Reals constructible from many countable sets of ordinals

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Abstract

We show that if there exist proper class many Woodin cardinals, then the set of reals x for which there is exists an ordinal α with $\{a \in \mathcal{P}_{\omega_1}(\alpha) \mid x \in L[a]\}$ stationary is countable. These results were announced in [2].

Given a real x and an ordinal α, we let $C(x, \alpha)$ denote the set

$$\{a \in \mathcal{P}_{\omega_1}(\alpha) \mid x \in L[a]\}.$$

We let $\mathcal{C}$ denote the set of reals x for which there exists an ordinal α such that $C(x, \alpha)$ is club.

We assume some familiarity with the stationary tower $\mathbb{Q}_{<\delta}$ (see [2]).

Theorem 0.1. *Suppose that there exists a proper class of Woodin cardinals. Then for every real x and every ordinal α, $C(x, \alpha)$ is either club or nonstationary, and $\mathcal{C}$ is countable.*

Proof. Fix a cardinal λ such that, for every real x,

- if there exists an ordinal α with $C(x, \alpha)$ stationary and costationary, then there is such an ordinal below λ,
- if there exists an ordinal α with $C(x, \alpha)$ club, then there is such an ordinal below λ.

Let G be V-generic for $Coll(\omega_1, \lambda)$, and note that the set of (x, α) such that $C(x, \alpha)$ is stationary (likewise, costationary, club) is the same in V and $V[G]$. Now let $V[G][H]$ be a semi-proper forcing extension of $V[G]$ in which $u_2 > \lambda$.

If δ is a Woodin cardinal, then since $u_2 \geq \lambda$, $j[\lambda]$ is the same whenever j is the elementary embedding induced by forcing with $\mathbb{Q}^{V[G][H]}_{<\delta}$. Fix a real x and an ordinal $\alpha < \lambda$. Then $x \in L[j[\alpha]]$ if and only if $C(x, \alpha)$ is in the $\mathbb{Q}^{V[G][H]}_{<\delta}$-generic. Since for all generic filters $x \in L[j[\alpha]]$ is decided in the same way, $C(x, \alpha)$ must be either club or nonstationary.

Suppose that (x_β, α_β) $(\beta < \omega_1)$ are pairs such that each x_β is a distinct real, each $\alpha_\beta < \lambda$ and each $C(x, \alpha_\beta)$ is stationary. Since $u_2 > \lambda$, there is a real z such that for each $\beta < \omega_1$ there is a bijection between ω_1^V and α_β in $L[z]$. Therefore, each x_β is in $L[z]$, which is impossible. $\square$

By contrast, Gitik [1] has shown that for every club subset C of $\mathcal{P}_{\omega_1}(\omega_2)$ and every real z there exist $a, b, c \in C$ such that $z \in L[a, b, c]$.

The Axiom of Determinacy implies that the club filter is an ultrafilter on $\mathcal{P}_{\omega_1}(\omega_2)$ [4]. It follows that if AD holds in $L(\mathbb{R})$, then for each real x the set $C(x, \underset{\sim}{\delta}^1_2)$ is either club or nonstationary. Using this fact in place of the stationary tower argument in the proof of Theorem 0.1, one can easily prove the following.

Theorem 0.2. *If AD holds in $L(\mathbb{R})$ in every forcing extension by a forcing of the form $Coll(\omega_1, \gamma)$, then for every real x and every ordinal α, $C(x, \alpha)$ is either club or nonstationary, and $\mathcal{C}$ is countable.*

The following then follows easily from the fact that the theory of $L(\mathbb{R})$ cannot be changed by set forcing in the presence of a proper class of Woodin cardinals.

Theorem 0.3. *If there exists a proper class of Woodin cardinals, then every inner model which is correct about ω_2 contains all the reals in $\mathcal{C}$.*

0.4 Question. What are the optimal hypothesis for the conclusions of Theorems 0.1 and 0.2?

0.5 Question. Does $\mathcal{C}$ have an independent characterization?

We imagine that the following is true for stronger mice, although we don't know how much stronger.

Theorem 0.6. *Suppose that there exists a Woodin cardinal, and $u_2 = \omega_2$. If x is a real and $C(x, \omega_2)$ contains a club, then $C(x^{\#}, \omega_2)$ contains a club.*

Proof. If δ is a Woodin cardinal and $u_2 = \omega_2$, then $j[\omega_2]$ is the same for all $\mathbb{Q}_{<\delta}$-generics, and so for each real x, $C(x, \omega_2)$ is either club or nonstationary. Fix a real x, and suppose that $C(x, \omega_2)$ is club. Then $x \in L[j[\omega_2]]$. But then there is a nontrivial elementary embedding from $L_{\omega_2^V}[x]$ to $L_{j(\omega_2^V)}[x]$ in $L[j[\omega_2]]$, so $x^{\#} \in L[j[\omega_2]]$ and thus $C(x^{\#}, \omega_2)$ is club. □

A simple analysis of Shelah's forcing to make the nonstationary ideal on ω_1 (NS_{ω_1}) saturated [3] shows that it cannot add any real from the ground model to $\mathcal{C}$. Since the saturation of NS_{ω_1} plus the existence of a measurable cardinal implies that $u_2 = \omega_2$ [5], we have the following corollary of Theorem 0.6.

Corollary 0.7. *If there exist proper class many Woodin cardinals, then the set $\mathcal{C}$ is closed under sharps.*

References

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