

## Small-sum pairs in abelian groups

par REZA AKHTAR et PAUL LARSON

RÉSUMÉ. Soit  $G$  un groupe abélien fini et  $A, B$  sous-ensembles de  $G$  tels que  $|A| = |B| = k$  et  $|A + A| = |A + B| = 2k - 1$ . Pour sous-ensembles  $X, Y$  de  $G$  et  $c \in G$ , signifions par  $\nu_c(X, Y)$  le nombre de couples  $(x, y) \in X \times Y$  tels que  $c = x + y$ . Nous résolvons une question de Bihani et Jin en montrant qu’il existe  $g \in G$  tel que  $A = g + B$  si  $A + B$  est aperiodique ou s’il existe  $a \in A$  et  $b \in B$  tels que  $\nu_{a+b}(A, B) = \nu_{a+a}(A, A) = 1$ . Nous caractérisons aussi les contre-exemples qui se présentent si ni l’une ni l’autre des ces hypothèses est satisfaite.

ABSTRACT. Let  $G$  be an abelian group and  $A, B$  two subsets of equal size  $k$  such that  $A + B$  and  $A + A$  both have size  $2k - 1$ . Answering a question of Bihani and Jin, we prove that if  $A + B$  is aperiodic or if there exist elements  $a \in A$  and  $b \in B$  such that  $a + b$  has a unique expression as an element of  $A + B$  and  $a + a$  has a unique expression as an element of  $A + A$ , then  $A$  is a translate of  $B$ . We also give an explicit description of the various counterexamples which arise when neither condition holds.

### 1. Introduction

Let  $G$  be an abelian group, written additively. If  $A$  and  $B$  are subsets of  $G$ , we write  $A + B$  to mean  $\{a + b : a \in A, b \in B\}$ , and similarly for  $A - B$ . We use  $A \setminus B$  to denote set difference. In their study [1] of sets of natural numbers with small upper Banach density, Bihani and Jin asked the following in the context of cyclic groups:

**Question 1.** Let  $G$  be an abelian group. Given a pair  $(A, B)$  of subsets of  $G$  such that  $|A| = |B| = k$  and  $|A + A| = |A + B| = 2k - 1$ , are  $A$  and  $B$  always a translates of each other; that is, does there exist  $h \in G$  such that  $B = A + h$ ?

It is easily seen that this question cannot always be answered in the affirmative. To see this, let  $G$  be any abelian group of odd order  $2k - 1 \geq 5$  and  $A, B$  any two subsets of  $G$  of order  $k$ . Since  $|A| + |B| > |G|$ , a rudimentary combinatorial result (cf. [7, Lemma 2.1] or Lemma 2.6 below) implies that  $A + B = G$ , regardless of whether or not  $A$  and  $B$  are translates of each other. It is far less obvious that all counterexamples are, in a suitable sense, based on this example.

The goal of this paper is to answer Question 1 in the affirmative under appropriate hypotheses and to provide explicit descriptions of counterexamples. We note that all our work is valid for arbitrary abelian groups  $G$ .

The key tool we use is the theory of *critical pairs*. A critical pair in an abelian group  $H$  is a pair  $(A, B)$  of subsets such that  $|A+B| = |A|+|B|-1$ ; an essentially complete description of all such pairs was given by Vosper [6] for groups of prime order and by Kemperman [5] (see also [2], [3], [4]) for general abelian groups. We show how to use Kemperman's results and Kneser's Theorem to give an essentially complete answer to Question 1. We will refer to the exposition of Kemperman's work in the paper of Gryniewicz [2], as the constructions used there suit our purposes better than that of the original work [5]. For a detailed discussion of Kneser's Theorem, we refer the reader to Chapter 4 of [7].

## 2. Preliminaries

Let  $H$  be an abelian group. A *small-sum pair*, or *SS-pair* for short, is a pair  $(X, Y)$  of subsets of  $H$  such that  $|X| = |Y| = k$  and  $|X+X| = |X+Y| = 2k-1$  for some  $k$ .

Given an abelian group  $H$ , subsets  $A, B \subseteq H$ , and  $c \in H$ , we define

$$\nu_c(A, B) = |\{(a, b) : a \in A, b \in B : a + b = c\}|.$$

We say that the pair  $(A, B)$  has the *unique expression property* (UEP) if there exist  $a \in A$  and  $b \in B$  such that  $\nu_{a+b}(A, B) = 1$  and that  $(A, B)$  has the *strong unique expression property* (SUEP) if there exist  $a \in A$  and  $b \in B$  such that  $\nu_{a+b}(A, B) = \nu_{a+a}(A, A) = 1$ .

If  $P \subseteq H$  is a subgroup, we denote by  $\phi_P : H \rightarrow H/P$  the canonical quotient map.

The *stabilizer* of a subset  $A \subseteq H$  is the subgroup of  $H$  defined by:

$$\text{Stab}(A) = \{h \in H : h + A = A\}$$

A subset  $A \subseteq H$  is called *periodic* if  $\text{Stab}(A) \neq \{0\}$ . For a nontrivial subgroup  $P$ , we say that  $A$  is  *$P$ -periodic* if  $A$  is a union of  $P$ -cosets; this is equivalent to the condition  $P \subseteq \text{Stab}(A)$ . We say that  $A \subseteq H$  is  *$P$ -subperiodic* if there exists  $h \in H - A$  such that  $A \cup \{h\}$  is  $P$ -periodic.

**Definition 2.1.** *Let  $H$  be an abelian group and  $P \subseteq H$  a nontrivial subgroup. A subset  $A \subseteq H$  is said to have a quasi-periodic decomposition with respect to the quasi-period  $P$  if there exists a partition of  $A$  into two disjoint subsets  $A_1 \cup A_0$  such that  $A_1$  is either empty or  $P$ -periodic and  $A_0$  is a subset of a  $P$ -coset.*

We refer to  $A_1$  as the periodic part of  $A$  and to  $A_0$  as the aperiodic part of  $A$ . We also say that  $A$  is quasi-periodic if  $A_1$  is nonempty. This notation (with the above meaning) will be used for quasi-periodic decompositions for the remainder of this article.

**Lemma 2.2.** *Suppose  $H$  is finite,  $P$  is a subgroup of  $H$  and  $A \subseteq H$  is any nonempty subset. If  $A = A_1 \cup A_0$  is a quasi-periodic decomposition with respect to  $P$  and  $|A_0| < |P|$ , such a decomposition is unique.*

*Proof.* With notation as above, suppose  $A$  has quasi-periodic decompositions  $A = A_1 \cup A_0$  and  $A = A'_1 \cup A'_0$  with respect to a common quasi-period  $P$ . Then  $A_1, A'_1$  are unions of  $P$ -cosets, and each of  $A_0, A'_0$  is a proper subset of a single  $P$ -coset. It follows that  $A_0 = A'_0$  and hence  $A_1 = A'_1$ .  $\square$

**Lemma 2.3.** *Suppose  $H$  is finite and  $A, B$  are subsets of  $H$ , and  $A = A_1 \cup A_0$  and  $B = B_1 \cup B_0$  are quasi-periodic decompositions with respect to a common quasi-period  $P$ . Suppose further that neither  $A$  nor  $B$  is  $P$ -periodic. If  $A = h + B$ , then  $A_0 = h + B_0$  and  $A_1 = h + B_1$ .*

*Proof.* If  $A = h + B$ , then  $(h + B_1) \cup (h + B_0)$  is also a quasi-periodic decomposition for  $A$ . By Lemma 2.2, it follows that  $A_1 = h + B_1$  and  $A_0 = h + B_0$ .  $\square$

The following results are elementary:

**Lemma 2.4.** *Suppose  $H$  is an abelian group and  $A \subseteq H$  is a nonempty subset such that  $|A + A| = 2|A| - 1$ . Then  $A$  is not periodic.*

*Proof.* Let  $k = |A|$ . If  $A$  is periodic, then there is a nontrivial subgroup  $P \subseteq H$  such that  $A$  is a disjoint union of  $P$ -cosets; in particular,  $|P|$  divides  $k$ . However,  $A + A$  is also a union of  $P$ -cosets, so  $|P|$  must divide  $2k - 1$ , which forces  $P$  to be the trivial subgroup.  $\square$

**Lemma 2.5.** *Let  $H$  be an abelian group. If  $A \subseteq H$  is  $P$ -periodic, then so is  $A + B$  for any  $B \subseteq H$ .*

Finally, we mention a rudimentary but well-known result:

**Lemma 2.6.** [7, Lemma 2.1] *Let  $H$  be a finite abelian group and  $A, B$  subsets of  $H$  such that  $|A| + |B| > |H|$ . Then  $A + B = H$ .*

### 3. Kemperman pairs

In this section, we review the theory of critical pairs in abelian groups, following the exposition of Kemperman's work in [2], and prove various results that will help in the solution of our problem.

**Definition 3.1.** *Let  $H$  be an abelian group. A Kemperman pair in  $H$  is a pair  $(A, B)$  of subsets of  $H$  such that  $A + B$  is not periodic or  $(A, B)$  has UEP.*

We now state Kemperman's Theorem [5, Theorem 5.1 and p. 82] in the form given by Grynkiewicz [2].

**Theorem 3.2.** ([2], p. 563) *Let  $G$  be an abelian group and  $(A, B)$  a Kemperman pair in  $G$ . Then  $|A + B| = |A| + |B| - 1$  if and only if there exist quasi-periodic decompositions  $A = A_1 \cup A_0$  and  $B = B_1 \cup B_0$  with nonempty aperiodic parts and common quasi-period  $P$  such that  $(A_0, B_0)$  is a Kemperman pair and:*

- (1)  $\nu_c(\phi_P(A), \phi_P(B)) = 1$ , where  $c = \phi_P(A_0) + \phi_P(B_0)$ ;
- (2)  $|\phi_P(A) + \phi_P(B)| = |\phi_P(A)| + |\phi_P(B)| - 1$ , and
- (3)  $|A_0 + B_0| = |A_0| + |B_0| - 1$  and the pair  $(A_0, B_0)$  is of one of the following (distinct) types:
  - (a)  $|A_0| = 1$  or  $|B_0| = 1$ .
  - (b)  $|A_0| \geq 2$ ,  $|B_0| \geq 2$ , and  $A_0, B_0$  are arithmetic progressions with common difference  $d \in H$  such that  $d$  has order at least  $|A_0| + |B_0| - 1$ . In this case,  $\nu_c(A_0, B_0) = 1$  for exactly two values  $c \in A_0 + B_0$ .
  - (c)  $|A_0| + |B_0| = |P| + 1$ , and there is exactly one element  $g \in G$  such that  $\nu_g(A_0, B_0) = 1$ .
  - (d)  $A_0$  is aperiodic,  $B_0$  is of the form  $B_0 = g - [(G \setminus A_0) \cap (g' + P)]$  for some  $g, g' \in A_0$ , and  $\nu_c(A_0, B_0) \neq 1$  for all  $c$ .

**Remark.**

We note (cf. [2], c.13 on p. 564) that if we select a different quasi-periodic decomposition for  $A$  or  $B$ , the type of the resulting pair of aperiodic parts does not change; hence we may speak of the type of  $(A, B)$  without reference to any choice of quasi-periodic decompositions.

If  $A = A_1 \cup A_0$  and  $B = B_1 \cup B_0$  are quasi-periodic decompositions of sets with respect to a common quasi-period  $P$ , we denote by  $(A + B)_P$  the  $P$ -periodic set  $(A_1 + B) \cup (A + B_1)$ . Clearly  $A + B = (A + B)_P \cup (A_0 + B_0)$ .

**Lemma 3.3.** *Suppose  $(A, B)$  is a Kemperman SS-pair. With notation as in Theorem 3.2:*

- (1)  $A + B$  is the disjoint union of  $A_0 + B_0$  and  $(A + B)_P$ . In particular,  $A + B$  is  $P$ -periodic if and only if  $A_0 + B_0$  is  $P$ -periodic.
- (2) If  $(A, B)$  has type (a) or (d), then  $A + B$  is not  $P$ -periodic. If  $(A, B)$  has type (c), then  $A + B$  is  $P$ -periodic.

*Proof.* For the first item, suppose  $A_0 \subseteq g_1 + P$  and  $B_0 \subseteq g_2 + P$  and let  $h_1 = \phi_P(g_1)$ ,  $h_2 = \phi_P(g_2)$ . By Theorem 3.2,  $\nu_{h_1+h_2}(\phi_P(A), \phi_P(B)) = 1$ .

Now if  $x \in (A + B)_P \cap (A_0 + B_0)$ , then  $x = a + b$ , where  $a \in A$ ,  $b \in B$  and  $a \in A_1$  or  $b \in B_1$ . Thus,  $\phi(x) = h_1 + h_2 = \phi(a) + \phi(b)$ , contradicting  $\nu_{h_1+h_2}(\phi_P(A), \phi_P(B)) = 1$ . The second assertion now follows immediately. If  $(A, B)$  has type (a), then  $|A_0| = |B_0| = 1$ , so  $|A_0 + B_0| = 1$ . Since  $P$  is nontrivial,  $A_0 + B_0$  is a strict subset of a  $P$ -coset and so  $A_0 + B_0$  (and hence also  $A + B$ ) is not  $P$ -periodic. If  $(A, B)$  has type (d), then, as noted in [2, p. 563],  $A_0 + B_0$  is  $P$ -subperiodic, so again  $A + B$  is not  $P$ -periodic. Finally, if  $(A, B)$  has type (c), then  $|A_0 + B_0| = |A_0| + |B_0| - 1 = |P|$ , so  $A_0 + B_0$  is a (full)  $P$ -coset; thus,  $A + B$  is  $P$ -periodic.  $\square$

The proof of the following is immediate:

**Lemma 3.4.** *Suppose  $A, B$  are subsets of an abelian group  $H$  and that  $A = A_1 \cup A_0$ ,  $B = B_1 \cup B_0$  are quasi-periodic decompositions with respect to some common quasi-period  $P$ . If  $A + B$  is  $P$ -periodic, then  $|A + B| = |\phi_P(A + B)||P|$ ; otherwise,  $|A + B| = (|\phi_P(A + B)| - 1)|P| + |A_0 + B_0|$ .*

**Lemma 3.5.** *Suppose  $(A, B)$  is a Kemperman  $SS$ -pair of subsets of an abelian group  $H$  with corresponding quasi-periodic decompositions  $A = A_0 \cup A_1$  and  $B = B_0 \cup B_1$  with respect to some common quasi-period  $P$ . Then  $|A_0| = |B_0|$ ,  $|\phi_P(A)| = |\phi_P(B)|$ ,  $|A_0 + B_0| = |A_0| + |B_0| - 1$ ,  $|\phi_P(A + B)| = |\phi_P(A)| + |\phi_P(B)| - 1$ ,  $|A_0 + A_0| = 2|A_0| - 1$ , and  $|\phi_P(A + A)| = 2|\phi_P(A)| - 1$ .*

*Proof.* Let  $p = |P|$ , and define  $x$  and  $y$  by  $|A_1| = xp$  and  $|B_1| = yp$ . We have  $|A| = xp + |A_0|$ ,  $|B| = yp + |B_0|$ ; since  $A_0$  and  $B_0$  are nonempty, we have  $1 \leq |A_0| \leq p$ ,  $1 \leq |B_0| \leq p$ . By properties of integer division,  $|A_0| = |B_0| = k$  and  $x = y$ , so  $|\phi_P(A)| = |\phi_P(A_1)| + 1 = x + 1 = y + 1 = |\phi_P(B_1)| + 1 = |\phi_P(B)|$ ; this establishes the first two formulas. The next two formulas follow from Theorem 3.2. If  $A + B$  is not  $P$ -periodic, then  $|A + A| = |A + B| \not\equiv 0 \pmod{|P|}$  so  $A + A$  is not  $P$ -periodic, either. It follows that  $A_0 + A_0$  is a strict subset of a single  $P$ -coset, and also that  $|A_0 + A_0| = |A_0 + B_0| = |A_0| + |B_0| - 1 = 2|A_0| - 1$ . If  $A + B$  is  $P$ -periodic, then  $|A + A| = |A + B| \equiv 0 \pmod{|P|}$ , so  $A + A$  is likewise  $P$ -periodic. Letting  $c = |A + A|/p = |\phi_P(A + A)|$ , we have  $cp = |A + A| = 2|A| - 1 = 2(xp + k) - 1$ . Thus,  $p(c - 2x) = 2k - 1$ . Since  $1 \leq k \leq p$ , it follows that  $2k - 1 = p$ , so  $c = 2x + 1$ ; that is,  $|\phi_P(A + A)| = 2|A_1|/p + 1 = 2(|\phi_P(A)| - 1) + 1 = 2|\phi_P(A)| - 1$ , as desired.  $\square$

**Proposition 3.6.** *Suppose  $(A, B)$  is a Kemperman  $SS$ -pair with quasi-periodic decompositions  $A = A_1 \cup A_0$  and  $B = B_1 \cup B_0$  with respect to a common quasi-period  $P$ .*

- *If  $(A, B)$  has type (a) or (b), or (d) then  $A_0$  is a translate of  $B_0$ .*
- *If  $(A, B)$  has type (c), then  $A + B$  is  $P$ -periodic,  $|P| \geq 3$  is odd, and  $|A_0| = |B_0| \geq 2$ . Furthermore, if  $(A, B)$  has SUEP, then there exist  $h_1, h_2 \in H$  and a set  $S \subseteq P$  such that  $A_0 = h_1 + S$ ,  $B_0 = h_2 + S$ , and  $h_1 + h_2 \notin (A_1 + B_1) \cup (A_1 + A_1)$ ; in particular,  $A_0$  is a translate of  $B_0$ .*

*Proof.* Since all properties mentioned in the proposition remain invariant when either  $A$  or  $B$  is replaced by a translate, we assume throughout this proof that  $A_0$  and  $B_0$  are subsets of  $P$ .

If  $(A, B)$  is of type (a) or (b), then clearly  $A_0$  is a translate of  $B_0$ . If  $(A, B)$  has type (c),  $A + B$  is  $P$ -periodic by Lemma 3.3. Moreover, because  $|A_0| = |B_0|$ , it follows that  $|P| = 2|A_0| - 1$  is odd. In particular, since  $P$  is nontrivial,  $|P| \geq 3$ , and so  $|A_0| = |B_0| \geq 2$ . Now suppose  $(A, B)$  has SUEP. By translating each appropriately, we may assume that  $\nu_0(A, B) = \nu_0(A, A) = 1$ . Then  $B_0 \cap -A_0 = A_0 \cap -A_0 = \{0\}$ ; hence  $A_0 = B_0 \subseteq P$ .

Finally, suppose  $(A, B)$  has type (d). As remarked in [2, p. 563], a pair of type (d) satisfies  $A_0 + B_0 = (h_0 + P) - \{h_0\}$  for some  $h_0 \in H$ , so  $|A + B| \equiv -1 \pmod{|P|}$ . In particular, letting  $k = |A_0|$ , we have  $|P| = |A_0 + B_0| + 1 = |A_0| + |B_0| = 2k$ . Since  $(A, B)$  is an  $SS$ -pair,  $A_0 + A_0$  is a subset of  $P$  of order  $2k - 1$ . Hence there exist  $x, y \in P$  such that  $A_0 + A_0 = P \setminus \{x\}$  and  $A_0 + B_0 = P \setminus \{y\}$ ; setting  $B'_0 = B_0 + x - y$ , we have  $A_0 + B'_0 = P \setminus \{x\}$ . Thus,  $A_0$  and  $B'_0$  are subsets of  $P$  of order  $k$ , each disjoint from  $-A_0 + x$ . This forces  $A_0 = B'_0$ , and hence  $A_0$  is a translate of  $B_0$ .  $\square$

**Lemma 3.7.** *Suppose  $(A, B)$  is a Kemperman  $SS$ -pair in  $H$  together with quasi-periodic decompositions  $A = A_1 \cup A_0$  and  $B = B_1 \cup B_0$  with respect to a common quasi-period  $P$ . If  $A + B$  is aperiodic or  $(A, B)$  has SUEP, then  $(\phi_P(A), \phi_P(B))$  has SUEP.*

*Proof.* For convenience, let  $C = \phi_P(A)$  and  $D = \phi_P(B)$ . By translating  $A$  and  $B$  appropriately, we may assume that  $A_0 \subseteq P$ ,  $B_0 \subseteq P$ , and hence that  $\phi_P(A_0) = \phi_P(B_0) = 0 \in C \cap D$ . Suppose  $(C, D)$  does not have SUEP. Since  $\nu_0(C, D) = 1$  by statement 1 of Theorem 3.2, it follows that  $\nu_0(C, C) > 1$ ; thus, we may select  $0 \neq c \in C$  such that  $-c \in C$ . Writing  $c = \phi_P(a)$  for some  $a \in A_1$ , observe that since  $\phi(-a) = -c \neq 0 \in C$ ,  $-a + P \subseteq A_1$ , so  $-a \in A_1$  also. Next, choose  $p \in P$ ,  $p \neq 0$ . Since  $A_1$  is  $P$ -periodic,  $a - p, -a + p \in A_1$ , and so  $(a - p) + (-a + p) = 0$  witnesses that  $\nu_0(A, A) \geq 2$ ; hence  $(A, B)$  does not have SUEP. Moreover,  $A + A$  is the union of a  $P$ -periodic set with  $A_0 + A_0$ . Since the latter is a subset of  $P$  and  $(a + P) + (-a + P) = P \subseteq A + A$ , it follows that  $A + A$  is  $P$ -periodic. Since  $|A + A| = |A + B|$  and  $A, B$  both have quasi-periodic decompositions with respect to  $P$ ,  $A + B$  is also  $P$ -periodic.  $\square$

Finally, we need the following ‘‘reconstruction lemma’’:

**Lemma 3.8.** *Suppose  $(A, B)$  is a critical Kemperman pair in an abelian group  $H$  with  $|A| = |B|$  and that  $A = A_0 \cup A_1$ ,  $B = B_0 \cup B_1$  are quasi-periodic decompositions with respect to a common quasi-period  $P$ . If  $A_0$  is*

a translate of  $B_0$  and  $\phi_P(A)$  is a translate of  $\phi_P(B)$  then  $A$  is a translate of  $B$ .

*Proof.* Note that the conclusion of Lemma 3.8 remains valid if we translate either  $A$  or  $B$  by any  $h \in H$ . Moreover, the assertion is automatically true if  $A_1$  and  $B_1$  are empty, so we assume henceforth that these are nonempty. Similarly, if  $A_0$  (and hence also  $B_0$ ) is a full  $P$ -coset, the assertion is clear, so we assume otherwise. If  $\phi_P(B) = \phi_P(A) + \bar{h}$  for some  $\bar{h} \in H/P$ , then picking  $h \in G$  such that  $\phi_P(h) = \bar{h}$ , we may replace  $B$  by  $B - h$  and assume henceforth that  $\phi_P(B) = \phi_P(A)$ . In particular, this means that there exist distinct cosets  $h_1 + P, \dots, h_s + P$  of  $P$  in  $G$  and decompositions  $A = \cup_{i=1}^s C_i$  and  $B = \cup_{i=1}^s D_i$  where for every  $i = 1, \dots, s$ ,  $C_i \subseteq h_i + P$  and  $D_i \subseteq h_i + P$  with  $C_i = h_i + P$  if and only if  $i \neq i_0$  and  $D_j = h_j + P$  if and only if  $j \neq j_0$ . With this notation,  $C_{i_0} = A_0$  and  $D_{j_0} = B_0$ . We claim that  $i_0 = j_0$ .

Note that  $A + B$  is the union of a  $P$ -periodic set, together with  $A_0 + B_0 \subseteq (h_{i_0} + h_{j_0}) + P$ . If  $i_0 \neq j_0$ , then  $A_1 + B_1 \supseteq C_{j_0} + D_{i_0} = (h_{i_0} + h_{j_0}) + P \supseteq A_0 + B_0$ , contradicting Lemma 3.3. Thus  $i_0 = j_0$ , and so  $A_1 = B_1$ . Since  $A_0 = p + B_0$  for some  $p \in P$ , we have that  $A = p + B$ , as desired.  $\square$

## 4. Results

**4.1. The Kemperman case.** In this section, we answer our question in the case that our original  $SS$ -pair  $(A, B)$  is a Kemperman pair.

**Theorem 4.1.** *Let  $G$  be an abelian group and  $(A, B)$  an  $SS$ -pair in  $G$ . If  $A + B$  is aperiodic or  $(A, B)$  has  $SUEP$ , then  $A = x + B$  for some  $x \in G$ .*

*Proof.* We prove the theorem by induction on  $|A| + |B|$ , the theorem being trivially true when  $|A| = |B| = 1$ . Since the hypothesis clearly implies that  $(A, B)$  is a Kemperman pair, fix quasi-periodic decompositions  $A = A_1 \cup A_0$  and  $B = B_1 \cup B_0$  as in Theorem 3.2 with respect to some common quasi-period  $P$ . By Proposition 3.6,  $A_0$  is a translate of  $B_0$ . If either of  $A_1, B_1$  is empty, the other is, too, so  $A = A_0, B = B_0$  and the claim is proven. Now suppose  $A_1$  and  $B_1$  are nonempty. Then  $(\phi_P(A), \phi_P(B))$  is an  $SS$ -pair by Lemma 3.5, so Lemma 3.7 implies that the hypotheses of the theorem are satisfied by  $(\phi_P(A), \phi_P(B))$ . By induction,  $\phi_P(A)$  is a translate of  $\phi_P(B)$ . Finally, Lemma 3.8 implies that  $A$  is a translate of  $B$ .  $\square$

**Corollary 4.2.** *Suppose  $G$  is a 2-group and  $(A, B)$  is an  $SS$ -pair of subsets of  $G$ . Then  $A$  is a translate of  $B$ .*

*Proof.* Let  $k = |A| = |B|$ . First, note that  $A + B$  cannot be periodic, because  $|A + B| = 2k - 1$ , which is odd, so it cannot be the union of cosets

of a nontrivial subgroup of  $G$ . Hence  $(A, B)$  is a Kemperman pair, and so  $A$  is translate of  $B$  by Theorem 4.1.  $\square$

**4.2. The periodic case.** In this section, we study the case in which  $(A, B)$  is an  $SS$ -pair with  $A + B$  periodic. If  $(A, B)$  is not a Kemperman pair, we cannot use Kemperman's Theorem, so we use Kneser's Theorem instead. We note that the results of this section also apply to the situation in which  $(A, B)$  has SUEP.

**Theorem 4.3.** (*Kneser, [7, Theorem 4.2]*)

Let  $G$  be an abelian group and  $A, B$  finite nonempty subsets of  $G$ ; set  $H = \text{Stab}(A + B)$ . If  $|A + B| < |A| + |B|$ , then

$$|A + B| = |A + H| + |B + H| - |H|$$

In particular, if  $|A + B| < |A| + |B| - 1$ , then  $H$  is nontrivial and so  $A + B$  is periodic.

One easily constructs examples of  $SS$ -pairs  $(A, B)$  such that  $\nu_c(A, B) > 1$  for all  $c \in A + B$ . We describe how, starting with one such pair, one may construct  $SS$ -pairs  $(A, B')$  where  $B'$  is not a translate of  $A$ .

**Lemma 4.4.** Let  $G$  be an abelian group,  $(A, B)$  an  $SS$ -pair in  $G$ , and  $P = \text{Stab}(A + B)$ . Let  $c_1 + P, \dots, c_k + P$  be distinct cosets whose union is  $A + P$  and  $d_1 + P, \dots, d_l + P$  distinct cosets whose union is  $B + P$ . Then:

- $|P|$  is odd.
- $|A + P| - |A| = |B + P| - |B| = \frac{|P| - 1}{2}$ . In particular,

$$|(c_i + P) \cap A| \geq \frac{|P| + 1}{2} \text{ and } |(d_j + P) \cap B| \geq \frac{|P| + 1}{2}$$

for all  $i = 1, \dots, k$  and  $j = 1, \dots, l$ .

- Let  $(A', B')$  be any pair such that  $A' \subseteq \cup_{i=1}^k (c'_i + P)$ ,  $B' \subseteq \cup_{j=1}^l (d'_j + P)$ , where the  $c'_i$  and  $d'_j$  are elements of  $G$  such that  $|(c'_i + P) \cap A'| \geq \frac{|P| + 1}{2}$  and  $|(d'_j + P) \cap B'| \geq \frac{|P| + 1}{2}$  for all  $i = 1, \dots, k$  and  $j = 1, \dots, l$ . Then  $A' + B' = \cup_{i=1}^k \cup_{j=1}^l ((c'_i + d'_j) + P)$ .

*Proof.* For convenience of notation, let  $C_i = c_i + P$ ,  $D_j = d_j + P$  for  $i = 1, \dots, k$ ,  $j = 1, \dots, l$ ; hence  $A + B = \cup_{i=1}^k \cup_{j=1}^l C_i + D_j$ . Because  $|A| = |B|$  and  $(A, B)$  is critical,  $|A + B| = |A| + |B| - 1 = 2|A| - 1$  is odd. Since  $A + B$  is a disjoint union of  $P$ -cosets,  $|P|$  must be odd. Now Kneser's Theorem gives  $|A| + |B| - 1 = |A + B| = |A + P| + |B + P| - |P|$ , or  $(|A + P| - |A|) + (|B + P| - |B|) = |P| - 1$ . Hence  $|A + P|$  (respectively,  $|B + P|$ ) is the smallest multiple of  $|P|$  greater than or equal to  $|A|$  (respectively



$|B|$ ). However, since  $|A| = |B|$ , we must have  $|A + P| = |B + P|$ , so  $2(|A + P| - |A|) = |P| - 1$ . Reduction modulo  $|P|$  gives  $2|A| \equiv 1 \pmod{|P|}$ , so since  $|P|$  is odd,  $|A| = |B| \equiv \frac{|P| + 1}{2} \pmod{|P|}$ . Thus,

$$|A + P| - |A| = |B + P| - |B| = \frac{|P| - 1}{2}$$

and

$$|C_i \cap A| = |C_i| - |C_i \setminus A| \geq |C_i| - |(A + P) \setminus A| = |P| - \frac{|P| - 1}{2} = \frac{|P| + 1}{2}.$$

A similar argument shows that  $|D_j \cap B| \geq \frac{|P| + 1}{2}$ , establishing the first statement. For the second statement, note that for every  $i$  and  $j$ ,

$$((c'_i + P) \cap A') + ((d'_j + P) \cap B') \subseteq (c'_i + d'_j) + P.$$

Moreover,

$$|(c'_i + P) \cap A'| + |(d'_j + P) \cap B'| \geq |P| + 1 > |P|$$

so by Lemma 2.6,  $((c'_i + P) \cap A') + ((d'_j + P) \cap B') = (c'_i + d'_j) + P$ . Thus,  $A' + B' = \cup_{i=1}^k \cup_{j=1}^l (c'_i + d'_j) + P$ .  $\square$

**Proposition 4.5.** *If  $G$  is an abelian group and  $(A, B)$  a critical pair in  $G$  with  $P = \text{Stab}(A + B)$ , then  $\phi_P(A + B)$  is aperiodic and  $(\phi_P(A), \phi_P(B))$  is a critical Kemperman pair of subsets of  $G/P$ . In particular, if  $(A, B)$  is an  $SS$ -pair in  $G$ , then  $(\phi_P(A), \phi_P(B))$  is a Kemperman  $SS$ -pair in  $G/P$  and  $\phi_P(A)$  is a translate of  $\phi_P(B)$ .*

*Proof.* Aperiodicity is clear in view of the definition of stabilizer. Let  $A' = A + P$  and  $B' = B + P$ ; then  $\phi_P(A) = \phi_P(A')$ ,  $\phi_P(B) = \phi_P(B')$  and  $A' + B' = (A + B) + P = A + B$ . By Kneser's Theorem,  $|A + B| = |A' + B'| = |A'| + |B'| - |P|$ . Dividing all terms by  $|P|$ , we have  $|\phi_P(A + B)| = |\phi_P(A)| + |\phi_P(B)| - 1$ , and so  $(\phi_P(A), \phi_P(B))$  is a critical Kemperman pair. This proves the first part of the Proposition.

Now assume that  $(A, B)$  is an  $SS$ -pair. We will prove that  $(\phi_P(A), \phi_P(B))$  is an  $SS$ -pair. Write  $A' = \cup_{i=1}^k c_i + P$  and  $B' = \cup_{j=1}^l d_j + P$  (as disjoint unions). By the second statement of Lemma 4.4,  $|A \cap (c_i + P)| \geq \frac{|P| + 1}{2}$ ,  $|B \cap (d_j + P)| \geq \frac{|P| + 1}{2}$  for each  $i$  and  $j$ , so by the third statement,  $A + A$  is  $P$ -periodic; that is,  $P \subseteq \text{Stab}(A + A)$ . Furthermore,  $A' + A' = (A + P) + (A + P) = (A + A) + P = A + A$ . Now Kneser's Theorem implies  $2|A| - 1 = |A + A| = 2|A'| - |\text{Stab}(A + A)|$ , or  $|A'| - |A| = \frac{|\text{Stab}(A + A)| - 1}{2}$ .

Because  $|A'| - |A| = \frac{|P| - 1}{2}$  by Lemma 4.4, it follows that  $P = \text{Stab}(A+A)$ . Finally, from  $|A' + A'| = |A + A| = 2|A'| - |P|$  we get  $|\phi_P(A) + \phi_P(A)| = |\phi_P(A + A)| = 2|\phi_P(A)| - 1$ . This establishes that  $(\phi_P(A), \phi_P(B))$  is an  $SS$ -pair; hence  $\phi_P(A)$  is a translate of  $\phi_P(B)$  by Theorem 4.1.  $\square$

Since Question 1 obviously has an affirmative answer if  $A$  and  $B$  have size 1 or 2, we assume henceforth without further mention that  $|A| \geq 3$ ,  $|B| \geq 3$ . The next result shows that in general there are “many”  $SS$ -pairs  $(A, B)$  such that  $A + B$  is periodic but  $A$  is *not* a translate of  $B$ . Since Proposition 4.5 implies that  $\phi_P(A)$  is always a translate of  $\phi_P(B)$ , we assume  $\phi_P(A) = \phi_P(B)$  in the following to simplify the discussion.

**Theorem 4.6.** *Suppose  $G$  is an abelian group and  $A \subseteq G$  is a subset such that  $|A + A| = 2|A| - 1$  and  $P = \text{Stab}(A + A) \neq \{0\}$ . Write  $A = \cup_{i=1}^l S_i$ , where each  $S_i$  is a subset of (the  $P$ -coset)  $c_i + P$ . Consider the collection  $\mathcal{B}$  of all subsets  $B \subseteq G$  such that  $|A| = |B| = k$ ,  $\phi_P(A) = \phi_P(B)$ , and  $|B \cap (c_i + P)| \geq \frac{|P| + 1}{2}$  for all  $i = 1, \dots, l$ . Then each  $(A, B)$  is an  $SS$ -pair in  $G$ . Moreover, at most  $|P|$  elements of  $\mathcal{B}$  are translates of  $A$ , and  $|\mathcal{B}| > |P|$ , so there is at least one  $B \in \mathcal{B}$  which is not a translate of  $A$ .*

*Proof.* Since  $\phi_P(A) = \phi_P(B)$ , any  $B \in \mathcal{B}$  must satisfy  $B \subseteq A + P$ . Observe that by Lemma 2.4,  $A$  must be a strict subset of  $A + P$ . Clearly  $P \subseteq \text{Stab}(A + P) = P'$ . Moreover, we have  $A + A + P' = (A + A) + P + P' = (A + P) + (A + P) + P' = (A + P) + (A + P) = (A + A) + P = A + A$ , so  $P' \subseteq \text{Stab}(A + A) = P$  and hence  $\text{Stab}(A + P) = P$ . For any  $B \in \mathcal{B}$ , Lemma 4.4 shows that  $A + B = A + A$ , so  $(A, B)$  is an  $SS$ -pair. If  $B = g + A$  for some  $g \in G$ , then (by definition)  $A + P = B + P = g + (A + P)$ , so  $g \in \text{Stab}(A + P) = P$ . Now  $|P|$  is odd by Lemma 4.4, so  $|P| \geq 3$ . Since  $A$  is a strict subset of  $A + P$ , there exists  $i_0$ ,  $1 \leq i_0 \leq l$  such that  $S_{i_0}$  is a strict subset of  $c_{i_0} + P$ . If  $l \geq 2$ , pick a nonzero element  $p \in P$  and consider  $B = \cup_{1 \leq i \leq l: i \neq i_0} S_i \cup (S_{i_0} + p)$ . Then  $B \in \mathcal{B}$ , so  $(A, B)$  is an  $SS$ -pair, but  $B$  is not a translate of  $A$ . If  $l = 1$ , then  $A \subseteq c_1 + P$  is a set of size  $n = |A| \geq 3$ , and  $A + A = 2c_1 + P$  has size  $2n - 1$ , so  $|P| = 2n - 1$ . There are at most  $|P| = 2n - 1$  translates of  $A$  within  $A + P = c_1 + P$  and  $\binom{|P|}{|A|} = \binom{2n - 1}{n}$  subsets of  $A + P$  of size  $n$ . Since  $n \geq 3$ ,  $\binom{2n - 1}{n} > 2n - 1$ , we may choose  $B$  to be a subset of  $A + P$  of size  $n$  which is not a translate of  $A$ .  $\square$

## References

- [1] P. Bihani and R. Jin. *Kneser's Theorem for Upper Banach Density*. Journal de Théorie des Nombres de Bordeaux. **18** (2006), no. 2, 323-343.
- [2] D. Gryniewicz. *Quasi-periodic decompositions and the Kemperman structure theorem*. European Journal of Combinatorics. **26** (2005), 559-575.
- [3] Y. O. Hamidoune. *Subsets with small sums in abelian groups. I. The Vosper property*. European Journal of Combinatorics. **18** (1997), no. 5, 541-556.
- [4] Y. O. Hamidoune. *Subsets with a small sum. II. The critical pair problem*. European Journal of Combinatorics. **21** (2000), no. 2, 231-239.
- [5] J. H. B. Kemperman. *On small subsets of an abelian group*. Acta Mathematica. **103** (1960), 63-88.
- [6] A. G. Vosper. *The critical pairs of subsets of a group of prime order*. J. London Math. Soc. **31** (1956), 200-205 and 280-282.
- [7] M. Nathanson. *Additive Number Theory*. Springer, 1996.

Reza Akhtar

Department of Mathematics

Miami University

Oxford, OH 45056

USA

*E-mail* : [reza@calico.mth.muohio.edu](mailto:reza@calico.mth.muohio.edu)

*URL*: <http://calico.mth.muohio.edu/reza/>

Paul Larson

Department of Mathematics

Miami University

Oxford, OH 45056

USA

*E-mail* : [larsonpb@muohio.edu](mailto:larsonpb@muohio.edu)

*URL*: <http://www.users.muohio.edu/larsonpb/>