Handbook of Set Theory

Foreman, Kanamori, and Magidor (eds.)

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Contents

I. Forcing over models of determinacy

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The *Axiom of Determinacy* (AD) is the statement that all integer games of perfect information of length ω are determined. This statement contradicts the Axiom of Choice, and presents a radically different view of the universe of sets. Nonetheless, determinacy was a subject of intense study by the mid 1960's, with an eye towards the possibility that some inner model of set theory satisfies AD (see, for example, the introductory remarks in [30]). Since strategies for these games can be coded by real numbers, the natural inner model to consider is $L(\mathbb{R})$, the smallest model of Zermelo-Fraenkel set theory containing the reals and the ordinals. This approach was validated by the following theorem of Woodin (see [13, 19]), building on work of Martin and Steel [23] and Foreman, Magidor and Shelah [6].

0.1 Theorem. *If there exists a measurable cardinal which is greater than infinitely many Woodin cardinals, then the Axiom of Determinacy holds in* $L(\mathbb{R})$.

A companion to Theorem 0.1, also due to Woodin (see [19]) and building on the work of Foreman, Magidor and Shelah [6], shows that the existence of a proper class of Woodin cardinals implies that the theory of $L(\mathbb{R})$ cannot be changed by set forcing. By Theorem 0.1, the Axiom of Determinacy is part of this fixed theory for $L(\mathbb{R})$.

0.2 Theorem. *If* δ *is a limit of Woodin cardinals and there exists a measurable cardinal greater than* δ*, then no forcing construction in* V_δ *can change the theory of* $L(\mathbb{R})$.

Theorem 0.2 has the following corollary. If P is a definable forcing construction in $L(\mathbb{R})$ which is homogeneous (i.e., the theory of the extension can be computed in the ground model), then the theory of the P-extension of $L(\mathbb{R})$ also cannot be changed by forcing (i.e., the P-extensions of $L(\mathbb{R})$ in all forcing extensions of V satisfy the same theory). This suggests that the absoluteness properties of $L(\mathbb{R})$ can be lifted to models of the Axiom of Choice, as Choice can be forced over $L(\mathbb{R})$.

In [33], Steel and Van Wesep made a major step in this direction, forcing over a model of a stronger form of determinacy than AD to produce a model of ZFC satisfying two consequences of AD, that $\underset{\sim}{\delta}^1_2$ (the supremum of the lengths of the Δ^1_2-definable prewellordings of the reals) is ω_2 and the nonstationary ideal on ω_1 (NS_{ω_1}) is saturated. Woodin [36] later improved the hypothesis to $\mathrm{AD}^{L(\mathbb{R})}$.

In the early 1990's, Woodin proved the following theorem, showing for the first time that large cardinals imply the existence of a partial order forcing the existence of a projective set of reals giving a counterexample to the Continuum Hypothesis. The question of whether ZFC is consistent with a projective witness to $\mathfrak{c} \geq \omega_3$ remains open.

0.3 Theorem. *If NS_{ω_1} is saturated and there exists a measurable cardinal then $\underset{\sim}{\delta}^1_2 = \omega_2$.*

One important point in this proof is the fact that if NS_{ω_1} is saturated then every member of $H(\omega_2)$ (those sets whose transitive closure has cardinality less than $\aleph_2$) appears in an iterate (in the sense of the next section) of a countable model of a suitable fragment of ZFC. Since these countable models are elements of $L(\mathbb{R})$, their iterations induce a natural partial order in $L(\mathbb{R})$. With certain technical refinements, this partial order, called $\mathbb{P}_{max}$, produces an extension of $L(\mathbb{R})$ whose $H(\omega_2)$ is the direct limit of the structures $H(\omega_2)$ of models satisfying every forceable theory (and more). In particular, the structure $H(\omega_2)$ in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ (assuming that AD holds in $L(\mathbb{R})$) satisfies every Π_2 sentence ϕ (in the language with predicates for NS_{ω_1} and each set of reals in $L(\mathbb{R})$) for $H(\omega_2)$ such that for some integer n the theory ZFC + "there exist n Woodin cardinals" implies that ϕ is forceable. Furthermore, the partial order $\mathbb{P}_{max}$ can be easily varied to produce other consistency results and canonical models.

The partial order $\mathbb{P}_{max}$ and some of its variations (and many other related issues) are presented in [37]. The aim of this chapter is to prepare the reader for that book. First, we attempt to give a complete account of the basic analysis of the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$, relative to published results. Then we briefly survey some of the issues surrounding $\mathbb{P}_{max}$, in particular $\mathbb{P}_{max}$ variations and forcing over larger models of determinacy. We also briefly introduce Woodin's Ω-logic, in order to properly state the maximality properties of the $\mathbb{P}_{max}$ extension. For the most part, though, our focus is primarily on the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$, and secondarily on $\mathbb{P}_{max}$-style forcing constructions as a means of producing consistency results. For other topics, such as the Ω-conjecture and the relationship between Ω-logic and the Continuum Hypothesis, we refer the reader to [40, 38, 39, 41, 2].

The material in this chapter is due to Woodin, except where noted otherwise. The author would like to thank Howard Becker and John Steel for advising him on parts of the material in Sections 4 and 9 respectively. He would also like to thank Andrés Caicedo, Neus Castells and Teruyuki

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1. Iterations

The fundamental construction in the $\mathbb{P}_{max}$ analysis is the iterated generic elementary embedding. These embeddings can have many forms, but we will concentrate on the following case. Suppose that I is a normal, uniform, proper ideal on ω_1 (so I is a proper subset of $\mathcal{P}(\omega_1)$ containing all the countable subsets, and such that whenever A is an I-positive set (i.e., in $\mathcal{P}(\omega_1)\setminus I$) and $f\colon A \to \omega_1$ is a regressive function, f is constant on an I-positive set; notationally, we are going to act as though "proper" and "uniform" are contained in the definition of *normal ideal*, and similarly for "measure" and "ultrafilter"). Then forcing with the Boolean algebra $\mathcal{P}(\omega_1)/I$ creates a V-normal ultrafilter U on ω_1^V. By convention, we identify the wellfounded part of the ultrapower $Ult(V, U)$ with its transitive collapse, and we note that this wellfounded part always contains ω_2^V. The corresponding elementary embedding $j\colon V \to Ult(V, U)$ has critical point ω_1^V, and since I is normal, for each $A \in \mathcal{P}(\omega_1)^V$, $A \in U$ if and only if $\omega_1^V \in j(A)$. Under certain circumstances, the corresponding ultrapower of V is wellfounded; if every condition in $\mathcal{P}(\omega_1)/I$ forces this, then I is *precipitous*.

For the most part, we will be concerned only with models of ZFC, but since occasionally we will want to deal with structures whose existence can be proved in ZFC, we define the fragment ZFC$^\circ$ to be the theory ZFC $-$ Powerset $-$ Replacement $+$ "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the following scheme, which is a strengthening of ω_1-Replacement: every (possibly proper class) tree of height ω_1 definable from set parameters has a maximal branch (i.e., a branch with no proper extensions; in the cases we are concerned with, this just means a branch of length ω_1). By the Axiom of Choice here we mean that every set is the bijective image of an ordinal. We will use ZFC$^\circ$ in place of the theory ZFC* from [37], which asserts closure under the Gödel operations (see page 178 of [9]) plus a scheme similar to the one above. One advantage of using ZFC* is that $H(\omega_2)$ satisfies it (and thus so do its elementary submodels). On the other hand, it raises some technical points that we would rather avoid here. Some of these points appear in Woodin's proof of Theorem 0.3. Our concentration is on $\mathbb{P}_{max}$, but we hope nonetheless that the reader will have no difficulty in reading the proofs of that theorem in [4, 37] after reading the material in this section.

With either theory, the point is that one needs to be able to prove the version of Łoś's theorem asserting that ultrafilters on ω_1 generate elementary embeddings, which amounts to showing the following fact. The fact follows immediately from the scheme above.

1.1 Fact. (ZFC°) Let n be an integer. Suppose that ϕ is a formula with $n+1$ many free variables and $f_0, \ldots, f_{n-1}$ are functions with domain ω_1. Then there is a function g with domain ω_1 such that for all $\alpha < \omega_1$,

$$\exists x \phi(x, f_0(\alpha), \ldots, f_{n-1}(\alpha)) \Rightarrow \phi(g(\alpha), f_0(\alpha), \ldots, f_{n-1}(\alpha)).$$

If M is a model of ZFC and κ is a cardinal of M of cofinality greater than ω_1^M (in M), then $H(\kappa)^M$ satisfies ZFC° if it has $|\mathcal{P}(\mathcal{P}(\omega_1))|^M$ as a member.

Suppose that M is a model of ZFC° , $I \in M$ is a normal ideal on ω_1^M and $\mathcal{P}(\mathcal{P}(\omega_1))^M$ is countable. Then there exist M-generic filters for the partial order $(\mathcal{P}(\omega_1)/I)^M$. Furthermore, if $j \colon M \to N$ is an ultrapower embedding of this form, then $\mathcal{P}(\mathcal{P}(\omega_1))^N$ is countable, and there exist N-generic filters for $(\mathcal{P}(\omega_1)/j(I))^N$. We can continue choosing generics in this way for up to ω_1 many stages, defining a commuting family of elementary embeddings and using this family to take direct limits at limit stages.

We use the following formal definition.

1.2 Definition. Let M be a model of ZFC° and let I be an ideal on ω_1^M which is normal in M. Let γ be an ordinal less than or equal to ω_1. An *iteration* of (M, I) of length γ consists of models M_α ($\alpha \leq \gamma$), sets G_α ($\alpha < \gamma$) and a commuting family of elementary embeddings $j_{\alpha\beta} \colon M_\alpha \to M_\beta$ ($\alpha \leq \beta \leq \gamma$) such that

- $M_0 = M$,
- each G_α is an M_α-generic filter for $(\mathcal{P}(\omega_1)/j_{0\alpha}(I))^{M_\alpha}$,
- each $j_{\alpha\alpha}$ is the identity mapping,
- each $j_{\alpha(\alpha+1)}$ is the ultrapower embedding induced by G_α,
- for each limit ordinal $\beta \leq \gamma$, M_β is the direct limit of the system $\{M_\alpha, j_{\alpha\delta} : \alpha \leq \delta < \beta\}$, and for each $\alpha < \beta$, $j_{\alpha\beta}$ is the induced embedding.

If $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1 \rangle$ is an iteration of a pair (M, I) and each $\omega_1^{M_\alpha}$ is wellfounded, then $\{\omega_1^{M_\alpha} : \alpha < \omega_1\}$ is a club subset of ω_1. Note also that if $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma \rangle$ is an iteration of a pair (M, I), then $j[\mathrm{Ord}^{M_0}]$ is cofinal in Ord^{M_γ}.

The models M_α in Definition 1.2 are called *iterates* of (M, I). If M is a model of ZFC° then an iteration of $(M, NS_{\omega_1}^M)$ is called simply an *iteration of* M and an iterate of $(M, NS_{\omega_1}^M)$ is called simply an *iterate* of M. When the individual parts of an iteration are not important, we sometimes call the elementary embedding $j_{0\gamma}$ corresponding to an iteration an iteration itself. For instance, if we mention an iteration $j\colon (M,I) \to (M^*,I^*)$, we mean that j is the embedding $j_{0\gamma}$ corresponding to some iteration

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma \rangle$$

of (M,I), and that M^* is the final model of this iteration and $I^* = j(I)$.

If M and I are as in Definition 1.2, then the pair (M,I) is *iterable* if every iterate of (M,I) is wellfounded. In this chapter, we are in general interested only in iterable pairs (M,I). Note that when checking for iterability it suffices to consider the countable length iterations, as any iteration of length ω_1 whose final model is illfounded contains an illfounded model at some earlier stage. The following two lemmas show that if

- M is a transitive model of ZFC°+ Powerset containing ω_1^V,
- $|\mathcal{P}(\mathcal{P}(\omega_1)/I)|^M$ is countable,
- κ is a cardinal of M greater than $|\mathcal{P}(\mathcal{P}(\omega_1)/I)|^M$ with cofinality greater than ω_1^M in M,
- $I \in M$ is a normal precipitous ideal on ω_1^M,

then the pair $(H(\kappa)^M, I)$ is iterable. In particular, Lemma 1.5 below shows that every such pair (M,I) is iterable, and then Lemma 1.4 shows that $(H(\kappa)^M, I)$ is iterable, as every iterate of $(H(\kappa)^M, I)$ embeds into an iterate of (M,I). This argument is our primary means of finding iterable models.

1.3 Remark. Note that the statement that a given pair (M,I) is iterable is Π^1_2 in any real x recursively coding the pair. One way to express this (not necessarily the most direct), is: for every countable model N of ZFC° with x as a member and every object $J \in N$ such that $N \models$ "J is an iteration of (M,I)" and every function f from ω to the "ordinals" of the last model of J, either N is illfounded (i.e., there exists an infinite descending sequence of "ordinals" of N) or $f(n+1)$"$\not\in$"$f(n)$ for some integer n, where "$\not\in$" is the negation of the $\in$-relation of the last model of J. Therefore, whether or not (M,I) is iterable is absolute between models of ZFC° containing the countable ordinals. Furthermore, assuming that $x^\#$ exists and letting γ denote $\omega_1^{L[x^\#]}$, any transitive model N of ZFC° containing $L_\gamma[x^\#]$ is correct about the iterability of (M,I), as $L[x^\#]$ is correct about it, and N thinks that $L_{\omega_1^N}[x^\#]$ is correct about it. Similarly, if γ and δ are countable ordinals coded by reals y and z, then the existence of an iteration of (M,I) of length γ which is illfounded is a Σ^1_1 fact about x and y, and the existence of an iteration of (M,I) of length γ such that the ordinals of the last model of the iteration have height at least (or, exactly) δ is a Σ^1_1 fact about x, y and z.

The first lemma is easily proved by induction. The last part of the lemma uses the assumption that N is closed under ω_1^M-sequences in M (this is the main way in which the lemma differs from the corresponding lemma in [37] (Lemma 3.8)). In our applications, N will often be $H(\kappa)^M$ for some cardinal κ of of M such that $M \models \mathrm{cf}(\kappa) > \omega_1^M$, in which case $H(\kappa)^M$ is indeed closed under ω_1^M-sequences in M.

1.4 Lemma. *Suppose that M is a model of ZFC° and $I \in M$ is a normal ideal on ω_1^M. Let N be a transitive model of ZFC° in M containing $\mathcal{P}(\mathcal{P}(\omega_1)/I)^M$ and closed under ω_1^M-sequences in M. Let $\gamma \leq \omega_1$ be an ordinal and let*

$$\langle N_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle$$

be an iteration of (N, I). Then there exists a unique iteration

$$\langle M_\alpha, G^*_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle$$

of (M, I) such that for all $\beta < \alpha \leq \gamma$, $G_\beta = G^*_\beta$ and

$$\mathcal{P}(\mathcal{P}(\omega_1)/I)^{N_\alpha} = \mathcal{P}(\mathcal{P}(\omega_1)/I)^{M_\alpha}.$$

Furthermore, $N_\alpha = j^*_{0\alpha}(N)$ for all $\alpha \leq \gamma$.

Given ordinals α, β, the partial order $\mathrm{Col}(\alpha, \beta)$ is the set of partial functions from α to β whose domain has cardinality less than that of α, ordered by inclusion. In particular, $\mathrm{Col}(\omega, \beta)$ makes β countable. Given ordinals α and β, $\mathrm{Col}(\alpha, {<}\beta)$ is the partial order consisting of all finite partial functions $p \colon \beta \times \alpha \to \beta$ such that for all $(\delta, \gamma) \in dom(p)$, $p(\delta, \gamma) \in \delta$, ordered by inclusion (we will not use this definition until the next section).

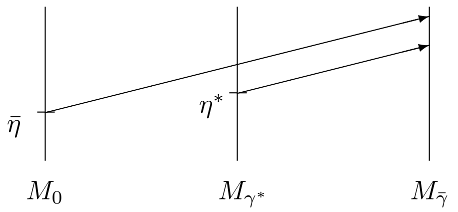
The proof of Lemma 1.5 is a modification of standard arguments.

1.5 Lemma. *Suppose that M is a transitive model of ZFC°+ Powerset and that $I \in M$ is a normal precipitous ideal on ω_1^M. Suppose that $j\colon (M, I) \to (M^*, I^*)$ is an iteration of (M, I) whose length is in $(\omega_1^V + 1) \cap M$. Then M^* is wellfounded.*

Proof. If j and M^* are as in the statement of the lemma, then M^* is the union of all sets of the form $j(H(\kappa)^M)$, where κ is a regular cardinal in M, and for each such $\kappa > |\mathcal{P}(\mathcal{P}(\omega_1))|^M$, $j{\restriction}H(\kappa)^M$ is an iteration of $(H(\kappa)^M, I)$. If the lemma fails, then, we may let $(\bar\gamma, \bar\kappa, \bar\eta)$ be the lexicographically least triple (γ, κ, η) such that

- κ is a regular cardinal in M greater than $|\mathcal{P}(\mathcal{P}(\omega_1)/I)|^M$,
- $\eta < \kappa$,
- there is an iteration $\langle N_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle$ of $(H(\kappa)^M, I)$ such that $j_{0\gamma}(\eta)$ is not wellfounded.

Since I is precipitous in M, $\bar{\gamma}$ is a limit ordinal, and clearly $\bar{\eta}$ is a limit ordinal as well. Fix an iteration $\langle N_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \bar{\gamma}\rangle$ of $(H(\bar{\kappa})^M, I)$ such that $j_{0\bar{\gamma}}(\bar{\eta})$ is not wellfounded, and let $\langle M_\alpha, G_\beta, j'_{\alpha\delta} : \beta < \alpha \leq \delta \leq \bar{\gamma}\rangle$ be the corresponding iteration of M as in Lemma 1.4. By the minimality of $\bar{\gamma}$ we have that M_α is wellfounded for all $\alpha < \bar{\gamma}$. Since $N_{\bar{\gamma}}$ is the direct limit of the iteration leading up to it, we may fix $\gamma^* < \bar{\gamma}$ and $\eta^* < j_{0\gamma^*}(\bar{\eta})$ such that $j_{\gamma^*\bar{\gamma}}(\eta^*)$ is not wellfounded. Note that by Lemma 1.4, $j'_{\gamma^*,\bar{\gamma}}(\eta^*) = j_{\gamma^*,\bar{\gamma}}(\eta^*)$ and $j'_{\gamma^*,\bar{\gamma}}(\bar{\eta}) = j'_{\gamma^*,\bar{\gamma}}(\bar{\eta})$.



The key point is that if N is a model of ZFC°, J is a normal ideal on ω_1^N in N, γ is an ordinal and η is an ordinal in N, then the statement positing an iteration of (N, J) of length γ whose last model is illfounded below the image of η is a Σ^1_1 sentence in a real parameter recursively coding N, η and γ, and so this statement is absolute between wellfounded models of ZFC° containing such a real. In particular, if

- N is a transitive model of ZFC° + Powerset,
- J is a normal ideal on ω_1^N in N,
- κ is a regular cardinal in N greater than $|\mathcal{P}(\mathcal{P}(\omega_1))|^N$,
- $\eta < \kappa$ and γ are ordinals in N and $\beta \in N$ is an ordinal greater than or equal to $\max\{(2^\kappa)^N, \gamma\}$,

then if G is N-generic for $\mathrm{Col}(\omega, \beta)$, then $N[G]$ satisfies the correct answer for the assertion that there exists an iteration of $(H(\kappa)^N, J)$ of length γ whose last model is illfounded below the image of η. Let $\phi(\gamma, \kappa, \eta, J)$ be the formula asserting that

- J is a normal ideal on ω_1,
- κ is a regular cardinal greater than $|\mathcal{P}(\mathcal{P}(\omega_1))|$,
- $\eta < \kappa$,
- letting $\beta = \max\{2^\kappa, \gamma\}$, every condition (equivalently, some condition) in $\mathrm{Col}(\omega, \beta)$ forces that there exists an iteration of $(H(\kappa), J)$ of length γ whose last model is illfounded below the image of η.

Then, in M, $(\bar{\gamma}, \bar{\kappa}, \bar{\eta})$ is the lexicographically least triple (γ, κ, η) such that $\phi(\gamma, \kappa, \eta, I)$ holds. Furthermore, since $j'_{0\gamma^*}$ is elementary, in M_{γ^*},

$$(j'_{0\gamma^*}(\bar{\gamma}), j'_{0\gamma^*}(\bar{\kappa}), j'_{0\gamma^*}(\bar{\eta}))$$

is the least triple (γ, κ, η) such that $\phi(\gamma, \kappa, \eta, j'_{0\gamma^*}(I))$ holds. However, the tail of the iteration $\langle N_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \bar{\gamma}\rangle$ starting with N_{γ^*} is an iteration of

$$(H(j_{0\gamma^*}(\bar{\kappa}))^{M_{\gamma^*}}, j_{0\gamma^*}(I))$$

(note that $j'_{0\gamma^*}(H(\bar{\kappa})^M) = j_{0,\gamma^*}(H(\bar{\kappa})^M)$) of length less than or equal to $\bar{\gamma}$ which in turn is less than or equal to $j'_{0\gamma^*}(\bar{\gamma})$. Furthermore, $\eta^* < j'_{0\gamma}(\bar{\eta}) = j_{0\gamma}(\bar{\eta})$, and $j_{\gamma^*\bar{\gamma}}(\eta^*)$ is not wellfounded, which, by the correctness property mentioned above (using the fact that M_{γ^*} is wellfounded) contradicts the minimality of $j'_{0\gamma^*}(\bar{\eta})$. ⊣

1.6 Example. Let M be any countable transitive model of ZFC in which there exists a measurable cardinal κ and a normal measure $\mu \in M$ on κ such that all countable iterates of M by μ are wellfounded. Iterating M by μ ω_1 times, we obtain a model N of ZFC containing ω_1 such that $(V_\kappa)^M = (V_\kappa)^N$. Now suppose that I is a normal precipitous ideal on ω_1^M in M. By Lemmas 1.4 and 1.5, $((V_\kappa)^M, I)$ iterable.

Before moving on, we prove an important fact about iterations of iterable models which will show up later (in Lemmas 3.3, 6.2 and 7.8). This fact is a key step in Woodin's proof of Theorem 0.3.

1.7 Lemma. *Suppose that M is a countable transitive model of ZFC° and $I \in M$ is a normal ideal on ω_1^M such that the pair (M, I) is iterable. Let x be a real coding the pair (M, I) under some recursive coding. Let*

$$\mathcal{I} = \langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$$

be an iteration of (M, I). Then every countable ordinal γ such that $L_\gamma[x]$ satisfies ZFC is on the critical sequence of $\mathcal{I}$.

Proof. Fix a countable ordinal γ such that $L_\gamma[x] \models$ ZFC. We want to see that for every $\eta < \gamma$ there is a $\delta < \gamma$ such that the ordinals of the final model of every iteration of (M, I) of length η are contained in δ. To see this, fix η and let $g \subset \mathrm{Col}(\omega, \eta)$ be $L_\gamma[x]$-generic. Then $L_\gamma[x][g] \models ZFC$, and in $L_\gamma[x][g]$, the set of ordertypes of the ordinals of iterates of (M, I) by iterations of length η is a Σ^1_1 set in a real coding (M, I) and g. By the boundedness lemma for $\utilde{\Sigma}^1_1$ sets of wellorderings (see [24]), then, there is a countable (in $L_\gamma[x][g]$) ordinal δ such that all of these ordertypes are less than δ. Furthermore, the nonexistence of an iteration of (M, I) of length η such that δ can be embedded in an order-preserving way into the ordinals of the final model is absolute between $L_\gamma[x][g]$ and V, by $\utilde{\Sigma}^1_1$-absoluteness. ⊣

Lemma 1.7 has the following useful corollary. The case where γ is countable follows immediately from Lemma 1.7. The case where $\gamma = \omega_1$ follows by applying the countable case to a forcing extension where ω_1 is collapsed.

1.8 Corollary. *Suppose that M is a countable transitive model of ZFC°, I is normal ideal on ω_1^M in M, (M, I) is iterable and x is a real coding (M, I). Suppose that γ is an x-indiscernible less than or equal to ω_1, and let $j\colon (M, I) \to (M^*, I^*)$ be an iteration of (M, I) of length γ. Then the ordinals of M^* have height less than the least x-indiscernible above γ.*

We note one more useful fact about sharps. The fact can be proved directly using the remarkable properties of sharps, or by noting that the two functions implicit in the fact necessarily represent the same ordinal in any generic ultrapower.

1.9 Fact. Let x be a real and let γ be the least x-indiscernible above ω_1. Let $\pi\colon \omega_1 \to \gamma$ be a bijection. Then the set of $\alpha < \omega_1$ such that $o.t.(\pi[\alpha])$ is the least x-indiscernible above α contains a club.[1]

1.10 Remark. Often in this chapter will we use recursive codings of elements of $H(\omega_1)$ by reals (by which we mean elements of ω^ω). The following coding is sufficient in all cases: fixing a recursive bijection $\pi\colon \omega \times \omega \to \omega$, let x be a real coding the set of sets coded by those $y \subseteq \omega$ for which there exists an $i < \omega$ such that $\pi(0, i) \in x$ and $y = \{j < \omega \mid \pi(j + 1, i) \in x\}$. Note that under this coding, the relations "$\in$" and "$=$" are both Σ^1_1, since permutations of ω can give rise to different codes for the same set.

1.11 Remark. If there exists a precipitous ideal on ω_1, then $A^\#$ exists for every $A \subseteq \omega_1$. To see this, note first of all that the existence of a precipitous ideal implies that for each real x there is an nontrivial elementary embedding from $L[x]$ to $L[x]$ in a forcing extension, which means that $x^\#$ exists already in the ground model. Furthermore, if I is a precipitous ideal on ω_1 and $j\colon V \to M$ is the generic embedding derived from a V-generic filter $G \subset \mathcal{P}(\omega_1)/I$, then $\mathcal{P}(\omega_1)^V \subseteq H(\omega_1)^M$. Therefore, for every $A \in \mathcal{P}(\omega_1)^V$, $A \in M$ and $M \models$ "$A^\#$ exists." Since M and $V[G]$ have the same ordinals, $V[G]$ and V then must also satisfy "$A^\#$ exists."

Similarly, if (M, I) is an iterable pair, then M is correct about the sharps of the reals of M, since M is elementarily embeddeded into a transitive model containing ω_1, and thus M is correct about the sharps of the members of $\mathcal{P}(\omega_1)^M$. In particular, if (M, I) is an iterable pair and A is in $\mathcal{P}(\omega_1)^M$, then $\mathcal{P}(\omega_1)^{L[A]} \subseteq M$, so M correctly computes $\omega_1^{L[A]}$.

[1] Given a function f and subset X of the domain of X, we let $f[X]$ denote the set $\{f(x) \mid x \in X\}$.

2. $\mathbb{P}_{max}$

We are now ready to define the partial order $\mathbb{P}_{max}$. We will make one modification of the definition given in [37] and require the conditions to satisfy ZFC$^\circ$ instead of the theory ZFC* defined in [37]. Our $\mathbb{P}_{max}$ is a dense suborder of the original; furthermore, the basic analysis of the two partial orders is the same, though the proofs of Lemma 7.10 and Theorem 7.11 are less elegant than they might otherwise be.

Recall that $\mathrm{MA}_{\aleph_1}$ is the version of Martin's Axiom for $\aleph_1$-many dense sets, i.e., the statement that whenever P is a c.c.c. partial order and D_α ($\alpha < \omega_1$) are dense subsets of P there is a filter $G \subset P$ intersecting each D_α.

2.1 Definition. The partial order $\mathbb{P}_{max}$ consists of all pairs $\langle (M, I), a\rangle$ such that

1. M is a countable transitive model of ZFC$^\circ$+ $\mathrm{MA}_{\aleph_1}$,
2. $I \in M$ and in M, I is a normal ideal on ω_1,
3. (M, I) is iterable,
4. $a \in \mathcal{P}(\omega_1)^M$,
5. there exists an $x \in \mathcal{P}(\omega)^M$ such that $\omega_1^M = \omega_1^{L[a,x]}$.

The order on $\mathbb{P}_{max}$ is as follows: $\langle (M, I), a\rangle < \langle (N, J), b\rangle$ if $N \in H(\omega_1)^M$ and there exists an iteration $j\colon (N, J) \to (N^*, J^*)$ such that

- $j(b) = a$,
- $j, N^* \in M$,
- $I \cap N^* = J^*$.

We say that a pair (M, I) is a ($\mathbb{P}_{max}$) *pre-condition* if there exists an a such that $\langle (M, I), a\rangle$ is in $\mathbb{P}_{max}$.

2.2 Remark. If $\langle (M, I), a\rangle$ is a $\mathbb{P}_{max}$ condition, then M is closed under sharps for reals (see Remark 1.11), and so a cannot be in $L[x]$ for any real x in M. Therefore, a is unbounded in ω_1^M, and this in turn implies that the iteration witnessing that a given $\mathbb{P}_{max}$ condition $\langle (M, I), a\rangle$ is stronger than another condition $\langle (N, J), b\rangle$ must have length ω_1^M.

2.3 Remark. To see that the order on $\mathbb{P}_{max}$ is transitive, let j_0 be an iteration witnessing that $\langle (M_1, I_1), a_1\rangle < \langle (M_0, I_0), a_0\rangle$ and let j_1 be an iteration witnessing that $\langle (M_2, I_2), a_2\rangle < \langle (M_1, I_1), a_1\rangle$. Then j_0 is an element if M_1, and it is not hard to check that $j_1(j_0)$ witnesses that $\langle (M_2, I_2), a_2\rangle < \langle (M_0, I_0), a_0\rangle$.

2.4 Remark. As we shall see in Lemma 2.7, the requirement that the models satisfy $\mathrm{MA}_{\aleph_1}$, along with condition (5) above, ensures that there is a unique iteration witnessing the order on each pair of comparable conditions. One can vary $\mathbb{P}_{max}$ by removing condition (5) and the requirement that $\mathrm{MA}_{\aleph_1}$ holds, and replace a with a set of iterations of smaller models into M, as in the definition of the order, satisfying this uniqueness condition. Alternately, one can require that the models satisfy the statement ψ_{AC} (see Definition 6.1 and Remark 6.4), which implies that the image of any stationary, co-stationary subset of ω_1 under an iteration determines the entire iteration.

2.5 Remark. Instead of using ideals on ω_1, we could use the stationary tower $\mathbb{Q}_{<\delta}$ (see [19]) to produce the iterations giving the order on conditions. This gives us another degree of freedom in choosing our models, since in this case a small forcing extension of a condition is also a condition, roughly speaking. The resulting extension is essentially identical.

2.6 Remark. Given a real x, $x^\dagger$ ("x dagger") is a real such that in $L[x^\dagger]$ there exists a transitive model M of ZFC containing $\omega_1^V \cup \{x\}$ in which some ordinal countable in $L[x^\dagger]$ is a measurable cardinal (see [13]; this fact about $x^\dagger$ does not characterize it, but it is its only property that we require in this chapter). By [10], if there exists a measurable cardinal, then there is a partial order forcing that NS_{ω_1} is precipitous. By [20, 12], c.c.c. forcings preserve precipitousness of NS_{ω_1}. Essentially the same arguments show that if κ is a measurable cardinal and P is a c.c.c. forcing in the $\mathrm{Col}(\omega,{<}\kappa)$-extension, then there is a normal precipitous ideal on ω_1 (which is κ) in the $\mathrm{Col}(\omega,{<}\kappa) * P$-extension. By Lemmas 1.4 and 1.5, then, the statement that $x^\dagger$ exists for each real x implies that every real exists in the model M of some $\mathbb{P}_{max}$ condition, and, by Lemma 2.8 below, densely many. However, the full strength of $\mathbb{P}_{max}$ will require the consistency strength of significantly larger cardinals.

We will now prove two facts about iterations which are central to the $\mathbb{P}_{max}$ analysis.

2.7 Lemma. *Let $\langle (M,I), a\rangle$ be a condition in $\mathbb{P}_{max}$ and let A be a subset of ω_1. Then there is at most one iteration of (M,I) for which A is the image of a. Furthermore, this iteration is in $L[\langle (M,I),a\rangle, A]$, if it exists.*

Proof. The consequence of $\mathrm{MA}_{\aleph_1}$ that we need is known as *almost disjoint coding* [11]. This says that if $Z = \{z_\alpha : \alpha < \omega_1\}$ is a collection of infinite subsets of ω whose pairwise intersections are finite (i.e., Z is an *almost disjoint family*), then for every $B \subseteq \omega_1$ there exists a $y \subseteq \omega$ such that for all $\alpha < \omega_1$, $\alpha \in B$ if and only if $y \cap z_\alpha$ is infinite.

Fix a real x in M such that $\omega_1^M = \omega_1^{L[a,x]}$, and let

$$Z = \langle z_\alpha : \alpha < \omega_1^M \rangle$$

be the almost disjoint family defined recursively from the constructibility order in $L[a,x]$ on $\mathcal{P}(\omega)^{L[a,x]}$ by letting z_α be the (constructibly, in $L[a,x]$) least infinite subset of ω almost disjoint from each z_β $(\beta < \alpha)$.

Suppose that

$$\mathcal{I} = \langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle$$

and

$$\mathcal{I}' = \langle M'_\alpha, G'_\beta, j'_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma'\rangle$$

are two iterations of (M, I) such that $j_{0\gamma}(a) = A = j'_{0\gamma'}(a)$. Then $j_{0\gamma}(Z) = j'_{0\gamma'}(Z)$ (again, this uses Remark 1.11 to see that the constructibility order on reals in $L[A,x]$ is computed correctly in M_γ and $M'_{\gamma'}$).

Since $j_{0\gamma}(\omega_1^M) = \sup(j_{0\gamma}(a)) = \sup(j'_{0\gamma'}(a)) = j'_{0\gamma'}(\omega_1^M)$, if either iteration is an initial segment of the other, then the two iterations are the same. Supposing that this is not the case, let $\bar{\gamma}$ be the length of the shortest initial segment of $\mathcal{I}$ which is not an initial segment of $\mathcal{I}'$ and let γ^* be the length of the shortest initial segment of $\mathcal{I}'$ which is not an initial segment of $\mathcal{I}$. Then $\bar{\gamma}$ and γ^* are both successor ordinals, and since the iterations up to their predecessors must be the same, they are both equal to $\eta + 1$ for some ordinal η, with $M_\eta = M'_\eta$.

Note that since $j_{0\gamma}(Z) = j'_{0\gamma'}(Z)$, and the critical points of $j_{(\eta+1)\gamma}$ and $j'_{(\eta+1)\gamma'}$ are both greater than $\omega_1^{M_\eta}$, $j_{0(\eta+1)}(Z)_{\omega_1^{M_\eta}} = j'_{0(\eta+1)}(Z)_{\omega_1^{M_\eta}}$. Let $\langle z_\alpha : \alpha < \omega_1^{M_\eta}\rangle$ list the members of $j_{0\eta}(Z)$ (and note that this is consistent with the definition of Z above). We now derive a contradiction by showing that G_η and G'_η are the same.

For every $B \in \mathcal{P}(\omega_1)^{M_\eta}$, $B \in G_\eta$ if and only if $\omega_1^{M_\eta} \in j_{\eta(\eta+1)}(B)$ and $B \in G'_\eta$ if and only if $\omega_1^{M_\eta} \in j'_{\eta(\eta+1)}(B)$. Fixing such a B, let $y \in \mathcal{P}(\omega)^{M_\eta}$ be such that for all $\alpha \in \omega_1^{M_\eta}$, $y \cap z_\alpha$ is infinite if and only if $\alpha \in B$. Then $\omega_1^{M_\eta} \in j_{\eta(\eta+1)}(B)$ if and only if $j_{0(\eta+1)}(Z)_{\omega_1^{M_\eta}} \cap y$ is infinite, which holds if and only if $j'_{0(\eta+1)}(Z)_{\omega_1^{M_\eta}} \cap y$ is infinite, which holds if and only if $\omega_1^{M_\eta} \in j'_{\eta(\eta+1)}(B)$. Thus G_η and G'_η are the same.

For the last part of the lemma, note that the argument just given gives a definition for each G_α in terms of A, x and the iteration up to α. ⊣

One consequence of Lemma 2.7 is that, if $G \subset \mathbb{P}_{max}$ is an $L(\mathbb{R})$-generic filter, and $A = \bigcup\{a \mid \langle (M,I), a\rangle \in G\}$, then $L(\mathbb{R})[G] = L(\mathbb{R})[A]$. Therefore, the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ satisfies the sentence "$V = L(\mathcal{P}(\omega_1))$" (see the discussion at the beginning of Section 5).

2.8 Lemma. *(ZFC°) If (M, I) is a pre-condition in $\mathbb{P}_{max}$ and J is a normal ideal on ω_1 then there exists an iteration $j\colon (M, I) \to (M^*, I^*)$ such that $j(\omega_1^M) = \omega_1$ and $I^* = J \cap M^*$.*

Proof. First let us note that if $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$ is any iteration of (M, I), then $j_{0\omega_1}(I) \subseteq J \cap M_{\omega_1}$. To see this, note that if $E \in j_{0\omega_1}(I)$, then $E \in M_{\omega_1}$ and $E = j_{\alpha\omega_1}(E')$ for some $\alpha < \omega_1$ and $E' \in j_{0\alpha}(I)$. Then for all $\beta \in [\alpha, \omega_1)$, $j_{\alpha\beta}(E') \notin G_\beta$, so $\omega_1^{M_\beta} \notin E$. Therefore, E is nonstationary, so $E \in J$ by the normality of J.

Now, noting that J is a normal ideal, let $\{A_{i\alpha} : i < \omega, \alpha < \omega_1\}$ be a collection of pairwise disjoint members of $\mathcal{P}(\omega_1) \setminus J$. We build an iteration $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$ by recursively choosing the G_β's. As we do this, for each $\alpha < \omega_1$ we let the set $\{B_i^\alpha : i < \omega\}$ enumerate $\mathcal{P}(\omega_1)^{M_\alpha} \setminus j_{0\alpha}(I)$. Given

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \gamma\rangle,$$

then, for some $\gamma \leq \omega_1$, if $\omega_1^{M_\gamma} \in A_{i\alpha}$ for some $i < \omega$ and $\alpha \leq \gamma$, then (noting that there can be at most one such pair (i, α)) we let G_γ be any M_γ-generic filter for $(\mathcal{P}(\omega_1)/j_{0\gamma}(I))^{M_\gamma}$ with $j_{\alpha\gamma}(B_i^\alpha)$ as a member. If $\omega_1^{M_\gamma}$ is not in $A_{i\alpha}$ for any $i < \omega$ and $\alpha \leq \gamma$, then we let G_γ be any M_γ-generic filter.

To see that this construction works, fix $E \in \mathcal{P}(\omega_1)^{M_{\omega_1}} \setminus j_{0\omega_1}(I)$. We need to see that E is not in J. We may fix $i < \omega$ and $\alpha < \omega_1$ such that $E = j_{\alpha\omega_1}(B_i^\alpha)$. Then $F = (A_{i\alpha} \cap \{\omega_1^{M_\beta} : \beta \in [\alpha, \omega_1)\}) \subseteq E$. Since F is the intersection of a club and set not in J, F is not in J, so E is not in J. $\dashv$

The construction in the proof of Lemma 2.8 appears repeatedly in the analysis of $\mathbb{P}_{max}$. In order to make our presentation of $\mathbb{P}_{max}$ more modular (i.e., to avoid having to write out the proof of Lemma 2.8 repeatedly), we give the following strengthening of the lemma in terms of games. We note that the games defined here (and before Lemmas 3.5 and 5.2 and at the end of Section 10.2) are not part of Woodin's original presentation of $\mathbb{P}_{max}$.

Suppose that (M, I) is a pre-condition in $\mathbb{P}_{max}$, let J be a normal ideal on ω_1 and let B be a subset of ω_1. Let $\mathcal{G}((M, I), J, B)$ be the following game of length ω_1 where Players I and II collaborate to build an iteration

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$$

of (M, I) of length ω_1. In each round α, if $\alpha \in B$, then Player I chooses a set A_α in $\mathcal{P}(\omega_1)^{M_\alpha} \setminus j_{0\alpha}(I)$ and then Player II chooses an M_α-generic filter G_α contained in $(\mathcal{P}(\omega_1)/j_{0\alpha}(I))^{M_\alpha}$ with $A_\alpha \in G_\alpha$. If $\alpha \notin B$, then Player II chooses any M_α-generic filter $G_\alpha \subseteq (\mathcal{P}(\omega_1)/j_{0\alpha}(I))^{M_\alpha}$. After all ω_1 many rounds have been played, Player I wins if $j_{0\omega_1}(I) = J \cap M_{\omega_1}$.

The proof of Lemma 2.9 is almost identical to the proof of Lemma 2.8.

2.9 Lemma. *(ZFC°) Suppose that (M, I) is a pre-condition in $\mathbb{P}_{max}$, let J be a normal ideal on ω_1, and let B be a subset of ω_1. Then Player I has a winning strategy in $\mathcal{G}((M, I), J, B)$ if and only if $B \notin J$.*

Using Remark 2.6 and Lemmas 2.7 and 2.8, we can show that $\mathbb{P}_{max}$ is homogeneous and countably closed. By *homogeneity* we mean the following property: for each pair of conditions p_0, p_1 in $\mathbb{P}_{max}$ there exist conditions q_0, q_1 such that $q_0 \leq p_0$, $q_1 \leq p_1$ and the suborders of $\mathbb{P}_{max}$ below the conditions q_0 and q_1 are isomorphic. The importance of this property is that it implies that the theory of the $\mathbb{P}_{max}$ extension can be computed in the ground model.

2.10 Lemma. *If $x^\dagger$ exists for each real x, then $\mathbb{P}_{max}$ is homogeneous.*

Proof. Let $p_0 = \langle (M_0, I_0), a_0 \rangle$ and $p_1 = \langle (M_1, I_1), a_1 \rangle$ be conditions in $\mathbb{P}_{max}$. By Remark 2.6, we can fix a pre-condition (N, J) with $p_0, p_1 \in H(\omega_1)^N$. Applying Lemma 2.8 in N, we may fix iterations $j_0 \colon (M_0, I_0) \to (M_0^*, I_0^*)$ and $j_1 \colon (M_1, I_1) \to (M_1^*, I_1^*)$ in N such that $I_0^* = J \cap M_0^*$ and $I_1^* = J \cap M_1^*$. Letting $a_0^* = j_0(a_0)$ and $a_1^* = j_1(a_1)$, then,

$$q_0 = \langle (N, J), a_0^* \rangle$$

and

$$q_1 = \langle (N, J), a_1^* \rangle$$

are conditions in $\mathbb{P}_{max}$ and j_0 and j_1 witness that $q_0 \leq p_0$ and $q_1 \leq p_1$ respectively.

Now, if $q_0' = \langle (N', J'), a' \rangle$ is a condition below q_0, then there is an iteration $j' \colon (N, J) \to (N^*, J^*)$ witnessing this. Then $a' = j'(a_0^*)$, and $q_1' = \langle (N', J'), j'(a_1^*) \rangle$ is a condition below q_1. Let π be the map with domain the suborder of $\mathbb{P}_{max}$ below q_0 which sends each $\langle (N', J'), a' \rangle$ to the corresponding $\langle (N', J'), j'(a_1^*) \rangle$ as above. By Lemma 2.7, this map is an isomorphism between the suborders below q_0 and q_1 respectively. $\dashv$

In order to show that $\mathbb{P}_{max}$ is countably closed, we must define a new class of iterations.

3. Sequences of models and countable closure

For each $i < \omega$, let $p_i = \langle (M_i, I_i), a_i \rangle$ be a $\mathbb{P}_{max}$ condition, and for each such i let $j_{i(i+1)} \colon (M_i, I_i) \to (M_i^*, I_i^*)$ be an iteration witnessing that $p_{i+1} < p_i$. Let $\{j_{ik} : i \leq k < \omega\}$ be the commuting family of embeddings generated by the $j_{i(i+1)}$'s. Let $a = \bigcup \{a_i : i < \omega\}$. By Lemma 2.7, for each $i < \omega$ there is a unique iteration $j_{i\omega} : (M_i, I_i) \to (N_i, J_i)$ sending a_i to a. Since each (M_i, I_i) is iterable, each N_i is wellfounded, and the structure $(\langle (N_i, J_i) : i < \omega \rangle, a)$ satisfies the following definition.

3.1 Definition. A *limit sequence* is a pair $(\langle (N_i, J_i) : i < \omega \rangle, a)$ such that the following hold for all $i < \omega$:

1. N_i is a countable transitive model of ZFC°+ $\text{MA}_{\aleph_1}$,
2. $J_i \in N_i$ and in N_i, J_i is a normal ideal on ω_1,

3. $\omega_1^{N_i} = \omega_1^{N_0}$,
4. for all $k < i$, $N_k \in H(\omega_2)^{N_i}$,
5. for all $k < i$, $J_k = J_i \cap N_k$,
6. $a \in \mathcal{P}(\omega_1)^{N_0}$,
7. there exists an $x \in \mathcal{P}(\omega)^{N_0}$ such that $\omega_1^{N_0} = \omega_1^{L[a,x]}$.

A structure $\langle (N_i, J_i) : i < \omega \rangle$ is a *pre-limit sequence* if there exists a set a such that $(\langle (N_i, J_i) : i < \omega \rangle, a)$ is a limit sequence.

If we write a sequence as $\langle N_k : k < \omega \rangle$ (as we will in Section 10.1), the ideals are presumed to be the nonstationary ideal on ω_1.

If p_i $(i < \omega)$ is a descending sequence of $\mathbb{P}_{max}$ conditions, then the *limit sequence corresponding to* p_i $(i < \omega)$ is the structure

$$(\langle (N_i, J_i) : i < \omega \rangle, a)$$

defined above. Note that in this case each $\langle (N_i, J_i), a \rangle$ is a condition in $\mathbb{P}_{max}$.

If $\langle (N_i, J_i) : i < \omega \rangle$ is a pre-limit sequence, then a filter

$$G \subset \bigcup \{\mathcal{P}(\omega_1)^{N_i} \setminus J_i : i < \omega\}$$

is a $\bigcup\{N_i : i < \omega\}$*-normal ultrafilter* for the sequence if for every regressive function f on $\omega_1^{N_0}$ in any N_i, f is constant on some member of G. Given such G and $\langle (N_i, J_i) : i < \omega \rangle$, we form the ultrapower of the sequence by letting N_i^* be the ultrapower of N_i formed from G and *all* functions $f : \omega_1^{N_0} \to N_i$ existing in any N_k (this ensures that the image of each N_i in the ultrapower of each N_k $(k > i)$ is the same as the ultrapower of N_i). As usual, we identify the transitive parts of each N_i^* with their transitive collapses. If (for each $i < \omega$) we let j_i^* be the induced embedding of N_i into N_i^* then for each $i < k < \omega$, $j_i^* = j_k^* \restriction N_i$, so we can let $j^* = \cup\{j_i^* : i < \omega\}$ be the embedding corresponding to the ultrapower of the pre-limit sequence.

3.2 Definition. Let $\langle (N_i, J_i) : i < \omega \rangle$ be a pre-limit sequence, and let γ be an ordinal less than or equal to ω_1. An *iteration* of $\langle (N_i, J_i) : i < \omega \rangle$ of length γ consists of pre-limit sequences $\langle (N_i^\alpha, J_i^\alpha) : i < \omega \rangle$ $(\alpha \leq \gamma)$, normal ultrafilters G_α $(\alpha < \gamma)$ and a commuting family of embeddings $j_{\alpha\beta}$ $(\alpha \leq \beta \leq \gamma)$ such that

- $\langle (N_i^0, J_i^0) : i < \omega \rangle = \langle (N_i, J_i) : i < \omega \rangle$
- for all $\alpha < \gamma$, $G_\alpha \subseteq \bigcup\{\mathcal{P}(\omega_1)^{N_i^\alpha} \setminus J_i^\alpha : i < \omega\}$ is a normal ultrafilter for the sequence $\langle (N_i^\alpha, J_i^\alpha) : i < \omega \rangle$, and $j_{\alpha(\alpha+1)}$ is the corresponding embedding,

- for each limit ordinal $\beta \leq \gamma$, $\langle (N_i^\beta, J_i^\beta) : i < \omega\rangle$ is the direct limit of the system $\{\langle (N_i^\alpha, J_i^\alpha) : i < \omega\rangle, j_{\alpha\delta} : \alpha \leq \delta < \beta\}$ and for each $\alpha < \beta$ $j_{\alpha\beta}$ is the induced embedding.

As with iterations of single models, we sometimes describe an iteration of a pre-limit sequence by fixing only the initial sequence, the final sequence and the embedding between them. An *iterate* of a pre-limit sequence $\bar{p}$ is a pre-limit sequence appearing in an iteration of $\bar{p}$. If every iterate of a pre-limit sequence is wellfounded, then the sequence is *iterable*.

By Lemma 1.7 and Corollary 1.8, pre-limit sequences derived from descending chains $\{\langle (M_i, I_i), a_i\rangle : i < \omega\}$ in $\mathbb{P}_{max}$ satisfy the hypotheses of the following lemma, letting x_i be any real in M_{i+1} coding M_i. Yet another way to vary $\mathbb{P}_{max}$ is to replace the model M in the definition of $\mathbb{P}_{max}$ conditions with sequences satisfying this hypothesis. This approach is used for the order $\mathbb{Q}^*_{max}$ defined in section 10.1.

3.3 Lemma. *Suppose that $\bar{p} = \langle (N_i, J_i) : i < \omega\rangle$ is a pre-limit sequence, and suppose that for each $i < \omega$ there is a real $x_i \in N_{i+1}$ such that $x_i^\# \in N_{i+1}$ and*

- *the least x_i-indiscernible above $\omega_1^{N_0}$ is greater than the ordinal height of N_i,*
- *every club subset of $\omega_1^{N_0}$ in N_i contains a tail of the x_i-indiscernibles below $\omega_1^{N_0}$.*

Then $\bar{p}$ is iterable.

Proof. First we will show that any iterate of $\bar{p}$ is wellfounded if its version of ω_1 is wellfounded. Then we will show that the ω_1 of each iterate of $\bar{p}$ is wellfounded.

For the first part, if $\langle (N_i^*, J_i^*) : i < \omega\rangle$ is an iterate of $\bar{p}$, then by elementarity the ordinals of each N_i^* embed into the least x_i-indiscernible above $\omega_1^{N_0^*}$. So, if $\omega_1^{N_0^*}$ is actually an ordinal (i.e., is wellfounded), then N_{i+1}^* constructs this next x_i-indiscernible correctly, and so N_i^* is wellfounded.

We prove the second part by induction on the length of the iteration, noting that the limit case follows immediately, and the successor case follows from the case of an iteration of length 1. What we want to see is that if G is a normal ultrafilter for $\bar{p}$ and j is the induced embedding, then $j(\omega_1^{N_0}) = \bigcup\{N_i \cap \mathrm{Ord} : i < \omega\}$. Notice that for each x_i, if $f_i \colon \omega_1^{N_0} \to \omega_1^{N_0}$ is defined by letting $f_i(\alpha)$ be the least x_i-indiscernible above α, then $j(f_i)(\omega_1^{N_0})$ is the least indiscernible of x_i above $\omega_1^{N_0}$. Thus

$$j(\omega_1^{N_0}) \geq \sup\{j(f_i)(\omega_1^{N_0}) : i < \omega\} = \bigcup\{N_i \cap \mathrm{Ord} : i < \omega\}.$$

For the other direction, let $h \colon \omega_1^{N_0} \to \omega_1^{N_0}$ be a function in some N_i. Then the closure points of h contain a tail of the x_i-indiscernibles, which means that $f_i > h$ on a tail of the ordinals below $\omega_1^{N_0}$, so $[f_i]_G > [h]_G$. Thus $j(\omega_1^{N_0}) = \bigcup\{N_i \cap \mathrm{Ord} : i < \omega\}$. ⊣

The following lemma has essentially the same proof as Lemma 2.8, and shows (given that $x^\dagger$ exists for each real x) that $\mathbb{P}_{max}$ is countably closed. The point is that if $\langle p_i : i < \omega\rangle$ is a descending sequence of $\mathbb{P}_{max}$ conditions, letting $\bar{p} = (\langle (N_i, J_i) : i < \omega\rangle, a)$ be the limit sequence corresponding to $\langle p_i : i < \omega\rangle$, if (M, I) is a $\mathbb{P}_{max}$ pre-condition with $\{p_i : i < \omega\}, \bar{p} \in H(\omega_1)^M$, then by letting j^* be an iteration of $\bar{p}$ resulting from applying Lemma 3.4 inside of M, the embedding $j^*(j_{i\omega})$ (where $j_{i\omega}$ is as defined in the first paragraph of this section) witnesses that $\langle (M, I), j^*(a)\rangle$ is below p_i in $\mathbb{P}_{max}$, for each $i < \omega$.

3.4 Lemma. *(ZFC°) Suppose that $\langle (N_i, J_i) : i < \omega\rangle$ is an iterable pre-limit sequence, and let I be a normal ideal on ω_1. Then there is an iteration*

$$j^* : \langle (N_i, J_i) : i < \omega\rangle \to \langle (N_i^*, J_i^*) : i < \omega\rangle$$

such that $j^*(\omega_1^{N_0}) = \omega_1$ and $J_i^* = I \cap N_i^*$ for each $i < \omega$.

Suppose that $\langle (N_i, J_i) : i < \omega\rangle$ is an iterable pre-limit sequence, let I be a normal ideal on ω_1, and let B be a subset of ω_1. Let

$$\mathcal{G}_\omega(\langle (N_i, J_i) : i < \omega\rangle, I, B)$$

be the following game of length ω_1 where Players I and II collaborate to build an iteration of $\langle (N_i, J_i) : i < \omega\rangle$ consisting of pre-limit sequences $\langle (N_i^\alpha, J_i^\alpha) : i < \omega\rangle$ ($\alpha \leq \omega_1$), normal ultrafilters G_α ($\alpha < \omega_1$) and a family of embeddings $j_{\alpha\beta}$ ($\alpha \leq \beta \leq \omega_1$), as follows. In each round α, let

$$X_\alpha = \bigcup\{\mathcal{P}(\omega_1)^{N_i^\alpha} \setminus J_i^\alpha : i < \omega\}.$$

If $\alpha \in B$, then Player I chooses a set $A \in X_\alpha$, and then Player II chooses a $\bigcup\{N_i^\alpha : i < \omega\}$-normal filter G_α contained in X_α with $A \in G_\alpha$. If α is not in B, then Player II chooses any $\bigcup\{N_i^\alpha : i < \omega\}$-normal filter G_α contained in X_α. After all ω_1 many rounds have been played, Player I wins if $J_i^{\omega_1} = I \cap N_i^{\omega_1}$ for each $i < \omega$.

Lemma 3.4 can be rephrased in terms of games as follows.

3.5 Lemma. *(ZFC°) Suppose that $\langle (N_i, J_i) : i < \omega\rangle$ is an iterable pre-limit sequence, let I be a normal ideal on ω_1 and let B be a subset of ω_1. Then Player I has a winning strategy in $\mathcal{G}_\omega(\langle (N_i, J_i) : i < \omega\rangle, I, B)$ if and only if $B \notin I$.*

At this point, we have gone as far with the $\mathbb{P}_{max}$ analysis as daggers can take us.

4. Generalized iterability

The following definition gives a generalized iterability property with respect to a given set of reals. In the $\mathbb{P}_{max}$ analysis, these sets of reals often code $\mathbb{P}_{max}$-names for sets of reals.

4.1 Definition. Let A be a set of reals. If M is a transitive model of ZFC$^\circ$ and I is an ideal on ω_1^M which is normal and precipitous in M, then the pair (M, I) is A-*iterable* if

- (M, I) is iterable,
- $A \cap M \in M$,
- $j(A \cap M) = A \cap M^*$ whenever $j\colon (M, I) \to (M^*, I^*)$ is an iteration of (M, I).

4.2 Remark. The definition of A-iterability in [37] is more general than this one, in ways which we won't require.

In order to achieve the full effects of forcing with $\mathbb{P}_{max}$ over a given model (for now we will deal with $L(\mathbb{R})$) we need to see (and in fact it is enough to see) that for each $A \subseteq \mathbb{R}$ in the model there exists a $\mathbb{P}_{max}$ pre-condition (M, I) such that

- (M, I) is A-iterable,
- $\langle H(\omega_1)^M, A \cap M\rangle \prec \langle H(\omega_1), A\rangle$.

As it turns out, the existence of such a condition for each $A \subseteq \mathbb{R}$ in $L(\mathbb{R})$ is equivalent to the statement that the Axiom of Determinacy holds in $L(\mathbb{R})$ (see pages 285-290 of [37]).

There are two basic approaches to studying the $\mathbb{P}_{max}$ extension. One can think of V as being a model of some form of determinacy, and use determinacy to analyze the $\mathbb{P}_{max}$ forcing construction and its corresponding extension. Alternately, one can assume that Choice holds and certain large cardinals exist and use these large cardinals to analyze the $\mathbb{P}_{max}$ extension of some inner model of ZF satisfying determinacy. Accordingly, the existence of A-iterable conditions (for a given set A) can be derived from determinacy or from large cardinals. We give here an example of each method, quoting some standard facts which we will briefly discuss.

The proof from large cardinals uses weakly homogeneous trees. Very briefly, a *countably complete tower* is a sequence of measures $\langle \sigma_i : i < \omega\rangle$ such that each σ_i is a measure on Z^i for some fixed underlying set Z and for every sequence $\{A_i \in i < \omega\}$ of sets such that each $A_i \in \sigma_i$ there exists a function $g \in Z^\omega$ such that $g{\restriction}i \in A_i$ for all $i < \omega$. Given a tree $T \subset (\omega \times Z)^{<\omega}$, for some set Z, the *projection* of T is the set

$$p[T] = \{y \in \omega^\omega \mid \exists z \in Z^\omega\ \forall i < \omega (x{\restriction}i, z{\restriction}i) \in T\}.$$

Given a set Z and a cardinal δ, a tree $T \subseteq \omega \times Z$ is *δ-weakly homogeneous* if there exists a countable family Σ of δ-complete measures on $Z^{<\omega}$ such that for each $x \in \omega^\omega$, $x \in p[T]$ if and only if there exists a sequence of measures $\{\sigma_i : i < \omega\} \subseteq \Sigma$ such that

- for all $i < \omega$, $\{z \in Z^i \mid (x{\restriction}i, z) \in T\} \in \sigma_i$,
- $\langle \sigma_i : i < \omega \rangle$ forms a countably complete tower.

A set of reals A is δ-weakly homogeneously Suslin if there exists a δ-weakly homogeneous tree T whose projection is A, and *weakly homogeneously Suslin* if it is δ-weakly homogeneously Suslin for some uncountable ordinal δ. The following fact is standard.

4.3 Theorem. *Let θ be a regular cardinal, suppose that $T \in H(\theta)$ is a weakly homogeneous tree on $\omega \times Z$, for some set Z. Let $\delta \geq 2^\omega$ be an ordinal such that there exists a countable collection Σ of δ^+-complete measures witnessing the weak homogeneity of T. Then for every elementary submodel Y of $H(\theta)$ of cardinality less than δ with $T, \Sigma \in Y$ there is an elementary submodel X of $H(\theta)$ containing Y such that $X \cap \delta = Y \cap \delta$, and such that, letting S be the image of T under the transitive collapse of X, $p[S] = p[T]$.*

Proof. Fixing θ, T, Σ and δ as in the statement of the theorem, the theorem follows from the following fact. Suppose that Y is an elementary submodel of $H(\theta)$ with $T, \Sigma \in Y$ and $|Y| < \delta$, and fix $x \in p[T]$. Fix a countably complete tower $\{\sigma_i : i < \omega\} \subseteq \Sigma$ such that for all $i < \omega$, $\{a \in Z^i : (x{\restriction}i, a) \in T\} \in \sigma_i$, and for each $i < \omega$, let $A_i = \bigcap(\sigma_i \cap Y)$. Then since $\{\sigma_i : i < \omega\}$ is countably complete, there exists a $z \in Z^\omega$ such that for all $i < \omega$, $z{\restriction}i \in A_i$. Then the pair (x, z) forms a path through T, and, letting

$$Y[z] = \{f(z{\restriction}i) \mid i < \omega \wedge f \colon Z^i \to H(\theta) \wedge f \in Y\},$$

$Y[z]$ is an elementary submodel of $H(\theta)$ containing Y and $\{z{\restriction}i : i < \omega\}$, and, since each σ_i is δ^+-complete, $Y \cap \delta = Y[z] \cap \delta$. Repeated application of this fact for each real in the projection of T proves the theorem. ⊣

Proofs of the following facts about weakly homogeneous trees and weakly homogeneously Suslin sets of reals appear in [19]. Some of these facts follow directly from the definitions, and none are due to the author. Theorem 4.6 derives ultimately from [21].

4.4 Fact. For every cardinal δ, the collection of δ-weakly homogeneously Suslin sets of reals is closed under countable unions and continuous images.

4.5 Theorem. *(Woodin) If δ is a limit of Woodin cardinals and there exists a measurable cardinal above δ then every set of reals in $L(\mathbb{R})$ is $<\delta$-weakly homogeneously Suslin (i.e., γ-weakly homogeneously Suslin for all $\gamma < \delta$).*

4.6 Theorem. *If δ is a cardinal and T is a δ-weakly homogeneous tree, then there is a tree S such that $p[T] = \omega^\omega \setminus p[S]$ in all forcing extensions by partial orders of cardinality less than δ (including the trivial one).*

4.7 Theorem. *(Woodin) If δ is a Woodin cardinal and A is a δ^+-weakly homogeneously Suslin set of reals, then the complement of A is $<\delta$-weakly homogeneously Suslin.*

Note also that if S and T are trees whose projections are disjoint, then they remain disjoint in all forcing extensions, as there is a ranking function on the tree of attempts to build a real in both projections. This fact plus Theorem 4.6 gives the following corollary.

4.8 Corollary. *If δ is a cardinal and T_0 and T_1 are δ-weakly homogeneous trees with the same projection, then T_0 and T_1 still have the same projection in all forcing extensions by forcings of cardinality less than δ.*

Given a set of reals A, a set of reals B is *projective in A* if it can be defined by a projective formula (i.e., all unbounded quantifiers ranging over reals) with A as a parameter. Fact 4.4 and Theorem 4.7 together imply that if δ is a limit of Woodin cardinals then the set of $<\delta$-weakly homogeneously Suslin sets of reals is projectively closed.

The following theorem is a generalized existence result which is useful in analyzing variations of $\mathbb{P}_{max}$.

4.9 Theorem. *Let γ be a strongly inaccessible cardinal, let A be a set of reals, and suppose that θ is a strong limit cardinal of cofinality greater than ω_1 such that every set of reals projective in A is γ^+-weakly homogeneously Suslin as witnessed by a tree and a set of measures in $H(\theta)$. Let X be a countable elementary submodel of $H(\theta)$ with $\gamma, A \in X$, and let M be the transitive collapse of $X \cap H(\gamma)$. Let N be any forcing extension of M in which there exists a normal precipitous ideal I on ω_1^N. Let $j\colon (N, I) \to (N^*, I^*)$ be any iteration of (N, I). Then*

- *N^* is wellfounded,*
- *$N \cap A \in N$,*
- *$j(N \cap A) = N^* \cap A$,*
- *$\langle H(\omega_1)^{N^*}, A \cap N^*, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$.*

Proof. Let $\{A_i : i < \omega\}$ be a listing in X of the sets of reals projective in A (with $A_0 = A$), and let $\{T_i : i < \omega\}$ and $\{\Sigma_i : i < \omega\}$ be sets in X such that each T_i is a γ^+-weakly homogeneous tree (as witnessed by Σ_i) projecting to A_i. By the proof of Theorem 4.3, there is an elementary submodel Y of $H(\theta)$ containing X such that $X \cap H(\gamma) = Y \cap H(\gamma)$ and such that, letting M^+ be the transitive collapse of Y, and letting, for each $i < \omega$, S_i be the image of T_i under this collapse, $p[S_i] = A_i$. Note that since there are sets projective in A which are not the projections of countable trees, $\omega_1 \subseteq M^+$. Now, let N be a forcing extension of M with I a normal precipitous ideal on ω_1^N in N, and let N^+ be the corresponding extension of M^+. Let $j\colon (N, I) \to (N^*, I^*)$ be an iteration of (N, I). By Lemmas 1.4 and 1.5, j extends to an iteration of (N^+, I) (which we will also call j), and N^* is wellfounded. Furthermore, for each $i < \omega$ there is a $j < \omega$ such that S_i and S_j project to complements. Then

- $p[S_i] \subseteq p[j(S_i)]$,
- $p[S_j] \subseteq p[j(S_j)]$,
- $p[j(S_i)] \cap p[j(S_j)] = \emptyset$,

which means that $p[S_i] = p[j(S_i)]$, so $j(N \cap A_i) = N^* \cap A_i$.

To verify the last part of the theorem, noting that the theory of $H(\omega_1)$ is recursive in the theory of $\mathbb{R}$ (see Remark 1.10), we need to see that for each formula ϕ (with a predicate for A) with all unbounded quantifiers ranging over the reals, if A_i is the set of reals satisfying ϕ, then in N^+, S_i projects to the set of reals satisfying ϕ (i.e., that this relationship is preserved in the forcing extension from M^+ to N^+). This is easily shown by induction on the complexity of formulas, using Theorem 4.6 and Corollary 4.8. Note first of all that this holds for all such ϕ whose quantifiers are all bounded, as $\langle H(\omega_1)^{N^+}, A \cap N^+, \in\rangle$ is elementary in $\langle H(\omega_1), A, \in\rangle$ for such formulas simply by virtue of being a substructure. The verification for $\wedge$ and $\exists$ follows from Corollary 4.8. Letting γ^* be the image of γ under the collapse of Y, fix integers i and j and formulas ϕ_i and ϕ_j such that S_i and S_j project to the sets of reals satisfying ϕ_i and ϕ_j respectively in both M^+ and N^+. Then, working in M^+, we can directly construct γ^*-weakly homogeneous trees T and T' projecting to the sets of reals satisfying $\exists x\phi_i(x)$ and $\phi_i \wedge \phi_j$ respectively in both M^+ and N^+. Then, letting k and k' be integers such that S_k and $S_{k'}$ project in M^+ to the sets of reals satisfying $\exists x\phi_i(x)$ and $\phi_i \wedge \phi_j$ respectively, Corollary 4.8 gives us that $p[T] = p[S_k]$ and $p[T] = p[S_{k'}]$ in N^+. The verification for $\neg$ follows similarly from Corollary 4.8 and Theorem 4.6. ⊣

Alternately, we can derive the existence of A-iterable conditions from determinacy. The proof from determinacy requires the following fact: if AD holds and Z is a set of ordinals, then there is an inner model of ZFC containing the ordinals with Z as a member in which some countable ordinal is a measurable cardinal. Note that a tree of finite sequences of ordinals can easily be coded as a set of ordinals (see [25], for instance).

The following unpublished theorem of Woodin is more than sufficient, but in the spirit of completeness we will not use it, since its proof is well beyond the scope of this chapter. Given a model M and a set X, HOD_X^M is the class of hereditarily ordinal definable sets (using X as a parameter), as computed in M. It is a standard fact that this model satisfies ZFC.

4.10 Theorem. *(AD) Suppose that Z is a set of ordinals. Then there exists a real x such that for all reals z with $x \in L[Z,z]$, $\omega_2^{L[Z,z]}$ is a Woodin cardinal in $\mathrm{HOD}_{\{Z\}}^{L[Z,z]}$.*

The following theorem is sufficient for our purposes.

4.11 Theorem. *(AD) For every subset Z of L, there is an inner model N of ZFC containing $\{Z\}$ and the ordinals such that some countable ordinal is measurable in N.*

Proof. For each increasing function $f\colon \omega \to \omega_1$, let $s(f)$ be the supremum of the range of f, and let $F(f)$ be the filter on $s(f)$ consisting of all subsets of $s(f)$ which contain all but finitely many members of the range of f. For each such f, let $N(f)$ be the inner model $L[Z, F(f)]$. We will find an f such that the restriction of $F(f)$ to $N(f)$ is a countably complete ultrafilter in $N(f)$, i.e., such that

(+) for every function g from $s(f)$ to ω in $N(f)$, g is constant on a set in $F(f)$.

Note the following facts.

1. If f_0 and f_1 are functions from ω to ω_1 whose ranges are the same modulo a finite set, then $F(f_0) = F(f_1)$ and so not only are the models $N(f_0)$ and $N(f_1)$ the same, but their canonical wellorderings are the same also.

2. Using the canonical wellordering of each $N(f)$, there is a function G choosing for each increasing $f\colon \omega \to \omega_1$ a function $G(f)\colon s(f) \to \omega$ failing condition (+) above, if one exists.

The key consequence of AD is the partition property $\omega_1 \to (\omega_1)^{\omega}_{\omega^\omega}$ (see [8] or pages 391-396 of [13]), which says that for every function from the set of increasing ω-sequences from ω_1 to ω^ω (the set of functions from ω to ω) there is an uncountable $E \subseteq \omega_1$ such that the function is constant on the set of increasing ω-sequences from E.

Now, for each increasing $f\colon \omega \to \omega_1$, let $P(f)$ be the constant function 0 if $F(f)$ satisfies conditions (+) in $N(f)$. If f fails condition (+) in $N(f)$, then let $P(f)(0)$ be 1 and let $P(f)(n+1) = G(f)(f(n))$ for all $n \in \omega$. Let E be an uncountable subset of ω_1 such that $P(f)$ is the same for all increasing $f\colon \omega \to E$. We show that the constant value is the constant function 0. If the constant value corresponds to a failure of (+), then there is an $i\colon \omega \to \omega$ such that for all increasing $f\colon \omega \to E$, for all $n \in \omega$, $G(f)(f(n)) = i(n)$.

Then i must be constant, since if $n \in \omega$ is such that $i(n) \neq i(0)$, then if f is an increasing function from ω to E and $g\colon \omega \to E$ is defined by letting $g(m) = f(m+n)$, then $G(f) \neq G(g)$, contradicting the fact that $F(f) = F(g)$. But if i is constant, then for every increasing $f\colon \omega \to E$, $G(f)$ is constant on a set in $F(f)$, contradicting the failure of (+). $\dashv$

4.12 Theorem. *Suppose that the Axiom of Determinacy holds in $L(\mathbb{R})$, and let A be a set of reals in $L(\mathbb{R})$. Then there exists a condition $\langle (M, I), a\rangle$ in $\mathbb{P}_{max}$ such that*

- *$A \cap M \in M$,*
- *$\langle H(\omega_1)^M, A \cap M\rangle \prec \langle H(\omega_1), A\rangle$,*
- *(M, I) is A-iterable.*
- *if M^+ is any forcing extension of M and J is a normal precipitous ideal on $\omega_1^{M^+}$ in M^+ then $A \cap M^+ \in M^+$ and (M^+, J) is A-iterable, and if $j\colon (M^+, J) \to (M^*, J^*)$ is any iteration of (M^+, J), then*

$$\langle H(\omega_1)^{M^*}, A \cap M^*\rangle \prec \langle H(\omega_1), A\rangle.$$

Proof. Work in $L(\mathbb{R})$. If there is an $A \subseteq \mathbb{R}$ which is a counterexample to the statement of the theorem, then we may assume that there exists such an A which is $\underset{\sim}{\Delta}^2_1$. This follows from the Solovay Basis Theorem (see [8]), which says (in ZF) that every nonempty $\underset{\sim}{\Sigma}^2_1$ collection of sets of reals has a member which is $\underset{\sim}{\Delta}^2_1$. We give a quick sketch of the proof. Note first of all that for any ordinal α the transitive collapse any elementary submodel of $L_\alpha(\mathbb{R})$ containing $\mathbb{R}$ is a set of the form $L_\beta(\mathbb{R})$ for some ordinal $\beta \leq \alpha$. Now, if α is any ordinal, there exist (in $L(\mathbb{R})$) an elementary submodel X of $L_\alpha(\mathbb{R})$ containing $\mathbb{R}$ and a surjection $\pi\colon \mathbb{R} \to X$, so if α is the least ordinal such that a member of a given $\underset{\sim}{\Sigma}^2_1$ set exists in $L_{\alpha+1}(\mathbb{R})$ then there is a surjection (in $L(\mathbb{R})$) from $\mathbb{R}$ onto $L_{\alpha+1}(\mathbb{R})$, and a formula ϕ and a real x such that some member of the set is defined over $L_\alpha(\mathbb{R})$ by ϕ from x. By the minimality of α, that member has $\underset{\sim}{\Sigma}^2_1$ and $\underset{\sim}{\Pi}^2_1$ definitions using x and incorporating ϕ.

Towards a contradiction, fix a $\underset{\sim}{\Delta}^2_1$ counterexample A. By [22], the pointclass Σ^2_1 has the scale property in $L(\mathbb{R})$, which means that every subset of $\mathbb{R} \times \mathbb{R}$ which is $\underset{\sim}{\Delta}^2_1$ in $L(\mathbb{R})$ is the projection of a tree in $L(\mathbb{R})$ on the product of ω and some ordinal, and can be uniformized by a function which is $\underset{\sim}{\Delta}^2_1$ in $L(\mathbb{R})$. (We refer the reader to [8, 13, 24] for a discussion of scales and their corresponding trees. Briefly, if $B \subseteq \mathbb{R} \times \mathbb{R}$ is the projection of a tree T on $\omega \times \omega \times \gamma$ (for some ordinal γ) then for each real x such that there exists a y with (x, y) in B, we can recursively define functions $f(x)\colon \omega \to \omega$ and $g(x)\colon \omega \to \gamma$ as follows: if (m, α) is the lexicographically least pair in $\omega \times \gamma$ such that there exist a real y and a function $a\colon \omega \to \gamma$ such that

- y extends $f(x)\restriction n$ and $y(n) = m$,
- a extends $g(x)\restriction n$ and $a(n) = \alpha$,
- (x, y, a) is a path through T,

then $f(x)(n) = m$ and $g(x)(n) = \alpha$. Then f uniformizes B, and if T is the tree corresponding to a $\undertilde{\Sigma}^2_1$ scale on B, then f is $\undertilde{\Delta}^2_1$.) Now, $\undertilde{\Delta}^2_1$ is closed under complements, projections and countable unions, so there exists a $\undertilde{\Delta}^2_1$ set $B \subseteq \mathbb{R} \times \mathbb{R}$ such that whenever $F \colon \mathbb{R} \to \mathbb{R}$ is a function uniformizing B and N is a transitive model N of ZF closed under F,

$$\langle H(\omega_1)^N, A \cap N, \in\rangle \prec \langle H(\omega_1), A, \in\rangle.$$

Fix such B and F, both $\undertilde{\Delta}^2_1$. Let S, S^*, T, T^* be trees (on $\omega \times \gamma$, for some ordinal γ) in $L(\mathbb{R})$ projecting to A, the complement of A, F and the complement of F respectively. Note that any transitive model of ZF with T as a member is closed under F.

Now by Theorem 4.11, we may fix a transitive model N of ZFC and a countable ordinal γ such that N contains the ordinals, S, S^*, T and T^* are elements of N and γ is a measurable cardinal in N. Since $N \subseteq L(\mathbb{R})$ and $L(\mathbb{R})$ satisfies AD, ω_1^V is a limit of strongly inaccessible cardinals in N. Let δ be any strongly inaccessible cardinal in N between γ and ω_1^V. Recall (Remark 2.6) that if we choose an N-generic filter G for the forcing consisting of $\mathrm{Col}(\omega, {<}\gamma)$ followed by the standard c.c.c. iteration to make Martin's Axiom hold, as defined in N, then if we let I be the normal ideal generated by an ideal in N dual to a fixed normal measure on γ in N, I is a precipitous ideal in $N[G]$ and $(N_\delta[G], I)$ is iterable, by Lemmas 1.4 and 1.5. It suffices now to fix a forcing extension M^+ of $N_\delta[G]$ in which there exists a normal precipitous ideal J on $\omega_1^{M^+}$ and to show that the second part of the conclusion of the theorem holds for M^+ and J. Let N^+ be the corresponding forcing extension of $N[G]$. Since $S \in N^+$, $A \cap M^+ \in M^+$. Fix an iteration $j \colon (M^+, J) \to (M^*, J^*)$. By Lemma 1.5 there is an iteration $j^* \colon (N^+, J) \to (N^*, J^*)$ such that $j^* \restriction M^+ = j$. Now, $p[S] \subseteq p[j^*(S)]$ and $p[S^*] \subseteq p[j^*(S^*)]$, and further, by absoluteness $p[j^*(S)] \cap p[j^*(S^*)] = \emptyset$, so $p[S] = p[j^*(S)]$. Similarly, $p[T] = p[j^*(T)]$. Then N^* is closed under F, so we have that

$$\langle H(\omega_1)^{M^*}, A \cap M^*, \in\rangle \prec \langle H(\omega_1), A, \in\rangle.$$

Furthermore, $j(A \cap M^+) = p[j^*(S)] \cap M^*$, so $j(A \cap M^+) = A \cap M^*$. This shows that A is not in fact a counterexample to the statement of the theorem. ⊣

Suppose that A is a set of reals and x is a real coding a condition p in $\mathbb{P}_{max}$ by some recursive coding, and let B be the set of reals coding members of $A \times \{x\}$. Then if (M, I) is a B-iterable pair such that

$$\langle H(\omega_1)^M, B \cap M, \in\rangle \prec \langle H(\omega_1), B, \in\rangle,$$

then (M, I) is A-iterable and $p \in H(\omega_1)^M$. Therefore, the existence, for each $A \subseteq \mathbb{R}$ in $L(\mathbb{R})$, of an A-iterable pair (M, I) such that

$$\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$$

implies that for each $A \subseteq \mathbb{R}$ in $L(\mathbb{R})$ the set of $\langle (M, I), a\rangle$ in $\mathbb{P}_{max}$ such that (M, I) is A-iterable and $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$ is dense.

5. The basic analysis

With the existence of A-iterable conditions (for all sets of reals A in $L(\mathbb{R})$) in hand, we can now prove the most important fact about the $\mathbb{P}_{max}$ extension, that every subset of ω_1 in the extension is the image of a member of the generic filter under the iteration of that member induced by the generic filter.

Formally, if $G \subset \mathbb{P}_{max}$ is a set of pairwise compatible conditions, then since the elementary embedding witnessing the order on a pair of $\mathbb{P}_{max}$ conditions has critical point the ω_1 of the smaller model, for each pair $\langle (M, I), a\rangle, \langle (N, J), b\rangle$ in G, $a \cap \gamma = b \cap \gamma$, where $\gamma = \min\{\omega_1^M, \omega_1^N\}$. For any such G, we let

$$A_G = \bigcup\{a \mid \exists (M, I)\ \langle (M, I), a\rangle \in G\}.$$

By Lemma 2.7, for any such G, for any member $\langle (M, I), a\rangle$ of G there is a unique iteration of (M, I) sending a to A_G. Using this fact, we define $\mathcal{P}(\omega_1)_G$ to be the collection of all E such that there exists a condition $\langle (M, I), a\rangle \in G$ and a set $e \in \mathcal{P}(\omega_1)^M$ such that $j(e) = E$, where j is the unique iteration of (M, I) sending a to A_G. Likewise, we define I_G to be the collection of all E such that there exists a condition $\langle (M, I), a\rangle \in G$ and a set $e \in I$ such that $j(e) = E$, where j is the unique iteration of (M, I) sending a to A_G.

We state the following theorem from the point of view of the ground model (so in particular, the universe V in the statement of the theorem does not satisfy AC). We have seen that large cardinals and determinacy each apply that the hypothesis of the theorem is satisfied in $L(\mathbb{R})$, but as we shall see, it can hold in other models as well.

5.1 Theorem. *(ZF) Assume that for every $A \subseteq \mathbb{R}$ there exists a $\mathbb{P}_{max}$ condition $\langle (M, I), a\rangle$ such that (M, I) is A-iterable and*

$$\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle.$$

Suppose that $G \subset \mathbb{P}_{max}$ is a V-generic filter. Then in $V[G]$ the following hold.

(a) $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$.

(b) $NS_{\omega_1} = I_G$.

(c) $\underset{\sim}{\delta}^1_2 = \omega_2$.

(d) NS_{ω_1} *is saturated.*

Before proving Theorem 5.1, we prove another iteration lemma in terms of games in order to separate out some commonly needed details.

Suppose that p is a $\mathbb{P}_{max}$ condition, let J be a normal ideal on ω_1 and let B be a subset of ω_1. Let $\mathcal{G}_{\omega_1}(p, J, B)$ be the game where Players I and II collaborate to build a descending ω_1-sequence of $\mathbb{P}_{max}$ conditions $p_\alpha = \langle (M_\alpha, I_\alpha), a_\alpha \rangle$ below p, where in round $\alpha < \omega_1$, I chooses p_α if $\alpha \notin B$, and II chooses p_α if $\alpha \in B$. At the end of the game, II wins if, letting $A = \bigcup\{a_\alpha : \alpha < \omega_1\}$ and letting $j_\alpha \colon (M_\alpha, I_\alpha) \rightarrow (M^*_\alpha, I^*_\alpha)$ (for each $\alpha < \omega_1$) be the iteration of (M_α, I_α) sending a_α to A, $j_\alpha(I_\alpha) = J \cap M^*_\alpha$ holds for each $\alpha < \omega_1$.

5.2 Lemma. *(ZFC°) Suppose that $x^\dagger$ exists for every real x. Let p be a condition in $\mathbb{P}_{max}$, let J be a normal ideal on ω_1 and let B be a subset of ω_1. Then II has a winning strategy in $\mathcal{G}_{\omega_1}(p, J, B)$ if and only if $B \notin J$.*

Proof. The interesting direction is showing that II has a winning strategy if $B \notin J$, and for this direction it suffices to consider the case where B consists entirely of limit ordinals (we have no use for the other direction and leave its proof to the reader). The strategy for II uses the usual trick. Partition B into J-positive sets $\{B^\alpha_i : \alpha < \omega_1, i < \omega\}$, and as the p_α are chosen, let $\{E^\alpha_i : i < \omega\}$ enumerate $\mathcal{P}(\omega_1)^{M_\alpha} \setminus I_\alpha$ for each α.

Fix a *ladder sytem* $\{h_\alpha : \alpha \in B\}$ (so each h_α is an increasing function from ω to α with cofinal range). Having constructed our sequence of p_α's up to some limit stage β in B, let

$$(\langle (N^\beta_i, J^\beta_i) : i < \omega \rangle, a^*_\beta)$$

be the limit sequence corresponding to the descending sequence

$$\langle p_{h_\beta(i)} : i < \omega \rangle,$$

and for each $i < \omega$ let $j'_{i\beta}$ be the unique iteration of $(M_{h_\beta(i)}, I_{h_\beta(i)})$ sending $a_{h_\beta(i)}$ to a^*_β. Since the dagger of each real exists, we may fix a $\mathbb{P}_{max}$ precondition (M_β, I_β) with

$$(\langle (N^\beta_i, J^\beta_i) : i < \omega \rangle, a^*_\beta) \in H(\omega_1)^{M_\beta}.$$

As in Lemma 3.4 (more precisely, using Lemma 3.5), we let j'_β be an iteration of $\langle (N^\beta_i, J^\beta_i) : i < \omega \rangle$ in M_β such that

$$j'_\beta(J^\beta_i) = I_\beta \cap j'_\beta(N^\beta_i)$$

for each $i < \omega$, with the extra stipulation that if

$$\omega_1^{N_0^\beta} \in B^\gamma_k$$

for some $\gamma < \beta$ and $k < \omega$, then, letting i' be the least element i of ω such that $h_\beta(i) \geq \gamma$,

$$j'_{i'\beta}(j_{\gamma h_\beta(i')}(E^\gamma_k))$$

is in the normal filter corresponding to the first step of this iteration of $\langle (N^\beta_i, J^\beta_i) : i < \omega \rangle$ (note that $j'_{i'\beta}(j_{\gamma h_\beta(i')}(E^\gamma_k))$ is $J^\beta_{i'}$-positive by the agreement of ideals imposed by the $\mathbb{P}_{max}$ order). Then, letting $a_\beta = j'_\beta(a^*_\beta)$, we have that

$$\omega_1^{N_0^\beta} \in j_{\gamma\beta}(E^\gamma_k).$$

Since for each $i < \omega$ and $\alpha < \omega_1$ the set of $\beta \in B^\alpha_i$ such that $\omega_1^{N_0^\beta} = \beta$ is J-positive, by playing in this manner Player II ensures that the image of each E^α_i is J-positive. ⊣

We separate out the following standard argument as well.

5.3 Lemma. *Suppose that $x^\dagger$ exists for every real x, and let $G \subset \mathbb{P}_{max}$ be an $L(\mathbb{R})$-generic filter. Let $p_0 = \langle (M, I), a \rangle$ be a $\mathbb{P}_{max}$ condition in G, and suppose that $P \in M$ is a set of $\mathbb{P}_{max}$ conditions such that $p \geq p_0$ for every $p \in P$. Let j be the unique iteration of (M, I) sending a to A_G. Then every member of $j(P)$ is in G.*

Proof. Let $\langle M_\alpha, G_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1 \rangle$ be the iteration corresponding to j, and fix $q = \langle (N_0, J_0), b_0 \rangle$ in $j(P)$. Fix $\alpha_0 < \omega_1$ such that $q \in j^*_{0\alpha_0}(P)$, and let j_q (in M_{α_0}) be the iteration of (N_0, J_0) sending b_0 to $j^*_{0\alpha_0}(a)$. By the genericity of G there is a condition $p_1 = \langle (N_1, J_1), b_1 \rangle$ in G such that $p_1 \leq p_0$ and $\alpha_0 < \omega_1^{N_1}$. Then $\langle M_\alpha, G_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1^{N_1} \rangle$ is in $M_{\omega_1^{N_1}}$ and is the unique iteration of (M, I) sending a to b_1. Since

$$j_q(J_0) = j^*_{0\alpha_0}(I) \cap j_q(N_0)$$

and

$$j^*_{0\omega_1^{N_1}}(I) = J_1 \cap M_{\omega_1^{N_1}},$$

$j^*_{\alpha_0\omega_1^{N_1}}(j_q)$ witnesses that $q \geq p_1$. ⊣

Proof of Theorem 5.1. (a) Let τ be a $\mathbb{P}_{max}$-name in $L(\mathbb{R})$ for a subset of ω_1, and let A be the set of reals coding (under some fixed recursive coding) the set of triples (p, α, i) such that $p \in \mathbb{P}_{max}$, $\alpha < \omega_1$, $i \in 2$ and, if $i = 1$ then $p \Vdash \check{\alpha} \in \tau$, otherwise $p \Vdash \check{\alpha} \notin \tau$. Let $p = \langle (N, J), d\rangle$ be any condition in $\mathbb{P}_{max}$ and let (M, I) be an A-iterable pre-condition such that

- $p \in H(\omega_1)^M$,
- $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$.

Applying Lemma 5.2 in M, there exists a descending sequence of $\mathbb{P}_{max}$ conditions $p_\alpha = \langle (N_\alpha, J_\alpha), d_\alpha\rangle$ $(\alpha < \omega_1^M)$ such that

(1) $p_0 = p$,

(2) each $p_{\alpha+1}$ decides the sentence "$\check{\alpha} \in \tau$,"

(3) letting $D = \bigcup\{d_\alpha : \alpha < \omega_1^M\}$, for each $\alpha < \omega_1$, $j_\alpha(J_\alpha) = I \cap j_\alpha(N_\alpha)$, where j_α is the unique iteration of (N_α, J_α) sending d_α to D.

Note that conditions (1) and (2) are easily satisfied, using the fact that $\langle H(\omega_1)^M, A\cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$, and we may apply Lemma 5.2 (letting B be the set of countable limit ordinals) inside M to meet Condition (3) since in M the dagger of each real exists.

Now, letting D be as in Condition (3) above, $\langle (M, I), D\rangle$ is a $\mathbb{P}_{max}$ condition below each p_α. Let e be the subset of ω_1^M in M such that for each $\alpha < \omega_1$, $\alpha \in e \Leftrightarrow p_{\alpha+1} \Vdash \check{\alpha} \in \tau$.

Suppose that $p' = \langle (M, I), D\rangle \in G$, and let

$$\langle M_\alpha, G_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$$

be the unique iteration of (M, I) sending D to A_G. We want to see that $j^*_{0\omega_1}(e) = \tau_G$. Let $\langle q_\alpha : \alpha < \omega_1\rangle = j^*_{0\omega_1}(\langle p_\alpha : \alpha < \omega_1^M\rangle)$. By the elementarity of $j^*_{0\alpha^*}$ and the A-iterability of (M, I), for each $\gamma < \omega_1$, $q_{\gamma+1} \Vdash \check{\gamma} \in \tau$ if $\gamma \in j_{0\alpha^*}(e)$ and $q_{\gamma+1} \Vdash \check{\gamma} \notin \tau$ if $\gamma \notin j_{0\alpha^*}(e)$. By Lemma 5.3, each q_γ is in G, so $j^*(e) = \tau_G$.

(b) The fact that $I_G = NS_{\omega_1}$ follows almost immediately. If $E \in I_G$, then there is a condition $\langle (M, I), a\rangle$ in G, an $e \in I$ and an iteration j of (M, I) sending e to E. Then E is disjoint from the critical sequence of this iteration and therefore nonstationary. On the other hand, if E is a nonstationary subset of ω_1 in $V[G]$, then there is a club C disjoint from E and a condition $\langle (M, I), a\rangle$ in G, sets $e, c \in \mathcal{P}(\omega_1)^M$ and an iteration j of (M, I) sending e and c to E and C respectively. Then c must be a club subset of ω_1^M in M, so $e \in I$, which means that E is in I_G.

(c) That $\underset{\sim}{\delta}^1_2 = \omega_2$ also follows almost immediately, using Corollary 1.8 and the standard fact that if $x^\#$ exists for every real x, then $\underset{\sim}{\delta}^1_2$ is equivalent to u_2, the second uniform indiscernible (the least ordinal above ω_1 which is an indiscernible of every real) (see [34, 37]). So, showing that $\underset{\sim}{\delta}^1_2 = \omega_2$ then amounts to showing that for every $\gamma < \omega_2$ there is a real x such that the least x-indiscernible above ω_1 is greater than γ. Working in $V[G]$, fix $\gamma \in [\omega_1, \omega_2)$ and a wellordering π of ω_1 of length γ. By the first part of this theorem, we may fix a condition $\langle (M, I), a\rangle \in G$ and an $e \in \mathcal{P}(\omega_1 \times \omega_1)^M$ such that $j(e) = \pi$, where j is the iteration of (M, I) sending a to A_G. Then γ is in $j(M)$, and so is less than the least indiscernible above ω_1 of any real coding (M, I), by Corollary 1.8.

(d) To show that NS_{ω_1} is saturated in $V[G]$, we show that for any set $D \subseteq \mathcal{P}(\omega_1)\backslash NS_{\omega_1}$ which is dense under the subset order, there is a subset D' of D of cardinality $\aleph_1$ whose diagonal union contains a club. So, following the proof of the first part of this theorem, let τ be a name for such a set D. Let A be the set of reals coding (by a fixed recursive coding) the set of pairs (p, e) such that $p = \langle (M, I), a\rangle$ is a condition in $\mathbb{P}_{max}$, $e \in \mathcal{P}(\omega_1)^M \setminus I$ and p forces that $j(\check{e}) \in \tau$, where j is the unique iteration of (M, I) sending a to A_G.

Let $p = \langle (N, J), b\rangle$ be any condition in $\mathbb{P}_{max}$ and let (M, I) be an A-iterable pre-condition such that

- $p \in H(\omega_1)^M$,
- $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$.

Fix a partition $\{B^\alpha_i : \alpha < \omega_1, i < \omega\}$ in M of pairwise disjoint I-positive sets whose diagonal union is I-large. Fix also a function $g\colon \omega_1^M \times \omega \to \omega_1^M$ in M such that $g(\alpha, i) \geq \alpha$ for all $(\alpha, i) \in dom(g)$.

Working in M, we are going to build a descending sequence of $\mathbb{P}_{max}$ conditions $p_\alpha = \langle (N_\alpha, J_\alpha), b_\alpha\rangle$ (with the order on conditions witnessed by a commuting family of embeddings $j_{\alpha\beta}$), enumerations $\{e^\alpha_i : i < \omega\}$ in M of each set $\mathcal{P}(\omega_1)^{N_\alpha} \setminus J_\alpha$ and sets d_α ($\alpha \leq \beta \leq \omega_1^M$) such that

(4) $p_0 = p$,

(5) each $d_\alpha \in \mathcal{P}(\omega_1)^{N_{\alpha+1}} \setminus J_{\alpha+1}$ and, if $\alpha = g(\beta, i)$ for some $\beta \leq \alpha$ and $i < \omega$, then $d_\alpha \subseteq j_{\beta(\alpha+1)}(e^\beta_i)$ and $(p_{\alpha+1}, d_\alpha)$ is coded by a real in A,

(6) for each $(\beta, i) \in dom(g)$, $B^\alpha_i \setminus j_{(g(\beta,i)+1)\omega_1^M}(d_{g(\beta,i)})$ is nonstationary.

Conditions (5) and (6) together imply that our sequence will satisfy Condition (3) from part (a) of this proof. Furthermore, Conditions (4) and (5) here are easily achieved, by the assumptions on τ. In particular, for each $\alpha < \omega_1^M$, by the assumptions on τ there exists a pair (p^*, d^*) such that $p^* \leq p_\alpha$ and Condition (5) holds with p^* in the role of $p_{\alpha+1}$ and d^* in the role of d_α, and we let $(p_{\alpha+1}, d_\alpha)$ be any such pair. Condition (6) implies that the diagonal union of the sets $j_{(g(\beta,i)+1)\omega_1^M}(d_{g(\beta,i)})$ will be I-large.

Condition (6) is achieved in almost exactly the same way as Condition (3) in the first part of the proof (but not exactly the same way; unfortunately we cannot quote Lemma 5.2). Fix a ladder system $\{h_\alpha : \alpha < \omega_1 \text{ limit}\}$ in M. Having constructed our sequence of p_α's up to some limit stage β, let $(\langle (N_i^\beta, J_i^\beta) : i < \omega\rangle, b_\beta^*)$ be the limit sequence corresponding to the descending sequence $\langle p_{h_\beta(i)} : i < \omega\rangle$, and again for each $i < \omega$ let $j'_{i\beta}$ be the unique iteration of $(N_{h_\beta(i)}, J_{h_\beta(i)})$ sending $b_{h_\beta(i)}$ to b_β^*. Fix a pre-condition (N_β, J_β) in M with

$$(\langle N_i^\beta : i < \omega\rangle, b_\beta^*) \in H(\omega_1)^{N_\beta}.$$

As in Lemma 3.4, we let j'_β be an iteration of $\langle (N_i^\beta, J_i^\beta) : i < \omega\rangle$ in N_β such that

$$j'_\beta(J_i^\beta) = J_\beta \cap j'_\beta(N_i^\beta)$$

for each $i < \omega$, with the extra stipulation that if

$$\omega_1^{N_0^\beta} \in B_k^\gamma$$

for some $\gamma < \beta$ and $k < \omega$ with $g(\gamma, k) < \beta$, then, letting i' be the least $i \in \omega$ such that $h_\beta(i) \geq g(\gamma, k)$,

$$j'_{i'\beta}(j_{(g(\gamma,k)+1)h_\beta(i')}(d_{g(\gamma,k)}))$$

is in the filter corresponding to the first step of this iteration of the sequence $\langle (N_i^\beta, J_i^\beta) : i < \omega\rangle$, ensuring (once we let $b_\beta = j'_\beta(b_\beta^*)$) that

$$\omega_1^{N_0^\beta} \in j_{(g(\gamma,k)+1)\beta}(d_{g(\gamma,k)}).$$

Then since $\{\omega_1^{N_0^\beta} : \beta < \omega_1 \text{ limit}\}$ is a club subset of ω_1^M, Condition (6) is satisfied.

Now, letting $B = \bigcup\{b_\alpha : \alpha < \omega_1^M\}$, $\langle (M, I), B\rangle$ is a $\mathbb{P}_{max}$ condition below each p_α. For each $\alpha < \omega_1$ and $i < \omega$, let $d'_{\alpha i} = j_{(g(\alpha,i)+1)\omega_1^M}(d_{g(\alpha,i)})$. Then the diagonal union of

$$\mathcal{A} = \{d'_{\alpha i} : \alpha < \omega_1^M, i < \omega\}$$

contains an I-large subset of ω_1^M in M.

Suppose that $\langle (M, I), B\rangle \in G$, and let $\langle M_\alpha, G_\beta, j^*_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$ be the unique iteration of (M, I) sending B to A_G. Then the diagonal union of $j^*_{0\omega_1}(\mathcal{A})$ contains the critical sequence of $j^*_{0\omega_1}$, which is a club. We want to see that $j^*_{0\omega_1}(\mathcal{A}) \subseteq \tau_G$.

Let $\langle q_\alpha : \alpha < \omega_1\rangle = j^*_{0\omega_1}(\langle p_\alpha : \alpha < \omega_1^M\rangle)$. By Lemma 5.3, each q_α is in G. Since (M, I) is A-iterable, each member of $j^*(\mathcal{A})$ is forced to be in τ_G by some q_α, so $j^*(\mathcal{A}) \subseteq \tau_G$. ⊣

5.4 Remark. It is shown in [16] that, under the hypothesis of Theorem 5.1, Todorcevic's Open Coloring Axiom [35] holds in the $\mathbb{P}_{max}$ extension. The proof in that paper can be greatly simplified by using Lemmas 5.2 and 5.3 to separate out the standard details.

6. ψ_{AC} and the Axiom of Choice

We have not yet shown that the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ satisfies the Axiom of Choice. We shall do this by showing (assuming that AD holds in $L(\mathbb{R})$) that the following axiom holds there.

We let $o.t.(X)$ denote the ordertype of a linear order X.

6.1 Definition. ψ_{AC} is the statement that for every pair A, B of stationary, co-stationary subsets of ω_1, there exists a bijection π between ω_1 and some ordinal γ such that the set $\{\alpha < \omega_1 \mid \alpha \in A \Leftrightarrow o.t.(\pi[\alpha]) \in B\}$ contains a club.

Using a partition $\{A_\alpha : \alpha < \omega_1\}$ of ω_1 into stationary sets, ψ_{AC} allows us to define an injection from 2^{ω_1} into ω_2. Since the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ satisfies the sentence "$V = L(\mathcal{P}(\omega_1))$," this is enough to see that AC holds there. Let B^* be any stationary, co-stationary subset of ω_1. For each $X \subseteq \omega_1$, let $A_X = \bigcup\{A_\alpha : \alpha \in X\}$, and let γ_X be the ordinal given by ψ_{AC}, where A_X is in the role of A, and B^* is in the role of B. Let X_0 and X_1 be distinct subsets of ω_1, and let E be the (stationary) symmetric difference of A_{X_0} and A_{X_1}. Supposing towards a contradiction that $\gamma_{X_0} = \gamma_{X_1}$, let π_0 and π_1 be bijections and C_0 and C_1 club subsets of ω_1 witnessing ψ_{AC} for the pairs A_{X_0}, B^* and A_{X_1}, B^* respectively. Then there is a club subset D of ω_1 such that $o.t.(\pi_0[\alpha]) = o.t.(\pi_1[\alpha])$ for all $\alpha \in D$. Then $E \cap C_0 \cap C_1 \cap D$ is nonempty, which gives a contradiction, since $\alpha \in A_{X_0} \Leftrightarrow o.t.(\pi_0[\alpha]) \in B \Leftrightarrow o.t.(\pi_1[\alpha]) \in B \Leftrightarrow \alpha \in A_{X_1}$ for all $\alpha \in C_0 \cap C_1 \cap D$. Therefore, ψ_{AC} implies that $2^{\omega_1} = \omega_2$. In fact, it also implies that $2^\omega = 2^{\omega_1}$, but we will not take the time to show this (it follows from a result of Shelah proved in Section 3.2 of [37]); we already know from Theorem 5.1 that the Continuum Hypothesis fails in the $\mathbb{P}_{max}$ extension (assuming AD in $L(\mathbb{R})$).

That ψ_{AC} holds in the $\mathbb{P}_{max}$ extension follows from part (a) of Theorem 5.1 and the following lemma.

6.2 Lemma. *(ZFC°) Suppose that (M, I) is a pre-condition in $\mathbb{P}_{max}$, and let $A, B \in M$ be I-positive subsets of ω_1^M whose complements in ω_1^M are also I-positive. Let J be a normal ideal on ω_1. Then there exist an iteration $j\colon (M, I) \to (M^*, I^*)$ of (M, I) of length ω_1, an ordinal $\gamma < \omega_2$, and a bijection $\pi\colon \omega_1 \to \gamma$ such that $I^* = J \cap M^*$ and*

$$\{\alpha < \omega_1 \mid \alpha \in j(A) \Leftrightarrow o.t.(\pi[\alpha]) \in j(B)\}$$

contains a club.

Proof. Let x be a real coding (M, I). Using Fact 1.9, it suffices to construct an iteration $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$ such that for every α which is a limit of countable x-indiscernibles, $j_{0\alpha}(A) \in G_\alpha$ if and only if $j_{0\alpha^*}(B) \in$

G_{α^*}, where α^* is the least x-indiscernible above α. Note that by the proof of Lemma 1.7, $\omega_1^{M_\gamma} = \gamma$ for each x-indiscernible γ, so in particular, each such γ is on the critical sequence.

We construct our iteration using the the winning strategy for Player I in $\mathcal{G}(\omega_1 \setminus E)$ from Lemma 2.9, where E is the set of countable ordinals of the form α^* as above, where α is a limit of x-indiscernibles. This ensures that $j_{0\omega_1}(I) = J \cap M_{\omega_1}$. To complete the construction, we recursively choose each G_{α^*} ($\alpha^* \in E$) in such a way that $j_{0\alpha^*}(B) \in G_{\alpha^*}$ if and only if $j_{0\alpha}(A) \in G_\alpha$. Fact 1.9 implies that any iteration satisfying these conditions satisfies the conclusion of the lemma. ⊣

Stated in the fashion of Theorem 5.1, we have shown the following.

6.3 Theorem. *(ZF) Assume that for every $A \subseteq \mathbb{R}$ there exists a $\mathbb{P}_{max}$ condition $\langle (M, I), a\rangle$ such that (M, I) is A-iterable and*

$$\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle.$$

Suppose that $G \subset \mathbb{P}_{max}$ is a V-generic filter. Then ψ_{AC} holds in $V[G]$.

6.4 Remark. Suppose that A, B are stationary, co-stationary subsets of ω_1, $\pi\colon \omega_1 \to \gamma$ is a bijection (for some $\gamma < \omega_2$) and the set

$$\{\alpha < \omega \mid \alpha \in A \Leftrightarrow o.t.(\pi[\alpha]) \in B\}$$

contains a club subset of ω_1. Then for any normal ideal I on ω_1, A is the Boolean value in the partial order $\mathcal{P}(\omega_1)/I$ that $\gamma \in j(B)$, where j is the induced embedding. It follows that if (M, I) is any iterable pair with M a countable transitive model of ZFC°+ ψ_{AC} and B is any stationary, costationary subset of ω_1^M in M, then the image of B under any iteration of (M, I) determines the entire iteration. This in turn implies that one can replace $\mathrm{MA}_{\aleph_1}$ with ψ_{AC} in the definition of $\mathbb{P}_{max}$ without significantly changing the corresponding analysis; in some cases the analysis is easier with ψ_{AC}.

7. Maximality and minimality

In this section we will show that if certain large cardinals exist in V then the $\mathbb{P}_{max}$ extension of the inner model $L(\mathbb{R})$ is maximal, in that all forceable Π_2 sentences for $H(\omega_2)$ hold there, and that it is minimal, in that every subset of ω_1 added by the generic filter for $\mathbb{P}_{max}$ generates the entire extension. We will also show that a certain form of this maximality characterizes the $\mathbb{P}_{max}$ extension.

The following theorem is an immediate consequence of part (a) of Theorem 5.1, and it implies in particular that $\mathrm{MA}_{\aleph_1}$ holds in the $\mathbb{P}_{max}$ extension. Corollary 7.7 below is the maximal version of this fact.

7.1 Theorem. *Assume that for every $A \subseteq \mathbb{R}$ there exists a $\mathbb{P}_{max}$ condition $\langle (M, I), a\rangle$ such that (M, I) is A-iterable and*

$$\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle.$$

Let ψ be a binary formula without unbounded quantifiers in a language with one additional unary predicate. Suppose that it is a theorem of ZFC + *"NS_{ω_1} is precipitous" that for every $X \in H(\omega_2)$ there is a partial order preserving stationary subsets of ω_1 and the precipitousness of NS_{ω_1} and forcing the formula $\exists Y \psi(X, Y)$ to hold in the structure $\langle H(\omega_2), NS_{\omega_1}, \in\rangle$ of the forcing extension. Then $\forall X \exists Y \psi(X, Y)$ holds in the $\mathbb{P}_{max}$ extension of V.*

To show that the $\mathbb{P}_{max}$ extension is Π_2-maximal, we will use the following theorem of Woodin (see [7, 5, 28, 37]; Foreman, Magidor and Shelah originally proved the theorem from the existence of supercompact cardinal [6]). An ideal I on ω_1 is *presaturated* if for any $A \in \mathcal{P}(\omega_1) \setminus I$ and any sequence $\langle \mathcal{A}_i : i < \omega\rangle$ of maximal antichains in $\mathcal{P}(\omega_1) \setminus I$ there exists a $B \in \mathcal{P}(A) \setminus I$ such that there are at most $\aleph_1$ many $X \in \bigcup\{\mathcal{A}_i : i < \omega\}$ such that $X \cap B \notin I$. It is straightforward to check that normal presaturated ideals on ω_1 are precipitous.

7.2 Theorem. *If δ is a Woodin cardinal, then every condition in the partial order* $\mathrm{Col}(\omega_1, {<}\delta)$ *forces that NS_{ω_1} is presaturated.*

Using this we can prove the following theorem.

7.3 Theorem. *Suppose that δ is a limit of Woodin cardinals, and let $\kappa > \delta$ be a measurable cardinal. Let A be a set of reals in $L(\mathbb{R})$. Let $p = \langle (M, I), a\rangle$ be a condition in $\mathbb{P}_{max}$, and let ψ be a binary formula without unbounded quantifiers in the language with two additional unary predicates, such that $\forall X \exists Y \psi(X, Y)$ holds in $\langle H(\omega_2), NS_{\omega_1}, A, \in\rangle$. Let x be an element of $H(\omega_2)^M$. Then there exists a $\mathbb{P}_{max}$ condition $q = \langle (N, J), b\rangle$ such that*

- *$q \leq p$,*
- *(N, J) is A-iterable,*
- *if $j : (M, I) \to (M^*, I^*)$ is the unique iteration of (M, I) sending a to b, then*
$$\langle H(\omega_2)^N, J, A \cap N, \in\rangle \models \exists y\ \psi(j(x), y).$$

Proof. By Theorem 4.5, since there exist infinitely many Woodin cardinals below a measurable cardinal, all sets of reals in $L(\mathbb{R})$ are $<\delta$-weakly homogeneously Suslin. Let γ be the least Woodin cardinal, let γ' be the least strongly inaccessible cardinal greater than γ, and let θ be a regular cardinal greater than κ. Let X be a countable elementary submodel of $H(\theta)$ with $\gamma, \gamma', A, p \in X$. Applying Lemma 2.8, there is an iteration $j\colon (M, I) \to (M^*, I^*)$ of (M, I) in X of length ω_1 such that $I^* = NS_{\omega_1} \cap M^*$. Then there is a $y \in H(\omega_2) \cap X$ such that

$$\langle H(\omega_2), NS_{\omega_1}, A, \in \rangle \models \psi(j(x), y).$$

Let M be the transitive collapse of $X \cap H(\gamma')$ and, letting γ^* be the image of γ under this collapse, let g be M-generic for $\mathrm{Col}(\omega_1, {<}\gamma^*)^M$. By Theorem 7.2, $NS_{\omega_1}^{M[g]}$ is precipitous in $M[g]$. Let h be $M[g]$-generic for the standard forcing to make $MA_{\aleph_1}$ hold. Then by Theorem 4.9, $(M[g][h], NS_{\omega_1}^{M[g][h]})$ is A-iterable. Letting $b = j(a)$ completes the proof of the theorem. ⊣

7.4 Definition. Given a cardinal κ, a set of reals A is *κ-universally Baire* if there exist trees S and T (contained in $\omega \times Z$ for some set Z) such that $p[S] = A$ and S and T project to complements in all extensions by forcing constructions of cardinality less than κ.

Theorem 4.6 shows that for any cardinal κ, κ-weakly homogeneously Suslin sets of reals are κ-universally Baire.

If κ is a cardinal, A is a κ-universally Baire set of reals and $V[G]$ is an extension of V by a forcing construction of cardinality less than κ, then we let $A(G)$ be the union of all sets of the form $(p[S])^{V[G]}$, where S is a tree in V whose projection in V is contained in A. (The notation A_G is often used here, but we are already using that for something else.) Note that for any pair of trees S and T in V witnessing that A is κ-universally Baire (i.e., such that $p[S] = A$ and S and T project to complements in all extensions by forcing constructions of cardinality less than κ), $(p[S])^{V[G]} = A(G)$.

The following theorem (due to Woodin) is proved in [19]. The set $\mathbb{R}^\#$ was introduced in [31]. All we need to know here is that $\mathbb{R}^\#$ is a set of reals coding the theory of $L(\mathbb{R})$ in the language with constants for each real and ω many ordinal indiscernibles, and that if $\mathbb{R}^\#$ exists then each set of reals in $L(\mathbb{R})$ is definable in $L(\mathbb{R})$ from a real and a finite set of these indiscernibles.

7.5 Theorem. *Suppose that δ is a limit of Woodin cardinals below a measurable cardinal. Then $\mathbb{R}^\#$ is ${<}\delta$-weakly homogeneous, and if M is any forcing extension of V by a forcing construction in V_δ then $(\mathbb{R}^\#)^V = (\mathbb{R}^\#)^M \cap V$.*

This gives the following strengthening of Theorem 0.2.

7.6 Theorem. *Suppose that δ is a limit of Woodin cardinals below a measurable cardinal, and let A be a set of reals in $L(\mathbb{R})$. Then if G is a V-generic filter contained in a partial order in V_δ, then in $V[G]$ there is an elementary embedding from $L(A, \mathbb{R}^V)$ to $L(A(G), \mathbb{R}^{V[G]})$ sending A to $A(G)$.*

By Theorems 5.1 and 7.6, Theorem 7.3 has the following corollary.

7.7 Corollary. *Suppose that δ is a limit of Woodin cardinals such that there exists a measurable cardinal above δ, and let A be a set of reals in $L(\mathbb{R})$. Suppose that ϕ is a Π_2 sentence in the language with two additional unary predicates and P is a partial order in V_δ which forces that ϕ holds in the structure $\langle H(\omega_2), NS_{\omega_1}, A(G), \in\rangle$ (where $A(G)$ is defined relative to P). Then ϕ holds in the structure $\langle H(\omega_2), NS_{\omega_1}, A, \in\rangle$ when computed in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ (of V).*

Theorem 7.12 below is a sort of converse to Corollary 7.7. First we will show that every new subset of ω_1 in the $\mathbb{P}_{max}$ extension generates the entire generic filter (Theorem 7.11).

Note that the conclusion of the following lemma corresponds to Condition 5 in the definition of $\mathbb{P}_{max}$.

7.8 Lemma. *Suppose that the dagger of each real exists. Let $\langle (M', I'), a'\rangle$ be a $\mathbb{P}_{max}$ condition and let e be an element of $\mathcal{P}(\omega_1)^{M'}$. Then there exist a $\mathbb{P}_{max}$ pre-condition (N, J) with $(M', I') \in H(\omega_1)^N$ and an iteration $j\colon (M', I') \to (M^*, I^*)$ in N such that*

- $j(\omega_1^{M'}) = \omega_1^N$,
- $I^* = J^* \cap M^*$,

and either

1. *for some $x \in \mathcal{P}(\omega)^N$, $j(e) \in L[x]$, or*
2. *for some $x \in \mathcal{P}(\omega)^N$, $\omega_1^N = \omega_1^{L[j(e),x]}$.*

Proof. Fix a limit sequence $(\langle (M_i, I_i) : i < \omega\rangle, a)$ corresponding to any descending ω-sequence in $\mathbb{P}_{max}$ starting with $\langle (M', I'), a'\rangle$, and let (N, J) be a $\mathbb{P}_{max}$ pre-condition with $\{(M', I'), (\langle (M_i, I_i) : i < \omega\rangle, a)\} \in H(\omega_1)^N$. Let j' be the iteration of (M', I') sending a' to a. Now one of two things must hold. Either there exist $i < \omega$, $\gamma < \omega_2^{M_i}$ and a bijection $f\colon \omega_1^{M_0} \to \gamma$ in M_i such that $\{\alpha < \omega_1^{M_0} : o.t.(f[\alpha]) \in j'(e)\}$ and $\{\alpha < \omega_1^{M_0} : o.t.(f[\alpha]) \notin j'(e)\}$ are both I_i-positive subsets of $\omega_1^{M_0}$ in M_i, or there are no such i, γ, f.

If there is no such triple, then the image of $j'(e)$ is the same under every iteration of $\langle (M_i, I_i) : i < \omega\rangle$ of length ω_1^N. Let x be a real in N coding $\langle (M_i, I_i) : i < \omega\rangle$. There exist iterations of $\langle (M_i, I_i) : i < \omega\rangle$ of length ω_1^N in forcing extensions of $L[x]$ by the partial order $\mathrm{Col}(\omega, {<}\omega_1^N)$, and since this partial order is homogeneous, this fixed image of $j'(e)$ exists in $L[x]$. Letting j be any suitable (for example, using a strategy for Player I in $\mathcal{G}(\omega_1)$ as in Theorem 3.5) such iteration in N of length ω_1^N, then, $j(j')$ is an iteration of (M', I') satisfying the first conclusion of the lemma.

If there is such a triple, note that there is a real y in M_{i+1} such that γ is definable in M_{i+1} (absolutely, in fact) from $\omega_1^{M_0}$ and y (for instance, we could let y be the sharp of any real whose least indiscernible above $\omega_1^{M_0}$ is greater than γ). In particular, we may fix a ternary formula ϕ such that γ is the unique ordinal such that $\phi(\gamma, y, \omega_1^{M_0})$ holds in $L[y]$. Let

$$A = \{\alpha < \omega_1^{M_0} : o.t.(f[\alpha]) \in j'(e)\}.$$

Then A is the Boolean value of the statement that γ is in the image of $j'(e)$. Let x be a real in N coding $(\langle (M_i, I_i) : i < \omega\rangle, a)$. Then just as in the proof of Lemma 1.7, the indiscernibles of x are on the critical sequence of any iteration of $\langle (M_i, I_i) : i < \omega\rangle$. Fix a set $B \subseteq \omega_1^N$ in N such that $\omega_1^N = \omega_1^{L[B]}$. Working in N, build an iteration j (with partial iterations $j_{\alpha\beta}$ ($\alpha \leq \beta \leq \omega_1^N$) and normal filters G_α ($\alpha < \omega_1^N$)) of $\langle (M_i, I_i) : i < \omega\rangle$, using a winning strategy for Player I in the game $\mathcal{G}(\omega_1^N \setminus E)$ from Theorem 3.5, where E is the set of countable x-indiscernibles which are not limits of x-indiscernibles (note that $x^\# \in N$, as (N, J) is iterable, so N contains the sharps for all its reals). When $j_{0\alpha}(\omega_1^{M_0})$ is in E, we put $j_{0\alpha}(A)$ in the normal filter G_α if and only if $\eta \in B$, where $j_{0\alpha}(\omega_1^{M_0})$ is the ηth successor x-indiscernible. Having completed the construction of our iteration, we have that B is constructible from $j(j'(e)), y$ and $x^\#$: B is the set of $\eta < \omega_1^N$ such that, letting ι_η be the η-th successor x-indiscernible, the unique ordinal γ^* satisfying $\phi(\gamma^*, y, \iota_\eta)$ in $L[y]$ is in $j(j'(e))$. Then $j(j')$ is an iteration of (M', I') satisfying the second conclusion of the lemma. ⊣

For the rest of this section we fix the following notation: if B is a subset of ω_1, we let F_B be the set of conditions $\langle (M, I), b\rangle$ in $\mathbb{P}_{max}$ such that there exists an iteration $j\colon (M, I) \to (M^*, I^*)$ such that $j(b) = B$ and $I^* = NS_{\omega_1} \cap M^*$.

Woodin defines the following axiom.

7.9 Definition. Axiom $(*)$ is the statement that AD holds in $L(\mathbb{R})$ and $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.

The proofs of Theorems 7.11 and 7.12 use the following lemma.

7.10 Lemma. *Assume that axiom $(*)$ holds, and let B be a subset of ω_1 such that there exists a real z such that $\omega_1 = \omega_1^{L[z,B]}$. Then the set F_B is a filter.*

Proof. Fix an $L(\mathbb{R})$-generic filter $G \subset \mathbb{P}_{max}$ such that $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G]$. Fix a real x such that $\omega_1 = \omega_1^{L[x,A_G]}$. As in the proof of Lemma 2.7, let $\{a_\alpha : \alpha < \omega_1\}$ be the almost disjoint family of subsets of ω constructed in $L[x, A_G]$ by recursively taking a_α to be the first real in the $L[x, A_G]$ constructibility order almost disjoint from each a_β ($\beta < \alpha$). Now let $y \subseteq \omega$ be such that for all $\alpha < \omega$, $a_\alpha \cap y$ is infinite if and only if $\alpha \in B$. Let

$$p_0 = \langle (M_0, I_0), b_0\rangle \text{ and } p_1 = \langle (M_1, I_1), b_1\rangle$$

be members of F_B, as witnessed by iterations j_0 and j_1 respectively, and let C_0 and C_1 be the respective critical sequences of j_0 and j_1. Let $\langle (N, J), a\rangle$ be a member of G with $x, y, z, p_0, p_1 \in H(\omega_1)^N$ and sets c_0 and c_1 in $\mathcal{P}(\omega_1)^N$ such that, for j the unique iteration of (N, J) sending a to A_G, $j(c_0) = C_0$ and $j(c_1) = C_1$. Then $c_0 = C_0 \cap \omega_1^N$, $c_1 = C_1 \cap \omega_1^N$, and c_0 and c_1 are both club subsets of ω_1^N. Since $\omega_1^{L[z,B]} = \omega_1$, $\omega_1^{L[z,b_0]} = \omega_1^{M_0}$ and $\omega_1^{L[z,b_1]} = \omega_1^{M_1}$. Since $x \in N$, $\{a_\alpha : \alpha < \omega_1^N\}$ is in N (it satisfies the same definition in N relative to x and a that $\{a_\alpha : \alpha < \omega_1\}$ satisfies relative to x and A_G). Since $y \in N$, $B \cap \omega_1^N \in N$, and the unique iterations

$$j_0^* \colon (M_0, I_0) \to (M_0^*, I_0^*)$$

and

$$j_1^* \colon (M_1, I_1) \to (M_1^*, I_1^*)$$

sending b_0 to $B\cap\omega_1^N$ and b_1 to $B\cap\omega_1^N$ respectively are in N, and furthermore $j(B \cap \omega_1^N) = B$. Once we see that $I_0^* = J \cap M_0^*$ and $I_1^* = J \cap M_1^*$ we will be done. The two proofs are the same. For (M_0, I_0), if $E \in J \cap M_0^*$, then since $E \in J$, $j(E) \in NS_{\omega_1}^{L(\mathbb{R})[G]}$. Now, $j(j^*)$ is an iteration of (M_0, I_0) sending b_0 to B, and so it is equal to j_0. Then $j(E)$ is the image of E under the tail of the iteration j_0 starting with (M_0^*, I_0^*). So $j(E) \in j_0(M_0)$, and since $j_0(I_0) = NS_{\omega_1}^{L(\mathbb{R})[G]} \cap j_0(M_0)$, $j(E) \in j_0(I_0)$, and so $E \in I_0^*$. ⊣

Note that if $G \subset \mathbb{P}_{max}$ is an $L(\mathbb{R})$-generic filter, then $F_{(A_G)}$ is a filter containing G, and so by the genericity of G, $F_{(A_G)} = G$.

Now we show that any new subset of ω_1 added by forcing with $\mathbb{P}_{max}$ generates the entire extension.

7.11 Theorem. *Suppose that (∗) holds. Then for every $B \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R})$, F_B is an $L(\mathbb{R})$-generic filter for $\mathbb{P}_{max}$, and $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[F_B]$.*

Proof. By Lemma 7.8, there is a real z such that $L[z, B]$ correctly computes ω_1. By Lemma 7.10, F_B is a filter. Now, let $\langle (M, I), a\rangle$ be a condition in G such that $z \in M$ and for some $b \in \mathcal{P}(\omega_1)^M$, $j(b) = B$, for j the unique iteration of (M, I) sending a to A_G. As in Lemma 2.10, the mapping π sending each condition $\langle (N, J), c\rangle$ below $\langle (M, I), a\rangle$ to the condition $\langle (N, J), b^*\rangle$, where b^* is the image of b by the iteration of (M, I) sending a to c, is an isomorphism. The image of G under π, F_B, is then an $L(\mathbb{R})$-generic filter in $\mathbb{P}_{max}$. Furthermore, π is in $L(\mathbb{R})$, so G is in $L(\mathbb{R})[F_B]$. ⊣

7.12 Theorem. *Suppose* AD *holds in $L(\mathbb{R})$, and that for every Π_2 sentence ϕ in the language with two additional unary predicates, if $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ and*

$$\langle H(\omega_2), A, NS_{\omega_1}, \in\rangle^{L(\mathbb{R})^{\mathbb{P}_{max}}} \models \phi$$

then

$$\langle H(\omega_2), A, NS_{\omega_1}, \in\rangle \models \phi.$$

Then for every $B \in \mathcal{P}(\omega_1) \setminus L(\mathbb{R})$, F_B is an $L(\mathbb{R})$-generic filter and

$$H(\omega_2) = H(\omega_2)^{L(\mathbb{R})[F_B]}.$$

Proof. Let $\mathcal{P}$ denote $\mathcal{P}(\omega_1)\setminus\bigcup\{L[x] : x \in \mathbb{R}\}$ (under $\mathrm{AD}^{L(\mathbb{R})}$ this is the same as $\mathcal{P}(\omega_1) \setminus L(\mathbb{R})$, but we want to make the relevant syntax more explicit). The sentence asserting that F_B is a filter for every $B \in \mathcal{P}$ is Π_2 in $H(\omega_2)$ with parameters for NS_{ω_1} and the set of $\mathbb{P}_{max}$ conditions, and by Lemma 7.10, this sentence holds in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$. If $X \in L(\mathbb{R})$ is a dense subset of $\mathbb{P}_{max}$, then the statement that $F_B \cap X$ is nonempty for every $B \in \mathcal{P}$ is Π_2 in $H(\omega_2)$ with parameters for NS_{ω_1}, X and the set of $\mathbb{P}_{max}$ conditions, and this sentence holds in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ by Theorem 7.11. Thus, for every $B \in \mathcal{P}$, F_B is $L(\mathbb{R})$-generic. Finally, the following statement is Π_2 in $H(\omega_2)$ with parameters for NS_{ω_1} and the set of $\mathbb{P}_{max}$ conditions, and holds in the $\mathbb{P}_{max}$ extension: for every $E \subseteq \omega_1$ and for every $B \in \mathcal{P}$ there is a $\mathbb{P}_{max}$ condition $\langle (M, I), b\rangle$ and an iteration $j : (M, I) \to (M^*, I^*)$ such that $E \in \mathcal{P}(\omega_1)^{M^*}$, $j(b) = B$ and $I^* = NS_{\omega_1} \cap M^*$. Fixing a set $B \in \mathcal{P}$, then, since $\{x, B\} \in L(\mathbb{R})[F_B]$, $H(\omega_2) \subseteq H(\omega_2)^{L(\mathbb{R})[F_B]}$. ⊣

Theorem 7.11 gives us another way to characterize the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$, this time without mention of $\mathbb{P}_{max}$. For the definition below, we fix the following notation. If g is a filter contained in $\mathrm{Col}(\omega, {<}\omega_1)$, then for each $\alpha < \omega_1$ we let

$$S^g_\alpha = \{\beta \mid \exists p \in g\ p(0, \beta) = \alpha\}$$

and, for each $\tau \subseteq \omega_1 \times \mathrm{Col}(\omega, {<}\omega_1)$, we let

$$I_g(\tau) = \{\alpha \mid \exists p \in g\ (\alpha, p) \in \tau\}.$$

7.13 Definition. Axiom $(^*_*)$ is the statement that $x^\#$ exists for every real number x and if X is a nonempty subset of $\mathcal{P}(\omega_1)$ which is definable from real and ordinal parameters then there exists a real x and a set

$$\tau \subseteq \omega_1 \times \mathrm{Col}(\omega, {<}\omega_1)$$

such that $\tau \in L[x]$ and such that for all filters $g \subset \mathrm{Col}(\omega, {<}\omega_1)$, if g is $L[x]$-generic and if for each $\alpha < \omega_1$, S^g_α is stationary, then $I_g(\tau) \in X$.

The converse of the following theorem also holds (see Sections 5.7 and 5.8 of [37]), though its proof is beyond the scope of this chapter.

7.14 Theorem. *Axiom $(*)$ implies that Axiom $(^*_*)$ holds in $L(\mathcal{P}(\omega_1))$.*

Proof. First note that AD implies that the sharp of every real exists. Now let G be an $L(\mathbb{R})$-generic filter such that $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G]$, and fix a set X as in the statement of Axiom $(^*_*)$. Let $p = \langle (M, I), a\rangle$ be a condition in G such that for some $b \in \mathcal{P}(\omega_1)^M$, p forces that $j(b) \in X$, for j the unique iteration of (M, I) sending a to A_G. Let x be a real such that $\langle (M, I), a\rangle \in H(\omega_1)^{L[x]}$. Now, if $g \subset \mathrm{Col}(\omega, {<}\omega_1)$ is $L[x]$-generic, then as in the proof of Lemma 2.8, in $L[x][g]$ there is an iteration

$$\{M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\}$$

of (M, I) such that for each $\beta < \omega_1$ and each set $e \in \mathcal{P}(\omega_1)^{M_\beta} \setminus j_{0\beta}(I)$ there is an $\alpha < \omega_1$ such that $S^g_\alpha \setminus j_{\beta\omega_1}(e)$ is nonstationary. Let σ be an $\mathrm{Col}(\omega, {<}\omega_1)$-name in $L[x]$ for the embedding $j_{0\omega_1}$ corresponding to such an iteration, and let τ be the set of pairs $(\alpha, p) \in \omega_1 \times \mathrm{Col}(\omega, {<}\omega_1)$ such that $p \Vdash \check{\alpha} \in \sigma(\check{b})$. Now suppose that $g \subset \mathrm{Col}(\omega, {<}\omega_1)$ is $L[x]$-generic and that each S^g_α is stationary. Then σ_g is an iteration of (M, I), and since there exists a real z such that

$$\omega_1^{L[z, \sigma_g(a)]} = \omega_1,$$

$\sigma_g(a)$ is not in $L(\mathbb{R})$. Then Theorem 7.11 implies that $G_{\sigma_g(a)}$ (as in the statement of that theorem) is an $L(\mathbb{R})$-generic filter for $\mathbb{P}_{max}$. Since each S^g_α is stationary, σ_g witnesses that $\langle (M, I), a\rangle$ is in $G_{\sigma_g(a)}$, which means that $\sigma_g(b)$ is in X. Since $\sigma_g(b) = I_g(\tau)$, we are done. ⊣

Theorem 7.14 has the following immediate corollary. By a perfect subtree of $2^{<\omega_1}$ we mean a tree of height ω_1 such that every node is extended by a pair of incompatible nodes, and such that every countable increasing sequence has a node extending it.

7.15 Corollary. *Suppose that* AD *holds in* $L(\mathbb{R})$ *and let* $G \subset \mathbb{P}_{max}$ *be an* $L(\mathbb{R})$*-generic filter. Let* ϕ *be a unary formula with parameters for elements of* $L(\mathbb{R})$ *and suppose that there exists a subset of* ω_1 *in* $L(\mathbb{R})[G] \setminus L(\mathbb{R})$ *satisfying* ϕ. *Then there is a perfect subtree* T *of* $2^{<\omega_1}$ *such that every subset of* ω_1 *corresponding to a path through* T *satisfies* ϕ.

Given an ordinal β, Martin's Maximum$^{+\beta}$ (MM$^{+\beta}$, derived from [6]) is the statement that whenever P is a partial order such that forcing with P preserves stationary subsets of ω_1, $\langle D_\alpha : \alpha < \omega_1\rangle$ is a sequence of dense subsets of P and $\langle \tau_\alpha : \alpha < \beta\rangle$ is a sequence of P-names for stationary subsets of ω_1, there is a filter $G \subset P$ such that $G \cap D_\alpha$ is nonempty for each $\alpha < \omega_1$ and $\{\gamma < \omega_1 \mid \exists p \in G\ p \Vdash \check{\gamma} \in \tau_\alpha\}$ is stationary for each $\alpha < \beta$.

It is shown in [15] that MM$^{+\omega}$ does not imply $(*)$, if the existence of a supercompact limit of supercompact cardinals is consistent with ZFC. The question of whether MM$^{+\omega_1}$ implies $(*)$ remains open. We mention the following two test cases, consequences of $(*)$ which have not been shown from large cardinals to be provably forceable by a semi-proper partial order. We omit the proofs, as they appear in full in [37] (Theorem 7.16 appears in [37] as Theorem 5.74(5) and Theorem 7.19 appears as Theorem 6.124).

7.16 Theorem. *Suppose that $\binom{*}{*}$ holds. Then for every $A \subseteq \omega_1$ which is not constructible from a real, there exist a real x and an $L[x]$-generic filter $g \subset \mathrm{Col}(\omega, {<}\omega_1)$ such that $L[x][g] = L[x, A]$.*

The statement of Theorem 7.19 requires the following definitions.

7.17 Definition. A tree $T \subseteq \{0,1\}^{<\omega_1}$ is *weakly special* if for all countable $X \prec \langle H(\omega_2), T, \in\rangle$, if $b\colon \omega_1 \cap X \to \{0,1\}$ is a cofinal branch of T_X not in M_X, then there is a bijection $\pi\colon \omega \to \omega_1^{M_X}$ definable in the structure $\langle M_X, T_X, b, \in\rangle$, where $\langle M_X, T_X, \in\rangle$ is the transitive collapse of X.

7.18 Definition. $\Phi^+_\Diamond$ is the statement that for each $A \subseteq \omega_1$ there exists a $B \subseteq \omega_1$ such that, letting $T_B = \{0,1\}^{<\omega_1} \cap L[B]$,

- $A \in L[B]$,
- T_B is weakly special,
- every branch of T_B is in $L[B]$.

7.19 Theorem. *Axiom $(*)$ implies $\Phi^+_\Diamond$.*

One consequence of Theorem 7.19 is that there are no weak Kurepa trees (subtrees of $\{0,1\}^{<\omega_1}$ of cardinality $\aleph_1$ with $\aleph_2$ many cofinal branches) in any $\mathbb{P}_{max}$ extension.

8. Larger models

The forcing construction $\mathbb{P}_{max}$ can be applied to larger models than $L(\mathbb{R})$, if they satisfy (ostensibly) stronger forms of determinacy.

8.1 Definition. A set of reals A is ∞-Borel if there exists a set of ordinals S, an ordinal α and a binary formula ϕ such that

$$A = \{y \in \mathbb{R} \mid L_\alpha[S, y] \models \phi(S, y)\}.$$

The ordinal Θ is defined to be the least ordinal which is not a surjective image of $\mathbb{R}$. The notion of continuity in the definition below refers to the discrete topology on λ, not the interval topology. *Dependent Choice* (DC) is a weak form of the Axiom of Choice saying that every tree of height ω with no terminal nodes has a cofinal branch; $\mathrm{DC}_\mathbb{R}$ (Dependent Choice for reals) is the restriction of DC to trees on the reals.

8.2 Definition. (ZF + $\mathrm{DC}_\mathbb{R}$) AD^+ is the conjunction of the following two statements.

- Every set of reals is ∞-Borel.

- If $\lambda < \Theta$ and $\pi\colon \lambda^\omega \to \omega^\omega$ is a continuous function, then $\pi^{-1}(A)$ is determined for every $A \subseteq \omega^\omega$.

It is an open question whether AD implies AD^+, though it is known that AD^+ holds in all models of AD of the form $L(A, \mathbb{R})$, where A is a set of reals (some of the details of the argument showing this appear in [8]).

The following consequences of AD^+ are enough to prove that $\mathbb{P}_{max}$ conditions exist in suitable generality.

8.3 Theorem. *(ZF + $\mathrm{DC}_\mathbb{R}$) If AD^+ holds and $V = L(\mathcal{P}(\mathbb{R}))$ then*

- *the pointclass Σ^2_1 has the scale property,*
- *every Σ^2_1 set of reals is the projection of a tree in* HOD*,*
- *every true Σ_1-sentence is witnessed by a $\underset{\sim}{\Delta}^2_1$ set of reals.*

Adapting the proof of Theorem 4.12, then, we have the following.

8.4 Theorem. *Suppose that $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a pointclass closed under continuous preimages and that $L(\Gamma, \mathbb{R}) \models \mathrm{DC}_\mathbb{R} + \mathrm{AD}^+$. Then for every set of reals A in $L(\Gamma, \mathbb{R})$ there is a $\mathbb{P}_{max}$ precondition (M, I) such that*

- *$A \cap M \in M$,*
- *$\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1), A, \in\rangle$,*
- *(M, I) is A-iterable.*

The corresponding parts of the proof of Theorem 5.1 then go through to give the following.

8.5 Theorem. *Suppose that $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a pointclass closed under continuous preimages such that $L(\Gamma, \mathbb{R}) \models \mathrm{DC}_\mathbb{R} + \mathrm{AD}^+$. Suppose that $G \subset \mathbb{P}_{max}$ is $L(\Gamma, \mathbb{R})$-generic. Then the following hold in $L(\Gamma, \mathbb{R})[G]$:*

- *$\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$,*
- *I_G is the nonstationary ideal,*
- *$\underset{\sim}{\delta}^1_2 = \omega_2$,*
- *I_G is saturated.*

If there is no surjection in $L(\Gamma, \mathbb{R})$ from $\mathbb{R} \times \mathrm{Ord}$ onto Γ, then Γ is not wellordered in the $\mathbb{P}_{max}$ extension of $L(\Gamma, \mathbb{R})$. Producing a model of Choice then requires the following step, which appears with proof in [37] as Theorem 9.36. The statement ω_2-DC says that $<\omega_2$-closed trees of height ω_2 with no terminal nodes have cofinal branches.

8.6 Theorem. *Suppose that* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *is a pointclass closed under continuous preimages such that* $L(\Gamma, \mathbb{R}) \models \mathrm{DC}_{\mathbb{R}} + \mathrm{AD}^+ +$ "Θ *is regular*". *Suppose that* $G \subset \mathbb{P}_{max}$ *is* $L(\Gamma, \mathbb{R})$*-generic. Then* $L(\Gamma, \mathbb{R})[G] \models \omega_2$*-DC.*

The axiom $\mathrm{AD}_{\mathbb{R}}$ is the statement that all two player games of perfect information of length ω where the players play real numbers are determined. This statement easily implies $\mathrm{DC}_{\mathbb{R}}$ and in the context of DC is properly stronger than AD^+. Theorem 8.7 below lists some properties of the $\mathbb{P}_{max}$ extension of a model of $\mathrm{AD}_{\mathbb{R}}$ plus "Θ is regular." Many of the corresponding proofs proceed by finding a $\mathbb{P}_{max}$ condition satisfying axiom $(*)$ and satisfying the conclusion of Theorem 8.4 for a suitable set A. We emphasize that part (1) of the conclusion of Theorem 8.7 says that in the $\mathbb{P}_{max}$ extension of $L(\Gamma, \mathbb{R})$, $L(\mathcal{P}(\omega_1))$ is a $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$, not (merely) $L(\Gamma, \mathbb{R})$.

Martin's Maximum$^{++}(\mathfrak{c})$ is the restriction of Martin's Maximum^{++} to partial orders of cardinality the continuum (which it implies is $\aleph_2$). The statement $\Diamond_\omega(\omega_2)$ says that there is a sequence

$$\{A_\gamma : \gamma < \omega_2 \wedge \mathrm{cf}(\gamma) = \omega\}$$

such that each A_γ is a subset of γ and such that for every $B \subseteq \omega_2$, the set of $\alpha < \omega_2$ of countable cofinality such that $B \cap \alpha = A_\alpha$ is stationary. Woodin shows in Section 5.2 of [37] that $\Diamond_\omega(\omega_2)$ follows from Martin's Maximum. Part (4) of the conclusion of Theorem 8.7 is due to Dan Seabold [26]. *Chang's Conjecture* is the statement that for each function $F\colon [\omega_2]^{<\omega} \to \omega_2$ there exists an $X \subseteq \omega_2$ of ordertype ω_1 such that $F[[X]^{<\omega}] \subseteq X$ (i.e., that the set of subsets of ω_2 of ordertype ω_1 is stationary, in the sense of [19]). It is an open question whether Chang's Conjecture holds in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ whenever $L(\mathbb{R})$ satisfies AD. This question has been resolved (negatively) for $\mathbb{Q}_{max}$ (see Remark 10.7).

Parts (5), (6) and (7) of Theorem 8.7 show that $\mathbb{P}_{max}$ can be used to produce consistency results at ω_2 as well as at ω_1. We let $NS^\omega_{\omega_2}$ denote the nonstationary ideal on ω_2 concentrating on the ordinals of cofinality ω. The ideal $NS^\omega_{\omega_2}$ is *weakly presaturated* if for every $S \in \mathcal{P}(\omega_2) \setminus NS^\omega_{\omega_2}$ and every function $f\colon S \to \omega_2$ there exist a ordinal $\gamma < \omega_3$ and a bijection $\pi\colon \omega_2 \to \gamma$ such that

$$\{\alpha \in S \mid f(\alpha) < o.t.(\pi[\alpha])\} \notin NS^\omega_{\omega_2}.$$

A normal ideal I on ω_2 is *semi-saturated* if whenever U is a set generic V-normal ultrafilter on ω_2 contained in $\mathcal{P}(\omega_2) \setminus I$, $Ult(V, U)$ is wellfounded.

Sections 9.6 and 9.7 of [37] contain stronger consistency results than parts (4), (5) and (6) of Theorem 8.7, but merely stating these facts would take us too far afield.

8.7 Theorem. *Suppose that* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *is a pointclass closed under continuous preimages such that* $L(\Gamma, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}} +$ "Θ *is regular.*" *Suppose that*

$G \subset \mathbb{P}_{max}$ is $L(\Gamma, \mathbb{R})$-generic, and let

$$H \subseteq \mathrm{Col}(\omega_3, H(\omega_3))^{L(\Gamma,\mathbb{R})[G]}$$

be an $L(\Gamma, \mathbb{R})[G]$-generic filter. Then the following hold in $L(\Gamma, \mathbb{R})[G][H]$:

1. *Axiom $(*)$,*
2. *Martin's Maximum$^{++}(\mathfrak{c})$,*
3. *$\diamondsuit_\omega(\omega_2)$,*
4. *Chang's Conjecture*
5. *$NS^{\omega}_{\omega_2}$ is precipitous,*
6. *$NS^{\omega}_{\omega_2}$ is weakly presaturated,*
7. *there is a normal semi-saturated ideal on ω_2 containing $NS^{\omega}_{\omega_2}$.*

9. Ω-logic

In this section we will briefly describe the relationship between $\mathbb{P}_{max}$ and Woodin's Ω-logic as presented in [37] (our presentation of Ω-logic, however, will follow the one in [42]). Let T be a set of sentences and let ϕ be a sentence, both in the language of set theory. Then $T \models_\Omega \phi$ (ϕ is Ω_T-valid) if for every forcing construction P and every ordinal α, if $V^P_\alpha \models T$ then $V^P_\alpha \models \phi$. We will define below the conjectured proof-theoretic complement to this model-theoretic notion.

A set of reals A is *universally Baire* if it is κ-universally Baire for all cardinals κ (see Definition 7.4). Woodin has shown that if δ is a limit of Woodin cardinals, then a set of reals is δ-universally Baire if and only if it is $<\delta$-weakly homogeneously Suslin (a proof is given in [19]). Given a universally Baire set of reals A, a transitive model N of ZFC is said to be *A-closed* if, whenever P is a partial order in N and $G \subset P$ is V-generic (not just N-generic), then $N[G] \cap A(G)$ is in $N[G]$. Lemmas 9.2 and 9.3 give useful reformulations of A-closure, and are relatively easy to prove. The proof of Lemma 9.2 uses the following fact, which will show up again in the proof of Theorem 9.4 and in Section 10.1. For a proof of Theorem 9.1, see page 516 of [9] or the appendix of [19].

9.1 Theorem. *(McAloon) If $\mathbb{P}$ is a partial order and forcing with $\mathbb{P}$ makes $\mathbb{P}$ countable, then $\mathbb{P}$ is forcing-equivalent to* $\mathrm{Col}(\omega, |\mathbb{P}|)$.

Theorem 9.1 implies that that every partial order $\mathbb{P}$ regularly embeds into $\mathrm{Col}(\omega, |\mathbb{P}|)$, which is forcing-equivalent to $\mathbb{P} \times \mathrm{Col}(\omega, |\mathbb{P}|)$.

9.2 Lemma. *Given a universally Baire set of reals A, a model M of* ZFC *is A-closed if and only if for all ordinals $\gamma \in M$, the set of pairs $(\tau, p) \in H(|\gamma|^+)^M$ such that τ is a $\mathrm{Col}(\omega, \gamma)$-name in M for a real, p is a condition in $\mathrm{Col}(\omega, \gamma)$ and p forces in V that the realization of τ is in $A(G)$ is in M.*

Lemma 9.3 shows that for countable models, it is not necessary to consider V-generic filters. The point is that, even without assuming the existence of large cardinals, if

- A is a universally Baire set of reals,
- M is a countable transitive model of ZFC,
- P is a partial order in M,
- p is a condition in P, and
- τ is a P-name in M for a real number

then p forces in V that τ_G is in $A(G)$ if and only if there exists (in V) a collection $\{D_i : i < \omega\}$ of dense subsets of P such that $\tau_g \in A$ for every M-generic filter $g \subset P$ containing p and intersecting each D_i.

9.3 Lemma. *Let A be a universally Baire set of reals and let M be a countable transitive model of* ZFC*. Then M is A-closed if and only if for each partial order P in M there exists (in V) a collection $\{D_i : i < \omega\}$ of dense subsets of P such that $M[G] \cap A \in M[G]$ for every M-generic filter $g \subset P$ intersecting each D_i.*

Note that if A is the set of reals coding wellorderings of ω (under some fixed recursive coding), then (expanding to the class of ω-models of ZFC) A-closure is equivalent to wellfoundedness.

Let T be a theory containing ZFC and let ϕ be a sentence, both in the language of set theory. Then $T \vdash_\Omega \phi$ (T implies ϕ in Ω-logic) if there exists a set of reals A such that

- $L(A, \mathbb{R}) \models \mathrm{DC}_\mathbb{R} + \mathrm{AD}^+$,
- every set of reals in $L(A, \mathbb{R})$ is universally Baire,
- for every countable A-closed model M and every ordinal $\alpha \in M$, if V_α^M satisfies T then V_α^M satisfies ϕ.

A sentence ϕ is Ω_{ZFC}-*consistent* if ZFC $\nvdash_\Omega \neg\phi$. The first two conditions above ensure that the set of reals A is sufficiently canonical, and hold of all universally Baire sets of reals in the presence of a proper class of Woodin cardinals. The third condition says that A serves as a sort of proof of ϕ, in the sense that ϕ holds in all models which are closed under a certain function corresponding to A.

The following theorem shows that statements which can be forced to hold (along with ZFC) in suitable initial segments of the universe are Ω_{ZFC}-consistent. Note that the proof shows the stronger fact that for every universally Baire set of reals A, all forceable statements hold in models N which are A-closed in the stronger sense that $N[G] \cap A \in N[G]$ for all N-generic filters G.

9.4 Theorem. *Suppose that A is a universally Baire set of reals and that κ is a strongly inaccessible cardinal. Then any forcing extension (in V) of any transitive collapse of any elementary submodel of V_κ containing A is A-closed.*

Proof. First note that A is universally Baire in V_κ. To see this, note that for any partial order P in V_κ, if S and T are trees witnessing the universal Baireness of A for P, then by taking a P-name τ in V_κ for all the reals of the P-extension and then taking an elementary submodel (of a sufficiently large $H(\theta)$) of size less than κ containing $\{S, T\} \cup \tau$, the images of S and T under the transitive collapse of this elementary submodel are in V_κ and witness the universal Baireness of A for P.

Now let X be an elementary submodel of V_κ with A as an element, and let M be the transitive collapse of X. Let P be a partial order in X, let $\bar{P}$ be the image of P under the transitive collapse of X, and let $g \subset \bar{P}$ be an M-generic filter. Let τ be a P-name in X for a partial order, and let $\bar{\tau}$ be the image of τ under the transitive collapse of X. We want to see that whenever $h \subseteq \bar{\tau}_g$ is a V-generic filter, then $M[g][h] \cap A(g,h)$ is in $M[g][h]$. Let $\gamma \in X \cap \kappa$ be a cardinal greater than $|P * \tau|$ and let S and T be trees in X witnessing the universal Baireness of A for $\mathrm{Col}(\omega, \gamma)$. Then S and T project to complements in any forcing extension of V by either $P * \tau$ or $\bar{\tau}_g$.

Let σ be a $\bar{\tau}_g$-name in $M[g]$ for a real. Let $\bar{S}$ and $\bar{T}$ be the images of S and T under the transitive collapse of X. Let $h \subseteq \bar{\tau}_g$ be V-generic. Then σ_h is in exactly one of $(p[S])^{V[h]}$ and $(p[T])^{V[h]}$, and by the elementarity of the collapsing map, σ_h is in exactly one of $(p[\bar{S}])^{M[g][h]}$ and $(p[\bar{T}])^{M[g][h]}$. Since $(p[\bar{S}])^{M[g][h]} \subseteq (p[S])^{V[h]}$ and $(p[\bar{T}])^{M[g][h]} \subseteq (p[T])^{V[h]}$, and since $A(h) = (p[S])^{V[h]}$, σ_h is in $A(h)$ if and only it is in $(p[\bar{S}])^{M[g][h]}$. Putting all of this together, we have that

$$M[g][h] \cap A(h) = (p[\bar{S}])^{M[g][h]},$$

which shows that $M[g]$ is A-closed. ⊣

Woodin has shown that the axiom $(*)$ is Ω_{ZFC}-consistent.

9.5 Theorem. *Suppose that there exists a proper class of Woodin cardinals and that there is an inaccessible cardinal which is a limit of Woodin cardinals. Then the theory*

$$\mathrm{ZFC} + (*)$$

is Ω_{ZFC}*-consistent.*

The proof of Theorem 9.5 requires one to force $(*)$ over larger models than $L(\mathbb{R})$, in particular, models of the form $L(S, \mathbb{R})$, where for some strongly inaccessible limit of Woodin cardinals κ, S is a $<\kappa$-weakly homogeneous tree. A proof that such models can satisfy AD^+ appears in [19]. Note that it is not known whether there are large cardinals whose existence implies that one can force over V to make $(*)$ hold. Woodin has conjectured that (ordertype) ω^2 many Woodin cardinals are sufficient. Of course, if MM^{++} implies $(*)$ (we discussed this question in Section 7) then one supercompact cardinal is sufficient. Woodin's Ω-conjecture asserts that if there exist proper class many Woodin cardinals then for every sentence ϕ, $\emptyset \models_\Omega \phi$ if and only if $\emptyset \vdash_\Omega \phi$.

Recall that for a set x, $x^\dagger$ is a set of the same cardinality as x coding a theory extending ZFC + "there exists a measurable cardinal" with constants for each member of x and for two classes of indiscernibles (above and below the measurable cardinal). If there exist proper class many Woodin cardinals, then the set D of reals coding (under some fixed recursive coding) pairs (x, i), where x is a real number, i is an integer and $i \in x^\dagger$ is universally Baire. Any D-closed model M then has the property that for any set x, $x^\dagger$ exists in any forcing extension of M where x is countable, which since $x^\dagger$ is unique means that $x^\dagger$ exists in M already (an easy way to say this uses the fact that $\mathrm{Col}(\omega, |x|)$ is homogeneous, though this is not necessarily the most direct way). Thus for every ordinal $\alpha \in M$, there exist an inner model N of M containing V^M_α (definable in M), an ordinal $\kappa > \alpha$ which is a measurable cardinal in N and a set μ which is a κ-complete nonprincipal measure on κ in N such that all iterates of N by μ are wellfounded. As in Example 1.6, then, if M is a D-closed model and I is a normal precipitous ideal on ω^M_1 in M, then every rank initial segment of M is a rank initial segment of a model N such that (N, I) is iterable, and so (M, I) is also iterable. Using this we have that every Π_2 sentence for $\langle H(\omega_2), NS_{\omega_1}, \in\rangle$ which is Ω_{ZFC}-consistent with the existence of a precipitous ideal on ω_1 holds in the $\mathbb{P}_{max}$ extension. Using the canonical inner models for Woodin cardinals we can do more, however. In the next few paragraphs we will sketch the proof of Theorem 9.6 below.

We are going to need a number of facts from inner model theory. Unfortunately, we do not have a reference for the exact facts that we need (though [32] is very close), which is why this is just a sketch. For each set x, let $M(x)$ denote the minimal model (i.e., sound, sufficiently iterable premouse) of ZFC plus "there exists an ordinal λ which is a limit of Woodin cardinals such that $V^\#_\lambda$ exists." This theory implies that $\mathbb{R}^\#$ is $<\lambda$-weakly homogeneously Suslin, and so there exist in $M(x)$ trees S and T on $\omega \times \lambda$ witnessing in $M(x)$ that $\mathbb{R}^\#$ and its complement are λ-universally Baire. Furthermore, from the point of view of V, S and T project to a subset of $\mathbb{R}^\#$ and a subset of $\mathbb{R} \setminus \mathbb{R}^\#$, respectively. The property of $M(x)$ that we need is the following: if δ is a Woodin cardinal in $M(x)$ below λ, γ is an ordinal below δ and y is a subset of ω, then there exist a partial order P (this partial order was discovered by Woodin and is typically called the *extender algebra*) of cardinality δ in $M(x)$ and an elementary embedding $j\colon M(x) \to M'$ with critical point greater than γ such that

- y is M'-generic for $j(P)$,
- $p[j(S)] \subseteq \mathbb{R}^\#$,
- $p[j(T)] \subseteq \mathbb{R} \setminus \mathbb{R}^\#$.

Furthermore, there is a canonical coding of $M(x)$ by a set of integers when x is itself a set of integers, giving rise to a universally Baire set B consisting of the pairs (x, i) such that i is part of the code for $M(x)$. A B-closed model of ZFC then contains $M(x)$ for every set $x \in M$.

Now let ϕ be a Π_2 sentence for $H(\omega_2)$ (of the form $\exists X \forall Y \psi(X, Y)$) with predicates for NS_{ω_1} and a given set of reals A in $L(\mathbb{R})$. Let z be a real number coding a given $\mathbb{P}_{max}$ condition and a real which codes A relative to $\mathbb{R}^\#$ (see the remarks before Theorem 7.5). Suppose that N is a countable B-closed model of ZFC satisfying ϕ and containing z. Let x be a wellordering of $H(\omega_2)^N$ in N. Then $H(\omega_2)^{M(x)} = H(\omega_2)^N$. Let γ be the least strongly inaccessible cardinal in $M(x)$ above the least Woodin cardinal. Let S and T be trees in $M(x)$ witnessing the λ-universal Baireness of $\mathbb{R}^\#$ and its complement, where λ is the least limit of Woodin cardinals in $M(x)$. We want to see that whenever we make NS_{ω_1} precipitous by any forcing in $V_\gamma^{M(x)}$ (getting a generic filter g) and then iterate $V_\gamma^{M(x)[g]}$ by NS_{ω_1}, we iterate correctly for $\mathbb{R}^\#$. Given this, if g is such a generic filter for a forcing preserving stationary subsets of $\omega_1^{M(x)}$ then $V_\gamma^{M(x)[g]}$ is an A-iterable model such that $\exists Y \psi(X, Y)$ holds in $H(\omega_2)^{V_\gamma^{M(x)[g]}}$ for all $X \in H(\omega_2)^{V_\gamma^{M(x)}}$, and by a density argument then, ϕ holds in the $\mathbb{P}_{max}$ extension.

Towards a contradiction, choose a bad generic filter g and bad iteration k. Let $j\colon M(x) \to M'$ be the embedding in the previous paragraph (with critical point above γ) such that we can add g and k to M' by forcing with the extender algebra for the image of the least Woodin cardinal in $M(x)$ above γ. Then $M'[g, k]$ has a bad iteration of $V_\gamma^{M(x)[g]}$ in it, and by Lemma 1.4 this iteration extends to an iteration of $M'[g]$ (which we will also call k), which means that

$$k(\mathbb{R}^\# \cap V_\gamma^{M(x)[g]}) = p[k(j(S))] \cap k(V_\gamma^{M(x)[g]})$$

and

$$k((\mathbb{R} \setminus \mathbb{R}^\#) \cap V_\gamma^{M(x)[g]}) = p[k(j(T))] \cap k(V_\gamma^{M(x)[g]}).$$

But $j(S)$ and $j(T)$ are $j(\lambda)$-universally Baire in M', so they project to complements in $M'[g,k]$. Furthermore,

$$p[j(S)] \subseteq \mathbb{R}^{\#}$$

and

$$p[j(T)] \subseteq \mathbb{R} \setminus \mathbb{R}^{\#}.$$

Since $p[j(S)] \subseteq p[k(j(S))]$ and $p[j(T)] \subseteq p[k(j(T))]$, $p[j(S)] = p[k(j(S))]$ and $p[j(T)] = p[k(j(T))]$, contradicting that k is a bad iteration. This argument shows the following.

9.6 Theorem. *If there exists a proper class of Woodin cardinals, then for every set of reals A in $L(\mathbb{R})$, every Ω_{ZFC}-consistent Π_2 sentence for $\langle H(\omega_2), NS_{\omega_1}, A, \in\rangle$ holds in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.*

Another strengthening of Theorem 0.2, using the absoluteness of $\mathbb{R}^{\#}$, is the following.

9.7 Theorem. *Suppose that there exists a proper class of Woodin cardinals. Then for every sentence ϕ, either* $\mathrm{ZFC} \vdash_\Omega L(\mathbb{R}) \models \phi$ *or* $\mathrm{ZFC} \vdash_\Omega L(\mathbb{R}) \not\models \phi$.

Since $\mathbb{P}_{max}$ is a homogeneous forcing extension of $L(\mathbb{R})$, this gives the following.

9.8 Theorem. *Suppose that there exists a proper class of Woodin cardinals. Then for every sentence ϕ, either*

$$\mathrm{ZFC} + (*) \vdash_\Omega L(\mathcal{P}(\omega_1)) \models \phi$$

or

$$\mathrm{ZFC} + (*) \vdash_\Omega L(\mathcal{P}(\omega_1) \not\models \phi.$$

Note that since $\mathbb{R}^{\#}$ is not in $L(\mathbb{R})$, the Continuum Hypothesis (plus the existence of $\mathbb{R}^{\#}$) implies that $L(\mathcal{P}(\omega_1))$ is not contained in a forcing extension of $L(\mathbb{R})$. Moreover, Woodin has shown (see Theorem 10.183 of [37]) that if ψ is any sentence for which Theorem 9.8 holds with ψ in the place of $(*)$, then ZFC + ψ implies in Ω-logic that the Continuum Hypothesis is false.

10. Variations

The $\mathbb{P}_{max}$ method is fairly flexible, and the partial order $\mathbb{P}_{max}$ can be varied in a number of ways. We present here two types of variations. The first is an example of the utility of $\mathbb{P}_{max}$ for manipulating ideals on ω_1. The second illustrates a method for producing extensions which are Π_2-maximal for $H(\omega_2)$ relative to a fixed Σ_2 sentence. Several other variations appear in [37, 43]. Still others appear in [3, 17].

10.1. Variations for NS_{ω_1}

An ideal I on ω_1 is $\aleph_1$-dense if the Boolean algebra $\mathcal{P}(\omega_1)/I$ has a dense subset of cardinality $\aleph_1$. In unpublished work, Woodin showed that starting from a huge cardinal one can force the existence of a normal $\aleph_1$-dense ideal on ω_1. Shelah later showed [27] that, starting from a supercompact cardinal, one can force that the nonstationary ideal restricted to a fixed stationary subset of ω_1 is $\aleph_1$-dense. The $\mathbb{P}_{max}$ variation $\mathbb{Q}^*_{max}$ discussed here, when applied to a model of the form $L(\mathbb{R})$ satisfying AD, produces a model of ZFC in which NS_{ω_1} is $\aleph_1$-dense; by unpublished work of Woodin, this shows that the Axiom of Determinacy and the $\aleph_1$-density of NS_{ω_1} are equiconsistent. To date, $\mathbb{P}_{max}$ variations are the only known means for producing models in which NS_{ω_1} is $\aleph_1$-dense.

Using the result of Shelah mentioned above, the partial order $\mathbb{Q}_{max}$ below can be used to obtain the $\aleph_1$-density of NS_{ω_1} from a supercompact cardinal below ω Woodin cardinals below a measurable. This hypothesis is not optimal, but unlike with $\mathbb{Q}^*_{max}$, we can give all the details here (aside from one argument, we have already done so).

By Theorem 9.1, the $\aleph_1$-density of a σ-ideal on ω_1 is witnessed by a function from ω_1 to $H(\omega_1)$ of the following form.

10.1 Definition. Given a normal $\aleph_1$-dense ideal I on ω_1, $Y_{Coll}(I)$ is the set of functions $f\colon \omega_1 \to H(\omega_1)$ satisfying the following conditions (where for each $p \in \mathrm{Col}(\omega, \omega_1)$ we let $S^f_p = \{\alpha < \omega_1 \mid p \in f(\alpha)\}$):

- for each $\alpha < \omega_1$, $f(\alpha)$ is a filter in $\mathrm{Col}(\omega, 1+\alpha)$
- for each $p \in \mathrm{Col}(\omega, \omega_1)$, $S^f_p \notin I$,
- for each $S \in \mathcal{P}(\omega_1)/I$, there exists a condition $p \in \mathrm{Col}(\omega, \omega_1)$ such that $S^f_p \setminus S \in I$.

10.2 Definition. The partial order $\mathbb{Q}_{max}$ consists of the set of pairs of the form $\langle (M, I), f\rangle$ satisfying the following conditions:

1. M is a countable transitive model of ZFC$^\circ$,
2. I is a normal $\aleph_1$-dense ideal on ω_1^M in M,
3. (M, I) is iterable,
4. $f \in (Y_{Coll}(I))^M$.

The order on $\mathbb{Q}_{max}$ is as follows: $\langle (N, J), g\rangle < \langle (M, I), f\rangle$ if $M \in H(\omega_1)^N$ and there exists an iteration $j\colon (M, I) \to (M^*, I^*)$ such that

- $j(f) = g$,
- $j, M^* \in N$,

- $I^* = M^* \cap J$.

If $\langle (M, I), f\rangle$ is a $\mathbb{Q}_{max}$ condition, then by the normality of I in M, the image of f under any iteration of (M, I) determines the entire iteration.

The only new argument we need to give in the $\mathbb{Q}_{max}$ analysis is the following. The corresponding versions for iterating sequences of models and for building descending ω_1-sequences of conditions are essentially the same.

10.3 Lemma. *Suppose that J is a normal $\aleph_1$-dense ideal on ω_1, and let g be a function in $Y_{Coll}(J)$. Suppose that $\langle (M, I), f\rangle$ is a condition in $\mathbb{Q}_{max}$. Then there is an iteration $j\colon (M, I) \to (M^*, I^*)$ of length ω_1 such that*

- $\{\alpha < \omega_1 \mid j(f)(\alpha) \neq g(\alpha)\} \in J$,
- $I^* = M^* \cap J$.

Proof. Note that the second conclusion follows from the first. Let

$$\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$$

be any iteration of (M, I) satisfying the condition that whenever $\beta < \omega_1$ is such that $j_{0\beta}(\omega_1^M) = \beta$ and $g(\beta)$ is M_β-generic for $\mathrm{Col}(\omega, \beta)$, then G_β is the corresponding filter in $\mathcal{P}(\omega_1)^{M_\beta}/j_{0\beta}(I)$, i.e., for each $p \in \mathrm{Col}(\omega, \beta)$, $S_p^{j_{0\beta}(f)} \in G_\beta$ if and only if $p \in g(\beta)$. It is immediate that G_β is M_β generic, and that the choice of G_β makes $j_{0(\beta+1)}(f)(\beta) = g(\beta)$. It remains to see that the set of $\beta < \omega_1$ such that $g(\beta)$ is not M_β-generic for $\mathrm{Col}(\omega, \beta)$ is in J. To see this, let A be subset of ω_1 coding $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \omega_1\rangle$ under some fixed recursive coding. Then for club many $\eta < \omega_1$, $j_{0\eta}(\omega_1)^M = \eta$ and $\langle M_\alpha, G_\beta, j_{\alpha\delta} : \beta < \alpha \leq \delta \leq \eta\rangle \in L[A \cap \eta]$. Every condition in $\mathcal{P}(\omega_1)/J$ forces that (letting k be the induced elementary embedding) $k(g)(\omega_1^V)$ is a V-generic (and thus $L[A]$-generic) filter in $\mathrm{Col}(\omega, \omega_1^V)$, which means that the set of $\eta < \omega_1$ such that $g(\eta)$ is not $L[A \cap \eta]$-generic is in J. Since $M_\eta \in L[A \cap \eta]$ for club many η, we are done. ⊣

Theorem 4.9 plus the result of Shelah mentioned above gives the following.

10.4 Theorem. *Suppose that there exists a supercompact cardinal below infinitely many Woodin cardinals below a measurable cardnal. Then for every set of reals A in $L(\mathbb{R})$ there exists a $\mathbb{Q}_{max}$ condition $\langle (M, I), f\rangle$ such that*

- $A \cap M \in M$,
- (M, I) *is A-iterable,*
- $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1, A, \in\rangle$.

The proof of the following is essentially the same as for $\mathbb{P}_{max}$. The $\aleph_1$-density of I_G follows immediately from $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$ and the definition of $\mathbb{Q}_{max}$ (letting I_G and $\mathcal{P}(\omega_1)_G$ have the definitions here analogous to those used for $\mathbb{P}_{max}$).

10.5 Theorem. *(*ZF*) Suppose that for every set of reals A there exists a $\mathbb{Q}_{max}$ condition $\langle (M, I), f\rangle$ such that*

- $A \cap M \in M$,
- (M, I) *is A-iterable,*
- $\langle H(\omega_1)^M, A \cap M, \in\rangle \prec \langle H(\omega_1, A, \in\rangle$.

Then $\mathbb{Q}_{max}$ is ω-closed and homogeneous. Furthermore, if G is an V-generic filter for $\mathbb{Q}_{max}$, then the following hold in $V[G]$:

- $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$,
- $NS_{\omega_1} = I_G$,
- ψ_{AC},
- $\underset{\sim}{\delta}^1_2 = \omega_2$,
- NS_{ω_1} *is $\aleph_1$-dense.*

To obtain the $\aleph_1$-density of NS_{ω_1} from the optimal hypothesis, we can use the partial order $\mathbb{Q}^*_{max}$ below. Conditions in $\mathbb{Q}^*_{max}$ are similar to the limit sequences used in the $\mathbb{P}_{max}$ analysis. The utility of this approach here is that the existence of $\mathbb{Q}^*_{max}$ conditions does not require the existence of a model with an $\aleph_1$-dense ideal on ω_1. The analyses of $\mathbb{Q}^*_{max}$ and $\mathbb{Q}_{max}$ are the same, once we show that $\mathbb{Q}^*_{max}$ conditions exist in suitable generality. Showing this, however, is beyond the scope of this chapter.

10.6 Definition. $\mathbb{Q}^*_{max}$ is the set of pairs $(\langle M_k : k < \omega\rangle, f)$ such that the following hold.

1. The set f is a function from $\omega_1^{M_0}$ to M_0 in M_0 such that for all $\alpha < \omega_1^{M_0}$, $f(\alpha)$ is a filter in $\mathrm{Col}(\omega, 1 + \alpha)$.
2. Each $M_k \models \mathrm{ZFC}^\circ$.
3. Each $M_k \in M_{k+1}$.
4. For all $k < \omega$, $\omega_1^{M_k} = \omega_1^{M_0}$.
5. For all $k < \omega$, $NS_{\omega_1}^{M_{k+1}} \cap M_k = NS_{\omega_1}^{M_{k+2}} \cap M_k$.
6. The sequence $\langle M_k : k < \omega\rangle$ is iterable.

7. For each $p \in \mathrm{Col}(\omega, \omega_1^{M_0})$, $\{\alpha < \omega_1^{M_0} \mid p \in f(\alpha)\} \notin NS_{\omega_1}^{M_1}$.

8. For each $k < \omega$ and for each $a \subseteq \omega_1^{M_0}$ such that $a \in M_k \setminus NS_{\omega_1}^{M_{k+1}}$, there exists a $p \in \mathrm{Col}(\omega, \omega_1^{M_0})$ such that

$$\{\alpha < \omega_1^{M_0} \mid p \in f(\alpha)\} \cap (\omega_1^{M_0} \setminus a) \in NS_{\omega_1}^{M_{k+1}}.$$

The ordering on $\mathbb{Q}^*_{max}$ is given by letting

$$(\langle N_k : k < \omega \rangle, g) < (\langle M_k : k < \omega \rangle, f)$$

if $\langle M_k : k < \omega \rangle \in H(\omega_1)^{N_0}$ and there exists an iteration

$$j \colon \langle M_k : k < \omega \rangle \to \langle M_k^* : k < \omega \rangle$$

in N_0 such that

- $j(f) = g$,
- $NS_{\omega_1}^{M^*_{k+1}} \cap M_k^* = NS_{\omega_1}^{N_1} \cap M_k^*$ for all $k < \omega$.

Condition (5) above says that the models in the sequence need not agree about stationary sets, but rather, each subset of $\omega_1^{M_0}$ in each M_k which is stationary in M_{k+1} is stationary in all further M_j's. This extra degree of freedom is essential in constructing $\mathbb{Q}^*_{max}$ conditions without presupposing the existence of $\mathbb{Q}_{max}$ conditions. Conditions (7) and (8) ensure that if

$$G \subset \bigcup \{\mathcal{P}(\omega_1^{M_0})^{M_k} \setminus NS_{\omega_1}^{M_{k+1}} : k < \omega\}$$

is a $\bigcup\{M_k : k < \omega\}$-normal filter, then (letting j be the induced embedding) $j(f)(\omega_1^{M_0})$ is a filter in $\mathrm{Col}(\omega, \omega_1^{M_0})$ meeting every dense set in each M_k, and vice-versa: if g is a filter in $\mathrm{Col}(\omega, \omega_1^{M_0})$ meeting every dense set in each M_k, then there is a $\bigcup\{M_k : k < \omega\}$-normal filter G contained in $\bigcup\{\mathcal{P}(\omega_1^{M_0})^{M_k} \setminus NS_{\omega_1}^{M_{k+1}} : k < \omega\}$ such that $j(f)(\omega_1^{M_0}) = g$.

10.7 Remark. If Γ is a pointclass closed under continuous images such that $L(\Gamma, \mathbb{R}) \models \mathrm{AD}_{\mathbb{R}}$ + "Θ is regular", then the $\mathbb{Q}_{max}$ extension of $L(\Gamma, \mathbb{R})$ satisfies Chang's Conjecture. However, for consistency strength reasons one cannot prove that Chang's Conjecture holds in the $\mathbb{Q}_{max}$ extension of $L(\mathbb{R})$ from the assumption that AD holds in $L(\mathbb{R})$ (see page 651 of [37]).

The utility of the $\mathbb{P}_{max}$ approach for manipulating ideals on ω_1 is applied in other several ways in [37], notably to create a model in which the saturation of NS_{ω_1} can be destroyed without adding a subset of ω_1. In [**?**], a variation of $\mathbb{P}_{max}$ is used to produce a model in which the saturation of NS_{ω_1} can be destroyed by forcing with a Suslin tree. As far as we know, these results have not been reproduced by other methods.

10.2. Conditional variations for Σ_2 sentences

As we saw in Section 7, the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$ (assuming that there exists a proper class of Woodin cardinals) satisfies all forceable Π_2 sentences for $H(\omega_2)$ with parameters for NS_{ω_1} and sets of reals in $L(\mathbb{R})$. In some cases, one can fix a Σ_2 sentence for this structure and produce a model satisfying all Π_2 sentences forceably consistent with it (and in some cases one cannot). If ϕ is a Σ_2 sentence of the form $\exists A \forall B \psi(A, B)$, where all quantifiers in ψ are bounded, the *optimal iteration lemma* for ϕ is the following statement: if

- M is a countable transitive model of ZFC°,
- I is normal ideal on ω_1^M in M,
- (M, I) is iterable,
- $a \in H(\omega_2)^M$ and $H(\omega_2)^M \models \forall b \psi(a, b)$,
- $H(\omega_2) \models \exists A \forall B \psi(A, B)$,
- J is a normal ideal on ω_1,

then there exists an iteration $j\colon (M, I) \to (M^*, I^*)$ of length ω_1 such that

- $I^* = J \cap M^*$,
- $H(\omega_2) \models \forall B \psi(j(a), B)$.

Roughly, the optimal iteration lemma for ϕ says that given a countable transitive iterable model of ϕ and a fixed witness for ϕ in this model, in order to prove that there is an iteration of this model mapping this witness to a witness for ϕ is V, we need assume only that ϕ holds in V. Since this assumption is necessary, in the cases where the lemma holds, it is optimal. In [29], the optimal iteration lemma is proved for the following sentences (the first four of which are defined in [1]; we direct the reader to [29] for the other two).

- The dominating number ($\mathfrak{d}$) is $\aleph_1$.
- The bounding number ($\mathfrak{b}$) is $\aleph_1$.
- The cofinality of the meager ideal is $\aleph_1$.
- The cofinality of the null ideal is $\aleph_1$.
- There exists a coherent Suslin tree.
- There exists a free Suslin tree.

Given a Σ_2 sentence ϕ as above, we can define the $\mathbb{P}_{max}$ variation $\mathbb{P}^{\phi}_{max}$ as follows. Since ϕ may contradict $MA_{\aleph_1}$, we remove the requirement that the models satisfy $MA_{\aleph_1}$ and ensure the uniqueness of iterations directly (alternately, we can usually replace $\mathrm{MA}_{\aleph_1}$ with ψ_{AC}). The partial order $\mathbb{P}^{\phi}_{max}$ is defined recursively on the ω_1 of the selected model M.

10.8 Definition. The partial order $\mathbb{P}^{\phi}_{max}$ consists of all pairs $\langle (M, I), a, X\rangle$ such that

1. M is a countable transitive model of ZFC°,
2. $I \in M$ and in M, I is a normal ideal on ω_1,
3. (M, I) is iterable,
4. $a \in \mathcal{P}(\omega_1)^M$ and $H(\omega_2)^M \models \forall b\psi(a, b)$,
5. $X \in M$ and X is a set (possibly empty) of pairs $(\langle (N, J), b, Y\rangle, j)$ such that
   - $\langle (N, J), b, Y\rangle \in \mathbb{P}^{\phi}_{max} \cap H(\omega_1)^M$,
   - j is an iteration of (N, J) of length ω_1^M such that $j(J) = I \cap j(N)$ and $j(b) = a$,
   - $j(Y) \subseteq X$,

   with the property that for each $p \in \mathbb{P}^{\phi}_{max}$ there is at most one j such that $(p, j) \in X$.

The order on $\mathbb{P}^{\phi}_{max}$ is implicit in the conditions on X:

$$\langle (M, I), a, X\rangle < \langle (N, J), b, Y\rangle$$

if there exists a j such that $(\langle (N, J), b, Y\rangle, j) \in X$.

For ϕ as above, we have games $\mathcal{G}^{\phi}_{\omega}$ and $\mathcal{G}^{\phi}_{\omega_1}$ which are strictly analogous to the games $\mathcal{G}_{\omega}$ and $\mathcal{G}_{\omega_1}$ for $\mathbb{P}_{max}$.

For $\mathcal{G}^{\phi}_{\omega}$, suppose that $\langle (N_i, J_i) : i < \omega\rangle$ is an iterable pre-limit sequence (in the sense of $\mathbb{P}_{max}$) and that there exists an $a \in \mathcal{P}(\omega_1)^{N_0}$ such that $H(\omega_2)^{N_i} \models \forall b\psi(a, b)$ for each $i < \omega$. Then given a normal ideal I on ω_1 and a set $E \subset \omega_1$, we define $\mathcal{G}^{\phi}_{\omega}(\langle (N_i, J_i) : i < \omega\rangle, I, E)$ to be the following game of length ω_1 where Players I and II collaborate to build an iteration of $\langle (N_i, J_i) : i < \omega\rangle$ consisting of pre-limit sequences $\langle (N^{\alpha}_i, J^{\alpha}_i) : i < \omega\rangle$ $(\alpha \leq \omega_1)$, normal ultrafilters G_{α} $(\alpha < \omega_1)$ and a commuting family of embeddings $j_{\alpha\beta}$ $(\alpha \leq \beta \leq \omega_1)$, as follows. In each round α, let

$$Q_{\alpha} = \bigcup\{\mathcal{P}(\omega_1)^{N^{\alpha}_i} \setminus J^{\alpha}_i : i < \omega\}.$$

If $\alpha \in E$, then Player I chooses a set $A \in Q_\alpha$, and then Player II chooses a $\bigcup\{N_i^\alpha : i < \omega\}$-normal filter G_α contained in Q_α with $A \in G_\alpha$. If α is not in E, then Player II chooses any $\bigcup\{N_i^\alpha : i < \omega\}$-normal filter G_α contained in Q_α. After all ω_1 many rounds have been played, Player I wins if $H(\omega_2) \models \forall B\psi(j_{0\omega_1}(a), B)$ and if $J_i^{\omega_1} = I \cap N_i^{\omega_1}$ for each $i < \omega$.

Similarly, given a $\mathbb{P}^\phi_{max}$ condition $p = \langle (M, I), a, X\rangle$, a normal ideal J on ω_1 and a subset of ω_1 E, we let $\mathcal{G}^\phi_{\omega_1}(p, J, E)$ be game of length ω_1 where players I and II collaborate to build a descending ω_1-chain of conditions $p_\alpha = \langle (M_\alpha, I_\alpha), a_\alpha, X_\alpha\rangle$ ($\alpha < \omega_1$) of $\mathbb{P}^\phi_{max}$ conditions below p as follows. In each round α, all p_β ($\beta < \alpha$) have been defined. If α is a successor ordinal (or 0), Player II chooses a condition $p_\alpha < p_{\alpha-1}$ ($< p$). If α is a limit ordinal, then Player I picks a condition p_α below each p_β ($\beta < \alpha$). Then, letting $j_{\alpha\beta}$ ($\alpha < \beta \leq \omega_1$) be the induced commuting family of embeddings (and letting j be the embedding witnessing that $p_0 < p$), Player I wins the game if $H(\omega_2) \models \forall B\psi(j_{0\omega_1}(j(a)), B)$, and if for all $\alpha < \omega_1$, $j_{\alpha\omega_1}(I_\alpha) = J \cap j_{\alpha\omega_1}(M_\alpha)$.

The arguments in [29] show that $\diamondsuit$ implies that Player I has a winning strategy in each game $\mathcal{G}^\phi_\omega(\langle (N_i, J_i) : i < \omega\rangle, I, E)$ and each game $\mathcal{G}^\phi_{\omega_1}(p, J, E)$ for each of the sentences listed before Definition 10.8 (typically these arguments are essentially the same as the proof of the corresponding optimal iteration lemma).

The proof of the following theorem then is a straightforward generalization of the arguments we have given for $\mathbb{P}_{max}$.

10.9 Theorem. *Assume that* AD *holds in* $L(\mathbb{R})$. *Let* ϕ *be an* Ω_{ZFC}*-consistent* Σ_2 *sentence for* $H(\omega_2)$. *Suppose that the optimal iteration lemma for* ϕ *holds, and that the following sentences are* Ω_{ZFC}*-consistent:*

- *for all iterable pre-limit sequences* $\langle (N_i, I_i) : i < \omega\rangle$ *and for all normal ideals* I *on* ω_1*, Player I has a winning strategy in*

$$\mathcal{G}^\phi_\omega(\langle (N_i, J_i) : i < \omega\rangle, I, \omega_1);$$

- *for all* $\mathbb{P}^\phi_{max}$ *conditions* p *and for all normal ideals* J *on* ω_1*, Player I has a winning strategy in* $\mathcal{G}_{\omega_1}(p, J, \omega_1)$.

Let $G \subset \mathbb{P}^\phi_{max}$ *be* $L(\mathbb{R})$*-generic. Then in* $L(\mathbb{R})[G]$ *the following hold:*

- ϕ,
- $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$,
- $NS_{\omega_1} = I_G$,
- NS_{ω_1} *is saturated.*

Furthermore, for every set of reals A in $L(\mathbb{R})$, $L(\mathbb{R})[G]$ satisfies every Π_2*-sentence for the structure* $\langle H(\omega_2), NS_{\omega_1}, A, \in \rangle$ *which is* Ω_{ZFC}*-consistent with* ϕ.

The variation $\mathbb{P}^{\phi}_{max}$ where ϕ asserts the existence of a coherent Suslin tree is studied in [14, 18].

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