Notes on getting presaturation from collapsing a Woodin cardinal

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1 Measurable cardinals

1.1 Definition. A *filter* on a set X is a set $F \subseteq \mathcal{P}(X)$ which is closed under intersections and supersets, i.e., such that

- for all A, B in F, $A \cap B \in F$;
- for all $B \subseteq \kappa$ and all $A \in F \cap \mathcal{P}(B)$, $B \in F$.

Given a cardinal κ, a filter F is κ-complete if $\bigcap \mathcal{A} \in \mathcal{F}$ whenever $\mathcal{A}$ is a subset of F of cardinality less than κ.

1.2 Definition. An *ultrafilter* on a set X is a filter on X such that for all $A \subseteq X$, exactly one of A and $X \setminus A$ is in U. The ultrafilter U is *nonprincipal* if no singleton is a member of U.

1.3 Definition. A cardinal κ is said to be *measurable* if there exists a κ-complete nonprincipal ultrafilter on κ.

1.4 Exercise. Show that if κ is measurable, then κ is a regular strong limit cardinal (i.e, κ is *strongly inaccessible*).

Suppose that U is an ultrafilter on a set X. Consider the class of functions with domain X, and the equivalence relation $\sim$ on this class defined by setting $f \sim g$ if and only if $\{x \in X \mid f(x) = g(x)\} \in U$. For each function f with domain X, let $[f]_U$ denote the $\sim$-equivalence class of f. Define the relation E on equivalence classes by setting $[f]_U E [g]_U$ if and only if

$$\{x \in X \mid f(x) \in g(x)\} \in U.$$

Note that this relation does not depend on the representatives chosen from $[f]_U$ and $[g]_U$. Then $\mathrm{Ult}(V, U)$, the ultrapower of V by U, consists of the class of all classes (though we could make these sets by restricting to those f such that the range of f is contained in the least V_α for which $\{x \in X \mid f(x) \in V_\alpha\} \in U$) of the form $[f]_U$, with the binary relation E.

Theorem 1.5. *Let U be an ultrafilter on a set X. For all function $f_1, \dots, f_n$ with domain X, and all n-nary formulas ϕ,*

$$(\mathrm{Ult}(V, U), E) \models \phi([f_1]_U, \dots, [f_n]_U)$$

if and only if

$$\{x \in X \mid \phi(f_1(x), \dots, f_n(x))\} \in U.$$

Proof. By induction on the complexity of formulas. For $=$ and $\in$ this is true by definition. For $\wedge$ it follows from the fact that U is a filter, and for $\neg$ it follows from the fact that U is an ultrafilter. For $\exists$, note that by Replacement and the Axiom of Choice, if

$$\{x \in X \mid \exists a \phi(a, f_1(x), \dots, f_n(x))\} \in U$$

then there is a function g with domain X such that

$$\{x \in X \mid \phi(g(x), f_1(x), \dots, f_n(x))\} \in U.$$

This gives the reverse implication for this step; the other direction is easier. □

1.6 Remark. Eventually, we will want to come back to this proof and think about the fragment of ZFC needed to carry it out.

Corollary 1.7. *Let U be an ultrafilter on a set X, and let $j\colon V \to \mathrm{Ult}(V, U)$ be the function which sends each set x to $[c_x]_U$, where c_x is the constant function from X to $\{x\}$. Then j is an elementary embedding.*

1.8 Remark. Let $\mathrm{tc}(x)$ denote the transitive closure of a set x, and let $\mathrm{tc}^E([f]_U)$ denote the transitive E-closure of an element $[f]_U$ of $\mathrm{Ult}(V, U)$. By convention, we identify each $[f]_U$ in $\mathrm{Ult}(V, U)$ for which $(\mathrm{tc}^E([f]_U), E)$ is wellfounded with its Mostowski collapse, i.e., the unique set a such that $(\{[f]_U\} \cup \mathrm{tc}^E([f]_U), E)$ is isomorphic to $(\{a\} \cup \mathrm{tc}(a), \in)$.

1.9 Exercise. Given an ultrafilter U on a set X, $\mathrm{Ult}(V, U)$ is wellfounded if and only if U is countably complete (i.e., closed under countable intersections).

Lemma 1.10. *Suppose that U is an ultrafilter on a set X, $j\colon V \to \mathrm{Ult}(V, U)$ is the corresponding embedding, i is the identity function on X and f is a function with domain X. Then $[f]_U = j(f)([i]_U)$.*

Proof. Let c_f be the constant function from X to $\{f\}$. Applying Theorem 1.5, we have that $[f]_U = j(f)([i]_U)$ if and only if $\{x \in X \mid f(x) = c_f(x)(i(x))\} \in U$. By the definitions of c_f and i, however, $c_f(x)(i(x)) = f(x)$ for all $x \in U$. □

The following is a second proof of Lemma 1.10 (the one given in class).

Proof. It suffices to see that for all functions f, g with domain X, and any relation R in $\{\in, =\}$, $[f]_U R [g]_U$ if and only if $j(f)([i]_U) R j(g)([i]_U)$. Since $j(f)$ and $j(g)$ are represented by the constant functions from X to $\{f\}$ and $\{g\}$ respectively, both expressions reduce to $\{x \in X \mid f(x) R g(x)\} \in U$. □

1.11 Definition. If $j\colon V \to M$ is an elementary embedding, the *critical point* of j is the least ordinal α such that $j(\alpha) > \alpha$, if one exists.

1.12 Exercise. Suppose that U is a κ-complete ultrafilter on a regular cardinal κ. Show that the critical point of j is κ.

1.13 Exercise (Scott). Show that there are no measurable cardinals in L. (Let κ be the least measurable cardinal in L, and consider the elementary embedding $j\colon L \to M$ given by a κ-complete ultrafilter U on κ (in L).)

1.14 Definition. Given a filter F on a set X, a set $A \subseteq X$ is said to be F-*positive* if A intersects each member of F.

For an ultrafilter U, being U-positive is the same as being in U, for arbitrary filters this is not the case.

1.15 Definition. A filter F on an ordinal γ is said to be *normal* if for each F-positive $A \subseteq \gamma$ and each regressive function $f\colon A \to \gamma$ (i.e., $f(\alpha) < \alpha$ for all $\alpha \in A \setminus \{0\}$) there is an $\alpha < \gamma$ such that $f^{-1}[\alpha]$ is F-positive. The filter F is said to be *uniform* if for all $\alpha < \gamma$, $\gamma \setminus \alpha \in F$.

1.16 Exercise. Show that if U is a normal uniform ultrafilter on κ, U is κ-complete.

1.17 Exercise. Show that if U is a normal uniform ultrafilter on κ and $i\colon \kappa \to \kappa$ is the identity function on κ, then $[i]_U = \kappa$ (under our identification of elements of $\mathrm{Ult}(V, U)$ with their Mostowski collapses).

1.18 Exercise. Suppose that $j\colon V \to M$ is an elementary embedding with critical point κ. Show that $\{A \subseteq \kappa : \kappa \in j(A)\}$ is a normal ultrafilter on κ.

The previous exercise shows that "κ is a measurable cardinal" is equivalent to "there exists an elementary embedding $j\colon V \to M$ with critical point κ" and also equivalent to "there exists a normal ultrafilter on κ."

1.19 Exercise. Suppose that U is a κ-complete ultrafilter on κ, and let $j\colon V \to M$ be the corresponding elementary embedding. Show that $\mathcal{P}(\kappa) \subseteq M$, but $U \notin M$. Show that M is closed under sequences of length κ. Show that $2^\kappa < j(\kappa) < (2^\kappa)^+$.

Theorem 1.20. *Suppose that U is a normal ultrafilter on κ, and that $\lambda > \kappa$ is a regular cardinal. Let X be an elementary submodel of V_λ of cardinality less than κ, with $U \in X$. Let γ be any element of $\bigcap(X \cap U)$, and let $Y = \{f(\gamma) \mid f\colon \kappa \to V_\lambda,\ f \in X\}$. Then $X \subseteq Y$, $Y \prec V_\lambda$, and $Y \cap \gamma = X \cap \gamma$.*

Proof. That $X \subseteq Y$ follows from the fact that there is a constant function in X for each element of X. The fact that $Y \cap \kappa = X \cap \kappa$ follows from normality, as follows. If $f\colon \kappa \to V_\lambda$ is in X, and $f(\gamma) < \gamma$, then f is regressive on a set in U, so there is an $\alpha < \gamma$ such that $f^{-1}[\{\alpha\}] \in U$. Then $\alpha \in X$ and $f(\gamma) = \alpha$.

For elementarity, by the Vaught-Tarski test we need to see only that if $a_1, \ldots, a_n$ are in Y and $V_\lambda \models \exists x \phi(x, a_1, \ldots, a_n)$, then there is a $b \in Y$ such that $V_\lambda \models \phi(b, a_1, \ldots, a_n)$. To see that this holds, fix functions $f_1, \ldots, f_n$ in X such that each $a_i = f_i(\gamma)$. There is in V_λ a function $g\colon \kappa \to V_\lambda$ such that, for all $\alpha < \kappa$, if $V_\lambda \models \exists y \phi(y, f_1(\alpha), \ldots, f_n(\alpha))$, then $V_\lambda \models \phi(g(\alpha), f_1(\alpha), \ldots, f_n(\alpha))$. By elementarity, there is such a g in X. Then $g(\gamma) \in Y$, and

$$V_\lambda \models \phi(g(\gamma), f_1(\gamma), \ldots, f_n(\gamma)).$$

□

1.21 Remark. Note that in the theorem above, the transitive collapse of Y is the ultrapower of the transitive collapse of X by the image of U under the transitive collapse of X.

1.22 Remark. By the previous theorem, applied repeatedly, if κ is a measurable cardinal, and $\lambda > \kappa$ is a regular cardinal, then for every countable $X \prec V_\lambda$ there is a $Y \prec V_\lambda$ such that $X \cap \omega_1 = Y \cap \omega_1$ and $Y \cap \kappa$ has ordertype ω_1. Taking the transitive collapse of Y, we get another contradiction to $V = L$, since for each $\alpha < \omega_1^L$ there is a $\beta < \omega_1^L$ such that α is countable in L_β.

1.23 Exercise (Levy-Solovay). Suppose that κ is a measurable cardinal, as witnessed by a κ-complete ultrafilter U on κ. Let P be any partial order of cardinality less than κ. Then, after forcing with P, the collection of subsets of κ containing an element of U is a κ-complete ultrafilter on κ.

2 Precipitous ideals

2.1 Definition. Given a set X, a set $I \subseteq \mathcal{P}(X)$ is an *ideal* on X if I is closed under subsets and unions (i.e., if $\{X \setminus A : A \in I\}$ is a filter). We call $\{X \setminus A : A \in I\}$ the filter *dual* to I. The ideal I is said to be *κ-complete* (for some cardinal κ) or *normal* or *uniform* if and only if its dual filter is. A subset of X is *I-positive* if it is not in I; I^+ denotes the collection of I-positive sets.

2.2 Exercise. If γ is an ordinal of cofinality κ, every normal uniform ideal on γ is κ-complete.

2.3 Definition. Given a model M of a sufficient fragment of ZFC, and a set $X \in M$, an *M-ultrafilter* is a filter U on X such that

- $U \subseteq \mathcal{P}(X) \cap M$;
- for all $A \in \mathcal{P}(X) \cap M$, $|U \cap \{A, X \setminus A\}| = 1$.

If X is an ordinal in M, U is said to be *M-normal* if for all $A \in U$ and all regressive $f\colon A \to X$, f is constant on a set in U.

2.4 Remark. If U is an M-ultrafilter, we can form $\mathrm{Ult}(M, U)$ as before, and get an elementary embedding from M into $\mathrm{Ult}(M, U)$, as in Theorem 1.5. If U is an normal M-ultrafilter on an ordinal κ of M, then the critical point of the corresponding embedding is κ. The corresponding versions of Lemma 1.10 and Exercise 5.9 also go through in this context.

Suppose that I is an ideal on a set X, and consider the Boolean algebra $\mathcal{P}(X)/I$, consisting of the powerset of X modulo the equivalence relation $A \sim B \Leftrightarrow A \triangle B \in I$. Removing the class corresponding to I, we consider this a forcing notion, under the order $[A] \leq [B] \Leftrightarrow A \setminus B \in I$. Often for the sake of convenience we identify this with the non-separative partial order $(\mathcal{P}(X) \setminus I, \subseteq)$, since as forcing notions they are equivalent. If I contains a cofinite set, then this partial order is trivial as a forcing construction, so in general we will assume otherwise.

2.5 Exercise. If I is an ideal on a set X, and I contains no cofinite sets, and $G \subseteq \mathcal{P}(X)/I$ is a generic filter, then $\{A \mid [A] \in G\}$ is a V-ultrafilter on X which is disjoint from I. If X is an ordinal and I is normal, then the generic ultrafilter added by $\mathcal{P}(X)/I$ is also normal.

2.6 Exercise. Suppose that I is a normal uniform ideal on a regular cardinal κ. For each $\alpha \in \kappa^+$, let $\pi_\alpha : \kappa \to \alpha$ be a surjection, and define the function $f_\alpha \colon \kappa \to \kappa$ by setting $f_\alpha(\beta) = \mathrm{ot}(\pi_\alpha[\beta])$ for all $\beta < \kappa$ (show that any two choices for π_α induce f_α's which agree on a club). Show that $[f_\alpha]_G = \alpha$ whenever G is a generic filter for $\mathcal{P}(\kappa)/I$ (such an f_α is called a *canonical function* for α). Show then that $j(\kappa) \geq \kappa^+$ for any generic elementary embedding induced by $\mathcal{P}(\kappa)/I$, and that a set $A \in I^+$ forces that $j(\kappa) = \kappa^+$ if and only if, for each $B \subseteq A$ in I^+ and every function $g \colon B \to \kappa$, there exist an $\alpha < \kappa^+$, a canonical function f_α for α, and an I-positive set $C \subseteq B$ such that $f_\alpha(\beta) \geq g(\beta)$ for all $\beta \in C$ (let's call this condition *canonical function bounding on I-positive sets*).

2.7 Exercise. Show that if κ is a limit cardinal, then for no normal uniform ideal I on κ does canonical function bounding on I-positive sets hold.[1]

2.8 Definition. We say that an ideal I on a set X is *precipitous* if whenever U is a V-ultrafilter added by forcing with $\mathcal{P}(X)/I$, $\mathrm{Ult}(V, U)$ is wellfounded.

Usually we identify U and G, and write $\mathrm{Ult}(V, G)$.

2.9 Remark. Suppose that I is an ideal on a set X, and $G \subseteq \mathcal{P}(X)/I$ is a generic filter for which $\mathrm{Ult}(V, G)$ is wellfounded. Then some condition $A \in G$ forces the ultrapower to be wellfounded, which implies that the ideal generated by $I \cup \{X \setminus A\}$ is precipitous.

We will not prove (or use) the following theorem, as its proof would take us too far afield.

Theorem 2.10 ([5])**.** *If there is a κ-complete precipitous ideal on an uncountable cardinal κ, then there is an inner model in which κ is a measurable cardinal.*

Theorem 2.11 ([5])**.** *If there is a measurable cardinal, then there is a forcing extension in which there is a normal precipitous ideal on ω_1.*

Before we begin the proof of Theorem 2.11, we introduce some terminology.

[1]Thanks to Giorgio Audrito and Alessandro Vignati for suggesting this example.

2.12 Definition. Given a set X and a cardinal κ, the partial order $\mathrm{Col}(\kappa, {<}X)$ consists of partial functions p of cardinality less than κ, from $(X \setminus \{0\}) \times \kappa$ to $\bigcup X$, with the requirement that $p(a, \beta) \in a$ for all (a, β) in the domain of p. The order is containment.

Forcing with $\mathrm{Col}(\kappa, {<}X)$ explicitly adds a surjection from κ onto each element of X.

2.13 Exercise. Show that if $\kappa < \lambda$ are regular cardinals, and $\gamma^{<\kappa} < \lambda$ for all $\gamma < \lambda$, then $\mathrm{Col}(\kappa, {<}\lambda)$ is λ-c.c.

For instance, if λ is a regular cardinal then $\lambda = \omega_1$ after forcing with $\mathrm{Col}(\omega, {<}\lambda)$.

2.14 Remark. If X and Y are disjoint sets, and κ is a cardinal, then $\mathrm{Col}(\kappa, {<}(X \cup Y))$ is isomorphic to $\mathrm{Col}(\kappa, {<}X) \times \mathrm{Col}(\kappa, {<}Y)$, so a generic filter for $\mathrm{Col}(\kappa, {<}X)$ can be extended to one for $\mathrm{Col}(\kappa, {<}(X \cup Y))$.

Proof of Theorem 2.11. Let κ be a measurable cardinal, and let U be a normal ultrafilter on κ. Let $j \colon V \to M$ be the elementary embedding induced by U. Let $H \subseteq \mathrm{Col}(\omega, {<}j(\kappa))$ be a V-generic filter, and let $G = H \cap \mathrm{Col}(\omega, {<}\kappa)$. Then G is V-generic for $\mathrm{Col}(\omega, {<}\kappa)$, H is M-generic for $\mathrm{Col}(\omega, {<}j(\kappa))$, and $H \setminus G$ is $V[G]$-generic for $\mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$.

The embedding j lifts to an elementary embedding $j^* \colon V[G] \to M[H]$, defined by setting $j^*(\tau_G) = j(\tau)_H$ for each $\mathrm{Col}(\omega, {<}\kappa)$-name τ. To see that this works, note that j is the identity function on $\mathrm{Col}(\omega, {<}\kappa)$, so if $p \in G$, $\tau_1, \ldots, \tau_n$ are $\mathrm{Col}(\omega, {<}\kappa)$-names and ϕ is a formula such that $p \Vdash^V_{\mathrm{Col}(\omega, <\kappa)} \phi(\tau_1, \ldots, \tau_n)$, then $j(p) = p$ is in H and $j(p) \Vdash^M_{\mathrm{Col}(\omega, <j(\kappa))} \phi(j(\tau_1), \ldots, j(\tau_n))$.

Let I be the set of $A \in \mathcal{P}(\kappa)^{V[G]}$ such that A is disjoint from some set in U. Then I is an ideal on ω_1 in $V[G]$. We claim that for each $A \in \mathcal{P}(\kappa)^{V[G]}$, $A \notin I$ if and only if there is an $s \in \mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$ such that

$$s \Vdash^{V[G]}_{\mathrm{Col}(\omega, <[\kappa, j(\kappa)))} \check{\kappa} \in j^*(\check{A}).$$

To see this, suppose that τ is a $\mathrm{Col}(\omega, {<}\kappa)$-name in V such that $A = \tau_G$. For the forward direction, since $A \notin I$, for each $p \in G$ the set

$$\{\alpha < \kappa \mid \exists q \leq p \;\; q \Vdash \check{\alpha} \in \tau\}$$

is in U. It follows then that for each $p \in G$, there is a $q \in \mathrm{Col}(\omega, {<}j(\kappa))$ extending p such that

$$q \Vdash^M_{\mathrm{Col}(\omega, <j(\kappa))} \check{\kappa} \in j(\tau).$$

By genericity, then, there is a $q \in \mathrm{Col}(\omega, {<}j(\kappa))$ such that $q \Vdash^M_{\mathrm{Col}(\omega, <j(\kappa))} \check{\kappa} \in j(\tau)$ and $q \cap \mathrm{Col}(\omega, {<}\kappa) \in G$. Then $q \restriction ([\kappa, j(\kappa)) \times \omega)$ is the desired condition s.

To see the reverse direction, since $A \in I$ there exist $B \in U$ and $p \in G$ such that $p \Vdash^V_{\mathrm{Col}(\omega, <\kappa)} \tau \cap \check{B} = \emptyset$. Since every condition in $\mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$ forces that $\check{\kappa} \in j^*(B)$, no condition in $\mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$ can force that $\check{\kappa} \in j^*(A)$.

Let us see that I is normal. Fix a set $A \in \mathcal{P}(\kappa)^{V[G]} \setminus I$, a regressive function $f\colon A \to \kappa$ in $V[G]$ and an $s \in \mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$ such that

$$s \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \check{\kappa} \in j^*(\check{A}).$$

We may strengthen s to a condition s' deciding $j^*(f)(\kappa)$ to be some fixed ordinal α. It follows then that

$$s' \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \check{\kappa} \in j^*(f^{-1}[\{\alpha\}]),$$

so $f^{-1}[\{\alpha\}] \notin I$.

Let $D = \{A \in \mathcal{P}(\kappa)^{V[G]} \mid \kappa \in j^*(A)\}$. We wish to see that $D \cap I = \emptyset$, that D is $V[G]$-generic for $\mathcal{P}(\kappa)/I$, and that $\mathrm{Ult}(V[G], D)$ is wellfounded. The first of these follows from the reverse direction of the claim above. To see that $\mathrm{Ult}(V[G], D)$ is wellfounded, note that it embeds into $M[H]$ via the function sending $[f]_D$ to $j^*(f)(\kappa)$.

Finally, we check that D is $V[G]$-generic for $\mathcal{P}(\kappa)/I$. Suppose that E is a subset of $\mathcal{P}(\kappa)/I$ in $V[G]$, and that $E \cap D = \emptyset$. It suffices to show that E is not dense in $\mathcal{P}(\kappa)/I$. Let r be a condition in $\mathrm{Col}(\omega, {<}[\kappa, j(\kappa))) \cap H$ such that

$$r \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \forall A \in \check{E}\ \check{\kappa} \notin j^*(A).$$

Let $f\colon \kappa \to \mathrm{Col}(\omega, {<}\kappa)$ be a function in V such that $[f]_U = r$, and let

$$B = \{\alpha \in \kappa \mid f(\alpha) \in G\}.$$

Then $B \in V[G]$ and $\kappa \in j^*(B)$, so $B \in D$ and $B \notin I$.

It suffices now to see that $B \cap A \in I$ for all $A \in E$. Fix such an A. If $B \cap A \notin I$, then there is an $s \in \mathrm{Col}(\omega, {<}[\kappa, j(\kappa)))$ such that

$$s \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \check{\kappa} \in j^*((A \cap B)\check{}).$$

However, $s \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \check{\kappa} \in j^*(\check{B})$ implies that

$$s \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} j^*(\check{f})(\check{\kappa}) \in G^*.$$

Since $j^*(f)(\kappa) = j(f)(\kappa) = r$, this implies that $s \leq r$, which means that $s \Vdash^{V[G]}_{\mathrm{Col}(\omega,<[\kappa,j(\kappa)))} \check{\kappa} \notin j^*(\check{A})$, giving a contradiction. □

2.15 Definition. Let A be a subset of an ordinal γ. The set A is *closed* if, for all $\alpha < \gamma$, $\sup(A \cap \alpha) \in A$. The set A is *cofinal* if, for all $\alpha < \gamma$, $A \setminus \alpha \neq \emptyset$. We say that A is *club* if it is both closed and cofinal. The set A is *stationary* if it intersects every club subset of γ, and *nonstationary* otherwise.

2.16 Definition. The collection of supersets of club subsets of an ordinal γ is called the *club filter* on γ, and the set of nonstationary sets is called the *nonstationary ideal.* The nonstationary ideal on γ is denoted NS_γ. If κ is a regular cardinal, the ideal generated by $\mathrm{NS}_\gamma \cup \{\alpha < \gamma \mid \mathrm{cof}(\alpha) \neq \kappa\}$ is called the *nonstationary ideal on γ restricted to cofinality κ* and denoted $\mathrm{NS}^\kappa_\gamma$.

2.17 Exercise. For any ordinal γ of uncountable cofinality κ, the club filter γ is a κ-complete filter (so the set of nonstationary subsets of γ is a κ-complete ideal).

2.18 Exercise. Every normal uniform ideal on a regular cardinal κ contains NS_κ.

2.19 Exercise. If κ is a measurable cardinal, then stationarily many $\gamma < \kappa$ are strongly inaccessible.

2.20 Exercise. If κ is a regular cardinal, $\theta \geq \kappa$ is a cardinal, $A \subseteq \kappa$ is stationary and $x \in H(\theta)$, then there is an elementary submodel X of $H(\theta)$ of cardinality less than κ such that $X \cap \kappa \in A$ and $x \in X$.

2.21 Remark. It is proved in [5] that the normal precipitous ideal on ω_1 in Theorem 2.11 can be made to be the nonstationary ideal, but again we will skip the proof.

2.22 Remark. Exercise 1.18 implies that if there is a κ-complete ultrafilter on a cardinal κ then there is a normal ultrafilter on κ. Gitik [4] has shown that the existence of a precipitous ideal on ω_1 does not imply the existence of a normal precipitous ideal on ω_1.

2.23 Remark. As the ideal dual to a countably complete ultrafilter is precipitous, and $\mathrm{Col}(\omega, {<}\kappa)$ is κ.c.c. when κ is a regular cardinal, Theorem 2.25 below strengthens Theorem 2.11.

2.24 Definition. A subset S of a partial order P is *predense* if every element of P is compatible with a member of S.

Theorem 2.25 (Kakuda [6], Magidor [9])**.** *If I is a precipitous κ-complete ideal on a regular cardinal κ, then I generates a precipitous ideal after any κ-c.c. forcing.*

Proof. Suppose that G is a V-generic filter for some κ-c.c. forcing P. Let J be the κ-complete ideal generated by I in $V[G]$, and suppose that U is a $V[G]$-ultrafilter added by forcing with $(\mathcal{P}(\kappa)/J)^{V[G]}$.

Let us see first that $U \cap V$ is V-generic for $(\mathcal{P}(\kappa)/I)^V$. First, note that since P is κ-c.c., and I is κ-complete, I-positive sets in V are J-positive in $V[G]$. Now, let $\mathcal{A}$ be any subset of $\mathcal{P}(\kappa) \setminus I$ in V which is predense in $\mathcal{P}(\kappa) \setminus I$ in the order of mod-I containment. Let τ be a name for a J-positive subset of κ which has intersection in J with each $A \in \mathcal{A}$. Since P is κ-c.c. and I is κ-complete, we can find for each $A \in \mathcal{A}$ a set $B_A \in I$ for which is it forced that $\tau \cap \check{A} \subseteq \check{B}_A$. Then τ is forced to be a subset of $\bigcap_{A \in \mathcal{A}}((\kappa \setminus A) \cup B_A)$. This set must be in I, however, since it has intersection in I with each element of $\mathcal{A}$.

We have then that $\mathrm{Ult}(V, U \cap V)$ is wellfounded. It suffices now to see that every function $f \colon \kappa \to \mathrm{Ord}$ in $V[G]$ is equal to a function $h \colon \kappa \to \mathrm{Ord}$ from V on a set in U. To see that this is the case, let τ be a P-name for a function from κ to the ordinals, and, for each $\alpha < \kappa$, let D_α be the set of conditions in

P deciding the value of τ at α. We need to find a $g\colon \kappa \to P$ in V for which $\{\alpha < \kappa \mid g(\alpha) \in D_\alpha \cap G\} \in U$. Let $\mathcal{B}$ be the collection of sets of the form $\{\alpha < \kappa \mid g(\alpha) \in D_\alpha \cap G\}$, for some $g\colon \kappa \to P$ in V. We claim that $\mathcal{B}$ is predense in $(\mathcal{P}(\kappa)/J)^{V[G]}$. Let σ be a P-name for an element of $\mathcal{P}(\kappa) \setminus J$ for which it is forced that for each $g\colon \kappa \to P$ in V, that

$$\{\alpha \in \sigma \mid \check{g}(\alpha) \in \check{D}_\alpha \cap G\} \in J.$$

By the κ-completeness of I and the κ-c.c. of P, we may fix for each such g a set $E_g \in I$ for which it is forced that

$$\{\alpha \in \sigma \mid \check{g}(\alpha) \in \check{D}_\alpha \cap G\} \subseteq E_g.$$

Suppose that there is a $p_0 \in P$ forcing that $\sigma \notin J$. Then for any $p \leq p_0$ the set F_p consisting of those $\alpha < \kappa$ for which there is a $p' \leq p$ forcing that $\check{\alpha} \in \sigma$ is not in I. For each such p we can find a function $g_p\colon \kappa \to P$ in V such that, for all $\alpha \in F_p$, $g_p(\alpha) \in D_\alpha$ and $g_p(\alpha) \Vdash \check{\alpha} \in \sigma$. Again for each such p, there is an $\alpha_p \in F_p \setminus E_{g_p}$. Then $g_p(\alpha_p) \leq p$ forces that $\check{\alpha}_p \in \sigma$. By genericity, some such $g_p(\alpha_p)$ is in G, giving a contradiction. $\square$

2.26 Exercise. Show that if I is a normal uniform ideal on a regular cardinal κ, then in any κ-c.c. forcing extension the ideal formed by closing I under subsets.

2.27 Remark. Kakuda's proof actually shows that $\mathrm{Ult}(V[G], U)$ is (isomorphic to) a forcing extension of $\mathrm{Ult}(V, U \cap V)$ via $i(P)$, where $i\colon V \to \mathrm{Ult}(V, U \cap V)$ is the canonical embedding.

3 Stationary sets

3.1 Definition. Let X be a nonempty set. A set $c \subset \mathcal{P}(X)$ is *club* in $\mathcal{P}(X)$ if there is a function $f\colon X^{<\omega} \to X$ for which c is the set of $A \subset X$ closed under f. Given a cardinal $\kappa \leq |X|$, c is *club* in $[X]^\kappa$ (or in $[X]^{<\kappa}$) if c is the set of $A \in [X]^\kappa$ (or $[X]^{<\kappa}$) closed under f. A set $a \subset \mathcal{P}(X)$ is *stationary* in $\mathcal{P}(X)$ if it intersects every club subset of $\mathcal{P}(X)$, and *stationary* in $[X]^\kappa$ (or $[X]^{<\kappa}$) if it intersects every c which is club in $[X]^\kappa$ (or $[X]^{<\kappa}$).

If c is club in $\mathcal{P}(X)$, then $\bigcup c = X$, so we can simply say that c is *club* if it is club in $\bigcup c$. Similarly, if a is stationary in $\mathcal{P}(X)$, then $\bigcup a = X$, so we can simply say that a is *stationary* if it is stationary in $\mathcal{P}(\bigcup a)$.

3.2 Exercise. If κ is a regular cardinal and A is a subset of κ, then A is stationary in the sense of Definition 2.15 if and only if it is stationary in $\mathcal{P}(\gamma)$ in the sense of Definition 3.1. A set $A \subseteq \omega_1$ contains a club in the sense of Definition 2.15 if and only if it contains club in $\mathcal{P}(\omega_1)$ in the sense of Definition 3.1.

3.3 Remark. For any first order structure on a nonempty set X, a Skolem function for the structure induces a club of elementary substructures.

Lemma 3.4 (The projection lemma for stationary sets)**.** *Suppose that $X \subseteq Y$ are nonempty sets, and $\kappa \leq |X|$ is a cardinal.*

1. *If a is stationary in $\mathcal{P}(Y)$, then $\{B \cap X \mid B \in a\}$ is stationary in $\mathcal{P}(X)$.*
2. *If a is stationary in $\mathcal{P}(X)$, then $\{B \subset Y \mid B \cap X \in a\}$ is stationary in $\mathcal{P}(Y)$.*
3. *If a is stationary in $[X]^\kappa$, then $\{B \in [Y]^\kappa \mid B \cap X \in a\}$ is stationary in $[Y]^\kappa$.*
4. *If a is stationary in $[X]^{<\kappa}$, then $\{B \in [Y]^{<\kappa} \mid B \cap X \in a\}$ is stationary in $[Y]^\kappa$.*

Proof. For the first part, given a function f on $X^{<\omega}$, extend it in any way to a function g on $Y^{<\omega}$. Then if $B \in a$ is closed under g, then $B \cap X$ is closed under f. For the other parts, given a function f on $Y^{<\omega}$, replace it with a function f' such that for all $A \subset Y$, the f'-image of $A^{<\omega}$ contains A and is closed under f. We may also fix a point $x \in X$ and assume that the f'-image of any nonempty set contains x. One way to do this is to fix a bijection $\pi \colon \omega \to \omega \times \omega$ (with projections π_0 and π_1, and $\pi_0(n) \leq n$ for all n) and let $f'(y_0, \ldots, y_{n-1})$ be the value of the $\pi_1(n)$-th term formed from compositions of f and $\pi_0(n)$ many variables, evaluated a $(y_0, \ldots, y_{\pi_0(n)-1})$. Finally, replace f' with a function f'' which agrees with f' when f' takes a value in X, and returns the value x otherwise. Then any subset of X closed under f'' is the intersection with X of a subset of Y closed under f. □

3.5 Remark. Exercise 2.20 is an instance of the Lemma 3.4. Moreover, suppose that X is a nonempty set which is a definable element of some $H(\theta)$, $\kappa \leq |X|$ is a cardinal, $C \subset [X]^\kappa$ is definable in $H(\theta)$ and C is club in $[X]^\kappa$. Then for every $Y \prec H(\theta)$ of cardinality κ, $Y \cap X \in C$.

3.6 Definition. Suppose that P is a partial order, θ is a regular cardinal greater than $2^{|P|}$, and X is a countable elementary submodel of $H(\theta)$ with $P \in X$. A condition $p \in P$ is (X, P)-*generic* if for every dense subset $D \subseteq P$ in X there is a condition $q \in D \cap X$ with $q \geq p$.

We note that the term (X, P)-*generic* is often used to mean something more general than the definition given here; our notion is sometimes called *completely* (X, P)-*generic*.

3.7 Definition. Given a cardinal κ, a partial order P is said to be κ-*closed* if whenever $\gamma < \kappa$ and $\{p_\alpha : \alpha < \gamma\} \subseteq P$ such that that $p_\alpha \geq p_\beta$ for all $\alpha < \beta < \gamma$, there exists a $p \in P$ such that $p \leq p_\alpha$ for all $\alpha < \gamma$. We usually say *countably closed* for ω_1-closed.

3.8 Remark. For any cardinal κ and any set X, the partial order $\mathrm{Col}(\kappa, {<}X)$ is κ-closed. It follows that if θ is a regular cardinal greater than $2^{|\mathrm{Col}(\kappa, <X)|}$, X is a countable elementary submodel of $H(\theta)$, $p \in P \cap X$ and $P \in X$, then there exists an (X, P)-generic condition $q \leq p$.

3.9 Definition. A sequence of sets $\langle N_\alpha : \alpha < \gamma \rangle$ is said to be $\subseteq$-*increasing* if $N_\alpha \subsetneq N_\beta$ for all $\alpha < \beta < \gamma$, and *continuous* if $N_\beta = \bigcup_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta < \gamma$.

3.10 Exercise. Suppose that X is a set, κ is a cardinal less than $|X|$, $a \subseteq [X]^{<\kappa}$ is stationary, and $\lambda > |X| \geq \kappa$. Then $\mathrm{Col}(\kappa, {<}\lambda)$ adds a continuous, $\subseteq$-increasing sequence $\langle N_\alpha : \alpha < \kappa \rangle$ of subsets of X of cardinality less than κ with $\bigcup\{N_\alpha : \alpha < \kappa\} = X$. If $\kappa = \aleph_1$, then the set $\{\alpha < \kappa \mid N_\alpha \in a\}$ will be stationary.

We will not be using the following definition, but we include it just to mention the issues that arise in the case $\kappa > \aleph_1$ in the previous exercise.

3.11 Remark. The last part of Exercise 3.10 is more complicated when κ is uncountable. Say that a set N is *internally approachable* (see page 33 of [3], for instance) if there exist a limit ordinal γ and a sequence $\langle N_\alpha : \alpha < \gamma \rangle$ such that $N = \bigcup_{\alpha<\gamma} N_\alpha$ and $\langle N_\alpha : \alpha < \beta \rangle \in N$ for all $\beta < \alpha$. Note that every countable elementary substructure of a set of the form $H(\theta)$ or V_θ (for infinite θ) is internally approachable. Exercise 3.10 holds for uncountable κ if one assumes, for some regular cardinal $\theta > 2^{|\mathrm{Col}(\kappa,<\lambda)|}$ that the set of internally approachable $M \prec H(\theta)$ of cardinality less than κ with $M \cap X \in a$ is stationary.

The following exercise uses Theorem 1.20 and Lemma 3.4.

3.12 Exercise. Suppose that κ is a measurable cardinal, $\lambda < \kappa$ is a regular cardinal, A is a stationary subset of λ, and $f \colon A \to \lambda$ is a function. Then the set of $X \in [\kappa]^{<\lambda}$ for which $X \cap \lambda \in \lambda$ and $\mathrm{ot}(X) > f(X \cap \lambda)$ is stationary.

Exercises 2.13, 3.10 and 3.12 give the following result. By Exercise 2.13, if λ is a strongly inaccessible cardinal, and $\kappa < \lambda$, then forcing with $\mathrm{Col}(\kappa, {<}\lambda)$ makes $\lambda = \kappa^+$. Furthermore, every subset of κ in the $\mathrm{Col}(\kappa, {<}\lambda)$-extension is added by some initial segment of the partial order (i.e., $\mathrm{Col}(\kappa, {<}\gamma)$ for some $\gamma < \lambda$).

3.13 Exercise. If λ is a strongly inaccessible limit of measurable cardinals, then NS_{ω_1} satisfies canonical function bounding for stationary sets after forcing with $\mathrm{Col}(\omega_1, {<}\lambda)$.

3.14 Remark. For $\kappa > \omega_1$, $\mathrm{NS}^{\omega}_{\kappa}$ satisfies canonical function bounding for stationary sets after forcing with $\mathrm{Col}(\kappa, {<}\lambda)$, but this consistently fails for $\mathrm{NS}^{\omega_1}_{\omega_2}$ by the argument for Theorem 2.13 of [2].

4 Presaturated ideals

4.1 Definition. Let $\gamma < \kappa$ be cardinals, with γ regular. Let I be a κ-complete ideal I on κ. We say that I is γ-*presaturated* if whenever $B \in I^+$ and $\mathcal{A}_\alpha$ ($\alpha < \gamma$) are antichains in $\mathcal{P}(\omega_1)/I$, there is a $C \in I^+ \cap \mathcal{P}(B)$ such that for each $\alpha < \gamma$, $[C]_I$ is compatible with at most κ many members of $\mathcal{A}_\alpha$. When $\kappa = \gamma^+$ we say simply that I is *presaturated.*

4.2 Exercise. Suppose that I is a κ-complete ideal on an uncountable cardinal κ. Let $G \subseteq \mathcal{P}(\kappa)/I$ be a V-generic filter and let $j\colon V \to \mathrm{Ult}(V,G)$ be the corresponding embedding. Show that if $\gamma < \kappa$ is an infinite cardinal and I is γ-presaturated, then $\mathrm{Ult}(V,G)$ is closed under γ-sequences in $V[G]$, and therefore wellfounded. Show that if $\kappa = \gamma^+$, then $j(\kappa) = \kappa^+$.

4.3 Exercise. Given a cardinal κ, find κ many partitions of κ^+ into κ^+ many sets such that no stationary subset of κ^+ has stationary intersection with just κ many members of each partition.

4.4 Remark. If there is a presaturated ideal on ω_1, then there is a normal presaturated ideal on ω_1 [1]. Presaturation of NS_{ω_1} is not necessarily preserved by c.c.c. forcing ([13, 8]).

4.5 Exercise. A theorem of Shelah says that if κ is a regular cardinal and P is a partial order such that forcing with P makes $\mathrm{cof}(\kappa) < \mathrm{cof}(|\kappa|)$ hold, then P collapses κ^+. Show that this implies that $\mathrm{NS}^{\omega}_{\omega_2}$ is not presaturated.

4.6 Remark. Woodin has shown (from determinacy hypotheses, see Section 9.7 of [14]) that it is possible to have a normal ω-presaturated ideal on ω_2.

4.7 Definition. Let κ be an uncountable cardinal. Given a collection $\mathcal{A}$ of subsets of κ and an ordinal $\beta > \kappa$, we let $\mathrm{sp}_\beta(\mathcal{A})$ be the set of $X \prec V_\beta$ for which there exist a $Y \prec V_\beta$ and a $B \in \mathcal{A} \cap Y$ such that $X \subseteq Y$, $X \cap \kappa = Y \cap \kappa$ and $X \cap \kappa \in B$. We say that a set Y *captures* a collection $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ if there is a $B \in \mathcal{A} \cap Y$ such that $Y \cap \kappa \in B$. A collection $\mathcal{A}$ of subsets of ω_1 is *semi-proper* if $\mathrm{sp}_{\omega_1+\omega}(\mathcal{A})$ contains a club subset of $[V_{\omega_1+\omega}]^{\aleph_0}$.

4.8 Remark. To generalize Definition 4.7 to an arbitrarily uncountable κ, say that $\mathcal{A}$ is *semi-proper* if for a club of $M \in [V_{\kappa+\omega}]^{<\kappa}$, if M is internally approachable, then $M \in \mathrm{sp}_{\kappa+\omega}(\mathcal{A})$.

Note that Y in the definition of semi-proper can be taken to have the same cardinality as X.

4.9 Exercise. If $\mathcal{A}$ is a subset of $\mathcal{P}(\omega_1)$ and $\mathcal{A}$ is not semi-proper, then $[V_\beta]^{\aleph_0} \setminus \mathrm{sp}_\beta(\mathcal{A})$ is stationary for all $\beta \geq \omega_1 + \omega$.

Lemma 4.10. *Let $\mathcal{A}$ be a collection of subsets of ω_1, and let $\beta > \omega_1 + \omega$ be an ordinal of cofinality greater than $\beth_{\omega_1+\omega}$. Then $\mathcal{A}$ is semi-proper if and only if for each countable $X \prec V_\beta$ of with $\mathcal{A} \in X$ there exists a $Y \prec V_\beta$ capturing $\mathcal{A}$ with $X \subseteq Y$ and $X \cap \omega_1 = Y \cap \omega_1$.*

Proof. The reverse direction follows from upwards projection of stationary sets, plus the fact that the set of countable elementary substructures of a structure contains a club. For the forward direction, recall that $\beth_{\omega_1+\omega} = |V_{\omega_1+\omega}|$. Suppose that $\mathrm{sp}(\mathcal{A})$ contains a club and $\mathcal{A}$ is an element of a countable elementary substructure X of V_β. Then there is a function $F\colon V_{\omega_1+\omega}^{<\omega} \to V_{\omega_1+\omega}$ in X such that every subset of $V_{\omega_1+\omega}$ closed under F is in $\mathrm{sp}(\mathcal{A})$. Then there is a countable $Y \prec V_{\omega_1+\omega}$ capturing $\mathcal{A}$ such that $X \cap V_{\omega_1+\omega} \subseteq Y$, and $X \cap \omega_1 = Y \cap \omega_1$. Let

$$X[Y] = \{f(y) \mid f\colon V_{\omega_1+\omega} \to V_\beta,\, y \in Y\}.$$

We want to see that $X[Y] \prec V_\beta$, and that $X[Y] \cap \omega_1 = X \cap \omega_1$. For the first of these, it suffices to see that if $f_1, \ldots, f_n$ are functions from $V_{\omega_1+\omega}$ to V_β, $y_1, \ldots, y_n$ are in Y and $V_\beta \models \exists x \phi(x, f_1(y_1), \ldots, f_n(y_n))$, then there is a function $g\colon (V_{\omega_1+\omega})^n \to V_\beta$ in X such that $V_\beta \models \phi(g(y_1, \ldots, y_n), f_1(y_1), \ldots, f_n(y_n))$. For the second part, if $f\colon V_{\omega_1+\omega} \to \omega_1$ is in X, $y \in Y$, then $y \in V_{\omega_1+m}$ for some m, and $f \restriction V_{\omega_1+m}$ is in $X \cap V_{\omega_1+\omega}$. □

4.11 Exercise. Let $\mathcal{A}_i$ $(i < \omega)$ be semi-proper collections of subsets of ω_1, and let $\beta > \omega_1 + \omega$ be an ordinal of cofinality greater than $\beth_{\omega_1+\omega}$. Then the set of countable $X \prec V_\beta$ which capture each $\mathcal{A}_i$ is stationary.

4.12 Remark. Suppose that $\kappa < \theta$ are cardinals. If $Y \prec H(\theta)$ with $Y \cap \kappa \in \kappa$, and A, B are two subsets of κ in Y with nonstationary intersection, it cannot be that $Y \cap \kappa \in A \cap B$.

Lemma 4.13. *If every predense set in $\mathcal{P}(\omega_1) \setminus NS_{\omega_1}$ is semi-proper, then NS_{ω_1} is precipitous.*

Proof. Suppose towards a contradiction that E is a stationary subset of ω_1 such that there exist $\mathcal{P}(E)/I$-names τ_i $(i \in \omega)$ for an ω-sequence of functions in V from ω_1 to the ordinals giving rise to a descending ω-sequence in a generic ultrapower via $\mathcal{P}(E)/\mathrm{NS}_{\omega_1}$. For each $i \in \omega$, let $\mathcal{A}_i$ be the union of $\{\omega_1 \setminus E\}$ with the collection of stationary sets $B \subseteq E$ such that B chooses a value $f_j : \omega_1 \to \mathrm{Ord}$ for each τ_j for $j < i$, and such that $f_0(\beta) > f_1(\beta) > \ldots > f_{i-1}(\beta)$ holds for each $\beta \in B$. Then each $\mathcal{A}_i$ is predense. Let $\eta > \beth_{\omega_1+\omega}$ be a regular cardinal, and let X be an elementary submodel of V_η of cardinality less than ω_1 with $X \cap \omega_1 \in E$ and each $\mathcal{A}_i$ in X. Iteratively apply the definition of semi-proper (and Lemma 4.10) to the predense sets $\mathcal{A}_i$ $(i \in \omega)$ to find a countable $Y \prec V_\eta$ and $B_i \in \mathcal{A}_i \cap Y$ $(i \in \omega)$ such that

- $X \subseteq Y$;
- $X \cap \omega_1 = Y \cap \omega_1$;
- $\omega_1 \cap X \in B_i$ for all $i \in \omega$.

The various sets B_i must be compatible in $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ (i.e., have stationary intersection), as they are all in X, and $\omega_1 \cap X$ is an element of all of them. Then each B_i decides the value of τ_i to be some f_i, and the values $f_i(X \cap \omega_1)$ $(i \in \omega)$ form a descending ω-sequence, giving a contradiction. □

Lemma 4.14. *Suppose that $\delta < \kappa$ are regular cardinals, $\gamma < \kappa$, and τ_α $(\alpha < \gamma)$ are $\mathcal{P}(\kappa)/\mathrm{NS}^\delta_\kappa$-names for a γ-sequence of elements of the generic ultrapower. For each $\alpha < \gamma$, let $\mathcal{A}_\alpha$ be the collection of $\mathrm{NS}^\delta_\kappa$-positive sets deciding that some* *fixed function from κ to V in V will give rise to the value of τ at α. Suppose that $\langle X_\alpha : \alpha < \kappa\rangle$ is a continuous, $\subseteq$-increasing sequence of subsets of $\kappa \cup \mathcal{P}(\kappa)$ of cardinality less than κ, and $B \subseteq \kappa$ is an $\mathrm{NS}^\delta_\kappa$-positive set such that, for all $\beta \in B$, $X_\beta \cap \kappa = \beta$, and X_β captures each $\mathcal{A}_\alpha$. Then $[B]_{\mathrm{NS}^\delta_\kappa}$ forces in $\mathcal{P}(\kappa)/\mathrm{NS}^\delta_\kappa$ that the realization of τ will be an element of the generic ultrapower.*

Proof. Define the function $f\colon B \to {}^\omega V$ by letting $f(\beta)(i)$ be $g(\beta)$, where g is forced by some $A \in \mathcal{A}_i \cap X_\beta$ for which $X_\beta \cap \omega_1 \in A$ to be the function giving rise to the ith element of the sequence represented by τ. We want to see that $[B]_{\mathrm{NS}^\delta_\kappa}$ forces that $[f]_G = \tau_G$, where G is the generic ultrafilter added by forcing with $\mathcal{P}(\kappa)/\mathrm{NS}^\delta_\kappa$.

If C is a stationary subset of B and h is a function on C which picks an element of X_β for each $\beta \in C$, then h is constant on an $\mathrm{NS}^\delta_\kappa$-positive set. Suppose now that C is an $\mathrm{NS}^\delta_\kappa$-positive subset of B, and i is an element of ω for which C has forced some function $g\colon \kappa \to V$ to represent the ith value of the sequence corresponding to τ. Let h be the function on C which picks for each $\beta \in C$ a set $A \in \mathcal{A}_i \cap X_\beta$ such that $\kappa \cap X_\beta \in A$. Recall also that $\beta = \kappa \cap X_\beta$ for $\beta \in C$. It follows that h is constant on an $\mathrm{NS}^\delta_\kappa$-positive $D \subseteq C$, and that D is contained in this constant value A. Since A and C have $\mathrm{NS}^\delta_\kappa$-positive intersection, they must choose the same function g, which means that $f(\beta)(i) = g(\beta)$ for all $\beta \in D$, and therefore that D forces the ith member of $[f]_G$ to be the ith member of τ_G. □

Lemma 4.15. *Suppose that κ is a regular cardinal, and $\mathcal{A}$ is a collection of NS_κ-positive sets, pairwise having intersection in NS_κ. Suppose that $\langle X_\alpha : \alpha < \kappa\rangle$ is a continuous, $\subseteq$-increasing sequence of subsets of $\kappa \cup \mathcal{P}(\kappa)$ of cardinality less than κ, and $B \subseteq \kappa$ is a stationary set such that, for each $\beta \in B$, $X_\beta \cap \kappa = \beta$, and X_β captures $\mathcal{A}$. Suppose that $C \in \mathcal{A}$ and $B \cap C$ is stationary. Then $C \in \bigcup_{\beta<\kappa} X_\beta$.*

Proof. Let h be the function on $B \cap C$ which picks for each β a set $A \in \mathcal{A} \cap X_\beta$ such that $\kappa \cap X_\beta \in A$. It follows that h is constant on a stationary $D \subseteq C$, and that D is contained in this constant value A. Since A and C have stationary intersection, and they are both members of $\mathcal{A}$, they must be the same set. □

4.16 Definition. A *Woodin cardinal* is a cardinal δ such that for every function $f\colon \delta \to \delta$ there exist a $\kappa < \delta$ closed under f and an elementary embedding $j\colon V \to M$ with critical point κ and $V_{j(f)(\kappa)} \subseteq M$.

4.17 Remark. Theorem 5.19 shows how to express the definition of *Woodin cardinal* in the language of set theory. It also shows that one can require that M is closed under sequences of length κ, without strengthening the definition. Similarly, by Remark 5.8, one can add the requirement that $j(\delta) = \delta$

4.18 Exercise. A Woodin cardinal is strongly inaccessible, and stationary limit of measurable cardinals. A stationary limit of Woodin cardinals is Woodin. The least Woodin cardinal is not measurable.

Theorem 4.19 (Shelah-Woodin[11])**.** *Suppose that δ is a Woodin cardinal. Let $G \subseteq \mathrm{Col}(\omega_1, {<}\delta)$ be a V-generic filter, and let $\{A^i_\alpha : i < \omega,\ \alpha < \delta\}$ be stationary* *subsets of ω_1 such that for each $i < \omega$, $\{A^i_\alpha : \alpha < \delta\}$ is predense in $\mathcal{P}(\omega_1)/NS_{\omega_1}$. Then there is a $\lambda < \delta$ such that, for each $i < \omega$, $\{A^i_\alpha : \alpha < \lambda\}$ is predense and semi-proper in $V[G \cap \mathrm{Col}(\kappa, {<}\lambda)]$.*

Proof. Fix a collection of $\mathrm{Col}(\omega_1, {<}\delta)$-names τ^i_α $(\alpha < \delta, i < \omega)$ such that for each $i < \omega$, $\langle \tau^i_\alpha : \alpha < \delta\rangle$ forms a $\mathrm{Col}(\omega_1, {<}\delta)$-name σ_i for a predense set in $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$. For each $\lambda < \delta$, let $\sigma_{i,\lambda}$ be the $\mathrm{Col}(\omega_1, {<}\lambda)$-name induced by $\langle \tau^i_\alpha : \alpha < \lambda\rangle$. We will find a $\lambda < \delta$ for which it is forced that in the $\mathrm{Col}(\omega_1, {<}\lambda)$-extension, the realization of each $\sigma_{i,\lambda}$ is presense in $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ and semi-proper.

Let us adopt the notation that H_α refers to the generic filter for $\mathrm{Col}(\omega_1, {<}\alpha)$ (we are not fixing these objects, just fixing notation for the forcing relation).

For any $\lambda < \delta$ and $i < \omega$, if $\sigma_{i,\lambda,H_\lambda}$ is not semi-proper in $V[H_\lambda]$, then

$$a = [V_{\lambda\cdot 2}]^{\aleph_0} \setminus \mathrm{sp}_{\lambda\cdot 2}(\sigma_{i,\lambda,H_\lambda})$$

is stationary in $V[H_\lambda]$, and the $\mathrm{Col}(\omega_1, {<}|V_{\lambda\cdot 2}|^+)$-extension adds a continuous, $\subseteq$-increasing sequence of countable sets, $\langle N_\alpha : \alpha < \omega_1\rangle$ with union $V^{V[H_\lambda]}_{\lambda\cdot 2}$, and $\{\alpha \mid N_\alpha \in a\}$ will be a stationary subset of ω_1. Moreover, for any two such sequences $\langle N_\alpha : \alpha < \omega_1\rangle$, the set $\{\alpha \mid N_\alpha \in a\}$ is the same modulo a nonstationary subset of ω_1.

Fix a function $f \colon \delta \to \delta$ with the following properties.

- For all $\alpha < \delta$, $f(\alpha)$ is a strongly inaccessible cardinal greater than α.
- If $\lambda < \delta$ is closed under f, then, for each $i \in \omega$, $\sigma_{i,\lambda}$ is forced to be a predense set in $\mathcal{P}(\omega_1)/\mathrm{NS}_{\omega_1}$ in the $\mathrm{Col}(\omega_1, {<}\lambda)$-extension.
- If $\lambda < \delta$ is closed under f, it is forced that if $i \in \omega$ is such that $\sigma_{i,\lambda,H_\lambda}$ is not semi-proper, then for some β with $\tau^i_\beta \in V_{f(\lambda)}$, $\tau^i_{\alpha,H_{f(\lambda)}}$ will have stationary intersection with any set of the form $\{\alpha < \omega_1 \mid N_\alpha \in a\}$ as above.

Since δ is Woodin, there exists an elementary embedding $j \colon V \to M$ with critical point λ, where λ is closed under f and $V_{j(f)(\lambda)} \subseteq M$. We may assume that that M is closed under countable sequences, so that $\mathrm{Col}(\omega_1, {<}\alpha)^V = \mathrm{Col}(\omega_1, {<}\alpha)^M$ for every ordinal α, and also (by Theorem 5.19) that for each $i \in \omega$,

$$j(\langle \tau^i_\alpha : \alpha < \delta\rangle) \restriction j(f)(\kappa) = \langle \tau^i_\alpha : \alpha < \delta\rangle \restriction j(f)(\kappa).$$

We claim that each $\sigma_{i,\lambda,H_\lambda}$ will be semi-proper in $V[H_\lambda]$.

Supposing otherwise, fix $i < \omega$ and a condition $p_0 \in \mathrm{Col}(\omega_1, {<}\lambda)$ forcing that a, the set of countable elementary submodels of $V_{\lambda\cdot 2}[H_\lambda]$ (which is equal to $V^{V[H_\lambda]}_{\lambda\cdot 2}$) which are not in $\mathrm{sp}_{\lambda\cdot 2}(\sigma_{i,\lambda,H_\lambda})$ is stationary. Then p_0 also forces this for the $\mathrm{Col}(\omega_1, {<}\lambda)$ extension of M, since $[V_{\lambda\cdot 2}]^{\aleph_0} \setminus \mathrm{sp}_{\lambda\cdot 2}(\sigma_{i,\lambda,H_\lambda})$ is the same in $V[H_\lambda]$ and $M[H_\lambda]$ (for the same H_λ). There are names ν_α $(\alpha < \omega_1)$ in the forcing $\mathrm{Col}(\omega_1, {<}|V_{\lambda\cdot 2}|^+)$ for a continuous, $\subseteq$-increasing sequence of countable sets whose union is the model $V_{\lambda\cdot 2}[H_\lambda]$, and a name ρ for the set of α for which

$\nu_{\alpha,H_{|V_{\lambda\cdot 2}|^+}} \notin \mathrm{sp}_{\lambda\cdot 2}(\sigma_{i,\lambda,H_\lambda})$. Then some condition $p_1 \leq p_0$ in $\mathrm{Col}(\omega_1, {<}j(f)(\lambda))$ forces in M that for some fixed $\beta < j(f)(\lambda)$ that the realizations of ρ and τ^i_β will have stationary intersection. As $V_{j(f)(\lambda)} \subseteq M$, p_1 forces this about τ^i_β in V as well.

Whenever $H_{j(\lambda)}$ is V-generic for $\mathrm{Col}(\omega_1, {<}j(\lambda))$ below p_1, then, there will be stationarily many countable elementary submodels Z of $V_\delta^{V[H_{j(\lambda)}]}$ such that $Z \cap \omega_1 \in \rho_{H_{|V_{\lambda\cdot 2}|^+}} \cap \tau^i_{\beta,H_{j(f)(\lambda)}}$ (since λ is closed under f, $j(\lambda)$ is a strongly inaccessible cardinal greater than $|V_{\lambda\cdot 2}|^+$ and $j(f)(\lambda))$ and such that every dense subset of $\mathrm{Col}(\omega_1, {<}j(\lambda))$ in $Z \cap V$ will intersect $Z \cap H_{j(\lambda)}$, which implies that

$$\nu_{Z\cap\omega_1,H_{|V_{\lambda\cdot 2}|^+}} = Z \cap V_{\lambda\cdot 2}^{V[H_\lambda]}.$$

Since $\mathrm{Col}(\omega_1, {<}j(\lambda))$ adds no countable subsets of V, the restriction of any such elementary submodel to V will be an element of V.

Then there is a countable elementary submodel X of V_δ such that

$$\sigma_i, p_1, \beta, j(V_{\lambda\cdot 2}), j \restriction V_{\lambda\cdot 2}, \leq_*$$

are all elements of X (where $\leq_*$ is a wellordering of $j(V_{\lambda\cdot 2})$ in M), and such that there is a condition $p_2 \leq p_1$ which is $\mathrm{Col}(\omega_1, {<}j(\lambda))$-generic for X and forces that

$$(X \cap V_{\lambda\cdot 2})[H_\lambda] \notin \mathrm{sp}_{\lambda\cdot 2}(\sigma_{i,\lambda,H_\lambda})$$

(this part is actually forced by $p_2 \cap \mathrm{Col}(\omega_1, {<}\lambda)$ and $X \cap \omega_1 \in \tau^i_\beta$. Note that p_2 also forces then that $X \cap \omega_1 \in \rho$.

It follows that, in M, $p_2 \cap \mathrm{Col}(\omega_1, {<}\lambda)$ forces in $\mathrm{Col}(\omega_1, {<}j(\lambda))$ that

$$j(X \cap V_{\lambda\cdot 2})[H_{j(\lambda)}] \in j(a).$$

Let Y be the Skolem closure of $\{\tau^i_\beta\} \cup j{\restriction}(X \cap V_{\lambda\cdot 2})$ in $j(V_{\lambda\cdot 2})$ according to $\leq_*$. We want to see that Y contradicts the previous paragraph. We have that $Y \in M$, $j(X \cap V_{\lambda\cdot 2}) \subseteq Y \subseteq X$, and $\tau^i_\beta \in Y$.

Let H be M-generic for $\mathrm{Col}(\omega_1, {<}j(\lambda))$, with $p_2 \in H$. Then since p_2 forces $X \cap \omega_1 = X[H] \cap \omega_1 = Y[H] \cap \omega_1$ (here we are using that $\mathrm{Col}(\omega_1, {<}j(\lambda))$ is the same partial order in V and M, so every $\mathrm{Col}(\omega_1, {<}j(\lambda))^M$-name in Y for an ordinal is a $\mathrm{Col}(\omega_1, {<}j(\lambda))^V$-name in X) into the realization of τ^i_β, which is in $Y[H]$, we have a contradiction. □

Corollary 4.20. *Suppose that δ is a Woodin cardinal. Then NS_{ω_1} is presaturated in the $\mathrm{Col}(\omega_1, {<}\delta)$-extension.*

Proof. By Lemmas 4.15 and Exercises 3.10 and 4.11 and Theorem 4.19, it suffices to show that if B is a stationary subset of ω_1 and $\mathcal{A}_i$ ($i < \omega$) are semi-proper subsets of $\mathcal{P}(\omega_1)$, then for any regular cardinal $\chi > |\beta_{\omega_1+\omega}|$, stationarily many countable $Y \prec V_\chi$ capture each $\mathcal{A}_i$. This follows from Lemma 4.10. □

4.21 Remark. If δ is a Woodin cardinal and κ is a regular cardinal below δ, $\mathrm{NS}^{\omega}_{\kappa}$ is ω-presaturated in the $\mathrm{Col}(\kappa, {<}\delta)$-extension. The proof for this is similar to the proof of Theorem 27 of [3]. Theorem 2.13 of [2] shows that this can fail to hold for $\mathrm{NS}^{\omega_1}_{\omega_2}$.

4.22 Remark. The results of this section were first proved by Foreman, Magidor and Shelah, using supercompact cardinals [3]. The improvement to Woodin cardinals came shortly afterwards.

Let us call Lebesgue measurability, the property of Baire, the perfect set property and the Ramsey property the *regularity properties*. Solovay [12] showed that if κ is a strongly inaccessible cardinal, then in the $\mathrm{Col}(\omega, {<}\kappa)$-extension every set of reals in $L(\mathbb{R})$ is Lebesgue measurable, and satisfies the property of Baire and the perfect set property. Mathias [10] later added the Ramsey property.

Theorem 4.23 (Woodin)**.** *Suppose κ is a weakly compact cardinal and there is a κ-c.c. partial ordering P such that whenever $G \subseteq P$ is a V-generic filter, in $V[G]$ there is an elementary embedding $j\colon V \to M$ with $j(\omega_1) = \kappa$ and $\mathbb{R}^{V[G]} \subseteq M$. Then in a generic extension there is a V-generic filter $H \subseteq \mathrm{Col}(\omega, {<}\kappa)$ such that $\mathbb{R}^{V[G]} = \mathbb{R}^{V[H]}$.*

Foreman, Magidor and Shelah [3] showed that (in more than one way) if δ is a supercompact cardinal, then there is a δ-c.c. forcing, not adding reals but producing a normal ideal I on ω_1 for which $\mathcal{P}(\omega_1)/I$ is $\aleph_2$-c.c. (i.e., I is *saturated*). One way in which they showed this was the following.

Theorem 4.24 (Foreman-Magidor-Shelah[3])**.** *If δ is a supercompact cardinal, then in the $\mathrm{Col}(\omega_1, {<}\delta)$-extension there is an $\aleph_2$-c.c. partial order forcing that NS_{ω_1} is saturated.*

It follows from this and the following exercise (due to Kunen [7]) that when δ is supercompact, $\mathrm{Col}(\omega_1, {<}\delta)$ forces the existence of a normal saturated ideal on ω_1.

4.25 Exercise. Suppose that there is an $\aleph_2$-c.c. partial order P forcing that NS_{ω_1} is saturated. Let I be the set of $B \subseteq \omega_1$ which are forced to be nonstationary by every condition in P. Show that I is a normal saturated ideal on ω_1.

Putting all of this together, we get that the existence of a supercompact cardinal implies that the regularity properties hold in $L(\mathbb{R})$, and, since forcings of cardinality less than a supercompact cardinal preserve the supercompact cardinals, that forcings of cardinality less than a supercompact cardinal cannot change the theory of $L(\mathbb{R})$.

5 Extenders

5.1 Definition. Given finite sets of ordinal $s \subseteq t$, define the projection map

$$\pi_{t,s}\colon [\mathrm{Ord}]^{|t|} \to [\mathrm{Ord}]^{|s|}$$

as follows. Suppose that $t = \{\gamma_0, \ldots, \gamma_{n-1}\}$ (listed in increasing order), and that $a \subseteq n$ is such that $s = \{\gamma_i : i \in a\}$. Then for each $\{\alpha_0, \ldots, \alpha_{n-1}\} \in [\kappa]^n$ (listed in increasing order), we let $\pi_{t,s}(\{\alpha_0, \ldots, \alpha_{n-1}\}) = \{\alpha_i : i \in a\}$.

5.2 Example. If $s = \{1, 3, 7\}$ and $t = \{0, 1, 3, 6, 7, 9\}$, then

$$\pi_{t,s}(\{3, 7, \omega, \omega + 2, \omega_1, \omega_1 \cdot 2\}) = \{7, \omega, \omega_1\}.$$

5.3 Definition. Given an uncountable cardinal κ and an ordinal $\gamma > \kappa$, a (κ, γ)-extender is a function

$$E\colon [\gamma]^{<\omega} \setminus \{\emptyset\} \to V_{k+2}$$

such that

1. each $E(s)$ is a κ-complete ultrafilter on $[\kappa]^{|s|}$;
2. (coherence) For all finite $s \subseteq t \subset \gamma$, for each $A \subseteq [\kappa]^{|s|}$,

$$A \in E(s) \Leftrightarrow \pi_{t,s}^{-1}[A] \in E(t).$$

3. (normality) for each s and each $f\colon [\kappa]^{|s|} \to \kappa$ such that

$$\{a \in [\kappa]^{|s|} \mid f(a) < \max(s)\} \in E(s),$$

   there exists a $t \supseteq s$ in $[\gamma]^{<\omega}$ such that

$$\{b \in [\kappa]^{|t|} \mid (f \circ \pi_{t,s})(b) \in b\} \in E(t).$$

We say that γ is the *length* of the extender E. Extenders satisfying condition (1) above are often called *short* extenders; these are sufficient for our needs.

5.4 Exercise. Suppose that $E\colon [\gamma]^{<\omega} \setminus \{\emptyset\} \to V_{k+2}$ is an extender. Prove the following facts directly from the definition of extender (i.e., without using the embedding defined below).

1. $E(\{0\})$ is the principal ultrafilter on κ generated by $\{0\}$.
2. For all $\alpha < \kappa$, $E(\{\alpha\})$ is the principal ultrafilter on κ generated by $\{\alpha\}$. (This is also true for each finite subset of κ, and these are the only principal $E(s)$'s.)
3. For all $\alpha < \gamma$ and all $A \subseteq \kappa$, $A \in E(\{\alpha\})$ if and only if

$$\{\beta + 1 \mid \beta \in A\} \in E(\{\alpha + 1\}).$$

4. For all $s \subseteq t \in [\gamma]^{<\omega}$, and all $A \subseteq [\kappa]^{|t|}$, $A \in E(t) \Rightarrow \pi_{t,s}[A] \in E(s)$. (The opposite direction is not necessarily true, see Example 5.16)
5. $E(\{\kappa\})$ is normal and uniform (modulo being a measure on sets of singleton ordinals as opposed to a measure on sets of ordinals).

Suppose that $E\colon [\gamma]^{<\omega} \setminus \{\emptyset\} \to V_{k+2}$ is an extender. For each $s \in [\gamma]^{<\omega}$, $E(s)$ induces an ultrapower embedding $j_s\colon V \to M_s$, as usual. Furthermore, given $s \subseteq t \in [\gamma]^{<\omega}$, there is an embedding $k_{s,t}\colon M_s \to M_t$ defined by setting

$$k_{s,t}([f]_{E(s)}) = [f \circ \pi_{t,s}]_{E(t)}.$$

Lemma 5.5. *Suppose that $E\colon [\gamma]^{<\omega} \setminus \{\emptyset\} \to V_{k+2}$ is an extender. For each $s \subseteq t \in [\gamma]^{<\omega}$, $k_{s,t}$ is elementary.*

Proof. By the coherence property of E, and Theorem 1.5, for any n-ary formula ϕ and any functions $f_1, \ldots, f_n$ on $[\kappa]^{|s|}$, $M_s \models \phi([f_1]_{E(s)}, \ldots, [f_n]_{E(s)})$ if and only if

$$\{a \in [\kappa]^{|s|} \mid \phi(f_1(a), \ldots, f_n(a))\} \in E(s)$$

if and only if

$$\pi_{t,s}^{-1}(\{a \in [\kappa]^{|s|} \mid \phi(f_1(a), \ldots, f_n(a))\}) \in E(t)$$

if and only if

$$\{b \in [\kappa]^{|t|} \mid \phi(f_1 \circ \pi_{t,s}(b), \ldots, f_n \circ \pi_{t,s}(b))\} \in E(t)$$

if and only if $M_t \models \phi([f_1 \circ \pi_{t,s}]_{E(t)}, \ldots, [f_n \circ \pi_{t,s}]_{E(t)})$. □

The directed system of models M_s ($s \in \gamma^{<\omega}$) with embeddings

$$k_{s,t}\colon M_s \to M_t \quad (s \subseteq t \in [\gamma]^{<\omega})$$

gives rise to a limit model $\mathrm{Ult}(V, E)$. Elements of $\mathrm{Ult}(V, E)$ are represented by pairs (f, s), where $s \in \gamma^{<\omega}$ and $f\colon \kappa^{|s|} \to V$. Given a relation $R \in \{\in, =\}$ and pairs (f, s) and (g, t), we set $[f, s]_E R [g, t]_E$ if and only if

$$\{a \in [\kappa]^{|s \cup t|} \mid f(\pi_{s\cup t,s}(a)) R g(\pi_{s\cup t,t}(b))\} \in E(s \cup t),$$

i.e., if and only if

$$[f \circ \pi_{s\cup t,s}]_{E(s\cup t)} R [g \circ \pi_{s\cup t,t}]_{E(s\cup t)}.$$

We let $j_E\colon V \to \mathrm{Ult}(V, E)$ be the embedding sending each x to $[c_x, \{\kappa\}]_E$, for $c_x\colon \kappa \to \{x\}$ constant (note : we could pick any element of γ in place of κ here). For each $s \in \gamma^{<\omega}$, there is an embedding $k_{s,\infty}\colon M_s \to \mathrm{Ult}(V, E)$ defined by setting $[f]_{E(s)} = [f, s]_E$.

5.6 Exercise. Given an n-ary formula, sets $s_1, \ldots, s_n \in [\gamma]^{<\omega}$ and functions $f_i\colon [\kappa]^{|s_i|} \to V$, $\mathrm{Ult}(V, E) \models \phi([f_1, s_1]_E, \ldots, [f_n, s_n]_E)$ if and only if, letting $t = s_1 \cup \ldots \cup s_n$,

$$\{a \in [\kappa]^{|t|} \mid \phi(f_1(\pi_{t,s_1}(a)), \ldots, f_n(\pi_{t,s_n}(a)))\} \in E(t).$$

It follows that the embeddings j_E and $k_{s,\infty}$ ($s \in [\gamma]^{<\omega}$) are all elementary.

5.7 Remark. While each M_s as above is wellfounded, $\mathrm{Ult}(V, E)$ need not be. It will be, however, in the cases we are interested in. When $\mathrm{Ult}(V, E)$ is wellfounded we denote it by M_E.

5.8 Remark. By cardinality considerations, it follows that if δ is a strongly inaccessible cardinal and E is a (κ, λ)-extender in V_δ for which $\mathrm{Ult}(V, G)$ is wellfounded past δ, then $j(\delta) = \delta$, for j the corresponding embedding.

5.9 Exercise. Let $E\colon [\gamma]^{<\omega} \setminus \{\emptyset\} \to V_{\kappa+2}$ be a (κ, γ)-extender. For each $n \in \omega$, let i_n be the identity function on $[\kappa]^n$. Show that for each set $s \in [\gamma]^{<\omega}$, the pair $(i_{|s|}, s)$ represents s in $\mathrm{Ult}(V, E)$. (Hint : It suffices to prove this for the singletons. Use induction, and normality.)

From Exercise 5.9 it follows that for each pair (f, s) representing an element of $\mathrm{Ult}(V, E)$, $[f, s]_E = j_E(f)(s)$, as

$$\{a \in [\kappa]^{|s|} \mid f(a) = c_f(a)(i_{|s|}(a))\} \in E(s).$$

Lemma 5.10. *Let $E\colon [\gamma]^{<\omega} \setminus \{\emptyset\} \to V_{\kappa+2}$ be a (κ, γ)-extender. For each nonempty finite $s \subseteq \lambda$, and each $A \subseteq [\kappa]^{|s|}$, $A \in E(s) \Leftrightarrow s \in j_E(A)$.*

Proof. Since $[i_{|s|}, s]_E = s$, $s \in j_E(A)$ if and only if

$$\{b \in [\kappa]^{|s \cup \{\kappa\}|} \mid (i_{|s|} \circ \pi_{s \cup \{\kappa\}, s})(b) \in (c_A \circ \pi_{s \cup \{\kappa\}, \{\kappa\}})(b)\} \in E(s),$$

which holds if and only if $A \in E(s)$. □

5.11 Definition. Suppose that $j\colon V \to M$ is an elementary embedding with critical point κ. For any $\gamma \in (\kappa, j(\kappa)]$, we define the (κ, γ)-extender *derived from* j by setting $E(s) = \{A \subseteq [\kappa]^{|s|} \mid s \in j(A)\}$, for each nonempty $s \in [\gamma]^{<\omega}$.

5.12 Exercise. Verify, for any elementary embedding $j\colon V \to M$ with critical point κ, and any $\gamma \in (\kappa, j(\kappa)]$, that the (κ, γ)-extender derived from j is a (κ, γ)-extender.

5.13 Definition. Let $j\colon V \to M$ be an elementary embedding with critical point κ, let $\gamma \in (\kappa, j(\kappa)]$ and let E be the (κ, γ)-extender derived from j. The *factor map* corresponding to j and E is the function $k\colon \mathrm{Ult}(V, E) \to M$ defined by setting $k([f, s]_E) = j(f)(s)$.

Lemma 5.14. *Let $j\colon V \to M$ be an elementary embedding with critical point κ, let $\gamma \in (\kappa, j(\kappa)]$ and let E be the (κ, γ)-extender derived from j. Then the* factor map *corresponding to j and E is elementary.*

Proof. Fix an n-ary formula ϕ, sets $s_1, \ldots, s_n$ in $[\gamma]^{<\omega}$ and functions $f_1, \ldots, f_n$, where each f_i has domain $[\gamma]^{|s_i|}$. Then $\mathrm{Ult}(V, E) \models \phi([f_1, s_1]_E, \ldots, [f_n, s_n]_E)$ if and only if, letting $t = s_1 \cup \ldots \cup s_n$,

$$\{a \in [\kappa]^{|t|} \mid \phi(f_1(\pi_{t,s_1}(a)), \ldots, f_n(\pi_{t,s_n}(a)))\} \in E(t),$$

which holds if and only if

$$t \in j(\{a \in [\kappa]^{|t|} \mid \phi(f_1(\pi_{t,s_1}(a)), \ldots, f_n(\pi_{t,s_n}(a)))\}),$$

which holds if and only if

$$M \models \phi(j(f_1)(\pi_{t,s_1}(t)), \ldots, j(f_n)(\pi_{t,s_n}(t)))$$

which holds if and only if

$$M \models \phi(j(f_1)(s_1), \ldots, j(f_n)(s_n)).$$

□

Again, let $j\colon V \to M$ be an elementary embedding, let $\gamma \in (\kappa, j(\kappa)]$ and let E be the (κ,γ)-extender derived from j. Since $\mathrm{Ult}(V,E)$ embeds into M, we have that $\mathrm{Ult}(V,E)$ is wellfounded if M is (which the notation M suggests that it is).

5.15 Example. Let $j\colon V \to M$ be an elementary embedding induced by a normal uniform ultrafilter U on κ. Let γ be any ordinal in the interval $(\kappa, j(\kappa)]$, and let E be the (κ,γ)-extender derived from j. Then $E(\{\kappa\}) = U$ and $j = j_E$. To see that $j = j_E$, note every element of M has the form $j(f)(\kappa)$ for some function f. It follows that the map $k([f,s]_E) = j(f)(s)$ is a surjective elementary embedding, and it therefore an isomorphism.

5.16 Example. Let U be a normal uniform ultrafilter on κ, and let $j_0\colon V \to M_0$ be the corresponding ultrapower embedding. In M_0, $j_0(U)$ is a normal uniform ultrafilter on $j_0(\kappa)$. Let $j_1\colon M_0 \to M_1$ be the corresponding ultrapower embedding. Let $j\colon V \to M_1$ be defined by setting $j = j_1 \circ j_0$. Then j is elementary. For each $\gamma \in (\kappa, j(\kappa)]$, let E_γ denote the (κ,γ)-extender corresponding to j. If $\gamma \in (\kappa, j_0(\kappa)]$, then $j_{E_\gamma} = j_0$, as for all $A \subseteq [\kappa]^n$ and all $s \in [\gamma]^n$, $s \in j(A)$ if and only if $s \in j_0(A)$. If $\gamma \in (j_0(\kappa), j(\kappa)]$, then $j_{E_\gamma} = j$, as every member of M_1 has the form $j(f)(\{\kappa, j_0(\kappa)\})$ for some function f in V with domain $[\kappa]^2$ (so we are in a case similar to the previous example).

Let $A = \{\{\alpha, \alpha+1\} : \alpha \in \kappa\}$. Then $A \notin E_{j_0(\kappa)+1}(\{\kappa, j_0(\kappa)\})$ but

$$\pi_{\{\kappa, j_0(\kappa)\}}[A] \in E_{j_0(\kappa)+1}(\{\kappa\}).$$

Lemma 5.17. *Suppose that $j\colon V \to M$ is an elementary embedding with critical point κ, let $\gamma \in (\kappa, j(\kappa)]$, and let E be the (κ,γ)-extender derived from j. Let $k\colon M_E \to M$ be the factor map. Then the following hold.*

1. *The critical point of k is at least γ.*
2. *If α is such that $M \models 2^\alpha < \gamma$, then $\mathcal{P}(\alpha)^M = \mathcal{P}(\alpha)^{M_E}$.*

Proof. For each $\alpha < \gamma$, $[i_1, \{\alpha\}]_E = \alpha$, so $k(\alpha) = j(i_1)(\{\alpha\}) = \alpha$. Since $k((2^\alpha)^{M_E}) = (2^\alpha)^M < \gamma$, $k((2^\alpha)^{M_E}) = (2^\alpha)^{M_E}$, and $k(\mathcal{P}(\alpha)^{M_E})$ is equal to both $\mathcal{P}(\alpha)^{M_E}$ and $\mathcal{P}(\alpha)^M$. □

Lemma 5.18. *Let $\delta \leq \kappa$ be cardinals. Suppose that γ is a strong limit cardinal of cofinality greater than δ, and that E is a (κ, γ)-extender with $V_\gamma \subseteq M_E$. Then M_E is closed under sequences of length δ.*

Proof. Suppose that γ is a strong limit cardinal of cofinality greater than δ, with $V_\gamma \subseteq M_E$. Each element of M_E has the form $j_E(f)(s)$ for some finite $s \subseteq \gamma$ and some function f with domain $[\kappa]^{|s|}$. Let (f_α, s_α) $(\alpha < \delta)$ be such that each s_α is a finite subset of γ and each f_α is a function with domain $[\kappa]^{|s_\alpha|}$. Since $\mathrm{cof}(\gamma) > \delta$, $\langle s_\alpha : \alpha < \delta \rangle \in V_\gamma$, and since $V_\gamma \subseteq M_E$, $\langle s_\alpha : \alpha < \delta \rangle \in M_E$. Let F be a function on $([\kappa]^{<\omega})^\delta$ such that $F(\langle a_\alpha : \alpha < \delta \rangle) = \langle f_\alpha(a_\alpha) : \alpha < \delta \rangle$ whenever each a_α has size $|s_\alpha|$. Then $j_E(F)(\langle s_\alpha : \alpha < \delta \rangle) \restriction \delta = \langle j_E(f_\alpha)(s_\alpha) : \alpha < \delta \rangle$ is an element of M_E. □

Theorem 5.19. *Suppose that δ is a Woodin cardinal, and fix $f \colon \delta \to \delta$ and $A \subseteq \delta$. Then there exist κ and λ below δ, and a (κ, λ)-extender F such that the following hold.*

1. *$f[\kappa] \subseteq \kappa$.*
2. *$j_F(f)(\kappa) = f(\kappa)$.*
3. *$V_{f(\kappa)} \subseteq M_F$.*
4. *$j_F(A) \cap f(\kappa) = A \cap f(\kappa)$.*
5. *M_F is closed under sequences of length κ.*

5.20 Exercise. Show that by replacing A with a set coding (A, f), and replacing f with a faster function, it suffices to prove the version of the theorem with conclusion (2) removed, and the assumption that f is an increasing function mapping into the strongly inaccessible cardinals below δ added. (For instance, let $H \colon L_\delta \to \delta$ be the function induced by the constructibility order, let $B = H[(A \times \{0\}) \cup (f \times \{1\})]$ and let g be an increasing function for which each value $g(\alpha)$ is a strongly inaccessible cardinal greater than α closed under f.)

Proof of Theorem 5.19. Applying Exercise 5.20, we prove the version of the theorem with conclusion (2) removed, and assume that f is an increasing function mapping into the strongly inaccessible cardinals below δ. Let $g \colon \delta \to \delta$ be the function defined by letting $g(\alpha)$ be the least strongly inaccessible cardinal above α which is closed under f. Applying the fact that δ is Woodin, let $j \colon V \to M$ be an elementary embedding whose critical point κ is closed under g, such that $V_{j(g)(\kappa)} \subseteq M$. Then κ is closed under f, and $V_{j(f)(\kappa)} \subseteq M$.

In M, $j(g)(\kappa)$ is closed under $j(f)$, and is a strongly inaccessible limit of strongly inaccessible cardinals. Let η be a strongly inaccessible cardinal of M in the interval $(j(f)(\kappa), j(g)(\kappa))$. Since $V_{j(g)(\kappa)} \subseteq M$, η is strongly inaccessible in V, and $V_\eta = V^M_\eta$. Let $E \colon [\eta]^{<\omega} \setminus \{0\} \to V_{\kappa+2}$ be the (κ, η)-extender derived from j. Then $E \in V_{\eta+1} = V^M_{\eta+1}$.

The critical point of the factor map $k_E \colon M_E \to M$ is at least η, which implies the following facts.

- $V_\eta^{M_E} = V_\eta^M$;
- $j_E(f)(\kappa) = j(f)(\kappa)$;
- $j_E(A) \cap j(f)(\kappa) = j(A) \cap j(f)(\kappa)$;
- $j_E(\kappa) > \eta$.

The first of these follow from Lemma 5.17, and the last from the fact that $\eta < j(g)(\kappa) < j(\kappa)$.

Let $j_E^M : M \to \mathrm{Ult}(M, E)$ be the embedding induced by applying E to M. By the elementarity of j, it suffices to show that, in M, E satisfies our desired properties with respect to $j(f)$ and $j(A)$. That is, we wish to see the following.

1. $j(f)[\kappa] \subseteq \kappa$.
2. $V^M_{j(f)(\kappa)} \subseteq \mathrm{Ult}(M, E)$.
3. $j_E^M(j(A)) \cap j(f)(\kappa) = j(A) \cap j(f)(\kappa)$.
4. In M, $\mathrm{Ult}(M, E)$ is closed under sequences of length κ.

The first of these follows from the fact that $j(f)\restriction\kappa = f\restriction\kappa$, and $f[\kappa] \subseteq \kappa$. For the second, note that $V^{M_E}_{j_E(\kappa)}$ is constructed from $V_{\kappa+1}$ and E. Since $V_{\kappa+1} \subseteq M$, this implies that

$$V^{M_E}_{j_E(\kappa)} = V^{\mathrm{Ult}(M,E)}_{j_E^M(\kappa)}.$$

Since $V_\eta^{M_E} = V_\eta^M$ and $j_E(\kappa) > \eta > j(f)(\kappa) = j_E(f)(\kappa)$, this gives that

$$V^M_{j(f)(\kappa)} = V^{M_E}_{j_E(f)(\kappa)} = V^{\mathrm{Ult}(M,E)}_{j_E(f)(\kappa)}.$$

For the third point,

$$j_E^M(j(A)) \cap j(f)(\kappa) = j_E^M(A) \cap j(f)(\kappa)$$

(since $j_E(\kappa) > j_E(f)(\kappa) = j(f)(\kappa)$ and $j(A) \cap \kappa = A \cap \kappa$), which is equal to

$$j_E(A) \cap j(f)(\kappa)$$

(since both are computed using $V_{\kappa+1}$ and E), which is equal to $j(A) \cap j(f)(\kappa)$, since the critical point of k_E is at least η, which is greater than $j(f)(\kappa)$.

The last item follows from Lemma 5.18, the fact that η is strongly inaccessible and the fact that $V_\eta^M = V_\eta^{M_E}$ is contained in $\mathrm{Ult}(M, E)$, which follows the fact that $V^{M_E}_{j_E(\kappa)} = V^{\mathrm{Ult}(M,E)}_{j_E^M(\kappa)}$. □

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