ is nonstationary. Let $h \colon (\bigcup a)^{<\omega} \to \bigcup a$ be a function such that $h(z, \sigma) = g_z(\sigma)$ for all $\sigma \in (\bigcup a)^{<\omega}$. Then any $Y \in a$ closed under h gives a contradiction. $\square$

Fact 1.5 gives the following.

Fact 1.6. *For any set Z, the identity function on $\mathcal{P}(Z)$ represents $j[Z]$ in the $\mathbb{P}_{<\kappa}$-generic ultrapower, and the identity function on $[Z]^{\aleph_0}$ represents $j[Z]$ in the $\mathbb{Q}_{<\kappa}$-generic ultrapower.*

It follows that the function on $\mathcal{P}(Z)$ (or $[Z]^{\aleph_0}$) sending X to the transitive collapse of X represents Z in the (corresponding) generic ultrapower, so each element of V_κ is in the image model of the generic embedding. In particular, the function on $\mathcal{P}(\alpha)$ sending X to the ordertype of X represents α.

Fact 1.7. *For each condition a in $\mathbb{P}_{<\kappa}$ or $\mathbb{Q}_{<\kappa}$, $a \in G$ if and only if $j[\bigcup a] \in j(a)$.*

Proof. We have that $a = \{z \subseteq \bigcup a \mid z \in a\}$. Consider the functions on $\mathcal{P}(\bigcup a)$ (or $[\bigcup a]^{\aleph_0}$) representing $j[\bigcup a]$ and $j(a)$ (call them f and g). The former is the identity function and the latter is the constant function taking the value a. Then $a \in G$ if and only if

$$\{z \subseteq \bigcup a \mid z \in a\} \in G$$

if and only if $\{z \subseteq \bigcup a \mid f(z) \in g(z)\} \in G$ if and only if $[f]_G \in [g]_G$ if and only if $j[\bigcup a] \in j(a)$. $\square$

It follows that for $\alpha < \beta < \kappa$, $j(\alpha) = \beta$ if and only if

$$\{Y \subseteq \beta \mid \alpha = o.t.(Y)\} \in G,$$

since this set is in G if and only if $j[\beta]$ has ordertype $j(\alpha)$.

Fact 1.8. *For any uncountable regular cardinal $\gamma < \kappa$, γ (as a condition) forces in $\mathbb{P}_{<\kappa}$ that the critical point of j is γ.*

Proof. Note that $j[\gamma] \in j(\gamma)$ if and only if γ is the critical point. Now apply Fact 1.7. $\square$

Note that for $\eta < \gamma$, if $\gamma \in G$ then $\{\eta\} \in G$, since $\gamma \setminus \eta$ is club in γ and $\{\eta\} \geq \gamma \setminus \eta$. Fact 1.7 also implies the following.

Fact 1.9. *For any nonempty set $X \in V_\kappa$, $[X]^{\aleph_0}$ forces (in both $\mathbb{P}_{<\kappa}$ and $\mathbb{Q}_{<\kappa}$) that X will be countable in the image model of the embedding.*

The following follows from Facts 1.7 and 1.9.

Fact 1.10. *If κ is a strongly inaccessible cardinal, ω_1^V is the critical point of any $\mathbb{Q}_{<\kappa}$-generic embedding j, and $j(\omega_1^V) \geq \kappa$.*

Example 1.11. In L, let define $f\colon \omega_1 \to \omega_1$ by letting $f(\alpha)$ be the least β such that $L_\beta \models |\alpha| \leq \aleph_0$. Then for any $\delta < \kappa$, club many countable $X \prec V_\delta$ satisfy $L_{o.t.(X\cap\delta)} \models |X \cap \omega_1| \geq \aleph_1$. Then $o.t.(X \cap \delta) < f(X \cap \omega_1)$ for these X, which means that $[f]_G > \delta$ whenever G is an L-generic filter for either $\mathbb{P}_{<\kappa}$ or $\mathbb{Q}_{<\kappa}$.

The following fact shows that the preceding example contrasts with the large cardinal case. Recall that a cardinal κ is measurable if there exists a nonprincipal κ-complete ultrafilter on κ.

Fact 1.12. *If κ is a limit of measurable cardinals, then $j(\omega_1^V) = \kappa$ for any $\mathbb{Q}_{<\kappa}$-generic embedding j.*

Proof. If δ is a measurable cardinal and $X \prec V_{\delta+2}$ has cardinality less than δ, then there is a $Y \prec V_{\delta+2}$ containing X such that $Y \cap \delta$ end-extends $X \cap \delta$ (see [11, Lemma 1.1.18], for instance). So, given a stationary set $a \in \mathbb{Q}_{<\delta}$ and a function $f\colon a \to \omega_1$, the set of countable $X \prec V_{\delta+2}$ such that $X \cap \bigcup a \in a$ and $o.t.(X \cap \delta) > f(X \cap \bigcup a)$ is a stationary set below a forcing in $\mathbb{Q}_{<\kappa}$ that $[f]_G < \delta$. □

Essentially the same argument shows that, given a measurable cardinal δ and a stationary set $a \in V_\delta$, the set of countable $X \prec V_{\delta+2}$ such that $X \cap \bigcup a \in a$ and $o.t.(X \cap \delta) = \delta$ is a stationary set below a forcing that $j(\delta) = \delta$. This gives the following.

Fact 1.13. *If κ is a limit of measurable cardinals, then for any $\mathbb{P}_{<\kappa}$-generic embedding j, $j(\kappa) = \kappa$ and $j(\delta) = \delta$ for cofinally many measurable cardinals $\delta < \kappa$.*

Definition 1.14. A cardinal δ is *Woodin* if it is strongly inaccessible, and if for each $f\colon \delta \to \delta$ there exist a $\kappa < \delta$ closed under f and an elementary embedding $j\colon V \to M$ with critical point κ such that $V_{j(f)(\kappa)} \subseteq M$.

Equivalently, δ is Woodin if for every $A \subseteq V_\delta$ there is a $\kappa < \delta$ which is *$<\delta$-A-strong*, i.e., for every $\gamma < \delta$ there is an elementary embedding $j\colon V \to M$ with critical point κ such that $j(\kappa) > \gamma$, $V_\gamma \subset M$ and $j(A) \cap V_\gamma = A \cap V_\gamma$ (see [7, 11], for instance).

Definition 1.15. A cardinal κ is *strong* if for each ordinal α there is an elementary embedding $j\colon V \to M$ with critical point κ such that $V_\alpha \subseteq M$.

The wellfoundedness of the image model of stationary tower embeddings, in the presence of a Woodin cardinal, is given by the following theorem.

Theorem 1.16. *If δ is a Woodin cardinal, then the image model of any $\mathbb{P}_{<\delta}$-generic embedding is closed under sequences of length less than δ in the forcing extension, and the image model of any $\mathbb{Q}_{<\delta}$-generic embedding is closed under countable sequences in the forcing extension.*

In the remainder of this section, we will sketch the proof of Theorem 1.16 for $\mathbb{P}_{<\delta}$. The proof uses the following definitions.

Definition 1.17. Given a set $D \subseteq V_\kappa$ (for some cardinal κ), $sp(D)$ is the set of $Z \prec V_{\kappa+1}$ of cardinality less than κ with $D \in Z$ such that there exists an $X \supseteq Z$, $X \prec V_{\kappa+1}$, $X \cap \bigcup(Z \cap V_\kappa) = Z \cap V_\kappa$ such that for some $a \in D \cap X$, $X \cup \bigcup a \in a$.

Definition 1.18. A set $D \subseteq V_\kappa$ is *semi-proper* in V_κ (for some cardinal κ) if $sp(D)$ contains a club relative to $[V_{\kappa+1}]^{<\kappa}$.

Theorem 1.16 follows from Theorem 1.19 (see [11, Theorem 2.5.8]).

Theorem 1.19. *If δ is Woodin and $\langle D_\alpha : \alpha < \delta\rangle$ is a sequence of predense sets in $\mathbb{P}_{<\delta}$, then there are cofinally many strongly inaccessible $\kappa < \delta$ such that for all $\gamma < \kappa$, $D_\gamma \cap V_\kappa$ is semi-proper in V_κ and predense in $\mathbb{P}_{<\kappa}$.*

Instead of proving Theorem 1.19 (which is Theorem 2.5.9 of [11]), we give prove the corresponding result in the context of supercompact cardinals, as it is simpler and contains the main ideas.

Definition 1.20. Given a cardinals κ and λ, κ is λ-supercompact if there exists an elementary embedding $j\colon V \to M$ with critical point κ such that M is closed under λ-sequences.

Definition 1.21. A subset D of a partial order P is *predense* if every element of P is compatible with an element of D.

Theorem 1.22. *If κ is a 2^κ-supercompact cardinal, then every predense subset of $\mathbb{P}_{<\kappa}$ is semi-proper in V_κ.*

Proof. Let D be such a predense set, and towards a contradiction, suppose that $[V_{\kappa+1}]^{<\kappa} \setminus sp(D)$ is stationary (call this set a, and note that $\bigcup a = V_{\kappa+1}$). Let $j\colon V \to M$ be an elementary embedding with critical point κ such that M is closed under 2^κ-sequences. Then a is a stationary set in M. Since $j(D)$ is predense, there is a $b \in j(D)$ such that a and b are compatible in $\mathbb{P}^M_{<j(\kappa)}$. Working in in M, let $X \prec V^M_{j(\kappa)+1}$ be closed under j, with a, b in X and $|X| < j(\kappa)$, such that $X \cap \bigcup a \in a$ and $X \cap \bigcup b \in b$. Let $Y = X \cap V_{\kappa+1}$. Then $Y \in a$, so $j(Y) \in j(a)$. Since $|Y| < \kappa$, $j(Y) = j[Y]$ and

$$j(Y) \cap V_{j(\kappa)} = Y \cap V_\kappa.$$

Then $j(Y) \subseteq X$,

$$X \cap \bigcup (j(Y) \cap V_{j(\kappa)}) = X \cap \bigcup (Y \cap V_\kappa) = Y \cap V_\kappa = j(Y) \cap V_{j(\kappa)},$$

$b \in X$ and $X \cap \bigcup b \in b$, giving a contradiction. $\square$

To see how Theorem 1.16 follows from Theorem 1.19, note first that if A is an antichain in $\mathbb{P}_{<\kappa}$ then

- for each $X \prec V_{\kappa+1}$ with $A \in X$, there can be at most one $a \in A \cap X$ with $X \cap \bigcup a \in a$.
- if $b \in \mathbb{P}_{<\kappa}$ is stationary, and for each $X \in b$ there is an $a \in A \cap X$ such that $X \cap \bigcup a \in a$, then for all $c \leq b$ there exist $c' \leq c$ and $a \in A$ such that for all $X \in c'$, $a \in X$ and $X \cap \bigcup a \in a$.

By Theorem 1.19, if A_α ($\alpha < \delta$) are maximal antichains in $\mathbb{P}_{<\delta}$, each deciding a name τ_α for an element of the ultrapower (we may assume that it does this in such a way for each $\alpha < \delta$ and each $a \in A_\alpha$ there is a function $g \colon a \to V$ such that $a \Vdash \tau_\alpha = [\check{g}]_G$), then there are cofinally many $\kappa < \delta$ such that for any $b \in \mathbb{P}_{<\kappa}$, the set c consisting of those $X \prec V_{\kappa+1}$ such that

- $X \cap \bigcup b \in b$,
- for each $\alpha \in X \cap \kappa$ there is a unique $a \in A_\alpha \cap X$ such that $X \cap \bigcup a \in a$

is stationary, and therefore a condition below b.

Define f on c by letting $f(X)$ be a sequence of length $o.t.(X \cap \kappa)$ such that whenever α is the βth member of $X \cap \kappa$, $(\beta, g(X \cap \bigcup a)) \in f(X)$, where $a \in X \cap A_\alpha$ with $X \cap \bigcup a \in a$, and $a \Vdash \tau_\alpha = [\check{g}]_G$.

Note that $\bigcup c = V_{\kappa+1}$. Then for any $c' \leq c$, and any $\alpha < \kappa$ such that $\alpha \in X$ for all $X \in c'$, there is for each $X \in c'$, since $X \cap V_{\kappa+1} \in c$, a unique $a \in X \cap V_{\kappa+1} \cap A_\alpha$ such that $X \cap \bigcup a \in a$, with a $g \colon a \to V$ such that $a \Vdash \tau_\alpha = [\check{g}]_G$. By pressing down we get an $a \in A_\alpha$ which is this unique a for all $X \in c'' \leq c'$, which means that $c'' \leq a$.

Then for all $X \in c''$, we get that if α is the βth member of $X \cap \kappa$, then $(\beta, g(X \cap \bigcup a)) \in f(X \cap \bigcup c)$, which means that c'' forces that the pair $(\alpha, \tau_{\alpha G})$ is in the sequence $[f]_G$ represented by f.

2. $\models_\Omega$

For every Σ_2 sentence φ there is a sentence φ' (e.g., φ + "there is no largest ordinal" + "$H(\kappa)$ exists for all cardinals κ") such that φ is equivalent to "there exists a α such that $V_\alpha \models \varphi'$". Similarly, for any φ the statement "there is an ordinal α such that $V_\alpha \models \varphi$" is Σ_2. We will often use these two classes of sentences interchangeably.

The following standard forcing fact can be found in the appendix to [11].

Fact 2.1. *If P and Q are partial orders, and forcing with P makes $\mathcal{P}(Q)^V$ countable, then there is a Q-name τ for a partial order such that P is forcing-equivalent to $Q * \tau$.*

The following theorem of Woodin shows that, in the presence of a proper class of Woodin cardinals, forceability of Σ_2-sentences is forcing-invariant.

Theorem 2.2. *Suppose that δ is a Woodin cardinal, φ is a sentence, $\alpha < \delta$ and $P \in V_\delta$ a partial order such that P forces the statement "$V_\alpha \models \varphi$." Then after forcing with any partial order Q in V_δ there exist a partial order P' and an ordinal α' such that P' forces the statement "$V_{\alpha'} \models \varphi$."*

Proof. Let G be a V-generic filter for $\mathbb{P}_{<\delta}$ below the condition $[\mathcal{P}(Q)]^{\aleph_0}$. Since $[\mathcal{P}(Q)]^{\aleph_0}$ forces that $\mathcal{P}(Q)^V$ is countable, $V[G]$ is a forcing extension of a generic extension of V by Q (below any given condition in Q). Let $j \colon V \to M$ be the corresponding embedding. By elementary, $j(P)$ forces in M that $V_{j(\alpha)} \models \varphi$. Since $j(\alpha) < \delta$ and $V^M_\delta = V^{V[G]}_\delta$, $j(P)$ forces this in $V[G]$ as well. $\square$

As an exercise, one can adapt the proof of Theorem 2.2 to show the following : assuming $2^{<\kappa} = \kappa$ for a cardinal κ below a Woodin cardinal δ, anything that can be forced to hold in a proper rank initial segment of V_δ by a forcing preserving all cardinals below κ^+ can still be forced to hold in such an initial segment by such a forcing after forcing with $2^{<\kappa}$.

Definition 2.3. Given a sentence φ and a theory T, we say that $T \models_\Omega \varphi$ if in every set forcing extension, for every ordinal α, if $V_\alpha \models T$ then $V_\alpha \models \varphi$. If $\emptyset \models_\Omega \varphi$, then we say that φ is Ω-*valid.* If $T \models_\Omega \varphi$ we say that T *implies φ in Ω-logic.*

By Theorem 2.2, the set of φ such that $\emptyset \models_\Omega \varphi$ cannot be changed by set forcing. Ω-logic is an attempt to understand this set (for instance, its complexity) by proposing a corresponding notion of "proof."

3. Universally Baire sets

The following definition is due to Feng, Magidor and Woodin [3].

Definition 3.1. A set of reals A is *universally Baire* if for all compact Hausdorff spaces X, and any continuous function $f\colon X \to \mathbb{R}$, $f^{-1}[A]$ has the property of Baire in X.

In the definition of universally Baire sets, "compact" can be replaced with "having a base consisting of regular open sets" without changing the corresponding class of sets.

Definition 3.2. A *tree* on a set X is a subset of $X^{<\omega}$ closed under initial segments. Given a tree T on a set X, $[T]$ is the set of all $f \in {}^{\omega}X$ such that $f \restriction n \in T$ for all $n \in \omega$. If T is a tree on $\omega \times X$, for some set X, then the *projection* of T, $p[T]$, is the set of $a \in {}^{\omega}\omega$ for which there exists an $f \in {}^{\omega}X$ with $(a, f) \in [T]$ (identifying pairs of sequences with sequences of pairs).

Definition 3.3. Given an ordinal γ, a set $A \subseteq {}^{\omega}\omega$ is γ-*Suslin* if it is the projection of a tree on $\omega \times \gamma$, and *Suslin* if it is γ-Suslin for some ordinal γ. If T is a tree on $\omega \times \gamma$, for some ordinal γ, and $p[T] = A$, we say that T is a *Suslin representation* for A.

In the interest of streamlining the presentation, we give the following nonstandard (relative to our definition of universally Baire) definition of λ-universally Baire.

Definition 3.4. Given a cardinal λ, $A \subseteq {}^{\omega}\omega$ is λ-*universally Baire* if there exists a pair of trees S, T on $\omega \times \gamma$, for some ordinal γ, such that $p[S] = A$, and $p[S]$ and $p[T]$ are complements in all forcings extensions by partial orders of cardinality less than or equal to λ.

Given a set X, the partial order $Coll(\omega, X)$ consists of all finite partial functions from ω to X, ordered by inclusion. It is a standard forcing fact (due to McAloon) that any partial order $\mathbb{P}$ forcing the statement $|\mathbb{P}| = \aleph_0$ is forcing-equivalent to $Coll(\omega, \mathbb{P})$. It follows that any partial order of the form $\mathbb{P} \times Coll(\omega, \mathbb{P})$ is forcing-equivalent to $Coll(\omega, \mathbb{P})$.

Definition 3.5. Given a partial order P, a *nice* name for a subset of the ground model is a set of pairs $(p, \check{x})$, where p is a condition in P and $\check{x}$ is the canonical name for an element x of the ground model.

By a theorem of Feng-Magidor-Woodin [3], a set A is universally Baire if and only if it is λ-universally Baire for all cardinals λ. If A is λ-universally Baire, this is witnessed by trees on $\omega \times 2^{\lambda}$ (since all partial orders of cardinality λ regularly embed into $Coll(\omega, \lambda)$, and there are just 2^λ many nice names for reals in this forcing). It follows that if A is universally Baire and κ is a strong limit, then A is universally Baire in V_κ.

In L, every set of reals is a continuous image of a universally Baire set. In fact this follows from the Continuum Hypothesis plus the existence of an ω_1-sequence of distinct reals such that the set of codes for initial segments of this sequence is universally Baire. In the current fine-structural inner models for Woodin cardinals, there exists such a sequence which is $<\delta$-universally Baire for δ the least Woodin cardinal. This is discussed in the introduction to [26].

Given a tree S on a set of the form $\omega \times X$, membership in the projection of S is upwards absolute to wellfounded models. Given trees S, T of this form, the statement $p[S] \cap p[T] = \emptyset$ is also upwards absolute (one can prove this either by considering a ranking function on the tree of attempts to build a path in common, or by forcing over a countable elementary submodel). From this fact one can see that if S_0, T_0, S_1, T_1 are two pairs of trees such that $p[S_0] = p[S_1]$ and the pairs S_0, T_0 and S_1, T_1 project to complements in all forcing extensions in given class, then S_0 and S_1 have the same projection in any such extension, as the following facts hold there:

- $p[S_0] = {}^\omega\omega \setminus p[T_0]$,
- $p[S_1] = {}^\omega\omega \setminus p[T_1]$,
- $p[S_0] \cap p[T_1] = \emptyset$,
- $p[S_1] \cap p[T_0] = \emptyset$.

If $A \subseteq {}^\omega\omega$ is λ-universally Baire and $V[G]$ is a forcing extension of V by a partial order of cardinality less than or equal to λ, then we reinterpret A in $V[G]$ as $p[S]$, for any such S as above, and call the reinterpreted set A_G.

As shown in [3], universally Baire sets are universally measurable and have the property of Baire. Furthermore, analytic sets are universally Baire, and the class of universally Baire sets is closed under complements and countable unions.

4. The Martin-Solovay tree

Suppose that X is a set, $i < j$ are integers, and σ and τ are ultrafilters on X^i and X^j respectively. Then τ *projects* to σ if for all $A \in \sigma$,

$$\{s \in X^j \mid s{\restriction}i \in A\} \in \tau$$

(equivalently, if for all $B \in \tau$, $\{s{\restriction}i \mid s \in B\} \in \sigma$). In this case, if $j_\sigma \colon V \to M_\sigma$ and $j_\tau \colon V \to M_\tau$ are the induced ultrapowers, there is a factor embedding $k\colon M_\sigma \to M_\tau$ defined by $k([f]_\sigma) = [f^*]_\tau$, where $f^*\colon X^j \to V$ is defined by $f^*(s) = f(s \restriction i)$.

Suppose that $\langle \sigma_n : n \in \omega \rangle$ is a sequence of ultrafilters such that, for some underlying set X, $\sigma_n(X^n) = 1$ for all $n \in \omega$. The tower is *countably complete* if whenever B_n is a σ_n-positive set for each $n \in \omega$, there is an f such that $f \restriction n \in B_n$ for all n. This is equivalent to saying that the measures project to one another, and that the direct limit of the ultrapowers of the σ_n's (via the factor maps) is wellfounded.

Definition 4.1. Given a cardinal κ, a set $A \subseteq {}^\omega\omega$ is *κ-homogeneously Suslin* if there is a collection of κ-complete measures μ_σ ($\sigma \in {}^{<\omega}\omega$) such that

- for each σ, μ_σ concentrates on $|\sigma|$-tuples,
- for any $a \in {}^\omega\omega$, $a \in A$ if and only if $\langle \mu_{a \restriction n} : n \in \omega \rangle$ is countably complete.

A set is *homogeneously Suslin* if it is $\aleph_1$-homogeneously Suslin.

There is an underlying tree here, where one concentrates on the measure one sets giving rise to witnesses for each real not in A (using the fact that if the measures are nontrivial they are $\mathfrak{c}^+$-complete).

Homogeneously Suslin sets are determined. If there exists a measurable cardinal, then Π^1_1 sets are homogeneously Suslin (using the same measure, plus principal measures). These facts are due to Martin [13].

Definition 4.2. Given a cardinal κ, a set $A \subseteq {}^\omega\omega$ is *κ-weakly homogeneously Suslin* if there is a collection of κ-complete measures $\mu_{\sigma,\tau}$ ($\sigma, \tau \in {}^{<\omega}\omega$, $|\sigma| = |\tau|$) such that

- for each such pair σ, τ, $\mu_{\sigma,\tau}$ concentrates on $|\sigma|$-tuples,
- for any $a \in {}^\omega\omega$, $a \in A$ if and only if there exists $b \in {}^\omega\omega$ such that $\langle \mu_{a \restriction n, b \restriction n} : n \in \omega \rangle$ is countably complete.

A set is *weakly homogeneously Suslin* if it is $\aleph_1$-weakly- homogeneously Suslin.

A set of reals is *κ-weakly homogeneously Suslin* if and only if it is the continuous image of a κ-homogeneously Suslin set.

Given a set of measures witnessing that some set A is weakly homogeneously Suslin, the Martin-Solovay tree [14] for the complement of A is the tree of attempts, for each real x, to produce a witness to the illfoundedness of all the towers corresponding to x. That is, fixing some enumeration $\langle s_m : m < \omega \rangle$ of $\omega^{<\omega}$, the tree is the set of sequences $\langle (i_m, \alpha_m) : m < n \rangle$ (for some $n \in \omega$) such that for all $m < m' < n$ such that $s_{m'}$ extends s_m,

$$k_{(\langle i_j : j < |s_m| \rangle, s_m)(\langle i_j : j < |s_{m'}| \rangle, s_{m'})}(\alpha_m) > \alpha_{m'}.$$

Again, there is an underlying tree here. Let X be a set such that each measure concentrates on n-tuples from X, for some $n \in \omega$. Fix for each non-countably complete tower of the form $\langle \mu_{a\restriction n, b\restriction n} : n \in \omega\rangle$ a witness $\langle A_{a,b,n} : n \in \omega\rangle$ to the fact that it is not countably complete, and for each pair σ, τ in $\omega^{<\omega}$ of the same length n, let $B_{\sigma,\tau}$ be the intersection of all corresponding $A_{a,b,n}$ such that $a\restriction n = \sigma$ and $b\restriction n = \tau$ (if there are any such $A_{a,b,n}$, and if not let $B_{\sigma,\tau} = X^n$). Then the set of pairs $(\sigma, (\tau, \rho))$ such that $\rho \in B_{\sigma,\tau}$ is a tree T projecting to A, and the Martin-Solovay tree projects to the complement of A.

To see this, note that for each $a \in A$ the Martin-Solovay tree is wellfounded, since there is a b such that $\langle \mu_{a\restriction n, b\restriction n} : n \in \omega\rangle$ is countably complete. On the other hand, if $a \notin A$, then there is a ranking function π on the tree

$$T_a = \{(\tau, \rho) : (a\restriction|\tau|, (\tau, \rho)) \in T\}.$$

Then if (τ, ρ) is initial segment of (τ', ρ') in T_a, $\pi(\tau, \rho) > \pi(\tau', \rho')$. For each $\tau \in \omega^{<\omega}$, let θ_τ be a function on $X^{|\tau|}$ (representing an ordinal in the τ-ultrapower) such that $\theta_\tau(\rho) = \pi(\tau, \rho)$ for each ρ such that $(\tau, \rho) \in T_a$. If α is the ordinal represented by θ_τ in the $\mu_{a\restriction|\tau|,\tau}$-ultrapower, then $k_{(a\restriction|\tau|,\tau),(a\restriction|\tau'|,\tau')}(\alpha)$ is represented by the function $\rho \mapsto \theta_\tau(\rho\restriction|\tau|)$ in the $\mu_{a\restriction|\tau'|,\tau'}$-ultrapower, which is greater than $\theta_{\tau'}$ on a set of measure one, since

$$\theta_\tau(\rho\restriction|\tau|) = \pi(\tau, \rho\restriction|\tau|) > \pi(\tau', \rho) = \theta_{\tau'}(\rho)$$

whenever $(\tau', \rho) \in T_a$.

If the system of measures witnesses the κ-weak homogeneity of A, then the Martin-Solovay tree witnesses that A is $<\kappa$-universally Baire, since in this case the ordinals of the $\mu_{\sigma,\tau}$-ultrapowers are represented by the same functions as in the ground model, so the construction of the Martin-Solovay tree is the same. The following theorem of Kunen [10] was used in his proof that if there exist uncountably many measurable cardinals, then $L(Ord^\omega)$ does not satisfy Choice.

Theorem 4.3 (Kunen). *Suppose that ν is a κ-complete measure on a cardinal κ, that $i < j$ are elements of ω, and that μ_i and μ_j are κ^+-complete measures on the i- and j-tuples from some set X, respectively, such that μ_j projects to μ_i. If $j \colon V \to M$ is the ν-embedding, then the maps using $j(\mu_i)$ and $j(\mu_j)$ in M, and the corresponding factor map, move ordinals in the same way as the corresponding maps using μ_i and μ_j in V.*

Since the definition of the Martin-Solovay tree uses only the images of the ordinals by the factor maps between ultrapowers, it follows from Kunen's theorem that the Martin-Solovay tree for any set of κ^+-complete measures is moved to itself by j as above. Steel proved that this situation persists to stationary tower embeddings (see [11, Theorem 3.3.17]).

Theorem 4.4 (Steel)**.** *Suppose that κ is a Woodin cardinal, that $i < j$ are elements of ω, and that μ_i and μ_j are κ^+-complete measures on the i- and j-tuples from some set X, respectively, such that μ_j projects to μ_i. If $j\colon V \to M$ is an embedding derived from forcing with $\mathbb{P}_{<\kappa}$ or $\mathbb{Q}_{<\kappa}$, then the maps using $j(\mu_i)$ and $j(\mu_j)$ in M, and the corresponding factor map, move ordinals in the same way as the corresponding maps using μ_i and μ_j in V.*

It follows that if δ is a Woodin cardinal, then the Martin-Solovay tree for a collection of δ^+-complete measures maps to itself under $\mathbb{P}_{<\delta}$ and $\mathbb{Q}_{<\delta}$ embeddings.

The following theorem, in conjunction with the results of [13] mentioned above, implies that the existence of n Woodin cardinals below a measurable cardinal implies the determinacy of all Π^1_{n+1} sets.

Theorem 4.5 (Martin-Steel [15])**.** *If δ is a Woodin cardinal and $A \subset {}^\omega\omega$ is δ^+-weakly homogeneously Suslin, then ${}^\omega\omega \setminus A$ is $<\delta$-homogeneously Suslin.*

It follows that in the presence of a proper class of Woodin cardinals, the three tree representation properties for sets of reals that we have introduced here are equivalent.

Theorem 4.6. *If δ is a limit of Woodin cardinals and $A \subseteq {}^\omega\omega$, then the following are equivalent*

(1) *A is $<\delta$-universally Baire,*
(2) *A is $<\delta$-weakly homogeneously Suslin,*
(3) *A is $<\delta$-homogeneously Suslin.*

(3) $\Rightarrow$ (2) is immediate, and (2) $\Rightarrow$ (1) follows from the construction of the Martin-Solovay tree. (2) $\Rightarrow$ (3) follow from the Martin-Steel theorem. (1) $\Rightarrow$ (2) follows from the following theorem of Woodin.

Theorem 4.7 (Woodin)**.** *Suppose that δ is a Woodin cardinal, and S, T are trees on $\omega \times \gamma$, for some ordinal γ such that S and T project to complements in all $\mathbb{Q}_{<\delta}$-extensions. Then $p[S]$ and $p[T]$ are $<\delta$-weakly homogeneous.*

Proof. Since every real added by $\mathbb{Q}_{<\delta}$ is the realization of a $\mathbb{Q}_{<\kappa}$-name for some $\kappa < \delta$ (even though the restriction of the generic filter to $\mathbb{Q}_{<\kappa}$ may not be generic), we may assume that S and T have cardinality δ, and even that they are trees on $\omega \times \delta$. There are arbitrarily large $\kappa < \delta$ which are $<\delta$-T-strong. It suffices to prove that for any such κ, $p[T]$ is κ-weakly homogeneously Suslin. For each $\lambda < \delta$, fix an elementary embedding $j_\lambda \colon V \to M_\lambda$ with critical point κ such that $j_\lambda(T) \cap (\omega \times \lambda) = T \cap (\omega \times \lambda)$ and $V_\lambda \subseteq M_\lambda$. For each pair $(s, u) \in T \cap (\omega \times \lambda)$, define a measure $\Sigma(s, u)$ on $\kappa^{|s|}$ by letting $X \in \Sigma(s, u)$ if and only if $u \in j_\lambda(X)$. Note that the set $A_s = \{v \in \kappa^{|s|} \mid (s, v) \in T\} \in \Sigma(s, u)$ for each $(s, u) \in T \cap (\omega \times \lambda)$ (recall that we are assuming that $|s| = |u|$ in this situation).

Let $i \colon V \to N$ be an embedding derived from $\mathbb{Q}_{<\delta}$. We claim that in N, the collection of all measures of the form $i(\Sigma_\lambda(s, u))$ witnesses that $p[i(T)]$ is $i(\kappa)$ weakly homogeneously Suslin. Every real of N is either in $p[T]$ or $p[S]$. For $x \in p[T]$, fix $\lambda < \delta$ and $f \in \lambda^\omega$ such that $(x, f) \in [T \cap (\omega \times \lambda)]$. Then $\langle i(\Sigma_\lambda(x{\restriction}n, f{\restriction}n)) : n \in \omega\rangle$ is countably complete, as its limit model embeds into the i-image of M_λ, and in N, since N is closed under countable sequences. If $x \in p[S]$, then $x \notin p[T]$, so the sets $A_{x\restriction n}$ witness that no tower of measures of the form $\langle i(\Sigma_\lambda(x{\restriction}n, u_n)) : n \in \omega\rangle$ will be countably complete. □

The following theorem is known as the Tree Production Lemma, and is one of the most useful ways to show that a set of reals is universally Baire.

Theorem 4.8 (Woodin)**.** *Suppose that δ is a Woodin cardinal, and let $A \subseteq {}^\omega\omega$. Then A is $<\delta$ universally Baire if there exist a formula ψ, and a set y, such that whenever $j \colon V \to M$ is an embedding derived from $\mathbb{Q}_{<\delta}$ and x is a real in M, $x \in j(A)$ if and only if $V[x] \models \psi(x, y)$.*

Equivalently, if δ is a Woodin cardinal and $A \subseteq {}^\omega\omega$, then A is $<\delta$ universally Baire if for every real x which is V-generic for a partial order in V_δ, either $x \in j(A)$ for all $\mathbb{Q}_{<\delta}$-generic embeddings $j \colon V \to M$ such that $x \in M$, or $x \in j(A)$ for no such embeddings.

One consequence of Theorem 4.4 plus the the Tree Production Lemma is that every universally Baire A set has a universally Baire scale (a set of reals coding a Suslin representation for A).

5. A-closed models

Definition 5.1. Given $A \subseteq {}^\omega\omega$, a countable transitive model M of ZFC is *A-closed* if for each partial order $\mathbb{P} \in M$ there is a countable set $\mathcal{D}$ consisting of dense open subsets of $\mathbb{P}$ (not necessarily members of M) such that for every filter $G \subseteq P$ intersecting every member of $\mathcal{D}$, $A \cap M[G] \in M[G]$.

Since statements of the form "$x \in p[T]$" are absolute between well-founded models of (a suitable fragment of) ZFC, we have the following.

Example 5.2. Suppose that κ is a cardinal, $A \subseteq {}^{\omega}\omega$, and $V_\kappa \models$ ZFC + "A is universally Baire." Let X be a countable elementary submodel of V_κ with $A \in X$, and let M be the transitive collapse of X. Then M is A-closed.

This can be generalized as follows.

Fact 5.3. *Suppose that*

- $A \subseteq {}^{\omega}\omega$,
- θ *is an uncountable cardinal*
- X *is a countable elementary submodel of* $H(\theta)$ *with trees* S, T *witnessing that* A *is* $\aleph_0$*-universally Baire as members,*
- N *is the transitive collapse of* X.

Then $A \cap N[G] = A_G^{N[G]}$ *for every* (N, P)*-generic filter* G *for a countable partial order* $P \in X$.

Given a universally Baire set of reals A, A-closure has several equivalent formulations. We give just three here ([1] has more).

Fact 5.4. *Given an* $\aleph_0$*-universally Baire set of reals* $A \subseteq {}^{\omega}\omega$ *and a countable transitive model* M *of ZFC, the following statements are equivalent.*

1. M *is* A*-closed.*
2. *For all partial orders* $\mathbb{P} \in M$, *for all* $(V, \mathbb{P})$*-generic filters* G, $A_G \cap M[G] \in M[G]$.
3. *For all partial orders* $\mathbb{P} \in M$, *the set of pairs* (τ, p) *such that*
   - τ *is a nice* $\mathbb{P}$*-name for a real,*
   - $p \in \mathbb{P}$ *forces in* V *that* $\tau_G \in A_G$

   is a member of M.
4. *For all infinite ordinals* $\gamma \in M$, *the set of pairs* (τ, p) *such that*
   - τ *is a nice* $Coll(\omega, \gamma)$*-name for a real,*
   - $p \in Coll(\omega, \gamma)$ *forces in* V *that* $\tau_G \in A_G$

   is a member of M.

Proof. (1) $\Leftrightarrow$ (2) follows from Fact 5.3, letting N be as described there, with $M \in N$. (1) implies (2) since if M is A-closed and $P \in M$ then dense open subsets of P witnessing the A-closure of M will be in N, so for every (N, P)-generic filter G, $A \cap M[G] \in M[G]$ and $A \cap N[G] = A_G^{N[G]}$, so

$$A_G^{N[G]} \cap M[G] = A \cap N[G] \cap M[G] = A \cap M[G] \in M[G].$$

For the reverse direction, for all (N, P)-generic filters, $A \cap N[G] = A_G^{N[G]}$ and $A_G^{N[G]} \cap M[G] \in M[G]$, so

$$A \cap M[G] = A \cap N[G] \cap M[G] = A_G^{N[G]} \cap M[G] \in M[G].$$

(3) $\Rightarrow$ (2) follows from the fact the $M[G]$ can use G and the set given to decode membership in A_G for the realization of any given name for a real. (3) $\Rightarrow$ (4) is immediate, and (4) $\Rightarrow$ (3) follows from the fact that $\mathbb{P}$ regularly embeds into $Coll(\omega, |\mathbb{P}|)$, and names in the former partial order can be translated through this embedding to names in the latter. To see (2) $\Rightarrow$ (4), note that if $A_G \cap M[G] \in M[G]$ holds in $V[G]$, then some condition $p \in Coll(\omega, \gamma)$ forces that some $Coll(\omega, \gamma)$-name σ in M represents $A_G \cap M[G]$. Since $Coll(\omega, \gamma)$ is homogeneous, σ can be converted into a name forced by the empty condition to represent $A_G \cap M[G]$. □

Fixing a recursive bijection between ω and $\omega \times \omega$, we can let WO denote the set of $x \subseteq \omega$ whose image under this bijection is a wellordering of ω. Then WO is Π^1_1, and in fact every Π^1_1 set is a continuous preimage of WO.

Officially, we are considering A-closure only for wellfounded models M. The following example shows that if we were to relax this restriction to require only that M be an ω-model we would get an equivalent notion for suitably complex A.

Fact 5.5. *An ω-model of ZFC is WO-closed if and only if it is wellfounded.*

Proof. Suppose that γ were an illfounded "ordinal" of a WO-closed ω-model M, and consider a filter $G \subset Coll(\omega, \gamma)$ such that $WO \cap M[G] \in M[G]$. Then $M[G]$ could identify exactly which ordinals below γ are wellfounded, which is impossible. □

A similar argument shows that A-closure for a certain set of reals A can imply closure under sharps for all sets (see [7] for more on sharps).

Fact 5.6. *Suppose that the sharp of every real exists, and let $A \subseteq \omega \times {}^\omega\omega$ be the set of pairs (i, x) such that $i \in x^\#$. Then if M is an A-closed countable transitive model of ZFC, then $a^\# \in M$ for all sets $a \in M$.*

Proof. We may assume that a is an infinite set of ordinals. Fix an A-closed model M, and consider a filter $G \subset Coll(\omega, |a|)$ such that $A \cap M[G] \in M[G]$. Then in $M[G]$ there is an $x \subseteq \omega$ coding a structure isomorphic to $(sup(a) + 1, a, \in)$. Then $x^\# \in M[G]$, so $a^\# \in M[G]$, which means that $a^\# \in M$. □

If there exist proper class many Woodin cardinals, then the set of reals $\mathbb{R}^{\#}$ is universally Baire (this can be seen by applying Corollary 3.1.19 of [11] with Theorem 4.8). An analogous version of the previous fact, using $\mathbb{R}^{\#}$, shows that there is a universally Baire set of reals A, simply definable from $\mathbb{R}^{\#}$, such that if M is an A-closed model of ZFC, then $H(\aleph_1)^M$ is an elementary submodel of $H(\aleph_1)$.

More generally, a recursive bijection between ω and $\omega \times \omega$ enables a coding of hereditarily countable sets by subsets of ω, in such a way that (letting $\pi(x)$ be the set coded by x; the code for $a \in H(\aleph_1)$ actually describes the transitive closure of $\{a\}$) the set of x coding a set in $H(\aleph_1)$ is Π^1_1 (call is C), and such that the relations $\pi(x) = \pi(y)$ and $\pi(x) \in \pi(y)$ are both Σ^1_1. Now suppose that $f : C \to C$ is a universally Baire function (i.e., the graph of f is universally Baire) such that whenever $\pi(x) = \pi(y)$, $\pi(f(x)) = \pi(f(y))$. This induces a function on $H(\aleph_1)$ which (in the presence of a proper class of Woodin cardinals) extends to a definable class function F on V, since it reinterprets uniquely to a function on $H(\aleph_1)$ in any forcing extension (we can use a countable elementary submodel plus Σ^1_1 absoluteness to show that the isomorphism property persists to forcing extensions). Moreover, any f-closed countable transitive model of ZFC is closed under F (on all sets, not just $H(\aleph_1)^M$).

6. $\vdash_\Omega$

We will develop the proof relation for Ω-logic in the context of a proper class of Woodin cardinals. This large cardinal assumption is not strictly necessary, but it makes the analysis simpler, and is natural since it is the hypothesis used for the invariance of $\models_\Omega$.

Definition 6.1. If T is a countable theory in the language of set theory and φ is a sentence, then $T \vdash_\Omega \varphi$ if and only if there exists a universally Baire set $A \subseteq {}^\omega\omega$ such that for all countable transitive A-closed models M of ZFC, if $T \in M$, then $M \models$ "$T \models_\Omega \varphi$."

We say T *proves* φ *in* Ω*-logic* if $T \vdash_\Omega \varphi$, that φ is Ω*-provable* if $\emptyset \vdash_\Omega \varphi$, and that φ is Ω*-consistent* if $\emptyset \nvdash_\Omega \neg\varphi$ (i.e., if $\neg\varphi$ is not Ω-provable).

Universally Baire sets of reals play the role of proofs in Ω-logic. In this sense, one "proof" can prove many statements; indeed, using Theorem 7.8 one can show that there is one universally Baire set serving as the proof all Ω-provable statements.

Given A, $B \subseteq {}^\omega\omega$, we say that A is *Wadge reducible* to B ($A \leq_W B$) if there is a continuous function $f : {}^\omega\omega \to {}^\omega\omega$ such that $A = f^{-1}[B]$. This notation comes from the following game (the Wadge game) for A

and B : I plays to build a real x, II builds a real y, and I wins if $x \in A$ if and only if $y \in B$. The determinacy of Wadge games implies that for all $A, B \subseteq {}^{\omega}\omega$, either $B \leq_W A$ (in the case I wins) or $A \leq_W {}^{\omega}\omega \setminus B$ (in the case II wins). Martin and Monk showed that if DC (the statement that every tree without terminal nodes has an infinite branch) holds and all sets of reals have the property of Baire, then $\leq_W$ is wellfounded; their proof shows that $\leq_W$ is a wellfounded relation on the universally Baire sets. Thus we can talk of the *Wadge rank* of a set $A \subseteq {}^{\omega}\omega$, and this corresponds roughly to the notion of length of proof in Ω-logic.

The following theorem can be proved using Theorems 4.4 and 4.8. The assumption of a proper class of Woodin cardinals is overkill; the theorem follows from the assumption that universally Baire sets have universally Baire scales and universally Baire sharps.

Theorem 6.2 (Woodin)**.** *Suppose that there exist proper class many Woodin cardinals. Let $A \subseteq {}^{\omega}\omega$ be universally Baire. Then in every set generic extension $V[G]$ of V there is an elementary embedding from $L(A, \mathbb{R}^V)$ to $L(A_G, \mathbb{R}^{V[G]})$ sending A to A_G.*

It follows from Theorem 6.2 that if $T \vdash_{\Omega} \varphi$ holds, then it holds in all set-forcing extensions. The following theorem, which follows from the fact that the Martin-Solovay tree maps to itself (as discussed after Theorem 4.4) implies (along with the fact that every forcing of cardinality less than δ regularly embeds into $\mathbb{P}_{<\delta}$) that if $T \vdash_{\Omega} \varphi$ holds in a set forcing extension, then it holds in V.

Theorem 6.3. *Suppose that there exists a proper class of Woodin cardinals, δ is Woodin, and $j \colon V \to M \subseteq V[G]$ is a generic elementary embedding induced by forcing with $\mathbb{P}_{<\delta}$. Then every universally Baire subset of ${}^{\omega}\omega$ in $V[G]$ is a universally Baire set in M.*

Proof. (Sketch) First note that if $\kappa < \lambda$ are cardinals, then every λ-complete measure (on an ordinal) in a forcing extension by a partial order of cardinality κ is the canonical extension of (i.e., the set of supersets of members of) such a measure in the ground model. To see this, first note that every measure one set must contain such a set from the ground model. Now assume that for every condition in the generic there is a set in the measure not forced to be there by this condition, and intersect these.

Suppose that A is universally Baire in $V[G]$. Fixing an arbitrary $\kappa > \delta$, we will show that A is κ^+-weakly homogeneously Suslin in M. Fix a countable set of measures $\{\mu_{\sigma,\tau} : \sigma, \tau \in \omega^{<\omega}\}$ witnessing that A is κ^+-weakly homogeneously Suslin in $V[G]$. These measures are each canonical extensions of measures $\nu_{\sigma,\tau}$ in V, and $\{j(\nu_{\sigma,\tau}) : \sigma, \tau \in$

$\omega^{<\omega}\}$ exists in M. By Theorem 4.4, the Martin-Solovay trees for the complements of these two sets are exactly the same. □

The following is the Soundness Theorem for Ω-logic.

Theorem 6.4. *Suppose that there exist proper class many Woodin cardinals, T is a theory and φ is a sentence. Then $T \vdash_\Omega \varphi$ implies $T \models_\Omega \varphi$.*

Proof. Suppose that A witnesses $T \vdash_\Omega \varphi$. Let P be a partial order and let α be an ordinal. Let κ be a strongly inaccessible cardinal such that $\{P, \alpha\} \in V_\kappa$, and let X be a countable elementary submodel of V_α with $\{P, \alpha\} \in X$. Let M be the transitive collapse of X, and let $\bar{P}$, $\bar{\alpha}$ be the images of P and α under this collapse. Then M is A-closed, so $M \models$ "$T \models_\Omega \varphi$", which means that in any forcing extension of M by $\bar{P}$, the $V_{\bar{\alpha}}$ of this extension satisfies φ if it satisfies T. Then in any forcing extension of V by P, the V_α of this extension satisfies φ if it satisfies T. □

The Ω-*conjecture* is the statement that if there exist proper class many Woodin cardinals, then for any sentence φ, $\emptyset \models_\Omega \varphi$ if and only if $\emptyset \vdash_\Omega \varphi$. The reverse direction is the Soundness Theorem above. The forward direction is the Completeness Theorem for Ω-logic. Since $\models_\Omega$ and $\vdash_\Omega$ are forcing absolute, the Ω-conjecture is as well.

7. AD^+

The following definition generalizes the notion of Suslin representations for sets of reals.

Definition 7.1. A set $A \subseteq {}^\omega\omega$ is ∞-*Borel* if for some ordinal α, some set of ordinals S, and some formula with two free variables $\varphi(x, y)$,

$$A = \{y \in \mathbb{R} \mid L_\alpha[S, y] \models \varphi(S, y)\}.$$

Equivalently, A is ∞-Borel if there is a set of ordinals giving a transfinite Borel construction of A (see [9]). Recall that $\mathrm{DC}_\mathbb{R}$ is the statement that for every $R \subseteq {}^\omega\omega \times {}^\omega\omega$ such that for every $x \in {}^\omega\omega$ there exists $y \in {}^\omega\omega$ with $(x, y) \in R$, there exists a function $f : \omega \to {}^\omega\omega$ such that for all $n \in \omega$, $(f(n), f(n + 1)) \in R$ (i.e., the restriction of DC to relations on the reals). Assuming $\mathrm{AD}+\mathrm{DC}_\mathbb{R}$, a set of reals A is ∞-Borel if and only if $A \in L(S, \mathbb{R})$, for some $S \subseteq Ord$ (again, see [9]).

Definition 7.2. Θ is the least ordinal α which is not the range of any function with domain $\mathbb{R}$.

If the reals can be well ordered, then $\Theta = \mathfrak{c}^+$. As shown by Solovay, in the context of determinacy, Θ is the Θ-th cardinal.

The following statement (originally called "within scales") was developed by Woodin as attempt to axiomatize the properties of a model whose sets of reals are all Suslin in a larger model of AD with the same reals.

Definition 7.3. The axiom AD^+ is the conjunction of the following three statements.

i) $\mathrm{DC}_{\mathbb{R}}$
ii) Every set of reals is ∞-*Borel,*
iii) If $\lambda < \Theta$ and $\pi\colon \lambda^\omega \to \omega^\omega$ is a continuous function (where λ has the discrete topology), then $\pi^{-1}[A]$ is determined for every $A \subseteq \omega^\omega$.

The inclusion of ordinal determinacy is explained by the following theorem.

Theorem 7.4 ([8])**.** *Assume ZF + AD + $DC_{\mathbb{R}}$, and fix $\lambda < \Theta$. If $A \subseteq {}^\omega\omega$ is Suslin and co-Suslin, and $\pi\colon \lambda^\omega \to {}^\omega\omega$ is continuous, then $\pi^{-1}[A]$ is determined.*

While the third part of AD^+ trivially implies AD, it is not known whether AD implies AD^+. Woodin has shown that if $L(\mathbb{R}) \models \mathrm{AD}$, then $L(\mathbb{R}) \models \mathrm{AD}^+$. More generally, he has shown the following in the context of large cardinals.

Theorem 7.5. *If there exists a proper class of Woodin cardinals and $A \subseteq \mathbb{R}$ is universally Baire then:*

1) $L(A, \mathbb{R}) \models \mathrm{AD}^+$,
2) *Every set in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is universally Baire.*

The following theorem says that all three parts of AD^+ reflect from models of AD^+ to inner models with the same reals. For $\mathrm{DC}_{\mathbb{R}}$ (and AD) this is immediate.

Theorem 7.6. *Assuming ZF + AD^+, any transitive inner model M of ZF with $\mathbb{R} \subseteq M$ satisfies* AD^+*.*

For the ∞-Borel property this follows from Theorem 7.4 below. Given $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, M_Γ is the collection of sets X for which there is a bijection π from $\mathbb{R}$ to the transitive closure of X such that the sets $\pi^{-1}[X]$ and $\{(a, b) \mid \pi(a) \in \pi(b)\}$ are both in Γ.

Theorem 7.7. *Assume ZF + AD + $DC_{\mathbb{R}}$. If A is ∞-Borel and Γ is collection of sets of reals projective in A, then A has an ∞-Borel code in M_Γ.*

Strategies in games on ordinals reflect via the Moschovakis Coding Lemma (see [16]), which says that, under ZF + AD + $\mathrm{DC}_{\mathbb{R}}$, if $\leq$ is a prewellordering of $\mathbb{R}$, then any union of $\leq$-degrees is $\Sigma^1_1(\leq)$ (so if $\leq$ is a prewellordering of ${}^\omega\omega$ of length γ, then every subset of γ is coded by a set of reals which is $\Sigma^1_1(\leq)$). It follows that the form of ordinal determinacy embodied in AD^+ is absolute to inner models containing all the reals.

The following is the Σ^2_1 basis theorem for AD^+.

Theorem 7.8. *Suppose that $V = L(\mathcal{P}(\mathbb{R}))$ and that AD^+ holds. Then for every real x, $M_{\Delta^2_1(x)} \prec_{\Sigma_1} L(\mathcal{P}(\mathbb{R}))$.*

It follows that if $A \subseteq \mathbb{R}$, $L(A, \mathbb{R}) \models \mathrm{AD}^+$ and there is a set B in $\mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ such that all B-closed models satisfy some statement, then there is such a B which is $\Delta^2_1(x)$ in $L(A, \mathbb{R})$, for some real x.

The $\mathbb{P}_{\max}$ axiom $(*)$ is an example of a sentence which is Ω-consistent but not known to be provably forceable from large cardinals (i.e., whose negation is not known not to be Ω-valid). To show that $(*)$ is Ω-consistent it suffices to show that for each universally Baire set of reals A there is a strongly inaccessible cardinal κ and a pair of trees S, T witnessing that A is $<\kappa$-universally Baire such that $L(S, T, \mathbb{R}) \models \mathrm{AD}^+$. Given this situation, we can force over V with $\mathbb{P}_{max}$, without adding reals, getting a generic filter G. Then in $V[G]$, transitive collapses of countable elementary substructures of $L_\kappa(S, T, \mathbb{R})[G]$ will be A-closed models of $(*)$.

To find such κ, S, T for a given A, we can let κ be the least strongly inaccessible cardinal. The following theorem is the key technical point.

Theorem 7.9. *Suppose that δ is a Woodin cardinal, $\kappa < \delta$ is limit of Woodin cardinals, and $\delta_0 < \kappa$ is Woodin. Let S and T be sets of ordinals which are moved to themselves by all $\mathbb{Q}_{<\delta}$-embeddings. Then in a forcing extension there exist a set G which is V-generic for $Coll(\omega, <\kappa)$, and elementary embeddings $j\colon V \to M$ and*

$$k\colon L(S, T, \mathbb{R})^M \to L(S, T, \mathbb{R}^*),$$

where $\mathbb{R}^* = \bigcup_{\gamma<\kappa} \mathbb{R}^{V[G\cap Coll(\omega,<\gamma)]}$ and j is derived from $\mathbb{Q}_{<\delta_0}$ below any given condition.

This is related to Woodin's Derived Model Theorem, which can be stated in a weak form as follows. A tree T is said to be $<\lambda$-absolutely complemented if there is a tree S such that $p[T] = \mathbb{R} \setminus p[S]$ in all forcing extensions by partial orders of cardinality less than λ.

Theorem 7.10 (Derived Model Theorem; Woodin (see [19])). *Let λ be a limit of Woodin cardinals. Let $G \subseteq Col(\omega, {<}\lambda)$ be a V-generic filter. Let*

- *$\mathbb{R}^*$ be $\bigcup_{\alpha<\lambda} \mathbb{R}^{V[G\restriction\alpha]}$;*
- *Hom^* be the collection of sets of the form $p[T] \cap \mathbb{R}^*$, for T a ${<}\lambda$-absolutely complemented tree in V.*

Then $L(Hom^*, \mathbb{R}^*) \models \mathrm{AD}^+$.

A variation of this theorem says that in the situation of Theorem 7.9, $L(S, T, \mathbb{R}) \models \mathrm{AD}^+$. The proof is essentially the same, and we give a quick sketch. From Theorem 7.9 and the Tree Production Lemma (Theorem 4.8) it follows that any set of reals definable in $L(S, T, \mathbb{R})$ from S, T and a real is ${<}\delta_0$-universally Baire, since for any real generic over V for a partial order in V_{δ_0}, we can figure out whether the real is in the image of the given set by applying the same definition in the $Coll(\omega, {<}\kappa)$-extension mentioned in the theorem via the forcing language. Therefore, $L(S, T, \mathbb{R})$ satisfies AD (if δ_0 is greater than a Woodin cardinal, by Theorem 8.1 below), since a supposed least counterexample to AD would be definable and thus ${<}\delta_0$-universally Baire. One can similarly show that the supposed least counterexamples to the two parts of AD^+ are universally Baire, thus Suslin in some larger model of AD (via a universally Baire scale), by the argument just given. Thus these are not counterexamples by Theorem 7.7 and the Moschovakis Coding lemma.

8. The definability of $\vdash_\Omega$

8.1. **Definability in $H(\delta_0^+)$.** For this section, let δ_0 denote the least Woodin cardinal. In this subsection we show that $\vdash_\Omega$ is definable in $H(\delta_0^+)$. This result relies on the following theorem.

Theorem 8.1 (Neeman [17, 18]). *If δ is a Woodin cardinal, then δ-universally Baire sets are determined.*

To see that $\vdash_\Omega$ is definable in $H(\delta_0^+)$, first note that the set of δ_0-universally Baire subsets of ${}^\omega\omega$ is definable in $H(\delta_0^+)$, since $A \subset {}^\omega\omega$ is δ_0-universally Baire if and only if for all partial orders P of cardinality $\leq \delta_0$ and all P-names for reals there exist a dense set of conditions $p \in P$ for which there exists a tree S on $\omega \times \delta_0$ such that $p \Vdash \tau \in p[S]$ and either $p[S] \subseteq A$ or $p[S] \subseteq {}^\omega\omega \setminus A$.

By Neeman's theorem, Wadge determinacy holds for δ_0-universally Baire sets. Since the class of universally Baire sets is closed under continuous preimages, the truly universally Baire sets are an initial segment of this hierarchy. For each δ_0-universally Baire set A, let T_A be the set of sentences φ such that for every A-closed countable transitive model M of ZFC, $M \models$ "$\emptyset \models_\Omega \varphi$." The union of the T_A's is a superset of the sentences φ such that $\vdash_\Omega \varphi$. If these two sets are equal, then we have given a definition of $\vdash_\Omega$ in $H(\delta_0^+)$. Otherwise, there is a least Wadge rank of a δ_0^+-universally Baire set A for which T_A includes sentences which are not Ω-provable. Let n_0 be the Gödel code of some such sentence. Then all truly universally Baire sets have rank less than A, and the union of the sets T_B, for B of Wadge rank less than A, is the set of Ω-provable sentences, which is again definable in $H(\delta_0^+)$, this time using the integer n_0.

8.2. **Gödel sentences.** Given that Ω-provability is defined in $H(\delta_0^+)$, and thus in a rank initial segment of the universe, one might expect to refute the Ω-conjecture by means of a Gödel sentence of the form "I am not Ω-provable." Assuming that there exist proper class many Woodin cardinals, let θ be the unary formula such that any sentence ψ is Ω-provable if and only if $H(\delta_0^+) \models \theta(\ulcorner\psi\urcorner)$. Then there is a unary formula φ such that $\varphi(n)$ says that if n is the code for a unary formula ψ, and there exists a least Woodin cardinal δ_0, then $H(\delta_0^+) \models \neg\theta(\ulcorner\psi(n)\urcorner)$. The corresponding Gödel sentence φ_0 is $\varphi(\ulcorner\varphi\urcorner)$.

If φ_0 were Ω-provable, then $H(\delta_0^+) \models \theta(\ulcorner\varphi_0\urcorner)$, and there would be a universally Baire set A such that all A-closed models satisfy "$\models_\Omega \varphi_0$". That is, all rank initial segments of forcing extensions (even for the trivial forcing) of A-closed models satisfy φ_0 and therefore satisfy the sentence "there is no rank initial segment of the universe containing a least Woodin cardinal δ such that $H(\delta^+) \models \theta(\ulcorner\varphi_0\urcorner)$." However, we can contradict this assertion by taking a countable elementary substructure of a model containing trees witnessing the universal Baireness of A up to some strongly inaccessible cardinal above δ (i.e., just by running the proof that Ω-provable sentence are true in all rank initial segments.)

So φ_0 is not Ω-provable. Assuming the Ω-conjecture, one can force to make it false in some initial segment of the universe. That is, one can force to make the sentence "there is a least Woodin cardinal δ_0 and $H(\delta_0^+) \models \theta(\ulcorner\varphi_0\urcorner)$" hold in some rank initial segment. Since the least Woodin cardinal in a rank initial segment of the universe really is the least Woodin cardinal, we appear to have forced to make φ_0 Ω-provable in this extension, violating the absoluteness of the Ω-provability relation. However, the definability of Ω-provability in $H(\delta_0^+)$ depends on an integer parameter, the least Gödel number of the sentence ψ which is not Ω-provable but for which there is a δ_0-universally Baire set A of least Wadge rank such that all A-closed models satisfy "$\models_\Omega \psi$"; $H(\delta_0^+)$ cannot identify this ψ. This parameter may change from one forcing extension to the next - indeed it must in some cases, since as we shall see there are models in which the Ω-conjecture holds. So the resolution of this apparent contradiction is that our Gödel sentence no longer has its desired meaning in this forcing extension.

8.3. **The $\mathbf{AD}^+$ conjecture.** Woodin's AD^+ *Conjecture* is the statement that whenever A and B are sets of reals such that $L(A,\mathbb{R})$ and $L(B,\mathbb{R})$ satisfy AD^+ and all sets of reals in $L(A,\mathbb{R}) \cup L(B,\mathbb{R})$ are $\aleph_1$-universally Baire, then one of

$$(\underset{\sim}{\Delta}^2_1)^{L(A,\mathbb{R})}$$

and

$$(\underset{\sim}{\Delta}^2_1)^{L(B,\mathbb{R})}$$

contains the other. Assuming that this conjecture holds, $\vdash_\Omega$ is definable in $H(\mathfrak{c}^+)$, by essentially the same argument that was used above for $H(\delta^+_0)$, using Theorem 7.8. The point is that under the AD^+ Conjecture, the members of the various classes $(\underset{\sim}{\Delta}^2_1)^{L(A,\mathbb{R})}$ as above fall into a Wadge hierarchy definable in $H(\mathfrak{c}^+)$, and, by the Σ^2_1 basis theorem for AD^+, every Ω-provable sentence is proved by one of them.

Woodin has shown that if there exists a measurable Woodin cardinal, then there exist sets of reals A, B such that $L(A,\mathbb{R})$ and $L(B,\mathbb{R})$ satisfy AD^+ but the Wadge game for the pair (A, B) is not determined.

8.4. **An argument against the Continuum Hypothesis.** As noted at the end of Section 5, in the presence of a proper class of Woodin cardinals there is a universally Baire set of reals A simply definable from $\mathbb{R}^\#$ witnessing that for every sentence ψ, either

$$ZFC \vdash_\Omega \text{“}H(\aleph_1) \models \psi\text{”}$$

or

$$ZFC \vdash_\Omega \text{“}H(\aleph_1) \models \neg\psi.\text{”}$$

Suppose that $\vdash_\Omega$ were definable in $H(\mathfrak{c}^+)$, and suppose that the analogous situation were to hold for $H(\aleph_2)$. That is, suppose that there were a true sentence φ giving an Ω-complete theory for $H(\aleph_2)$, in the sense that for every sentence ψ, either

$$ZFC + \varphi \vdash_\Omega \text{“}H(\aleph_2) \models \psi\text{”}$$

or

$$ZFC + \varphi \vdash_\Omega \text{“}H(\aleph_2) \models \neg\psi.\text{”}$$

Aside from the requirement that φ be true, we know that there are such statements φ, for example, the $\mathbb{P}_{\max}$ axiom $(*)$ and its variants, again using this simple variant of $\mathbb{R}^\#$ as the universally Baire "proof." The known examples of such axioms all imply the failure of the Continuum

Hypothesis. In fact, if $\vdash_\Omega$ is definable in $H(\mathfrak{c}^+)$, this must be the case. Otherwise, if the Continuum Hypothesis held and V satisfied a statement which is Ω-complete for $H(\aleph_2)$, in the sense above, then truth in $H(\aleph_2)$ would be definable in $H(\aleph_2) = H(\mathfrak{c}^+)$, which contradicts Tarski's theorem on the definability of truth. Woodin has shown more, that every Ω-complete sentence for $H(\aleph_2)$ would imply that u_2, the least ordinal above ω_1 which is a Silver indiscernible relative to every real number, is equal to ω_2.

The application of Tarski's theorem can be extended to arbitrary uncountable cardinals. Suppose that λ is a definable cardinal, and that there is a true statement ψ which gives an Ω-complete theory for $H(\lambda)$, in the sense that for every sentence ψ, either

$$ZFC + \varphi \vdash_\Omega \text{``}H(\lambda) \models \psi\text{''}$$

or

$$ZFC + \varphi \vdash_\Omega \text{``}H(\lambda) \models \neg\psi.\text{''}$$

Then we get a contradiction to Tarski's theorem by supposing that $\vdash_\Omega$ is definable in $H(\lambda)$. Since $\vdash_\Omega$ is definable in $H(\delta_0^+)$, it follows that there is no statement which gives an Ω-complete theory for $H(\delta_0^+)$.

8.5. **An argument against the vagueness of the Continuum Hypothesis.** It is a commonly held view that the Continuum Hypothesis is in some sense too vague to have a truth value. One argument in support of this view appeals to the fact that it is a theorem of ZFC that the truth value of CH can be changed by forcing. Formalizing this view, suppose then that one were to take the set of true sentences to be those which hold in all forcing extensions. Provability in Ω-logic, being definable in $H(\delta_0^+)$, is Δ_2 in the language of set theory. If the Ω-conjecture holds, then the set of Π_2 sentences true in all forcing extensions is then Δ_2. Thus if the set of true Π_2 sentences is the set of such sentences true in every forcing extension, we have a collapse of the complexity of the definability hierarchy. If there is no such collapse, then the fact that one can force the Continuum Hypothesis to be either true or false is not the reason that it doesn't have a truth value.

9. Proving the Ω-conjecture

As mentioned above, the Ω-conjecture is known to be consistent. In this final section we outline a method for proving the Ω-conjecture. This method works in certain inner models, and it is not known whether it can be used to prove the that Ω-conjecture holds in V. We begin by introducing the partial order (due to Woodin) known as the *extender algebra* (following the presentation in [4]).

9.1. $\mathcal{L}_{\delta,\gamma}$. For regular cardinals $\gamma \leq \delta$, let $\mathcal{L}_{\delta,\gamma}$ be the propositional logic with variables a_ξ ($\xi < \gamma$), the usual connectives $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ and $\neg$, plus infinitary conjunctions and disjunctions of size less than δ. In addition to the standard rules of inference for finitary propositional logic, proofs in $\mathcal{L}_{\delta,\gamma}$ can use the infinitary version of DeMorgan's Laws, can deduce any individual element from an infinitary conjunction, and can deduce an infinitary conjunction from all of its parts.

We let $\mathcal{B}_{\delta,\gamma}$ denote the equivalence classes of $\mathcal{L}_{\delta,\gamma}$ under mutual provability, and, for any $\mathcal{L}_{\delta,\gamma}$ theory $\mathcal{T}$ we define the quotient algebra $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ by letting $\varphi \sim \psi$ if $T \vdash \varphi \leftrightarrow \psi$.

A subset x of γ is naturally interpreted as a model in $\mathcal{L}_{\delta,\gamma}$ by letting each a_ξ have the value True if and only if $\xi \in x$. Similarly, forcing with $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ produces a generic subset of γ. If M is a transitive model of (a suitable fragment of) ZFC and $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$ (as defined in M) satisfies the δ-chain condition in M, then any subset x of γ (in M or not) which satisfies all the sentences of $\mathcal{T}$ is M-generic for $\mathcal{B}_{\delta,\gamma}/\mathcal{T}$. This follows from the fact that any maximal antichain has size less than δ, which means that $\mathcal{T}$ proves its disjunction, which means that x satisfies its disjunction.

9.2. **Extenders.** Given uncountable cardinals $\kappa < \lambda$, a (κ, λ)-extender is a function $E \colon [\lambda]^{<\omega} \setminus \{\emptyset\} \rightarrow V_{k+2}$ such that

(1) each $E(s)$ is a nonprincipal κ-complete ultrafilter on $\kappa^{|s|}$;
(2) (coherence) whenever $s \subset t$ are finite subsets of λ and $a \subset |t|$ is the set of i such that the ith element of t is in s, $X \in E(s)$ if and only if the set of $\langle \xi_i : i < |t| \rangle$ such that $\langle \xi_i : i \in a \rangle \in X$ is in $E(t)$;
(3) (normality) for each s and each $f \colon \kappa^{|s|} \rightarrow \lambda$ such that
$$\{a \in \kappa^{|s|} \mid f(a) < \max(s)\} \in E(s),$$
there exists a $t \supseteq s$ in $[\lambda]^{<\omega}$ such that
$$\{b \in \kappa^{|s|} \mid f(b) \in t\} \in E(t).$$

We say that λ is the *length* of the extender E. Extenders satisfying condition (1) above are often called *short* extenders; these are sufficient for our needs. An extender gives rise to a directed system of wellfounded embeddings whose limit model may or may not be wellfounded, but does contain V_λ, if λ is a strongly inaccessible cardinal.

A *system of extenders* is a set of extenders indexed by some set contained in $Ord \times Ord$, though for our purposes we can think of a system of extenders as simply a set of extenders.

Fixing a cardinal δ, we say that a set $A \subseteq V_\delta$ *reflects* to a cardinal $\kappa < \delta$ if for every $\lambda < \delta$ there is an elementary embedding $j\colon V \to M$ with critical point κ such that $j(A) \cap V_\lambda = A \cap V_\lambda$. A cardinal δ is Woodin if for each $A \subseteq V_\delta$ there is a $\kappa < \delta$ such that A reflects to κ.

Given a system $\vec{E}$ of extenders, let $\mathcal{T}_{\delta,\gamma}(\vec{E})$ be the deductive closure of the set of sentences of the form

$$\bigvee_{\xi<\kappa} \varphi_\xi \leftrightarrow \bigvee_{\xi<\lambda} \varphi_\xi,$$

where $\langle \varphi_\xi : \xi < \lambda \rangle$ is a sequence of sentences in $\mathcal{L}_{\delta,\gamma}$ which reflects to κ.

If $\vec{E}$ witnesses that δ is Woodin, then $\mathcal{W}_{\delta,\gamma}(\vec{E}) = \mathcal{B}_{\delta,\gamma}/\mathcal{T}_{\delta,\gamma}(\vec{E})$ is δ-c.c.. To see this, note that if $\{[\varphi_\xi] : \xi < \delta\}$ is a set of elements of $\mathcal{W}_{\delta,\gamma}(\vec{E})$, then there is a $\kappa < \delta$ such that $\langle [\varphi_\xi] : \xi < \delta \rangle$ reflects to κ, which means that for some $\lambda \in (\kappa, \delta)$,

$$\bigvee_{\xi<\kappa} \varphi_\xi \leftrightarrow \bigvee_{\xi<\lambda} \varphi_\xi,$$

so none of the members of $\{[\varphi_\xi] : \kappa < \xi < \delta\}$ is incompatible with all of the members of $\{[\varphi_\xi] : \xi < \kappa\}$.

9.3. **Iteration trees.** Given a transitive model M and an extender sequence $\vec{E}$ in M, an *iteration tree* is a tree T on some ordinal η, along with models M_ξ ($\xi < \eta$) and extenders E_ξ (for ξ such that $\xi + 1 < \eta$) and commuting elementary embeddings $j_{\xi\nu}\colon M_\xi \to M_\mu$ for $\xi \leq_T \nu$ such that

- $M_0 = M$,
- $0 \leq_T \xi$ for all $\xi < \eta$,
- each $E_\xi \in j_{0\xi}(\vec{E})$,
- if $\alpha + 1 < \eta$, then α has an immediate predecessor α^* in T, and $M_{\alpha+1}$ is the ultrapower of M_{α^*} by E_α, with $j_{\alpha^*(\alpha+1)}$ the induced embedding (we require that M_{α^*} and M_α have a long enough initial segment in common for this to make sense, i.e. that all subsets of the critical point of E_α in M_{α^*} are in M_α),
- if $\beta < \lambda$ is a limit ordinal, M_β is the direct limit of the M_α's along some cofinal branch of $T{\restriction}\beta$

We can construe the construction of an iteration tree as a game between player I and player II, where I constructs at successor stages, and II constructs at limit stages and the first player unable to continue the construction with a wellfounded model loses. We say that M is η-*iterable* if II has a strategy which avoids a loss in the version of this game that runs for η-stages.

The following remarkable theorem of Woodin says that every subset of η is generic over some iterate of any $(\eta^+ + 1)$-iterable model.

Theorem 9.1. *Let η be an infinite cardinal, let δ be an ordinal greater than η, and suppose that $(M, \vec{E})$ is $(\eta^+ + 1)$-iterable, where M is a transitive model of ZFC of cardinality η and $\vec{E}$ is an extender sequence witnessing that δ is a Woodin cardinal in M. Suppose that for every (κ, λ)-extender in $\vec{E}$, $\kappa > \eta$ and λ is a strongly inaccessible cardinal in M. Then for each $x \subseteq \eta$ there exists an elementary embedding $j \colon M \to M^*$ induced by an iteration tree on M such that x is M^*-generic for $j(\mathcal{W}^M_{\delta,\eta}(\vec{E}))$.*

Proof. We construct an iteration tree on M of length at most $\eta^+ + 1$, using $\vec{E}$. We fix an $\eta^+ + 1$-iteration strategy for $(M, \vec{E})$ and use it to pick the models a limit stages. At each successor stage, given the last model M_α, we are done if x is M_α-generic for $j_{0\alpha}(\mathcal{W}_{\delta,\delta}(\vec{E}))$. If not, there exist a $(\kappa_\alpha, \lambda_\alpha)$ extender $E_\alpha \in j_{0\alpha}(\vec{E})$ and a sequence $\vec{\varphi}$ in M_α such that x fails to satisfy the corresponding axiom from $j_{0\alpha}(\mathcal{T}_{\delta,\gamma}(\vec{E}))$. Let $\beta \leq \alpha$ be minimal such that M_α and M_β have the same subsets of κ_α, apply E_α to M_β, and let $M_{\alpha+1}$ be the corresponding ultrapower. This completes the construction.

The requirement of applying each E_α to the earliest possible model gives that $\lambda_{\alpha_1} \leq \kappa_{\alpha_2}$ that whenever $\alpha_0 <_T \alpha_1 + 1 <_T \alpha_2 + 1$ are three consecutive nodes of T. Otherwise, since $V^{M_{\alpha_0}}_{\lambda_{\alpha_1}} = V^{M_{\alpha_1+1}}_{\lambda_{\alpha_1}}$, M_{α_0} and M_{α_1+1} would have the same subsets of κ_{α_2}, and E_{α_2} could have been applied to M_{α_0}. It follows that for all $\alpha <_T \alpha'$, $\lambda_\alpha \leq \kappa_{\alpha'}$.

Now suppose that the construction ran for $\eta^+ + 1$ stages. Take an elementary submodel X of a large enough $H(\theta)$ of cardinality η with M, x and the strategy used above as elements, and let $\alpha = X \cap \eta^+$. Then the image of M_{η^+} under the transitive collapse of X is M_α, and $j_{\alpha(\eta^+ + 1)}$ is the inverse π of the transitive collapse restricted to M_α. Let $\beta + 1$ be the successor of α along the cofinal branch leading to M_{η^+}, and let $\vec{\varphi}$ be the sequence of formulas used at stage β. Then x fails to satisfy $\bigvee \vec{\varphi} \restriction \kappa_\beta$, but satisfies $\bigvee \vec{\varphi} \restriction \lambda_\beta$. Since $\vec{\varphi} \restriction \kappa_\beta \in M_\alpha$, $\pi(\vec{\varphi} \restriction \kappa_\beta) = j_{\alpha(\eta^+ + 1)}(\vec{\varphi} \restriction \kappa_\beta)$, By the elementarity of π, $\pi(x) = x$ fails to satisfy $\bigvee \pi(\vec{\varphi} \restriction \kappa_\beta)$. But $\vec{\varphi} \restriction \lambda_\beta$ is an initial segment of $j_{\alpha(\eta^+ + 1)}(\vec{\varphi} \restriction \beta)$, since all subsequent embeddings along the cofinal branch after β have critical point at least λ_β, giving a contradiction. $\square$

9.4. **Proving the Ω-conjecture.** At the end of Section 5 we discussed a definable function π coding elements of $H(\aleph_1)$ by reals. Suppose that M is a countable transitive model of (a suitable fragment of) ZFC. We say that M has a *universally Baire iteration strategy* if $(M, \vec{E})$ has an ω_1-iteration strategy which is represented by a universally Baire set of reals A via π. The existence of such a strategy shows that $(M, \vec{E})$ is fully iterable (i.e., there is a definable class strategy for player II that works for games of all lengths). Moreover, if N is any A-closed model of ZFC with $(M, \vec{E}) \in H(\aleph_1)^N$, then for any ordinal γ in N, N contains the fragment of this strategy for all iteration trees on $(M, \vec{E})$ in N of length γ or less. These facts follow from Theorem 6.2 plus the remarks at the of Section 5. Combined with Theorem 9.1, these facts present a possible route for proving the Ω-conjecture.

Theorem 9.2. *Suppose that there exist proper class many Woodin cardinals. Let φ be a sentence such that $\neg\varphi$ is not Ω-provable. Let δ be a Woodin cardinal, let $\vec{E}_0$ be an extender sequence witnessing that δ is Woodin, let $\kappa > \delta$ be a strongly inaccessible cardinal, and suppose that X is a countable elementary substructure of V_κ such that $(M, \vec{E})$ has a universally Baire iteration strategy, where M is the transitive collapse of X and $\vec{E}$ is the image of $\vec{E}_0$ under this collapse. Then there is a forcing extension in which some rank initial segment satisfies φ.*

Proof. Let A be a universally Baire set of reals giving rise to an ω_1-iteration strategy for $(M, \vec{E})$ via π. Let N be a transitive A-closed model of ZFC with $(M, \vec{E}) \in H(\aleph_1)^N$, such that for some limit ordinals $\alpha < \beta$ in N and some partial order $P \in V_\beta^N$, P forces in N that $V_\alpha \models \varphi$. Since $(M, \vec{E})$ is fully iterable in N, we may iterate $(M, \vec{E})$ in N to make V_β^N generic for this iterate M^* of M (to apply Theorem 9.1, we need to first iterate M to a model M_0 such that $|V_\beta|^N \subset M_0$, and then apply the theorem to this model). Then V_β^N is the V_β of this forcing extension of M^*, so this forcing extension thinks that there exist such α and P. Since M is the transitive collapse of an elementary submodel of a suitable rank initial segment of V, this means that it is possible to force over V to make some rank initial segment satisfy φ. $\square$

All of the currently known canonical inner models for large cardinals have the property that transitive collapses of countable elementary submodels of rank initial segments have universally Baire iteration strategies. The previous theorem then says that these models satisfy the Ω-conjecture. For all we know, it may be a theorem of ZFC that every iteration tree on a rank initial segment of the the universe has a unique cofinal wellfounded branch, if one uses short extenders (i.e., extenders as defined here) whose length is strongly inaccessible. If this is the case, then the Tree Production Lemma would imply that transitive collapses of rank initial segments of the universe are fully iterable with universally Baire iteration strategies (to see this, note that if (x, y) are reals such that x codes an iteration tree on a structure and y codes a cofinal branch through the tree, it is absolute to all wellfounded models containing x and y whether the corresponding limit model is wellfounded), and that the Ω-conjecture holds.

This scenario may not be the likeliest approach to proving the Ω-conjecture. There are weaker iteration hypotheses which might hold for V and which would suffice. In another direction, Woodin has recently been developing an inner model which, in the presence of a supercompact cardinal, would contain all the large cardinals of V, and which would satisfy the Ω-conjecture [26]. This would show that the Ω-conjecture is consistent with all large cardinals.

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