Six lectures on the stationary tower

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1 The stationary tower

1.1 Definition. Let X be a nonempty set. A set $c \subset \mathcal{P}(X)$ is *club* in $\mathcal{P}(X)$ if there is a function $f \colon X^{<\omega} \to X$ for which c is the set of $A \subset X$ closed under f. Given a cardinal $\kappa \leq |X|$, c is *club* in $[X]^\kappa$ (or in $[X]^{<\kappa}$) if c is the set of $A \in [X]^\kappa$ (or $[X]^{<\kappa}$) closed under f. A set $a \subset \mathcal{P}(X)$ is *stationary* in $\mathcal{P}(X)$ if it intersects every club subset of $\mathcal{P}(X)$, and *stationary* in $[X]^\kappa$ (or $[X]^{<\kappa}$) if it intersects every c which is club in $[X]^\kappa$ (or $[X]^{<\kappa}$).

If c is club in $\mathcal{P}(X)$, then $\bigcup c = X$, so we can simply say that c is *club* if it is club in $\bigcup c$. Similarly, if a is stationary in $\mathcal{P}(X)$, then $\bigcup a = X$, so we can simply say that a is *stationary* if it is stationary in $\mathcal{P}(\bigcup a)$.

The two following facts were discussed previously.

1.2 Remark. For any first order structure on a nonempty set X, a Skolem function for the structure induces a club of elementary substructures.

Lemma 1.3 (The projection lemma for stationary sets)**.** *Suppose that $X \subseteq Y$ are nonempty sets, and $\kappa \leq |X|$ is a cardinal.*

1. *If a is stationary in $\mathcal{P}(Y)$, then $\{B \cap X \mid B \in a\}$ is stationary in $\mathcal{P}(X)$.*
2. *If a is stationary in $\mathcal{P}(X)$, then $\{B \subset Y \mid B \cap X \in a\}$ is stationary in $\mathcal{P}(Y)$.*
3. *If a is stationary in $[X]^\kappa$, then $\{B \in [Y]^\kappa \mid B \cap X \in a\}$ is stationary in $[Y]^\kappa$.*
4. *If a is stationary in $[X]^{<\kappa}$, then $\{B \in [Y]^{<\kappa} \mid B \cap X \in a\}$ is stationary in $[Y]^\kappa$.*

Lemma 1.4 (Normality for stationary sets)**.** *Suppose that a is a stationary set, and that $f \colon a \to \bigcup a$ is such that $f(X) \in X$ for all $X \in a$. Then there is a $z \in \bigcup a$ such that $f^{-1}[\{z\}]$ is stationary.*

Proof. Otherwise, for each $z \in \bigcup a$ there is a function $g_z \colon (\bigcup a)^{<\omega} \to \bigcup a$ such that no $X \in f^{-1}[\{z\}]$ is closed under g_z. Fix a function $h \colon (\bigcup a)^{<\omega} \to \bigcup a$ with the property that $h(z, \bar{y}) = g_z(\bar{y})$ for all $z \in \bigcup a$ and $y \in (\bigcup a)^{<\omega}$. Then if $X \in a$ is closed under h we get a contradiction by considering $z = f(X)$. $\square$

1.5 Definition. We define the following order on stationary sets : $a \leq b$ if $\bigcup b \subseteq \bigcup a$, and

$$\{X \cap \bigcup b \mid X \in a\} \subseteq b.$$

Note that if two stationary sets a, b are compatible in this order, they have a greatest lower bound : $\{X \subseteq (\bigcup a) \cup (\bigcup b) \mid X \cap \bigcup a \in a \wedge X \cap \bigcup b \in b\}$.

1.6 Definition (The stationary tower). Given a limit ordinal $\alpha > \omega$, we let $\mathbb{P}_{<\alpha}$ the (*full tower*) be the restriction to the stationary sets in V_α of the order given in Definition 1.5. Given a cardinal $\kappa < |V_\alpha|$, we let $\mathbb{Q}^\kappa_{<\alpha}$ (the *size-*κ tower, or *countable tower* in the case where $\kappa = \aleph_0$) be the restriction of $\mathbb{P}_{<\alpha}$ to those stationary sets a for which $a \subseteq [\bigcup a]^\kappa$. We write $\mathbb{Q}_\alpha$ for $\mathbb{Q}^{\aleph_0}_\alpha$.

Suppose that $\alpha > \omega$ is a limit ordinal, and that $G \subseteq \mathbb{P}_{<\alpha}$ (or $\mathbb{Q}^\kappa_{<\alpha}$, for some cardinal $\kappa < |V_\alpha|$), we form $\mathrm{Ult}(V, G)$ as follows. Elements of the generic ultrapower are represented by functions $f\colon a \to V$, where $a \in G$ and $f \in V$. Given such functions f and g and a relation $R \in \{\in, =\}$, we say that $[f]_G R [g]_G$ if and only if $\{X \subseteq (\bigcup a) \cup (\bigcup b) \mid X \cap \bigcup a \in a \wedge X \cap \bigcup b \in b \wedge f(X \cap \bigcup a) R g(X \cap \bigcup b)\} \in G$. As usual, we identify the wellfounded part of $\mathrm{Ult}(V, G)$ with its Mostowski collapse, and denote $\mathrm{Ult}(V, G)$ with a single capital letter if it is wellfounded.

1.7 Remark. Suppose that $c \in \mathbb{P}_{<\alpha}$ is club or that $c \in \mathbb{Q}^\kappa_{<\alpha}$ is the intersection of some club subset of $\bigcup c$ with $[\bigcup c]^\kappa$. Then c is an element of every generic filter for its corresponding partial order. The generic ultrapower $\mathrm{Ult}(V, G)$ could alternately defined using functions of the form $f\colon c \to V$ for c's of this type.

We have seen the proof of the following lemma several times already.

Lemma 1.8. *Suppose that G is as in the previous paragraph. For each stationary a and each set x, let $c^a_x\colon a \to \{x\}$ be the corresponding constant function. Then for every set x, $[c^a_x]_G = [c^b_x]_G$ for all $a, b \in G$. The map $j\colon V \to \mathrm{Ult}(V, G)$ sending each x to the common value of $[c^a_x]_G$ for all $a \in G$ is elementary.* x

1.9 Remark. Forcing with $\mathbb{P}_{<\alpha}$ (or $\mathbb{Q}^\kappa_{<\alpha}$) adds a V-ultrafilter $G_X = G \cap \mathcal{P}(\mathcal{P}(X))$ on $\mathcal{P}(X)$, for each suitable $X \in V_\alpha$ (note that these G_X are not necessarily V-generic for the (suitable version of the) partial order of stationary subsets of $\mathcal{P}(X)$ modulo nonstationarity). Let $j_X\colon V \to \mathrm{Ult}(V, G_X)$ be the corresponding elementary embedding for each X. Then if $X \subseteq Y$ are in V_α, there are elementary embeddings $k_{X,Y}\colon \mathrm{Ult}(V, G_X) \to \mathrm{Ult}(V, G_Y)$ defined by setting $k_{X,Y}([f]_{G_X}) = [f^Y]_{G_Y}$, where f^Y is defined by setting $f^Y(Z) = f(Z \cap X)$. Similarly, there are elementary embeddings $k_{X,\infty}\colon \mathrm{Ult}(V, G_X) \to \mathrm{Ult}(V, G)$ defined by setting $k_{X,\infty}([f]_{G_X}) = [f]_G$. The word *tower* refers to the fact that the stationary tower ultrapower can be seen as a direct limit of generic ultrapowers via the G_X's.

1.10 Remark. For any nonempty set X, $\{X\}$ is stationary. In any suitable $\mathbb{P}_{<\alpha}$ or $\mathbb{Q}^\kappa_{<\alpha}$, the condition $\{X\}$ is in the generic filter G if and only if the filter G_X as defined in the previous remark is principal.

Lemma 1.11. *Let $\alpha > \omega$ be a limit ordinal, and let $G \subset \mathbb{P}_{<\alpha}$ (or $\mathbb{Q}^{\kappa}_{<\alpha}$) be a V-generic filter.*

1. *For each set $X \in V_\alpha$, and any $a \in G$ with $X \subseteq \bigcup a$ the function i_x on a defined by setting $i_x(Y) = X \cap Y$ represents $j[X]$ in $\mathrm{Ult}(V, G)$.*
2. *The wellfounded part of $\mathrm{Ult}(V, G)$ contains V_α.*
3. *For each $a \in \mathbb{P}_{<\alpha}$ (or $\mathbb{Q}^{\kappa}_{<\alpha}$), $a \in G$ if and only if $j[\bigcup a] \in j(a)$.*
4. *For each $\beta < \alpha$, $G \cap V_\beta \in \mathrm{Ult}(V, G)$.*

Proof. The first part follows from normality. The second part follows from the fact that for each transitive set X, the transitive collapse of $j[X]$ is X. For the third part, note that $j[\bigcup a]$ is represented by the identity function i on $\mathcal{P}(\bigcup a)$ (or on $[\bigcup a]^{\kappa}$) and $j(a)$ is represented by the constant function c_a from $\mathcal{P}(\bigcup a)$ (or $[\bigcup a]^{\kappa}$) to $\{a\}$. Then in the case of $\mathbb{P}_{<\alpha}$, $j[\bigcup a] \in j(a)$ if and only if

$$\{X \subseteq \bigcup a \mid i(X) \in c_a(a)\} \in G,$$

which holds if and only if $a \in G$. The case of $\mathbb{Q}^{\kappa}_{<\alpha}$ is essentially the same.

For the last part, note that for each $\beta < \alpha$, $j \restriction V_\beta$ is in $\mathrm{Ult}(V, G)$, and $G \cap V_\beta$ is equal to the set of $X \in V_\beta$ for which $j[X] \in j(X)$. ☐

1.12 Remark. The previous lemma shows that each element X of V_α is represented by the function (on any suitable domain) which maps each set Y to the transitive collapse of $X \cap Y$. Similarly, it shows that each set of the form $[f]_G$ also has the form $j(f)(j[\bigcup a])$, for j the generic elementary embedding.

1.13 Exercise. Let $\alpha > \omega$ be a limit ordinal, let $G \subset \mathbb{P}_{<\alpha}$ (or $\mathbb{Q}^{\kappa}_{<\alpha}$) be a V-generic filter, and let $j \colon V \to \mathrm{Ult}(V, G)$ be the corresponding embedding. Then, for any pair of ordinals $\beta \leq \delta$ below α, and any relation $R \in \{\leq, =, \geq\}$, $j(\beta)\, R\, \delta$ if and only if

$$\{X \subset \delta \mid \beta\, R\, \mathrm{ot}(X \cap \delta)\} \in G$$

(or $\{X \in [\beta \cup \kappa]^{\kappa} \mid \beta\, R\, \mathrm{ot}(X \cap \delta)\} \in G$). Show that in the case of a generic embedding j induced by $\mathbb{Q}^{\kappa}_{<\alpha}$, $j(\kappa^+)$ is at least α.

1.14 Exercise. If λ is a cardinal of uncountable cofinality, then λ is a stationary subset of $\mathcal{P}(\bigcup \lambda)$ (but club only in the case $\lambda = \omega_1$). If κ and λ are infinite cardinals, then $\lambda \cap [\bigcup \lambda]^{\kappa}$ is a stationary subset of $[\bigcup \lambda]^{\kappa}$ only in the case $\lambda = \kappa^+$.

1.15 Remark. Even in the case $\lambda = \omega_2$ and $\kappa = \omega_1$, $[\bigcup \lambda]^{\kappa} \setminus \lambda$ can be stationary, i.e., if Chang's Conjecture, the statement that $[\omega_2]^{\omega_1}$ (the set of subsets of ω_2 of ordertype ω_1) is stationary, holds. By Exercise 1.13, the condition $[\omega_2]^{\omega_1}$ (if stationary) forces (in $\mathbb{P}_{<\alpha}$ or $\mathbb{Q}^{\aleph_1}_{\alpha}$) that $j(\omega_1) = \omega_2$.

Lemma 1.16. *If λ is a cardinal of uncountable cofinality, $\alpha > \lambda$ and $G \subseteq \mathbb{P}_{<\alpha}$ is a V-generic filter, then $\lambda \in G$ if and only if the critical point of the induced* *elementary embedding is λ. Similarly, if κ is a cardinal, $\lambda \leq \kappa^+$ is a cardinal, $\alpha > \kappa^+$ is a limit ordinal and $G \subseteq \mathbb{Q}^\kappa_{<\alpha}$ is a V-generic filter, then*

$$\{X \in [\kappa^+]^\kappa \mid X \cap \lambda \in \lambda\} \in G$$

if and only if λ is the critical point of the induced elementary embedding.

Proof. For the first part, $\lambda \in G$ if and only if $j[\bigcup \lambda] \in j(\lambda)$, which holds if and only if $j[\lambda]$ is an ordinal below $j(\lambda)$, which holds if and only if λ is the critical point of j. Similarly, for the second part, $\{X \in [\kappa^+]^\kappa \mid X \cap \lambda \in \lambda\} \in G$ if and only if $j[\kappa^+] \cap j(\lambda) \in j(\lambda)$, which holds if and only if $j[\kappa^+] \cap j(\lambda)$ is an ordinal below $j(\lambda)$, which holds if and only if λ is the critical point of j. □

Recall that the cardinals $\beth_\alpha$ are defined by $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$ and $\beth_\beta = \sup_{\alpha<\beta} \beth_\beta$ when β is a limit ordinal. For any ordinal α, $|V_{\omega+\alpha}| = \beth_\alpha$. A $\beth$-*fixed point* is a cardinal κ for which $\beth_\kappa = \kappa$, i.e., for which $|V_\kappa| = \kappa$.

Lemma 1.17. *Suppose that κ is a $\beth$-fixed point, and that $G \subseteq \mathbb{P}_{<\kappa}$ is a V-generic filter. Then there is an uncountable regular cardinal $\lambda < \kappa$ such that $\lambda \in G$.*

Proof. Let $a \in V_\kappa$ be a stationary set. By genericity, it suffices to find a regular cardinal $\lambda < \kappa$ such that λ is compatible with a. Applying the fact that κ is a $\beth$-fixed point, choose a cardinal $\gamma \in (|\bigcup a|, \kappa)$. Let

$$b = \{X \prec V_{\gamma^+} : |X| < \gamma \wedge \gamma \in X \wedge X \cap \bigcup a \in a\}.$$

Then b is stationary, and $b \leq a$. Define the function $f\colon b \to (\gamma + 1)$ by letting $f(X)$ be the unique ordinal $\beta \in X$ for which $X \cap \beta \in \beta$. Each value taken by f is a regular cardinal, and there is a cardinal λ such that $c = \{X \in b \mid f(X) = \lambda\}$ is stationary. Then $c \leq b$ and $c \leq \lambda$. □

1.18 Exercise. Show that if $\alpha > \omega$ is a limit ordinal, then the generic elementary embedding induced by $\mathbb{P}_{<\alpha}$ always has a critical point.

1.19 Exercise. Suppose that κ is a cardinal, $\alpha > \kappa^+$ is a limit ordinal and $G \subseteq \mathbb{Q}^\kappa_{<\alpha}$ is a V-generic filter. Show that there exists a $\lambda \leq \kappa^+$ such that $\{X \in [\kappa^+]^\kappa \mid X \cap \lambda \in \lambda\} \in G$.

1.20 Exercise. Suppose that κ is a cardinal and $\alpha > \kappa^+$ is a limit ordinal. Show that forcing with $\mathbb{Q}^\kappa_{<\alpha}$ below the condition $\kappa^+ \setminus \kappa$ adds surjections from κ onto each V_β with $\beta < \alpha$.

1.21 Exercise. Suppose that $V{=}L$ and that $\alpha > \omega_1$ is a limit ordinal. Show that there is a function $f\colon \omega_1 \to \omega_1$ such that $[f]_G > \beta$ for all $\beta < \alpha$, whenever $G \subseteq \mathbb{Q}_{<\alpha}$ is V-generic and j is the associated embedding (see Remark 1.22 of the previous set of notes). Note that $j(\omega_1) > [f]_G$.

1.22 Remark. More generally, whenever we force over L with any forcing of the form $\mathbb{P}_{<\alpha}$ or $\mathbb{Q}^{\kappa}_{<\alpha}$, if the critical point of the associated embedding j is a successor cardinal γ^+, then $j(\gamma^+)$ is illfounded, as $\mathrm{Ult}(V, G)$ will satisfy $V{=}L$ but will contain new subsets of γ.

1.23 Extra Credit. Suppose that $\alpha > \omega$ is a limit ordinal but not a $\beth$-fixed point. Must there be a condition $a \in \mathbb{P}_{<\alpha}$ forcing that the critical point of the induced elementary embedding is at least α?

2 Completely Jónsson cardinals

2.1 Definition. A strongly inaccessible cardinal κ is *completely Jónsson* if for all $a \in \mathbb{P}_{<\kappa}$,

$$\{X \subseteq V_\kappa \mid X \cap \bigcup a \in a \wedge |X \cap \kappa| = \kappa\}$$

is stationary.

2.2 Remark. For any ordinal α and any $a \in \mathbb{P}_{<\alpha}$, the set

$$\{X \subseteq V_\kappa \mid X \cap \bigcup a \in a \wedge |X \cap \kappa| \leq |\bigcup a|\}$$

is stationary. If κ is completely Jónsson, then for any cardinal λ such that $|\bigcup a| \leq \lambda \leq \kappa$, the set

$$\{X \subseteq V_\kappa \mid X \cap \bigcup a \in a \wedge |X \cap \kappa| = \lambda\}$$

is stationary.

Note that the statement "κ is completely Jónsson" is computed in $V_{\kappa+1}$.

2.3 Exercise. Show that if we removed from the definition of completely Jónsson the requirement that κ be strongly inaccessible, but required it to be uncountable, it would still follow that κ is a strong limit (indeed, a $\beth$-fixed point), but not that it has uncountable cofinality.

The argument for the following is based on the end-extension property of elementary submodels with measurable cardinals (Theorem 1.20 of the previous set of notes).

Theorem 2.4. *Measurable cardinals are completely Jónsson, and are limits of completely Jónsson cardinals.*

The following follows from Exercise 1.13 and genericity.

Theorem 2.5. *If κ is a limit of completely Jónsson cardinals, and j is a generic elementary embedding derived from forcing with $\mathbb{P}_{<\kappa}$, then $j(\lambda) = \lambda$ for cofinally many $\lambda < \kappa$. If κ is regular then it follows that $j(\kappa) = \kappa$.*

2.6 Remark. In contrast to Lemma 1.11, let us see that if κ is a limit of completeley Jónsson cardinals, and if $G \subseteq \mathbb{P}_{<\kappa}$ is a V-generic filter, then V_κ is not a member of $\mathrm{Ult}(V, G)$. Supposing otherwise, there is for some $\alpha < \kappa$ a stationary $a \subseteq \mathcal{P}(V_\alpha)$ and a function f with domain a forced by a to represent V_κ. We may assume that for each $X \in a$, $f(X)$ is a transitive structure satisfying "there are cofinally many Jónsson cardinals." Increasing α if necessary, we may assume that α is completely Jónsson and that $\mathrm{ot}(X \cap \alpha) = \alpha$ and $\alpha + 1 \subset f(X)$ hold for all $X \in a$.

At least one of the two following sets is stationary.

- b_1, the set of $X \in a$ such that, letting $\bar{\beta}$ be the least completely Jónsson cardinal of $f(X)$ above α, $|f(X) \cap \bar{\beta}| = \alpha$.
- b_2, the set of $X \in a$ such that, letting $\bar{\beta}$ be the least completely Jónsson cardinal of $f(X)$ above α, $|f(X) \cap \bar{\beta}| > \alpha$.

Let β be the least completely Jónsson cardinal above α. If b_1 is stationary, then so is c_1, the set of $Y \prec V_{\beta+1}$ such that $Y \cap V_\alpha \in b_1$ and $|Y \cap \beta| > \alpha$. If b_2 is stationary, then so is c_2, the set of $Y \prec V_{\beta+1}$ such that $Y \cap V_\alpha \in b_2$ and $|Y \cap \beta| = \alpha$.

The function h on $\mathcal{P}(V_{\beta+1})$ sending Y to its transitive collapse represents $V_{\beta+1}$. We have then for a (relative) club of $Y \subseteq V_{\beta+1}$ with $Y \cap \bigcup a \in a$ that the transitive collapse of Y is a rank initial segment of $f(Y \cap V_\alpha)$.

However, if b_1 is stationary, then c_1 contradicts this, as for each $Y \in c_1$, letting $\bar{Y}$ be the transitive collapse of Y and letting $\bar{\beta}$ be the least completely Jónsson cardinal of $\bar{Y}$ above α, $|\bar{Y} \cap \bar{\beta}| > \alpha$. Similarly, if b_2 is stationary, then c_2 gives a contradiction, as for each $Y \in c_2$, letting $\bar{Y}$ be the transitive collapse of Y and letting $\bar{\beta}$ be the least completely Jónsson cardinal of $\bar{Y}$ above α, $|\bar{Y} \cap \bar{\beta}| = \alpha$.

2.7 Exercise. Prove that if δ is a limit of completely Jónsson cardinals, then $\mathbb{P}_{<\delta}$ is not δ-.c.c.. (Hint : what are the possibilities for $j(\omega_1)$?)

2.8 Example. If κ and δ are cardinals, $\alpha > \kappa$ is a limit ordinal and there exist δ many measurable cardinals in the interval (κ, α), then $\mathbb{Q}^\kappa_{<\alpha}$ is not δ-c.c.. To see this, let I be a set of δ many measurable cardinals in the interval (κ, δ) such that no member of I is a limit of members of I. For each λ in I, fix a stationary set A_λ consisting of points whose cofinality is the same as that of κ. By using our usual end-extension trick for measurable cardinals (Theorem 1.20 of the previous set of notes), one can show for each regular cardinal ρ in (κ, δ), that for stationarily many elementary submodels X of V_ρ of cardinality κ, $\sup(X \cap \lambda) \in A_\lambda$ for each λ in $I \cap X$. To make this work for a fixed λ, end-extend λ many times. By the stationarity of A_λ, some supremum was in the desired A_λ; then take a hull of some cofinal sequence in this sup of cardinality κ.

Now, if each A_λ were also costationary on the same cofinality, for each λ in I the set b_λ of $X \prec V_{\lambda^+}$ of cardinality κ for which λ is least in $X \cap I$ with sup$(X \cap \lambda)$ not in A_λ is also stationary, by the same argument. Then the b_λ's form an antichain.

2.9 Exercise. Let $\alpha > \omega$ be a limit ordinal, let $\mathcal{D}$ be a subset of $\mathbb{P}_{<\alpha}$, and let $\beta < \alpha$ be such that $\mathcal{D} \subseteq V_\beta$. Let a be the set of $X \subseteq V_{\beta+1}$ of cardinality less than $|V_\beta|$ such that $\mathcal{D} \in X$ and, for all $d \in \mathcal{D}$, $X \cap \bigcup d \notin d$. Then a is stationary if and only if $\mathcal{D}$ is not predense in $\mathbb{P}_{<\alpha}$, and, if a is stationary, it is incompatible with each element of $\mathcal{D}$. Show that the same construction works for each partial order of the form $\mathbb{Q}^\kappa_\alpha$, requiring instead that $|X| = \kappa$.

2.10 Exercise. Let δ be a strongly inaccessible cardinal, let $\{a_\alpha : \alpha < \delta\}$ be an antichain in $\mathbb{P}_{<\delta}$, and let $C \subseteq \delta$ be a club set of limit ordinals. For each $\alpha < \delta$, let $b_\alpha = \{X \subseteq V_\gamma \mid X \cap \bigcup a_\alpha \in a_\alpha\}$, for γ minimal such that $\gamma \in C$ and $\bigcup a_\alpha \subseteq V_\gamma$. Then each a_α is equivalent to b_α as a condition in $\mathbb{P}_{<\delta}$, so $\mathcal{B} = \{b_\alpha : \alpha < \delta\}$ is an antichain. However, if $\kappa \in \delta \setminus C$, then $\mathcal{B} \cap \mathbb{P}_{<\kappa}$ is not predense in $\mathbb{P}_{<\kappa}$. (Hint : Let β be maximal element of C below κ and use the previous exercise.) Show that the same argument works for partial orders of the form $\mathbb{Q}^\lambda_{<\delta}$ with the appropriate changes.

3 Wellfoundedness

We say that a set Y *end-extends* a set X if $X \subseteq Y$ and $X = Y \cap V_\beta$, for β least such that $X \subseteq V_\beta$.

3.1 Definition. Let κ be a $\beth$-fixed point, and let D be a subset of $\mathbb{P}_{<\kappa}$. We let sp(D) be the set of $X \prec V_{\kappa+1}$ of cardinality less than κ with $D \in X$ for which there exists a $Y \prec V_{\kappa+1}$ satisfying the following.

- $X \subseteq Y$;
- $Y \cap V_\kappa$ end-extends $X \cap V_\kappa$;
- for some $d \in D \cap Y$, $Y \cap (\bigcup d) \in d$.

We say that D is *semi-proper* if sp(D) is club in $[V_\kappa]^{<\kappa}$.

Note that $\mathbb{Q}^\gamma_{<\kappa} \subseteq \mathbb{P}_{<\kappa}$, and we can assume that $|Y| = |X|$, so we do not need a separate definition of semi-properness for $\mathbb{Q}^\gamma_{<\kappa}$.

3.2 Exercise. Show that if $\alpha > \omega$ is not a $\beth$-fixed point, and $\mathbb{P}_{<\alpha}$ contains an antichain not contained in $\mathbb{P}_{<\beta}$ for any $\beta < \alpha$, then there is a predense $D \subseteq \mathbb{P}_{<\alpha}$ which is not semi-proper.

3.3 Exercise. Let κ be a $\beth$-fixed point. Show that every semi-proper subset of $\mathbb{P}_{<\kappa}$ is predense.

3.4 Exercise. Let κ be a $\beth$-fixed point. Show that every predense subset of $\mathbb{P}_{<\kappa}$ in V_κ is semi-proper (with $X = Y$).

The following is the stationary tower version of Lemma 4.10 from the first set of notes.

Lemma 3.5. *Supppose that κ is a $\beth$-fixed point, and let D be a subset of $\mathbb{P}_{<\kappa}$. The following are equivalent.*

- *D is semi-proper;*
- *For any regular cardinal $\lambda > \kappa$, and any $X \prec V_\lambda$ with $\kappa, D \in X$ and $|X| < \kappa$, there is a $Y \prec V_\lambda$ such that*
  - *$X \subseteq Y$;*
  - *$Y \cap V_\kappa$ end-extends $X \cap V_\kappa$;*
  - *$Y \cap (\bigcup d) \in d$ for some $d \in Y \cap D$.*

Proof. The reverse direction follows from upwards projection of stationarity. For the forward direction, fix such an X as given, and let $W \prec V_{\kappa+1}$ be such that the following hold.

- $X \cap V_{\kappa+1} \subseteq W$;
- $W \cap V_\kappa$ end-extends $X \cap V_\kappa$;
- for some $d \in D \cap W$, $W \cap (\bigcup d) \in d$.

Now let Y be the set of values $f(a)$, for f a function in X with domain V_κ, and $a \in W \cap V_\kappa$. Then Y is as desired. To show that $Y \cap V_\kappa$ end-extends $X \cap V_\kappa$, note that every $f\colon V_\kappa \to V_\kappa$ in X is in $X \cap V_{\kappa+1}$ and thus in W. $\square$

3.6 Exercise. Verify the previous lemma in the case where we let λ be $\kappa + \omega$.

3.7 Exercise. Let α be a limit ordinal of uncountable cofinality, and let $C \subseteq \alpha$ be club. Let D be the set of elements $a \in \mathbb{P}_\alpha$ such that $X \cap C$ is cofinal in $X \cap \mathrm{Ord}$ for every $X \in a$. Show that D is predense, and that $\kappa \in C$ for any κ for which $D \cap \mathbb{P}_\kappa$ is semi-proper. Show that the same holds for towers of the form $\mathbb{Q}^{\lambda}_{<\alpha}$.

Lemma 3.8. *Suppose that κ is a strongly inaccessible cardinal and γ is a cardinal below κ such that whenever $\{D_\alpha : \alpha < \gamma\}$ are predense subsets of $\mathbb{P}_{<\kappa}$, there exists a $\beth$-fixed point λ such that each $D_\alpha \cap V_\lambda$ is semi-proper. Then whenever $G \subseteq \mathbb{P}_{<\kappa}$ is a V-generic filter, $\mathrm{Ult}(V, G)$ is closed under sequences of length γ.*

Proof. Let τ_α $(\alpha < \gamma)$ be names for elements of $\mathrm{Ult}(V, G)$. For each $\alpha < \gamma$, let D_α be the set of $b \in \mathbb{P}_{<\kappa}$ for which there exists a function f^α_b with domain b such that b forces τ_α to be the element of $\mathrm{Ult}(V, U)$ represented by f^b_α. Then each D_α is predense. Fix $a \in \mathbb{P}_{<\kappa}$, and, applying Exercise 3.7, fix λ as in the statement of the lemma with $a \in V_\lambda$. Let $\delta > \lambda$ be a regular cardinal. Let c be the set of $X \prec V_\delta$ for which

- $\lambda \in X$;

- $X \cap \bigcup a \in a$;
- for each $\alpha \in X \cap \gamma$ there exists a $b \in D_\alpha \cap X$ such that $X \cap \bigcup b \in b$.

Applying Lemma 3.5, we have that c is stationary, and $c \leq a$. For each $\alpha \in \gamma$, choose a function $h_\alpha\colon c \to V_\lambda$ such that for each $X \in C$ and each $\alpha \in X \cap \gamma$, $h_\alpha(X) \in D_\alpha \cap X$ and $X \cap \bigcup h_\alpha(X) \in h_\alpha(X)$. Define a function g on c by letting each $g(X)$ be the function with domain $\mathrm{ot}(X \cap \gamma)$, such that whenever $\alpha \in X \cap \gamma$ and $\mathrm{ot}(X \cap \alpha) = \beta$, $g(X)(\beta) = f_\alpha^{h_\alpha(X)}$.

Then g represents a function with domain γ, and it suffices to see that for each $\alpha < \gamma$, c forces that the α-th member of the sequence represented by g is equal to the realization of τ_α. To see that this holds, fix α, and fix a condition $d \leq c$. We may assume that $\alpha \in Y$ for all $Y \in d$, and by strengthening d if necessary (via normality) that $h_\alpha(Y \cap V_\lambda)$ is the same value b for all $Y \in d$. Since $X \cap \bigcup b \in b$ for all $X \in d$, $b \leq d$, so d forces that the realization of τ_α is represented by f_α^b. Now, the α-th member of the sequence represented by g is represented by the function g_α on c for which $g_\alpha(X)$ is the $\mathrm{ot}(X \cap \alpha)$-th member of $g(X)$, i.e., $f_\alpha^{h_\alpha(X)}$. It follows that $g_\alpha(Y \cap V_\lambda) = f_\alpha^b(Y \cap V_\lambda)$ for all $Y \in d$, and therefore that the α-th member of the sequence represented by g is equal to the realization of τ_α. □

3.9 Exercise. Prove Lemma 3.8 for towers of the form $\mathbb{Q}^\chi_{<\kappa}$ and $\gamma \leq \chi$.

Theorem 3.10. *Suppose that δ is a Woodin cardinal, and let D_α ($\alpha < \delta$) be predense subsets of $\mathbb{P}_{<\delta}$. Then there is a measurable cardinal $\lambda < \delta$ such that $D_\alpha \cap V_\lambda$ is semi-proper for each $\alpha < \lambda$.*

Proof. Let $f\colon \delta \to \delta$ be a function such that

- for each $\alpha < \delta$, $f(\alpha)$ is a $\beth$-fixed point and each D_β ($\beta < f(\alpha)$) is predense in $\mathbb{P}_{<f(\alpha)}$;
- for each $\beth$-fixed point $\alpha < \delta$ and each $\beta < \alpha$, if $D_\beta \cap V_\alpha$ is not semi-proper, then there is an element of $D_\beta \cap V_{f(\alpha)}$ compatible with

$$[V_{\alpha+1}]^{<\alpha} \setminus \mathrm{sp}(D_\beta \cap V_\alpha).$$

Applying Theorem 5.19 from the first set of notes, fix $\lambda < \delta$ and an elementary embedding $j\colon V \to M$ with critical point λ such that

- $j(f)(\lambda) = f(\lambda)$;
- $V_{f(\lambda)+\omega} \subseteq M$;
- $j(D_\alpha) \cap V_{f(\lambda)} = D_\alpha \cap V_{f(\lambda)}$ for all $\alpha < \lambda$;
- M is closed under sequences of length λ;
- $j(\delta) = \delta$.

We want to see that each $D_\alpha \cap V_\lambda$ is semi-proper. Towards this end, fix an $\alpha < \lambda$. Let $a = [V_{\lambda+1}]^{<\lambda} \setminus \mathrm{sp}(D_\alpha \cap V_\lambda)$. If $D_\alpha \cap V_\lambda$ is not semi-proper, then this holds in M also, and there is a condition $b \in j(D_\alpha) \cap V_{f(\lambda)}$ compatible with a in M. Then $b \in D_\alpha$, and a and b are compatible in V also.

Let X be an elementary submodel of V_δ with $X \cap \bigcup a \in a$, $X \cap \bigcup b \in b$ and the sets D_α, a, b, $j \restriction V_{\lambda+1}$, $j(V_{\lambda+4})$ all in X. Since $\bigcup a = V_{\lambda+1}$ and each element of a has cardinality less than λ, $X \cap V_{\lambda+1}$ has cardinality less than λ. We have then that

- $j(X \cap V_{\lambda+1}) = j[X \cap V_{\lambda+1}]$;
- $j[X \cap V_{\lambda+1}] \in j(a)$;
- $j[X \cap V_{\lambda+1}] \notin j(\mathrm{sp}(D_\alpha \cap V_\lambda))$;
- $j \restriction (X \cap V_{\lambda+1}) \in M$.

Let $\leq^*$ be a wellordering of $j(V_{\lambda+1})$ in both M and X. Since $j \restriction V_{\lambda+1} \in X$, $j[X \cap V_{\lambda+1}] \subseteq X$. Let Y be the Skolem closure in $j(V_{\lambda+1})$ according to $\leq^*$ of

$$\{a, b\} \cup j \restriction (X \cap V_{\lambda+1}) \cup (X \cap (\bigcup a \cup \bigcup b)).$$

Since these set are all in M, Y is in M, and since they are all subsets of X, $Y \subseteq X$. We derive a contradiction by showing that Y witnesses in M that

$$j(X \cap V_{\lambda+1}) \in j(\mathrm{sp}(D_\alpha \cap V_\lambda)).$$

Since the critical point of j is λ, and $Y \subseteq X$, Y end-extends $j(X \cap V_{\lambda+1})$ below $j(\lambda)$. We have already that $j(X \cap V_{\lambda+1}) \subseteq Y$. Finally, we have that $b \in Y$ and $X \cap \bigcup b \in b$. Since $j(\lambda) > f(\lambda)$, $b \in j(D_\alpha \cap V_\lambda)$. Therefore, $b \in Y \cap j(D_\alpha)$ and $Y \cap \bigcup b \in b$, finishing the proof. □

Corollary 3.11. *Suppose that δ is a Woodin cardinal, $G \subseteq \mathbb{P}_{<\delta}$ is a V-generic filter and $j \colon V \to M$ is the corresponding elementary embedding. Then $j(\delta) = \delta$, M is closed under sequences of length less than δ and $V_\delta^M = V_\delta^{V[G]}$.*

Proof. That M is closed under sequences of length less than δ follows from Lemma 3.8 and Theorem 3.10. That $j(\delta) = \delta$ follows from the fact that δ is a limit of measurable cardinals, and thus a limit of completely Jónsson cardinals. It follows then that δ is strongly inaccessible in M. Since M is closed under sequences of length less than δ in $V[G]$, one can prove by induction on $\alpha < \delta$ that $V_\alpha^M = V_\alpha^{V[G]}$ for all $\alpha < \delta$. □

3.12 Exercise. Prove that if δ is a Woodin cardinal, and A_α ($\alpha < \delta$) are antichains in $\mathbb{P}_{<\delta}$, then for densely many $b \in \mathbb{P}_{<\delta}$, the set of $a \in A_\alpha$ compatible with b has cardinality less than δ, for each $\alpha < \delta$.

3.13 Exercise. Prove Theorem 3.10 for $\mathbb{Q}^\chi_{<\delta}$ and D_α ($\alpha < \chi$), for any cardinal $\chi < \delta$. It follows then that generic ultrapower is closed under sequences of length χ. Prove the corresponding version of Exercise 3.12 for χ many antichains. Conclude then that $j(\chi^+) = \delta$, so $V_\delta^M = V_\delta^{V[G]}$.

It is an open question whether the image of the ordinals under the generic embedding induced by $\mathbb{Q}_{<\delta}$ (for δ a Woodin cardinal) is independent of the generic filter, or whether it even can be.

3.14 Definition. Given cardinals κ and λ, κ is λ-*supercompact* if there exists an elementary embedding $j\colon V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and M is closed under sequences of length λ.

3.15 Exercise. Prove that if κ is 2^κ-supercompact, then every predense subset of $\mathbb{P}_{<\kappa}$ (or $\mathbb{Q}^\lambda_{<\kappa}$, for any cardinal $\lambda < \kappa$) is semi-proper. (Hint: Fix an embedding $j\colon V \to M$ witnessing that κ is 2^κ semiproper, and fix a regular cardinal $\eta > j(\kappa)$. Suppose that some predense $D \subseteq \mathbb{P}_{<\kappa}$ is not semi-proper, and let $a = [V_{\kappa+1}]^{<\kappa} \setminus \mathrm{sp}(D)$. Find a $b \in j(D)$ compatible with a in $j(\mathbb{P}_{<\kappa})$ and an elementary submodel $Y \prec V^M_\eta$ in M with $Y \cap \bigcup a \in a$, $Y \cap \bigcup b \in b$ and $\{a, b, j{\restriction}V_{\kappa+1}\} \in M$. Let $Y = X \cap V_{\kappa+1}$. Show that $j(Y) \in j(a)$ and that $X \cap j(V_{\kappa+1})$ contradicts this.

Exercise 3.15 holds for strongly compact cardinals as well.

4 Forcing applications

4.1 Remark. It is a classical forcing fact that if $\mathbb{B}$ and $\mathbb{C}$ are complete Boolean algebras, and there exists a $\mathbb{B}$-name σ for a V-generic filter for $\mathbb{C}$ with the property that for each $A \in \mathbb{C}$, $[\![\check{A} \in \sigma]\!] \neq 0_{\mathbb{B}}$, then there is a $\mathbb{C}$-name τ for a complete Boolean algebra such that $\mathbb{B}$ is forcing-equivalent to $\mathbb{C} * \tau$. To see this, let $\pi\colon \mathbb{C} \to \mathbb{B}$ be the embedding defined by letting $\pi(A) = [\![\check{A} \in \sigma]\!]$. Let τ be a $\mathbb{C}$-name for the Boolean algebra formed by taking the quotient of $\mathbb{B}$ by the π-image of the complement of the generic filter. That is, let τ be such that whenever G is V-generic for $\mathbb{C}$, τ_G is the set of equivalence classes of the relation $\sim$ on $\mathbb{B}$ defined by setting $E \sim F$ if and only if $E \triangle F \leq \pi(A)$ for some $A \in \mathbb{C} \setminus G$. Then conditions in $\mathbb{C} * \tau$ can be represented as pairs (A, ρ_E), where $A \in \mathbb{C}$, $E \in \mathbb{B}$, $E \cap \pi(A') \neq 0_{\mathbb{B}}$ for all $A' \leq A$ in $\mathbb{C}$ and ρ_E is a $\mathbb{C}$-name for the $\sim$-class of E. We can map such pairs into $\mathbb{B}$ by setting $\nu(A, \rho_E) = E \cap \pi(A)$.

4.2 Exercise. Show that the embedding ν defined in Remark 4.1 maps V-generic filters for $\mathbb{C} * \tau$ to V-generic filters for $\mathbb{B}$, and that ν^{-1} does the reverse.

4.3 Remark. If P and Q are partial orders, and forcing with $\mathbb{Q}$ makes $\mathcal{P}(P)^V$ countable, then there is a Q-name for a V-generic filter for P with the property that every condition in P is forced by some condition in Q to be in this filter. By Remark 4.1, then, there is a P-name τ for a partial order such that $P * \tau$ is forcing equivalent to Q.

By Remark 1.20, or the last part of Remark 1.13, if P is a partial order in V_α (for α a limit ordinal greater than ω), then forcing with $\mathbb{Q}_{<\alpha}$ makes $\mathcal{P}(P)^V$ countable, as does forcing with $\mathbb{P}_{<\alpha}$ below the condition $[\mathcal{P}(P)]^{\aleph_0}$.

4.4 Exercise. Show that every set which is generic over L exists (in a forcing extension) in a class sized model of $V{=}L$. While the model must be illfounded if the set in question is not in L, the model can be made wellfounded up to any desired ordinal, though its ω_1 will be illfounded if it is an generic ultrapower of L via a partial order of the form $\mathbb{Q}_{<\alpha}$.

The following theorem is a central part of Woodin's theory of Ω-logic. One interesting application of the theorem is the case where there exists exactly one huge cardinal or extendible cardinal or cardinal κ which is 2^κ-supercompact, and Q makes this cardinal countable.

Theorem 4.5. *Suppose that δ is a Woodin cardinal, $\alpha < \delta$, P and Q are partial orders in V_δ and P forces that $V_\alpha \models T$, for some theory T. Then there is a Q-name τ for a partial order such that, in the $Q * \tau$ extension there is an ordinal β such that $V_\beta \models T$.*

Proof. By Remark 4.3, there is a Q-name τ_0 such that $Q * \tau_0$ is forcing-equivalent to the partial order $\mathbb{P}_{<\delta}$ below $[\mathcal{P}(Q)]^{\aleph_0}$. Let $G \subset \mathbb{P}_{<\delta}$ be a V-generic filter with $[\mathcal{P}(Q)]^{\aleph_0} \in G$, and let $j \colon V \to M$ be the corresponding embedding. Then $j(\delta) = \delta$, and, by Corollary 3.11, $V^M_\delta = V^{V[G]}_\delta$. In M, $j(P)$ forces that $V_{j(\alpha)}$ models T. Since δ is strongly inaccessible in $V[G]$, in $V[G]$ also $j(P)$ forces that $V_{j(\alpha)}$ models T. □

4.6 Example. Suppose that $\gamma < \lambda < \kappa$ are regular cardinals below a Woodin cardinal κ. Forcing with $\mathbb{P}_{<\kappa}$ below the condition $b = \{\alpha < \lambda \mid \mathrm{cof}(\alpha) = \gamma\}$, we get that $j[\cup b] \in j(b)$, i.e., that in M, $j[\lambda]$ is an ordinal below $j(\lambda)$ of cofinality $j(\gamma)$. This means that the critical point of j is λ, and that $\mathrm{cof}(\lambda) = \gamma$ in M. Furthermore, if α is such that $2^\alpha < \lambda$, then all subsets of α in M are in V. Since $V^M_\kappa = V^{V[G]}_\kappa$, these facts hold in $V[G]$ also.

4.7 Example. Suppose that κ is a measurable cardinal, and let $\lambda > \kappa$ be a regular cardinal. Let a be the set of $X \prec V_\kappa$ such that $\mathrm{ot}(X \cap \kappa)$ and such that the transitive collapse of X is constructible from a real. Then a is stationary. To see this, fix a regular cardinal $\lambda > \kappa$ and a function $F \colon V^{<\omega}_\kappa \to V_\kappa$. Let $Y \prec V_\lambda$ be countable with κ and F in Y, and let U be a normal uniform ultrafilter on κ in Y. Let M be the transitive collapse of X. As in Theorem 1.20 from the first set of notes, we can successively produce Y_α ($\alpha \leq \kappa$) such that $Y_0 = Y$ and each $Y_{\alpha+1}$ has the form $Y_\alpha[\gamma]$ for some $\gamma \in \bigcap(Y_\alpha \cap U)$, taking unions at limits. Then Y_κ will be an elementary submodel of V_λ with $\mathrm{ot}(Y_\kappa \cap \kappa) = \kappa$, and Y_κ will be closed under F. Furthermore, letting M_α ($\alpha \leq \kappa$) be the transitive collapse of Y_α, the sequence $\langle M_\alpha : \alpha < \kappa \rangle$ is the length κ-iteration of M by the image of U under the transitive collapse of Y. It follows that each M_α is in $L[M]$, including the transitive collapse of Y_κ.

Let $\delta > \lambda$ be a Woodin cardinal. Forcing with $\mathbb{P}_{<\delta}$ below a then produces a real from which V^V_κ is constructible. The associated elementary embedding maps κ to κ, so κ is still measurable in M and thus in $V[G]$.

5 Factoring

Lemma 5.1. *Suppose that δ is a Woodin cardinal, and $\alpha > \delta$ is a limit ordinal. Let a be the set of $X \prec V_{\delta+1}$ such that for every predense $D \subseteq \mathbb{Q}_{<\delta}$ in X there exists a $d \in X \cap D$ with $X \cap \bigcup d \in d$. Then a is stationary, and a is compatible with every element of $\mathbb{Q}_{<\delta}$. Furthermore, if $G \subseteq \mathbb{P}_{<\alpha}$ is a V-generic filter, then $a \in G$ if and only if $G \cap \mathbb{Q}_{<\delta}$ is a V-generic filter for $\mathbb{Q}_{<\delta}$.*

Proof. To show that a is stationary and compatible with every element of $\mathbb{Q}_{<\delta}$, fix a set $b \in \mathbb{Q}_{<\delta}$, a function $F\colon V_{\delta+1}^{<\omega} \to V_{\delta+1}$, a regular cardinal $\lambda > \delta$ and a countable $X \prec V_\lambda$ with $F \in X$ and $X \cap \bigcup b \in b$. Fix a bijection $\pi\colon \omega \to \omega \times \omega$ such that $\pi(i) \in (i+1)^2$ for all $i \in \omega$, and let π_0 and π_1 be functions on ω such that $\pi(n) = (\pi_0(n), \pi_1(n))$ for all $n \in \omega$. Recursively build a sequence $\langle X_\alpha : \alpha \leq \omega\rangle$ of countable elementary submodels of V_λ, a sequence $\langle e_i : i < \omega\rangle$ and an increasing sequence of ordinals $\langle \xi_i : i < \omega\rangle$ such that

1. $X_0 = X$;
2. $b \in V_{\xi_0}$;
3. for each $i < \omega$ e_i is a surjection from ω onto the predense subsets of $\mathbb{Q}_{<\delta}$ in X_i;
4. for all $i < j < \omega$, $X_i \cap V_{\xi_i} = X_j \cap V_{\xi_i}$;
5. for each $i < \omega$, for some $d \in e_{\pi_0(i)}(\pi_1(i))$, $X_{\alpha+1} \cap \bigcup d \in d$;
6. $X_\omega = \bigcup_{i<\omega} X_i$.

To achieve item (5), note that by Theorem 3.10, there exists in X_i an ordinal ξ, greater than ξ_j for all $j < i$, such that $e_{\pi_0(i)}(\pi_1(i)) \cap \mathbb{P}_{<\xi}$ is semi-proper.

For the last part of the theorem, suppose that $G \subseteq \mathbb{P}_{<\alpha}$ is a V-generic filter. Then $a \in G$ if and only if $j[\bigcup a] \in j(a)$ if and only if $j[V_{\delta+1}] \in j(a)$ if and only if, in M, for each predense $D \subseteq j(\mathbb{Q}_{<\delta})$ in $j[V_{\delta+1}]$ there is a $d \in D \cap j[V_{\delta+1}]$ with $j[V_{\delta+1}] \cap \bigcup d \in d$ if and only if for each predense $D \subseteq \mathbb{Q}_{\delta+1}$ there is a $d \in D$ such that $j[V_{\delta+1}] \cap \bigcup j(d) \in j(d)$. Finally, note that

$$j[V_{\delta+1}] \cap \bigcup j(d) = j[V_{\delta+1}] \cap j(\bigcup d) = j[\bigcup d],$$

and $j[\bigcup d] \in d$ if and only if $d \in G$. □

5.2 Remark. In the previous lemma, $\mathbb{P}_{<\alpha}$ can be replaced with a partial order of the form $\mathbb{Q}^\kappa_{<\alpha}$.

5.3 Exercise. Suppose that $\langle \delta_i : i < \omega\rangle$ is an increasing sequence of Woodin cardinals, with supremum λ. Let a be the set of $X \prec V_{\lambda+1}$ of cardinality less than δ_0 such that $\langle \delta_i : i < \omega\rangle \in X$, and for all $i < \omega$ and every predense $D \subseteq \mathbb{P}_{<\delta_i}$ in X there exists a $d \in X \cap D$ with $X \cap \bigcup d \in d$. Show that a is stationary.

5.4 Remark. Suppose that $\alpha < \beta$ are limit ordinals greater than ω, and that $G \subseteq \mathbb{P}_{<\alpha}$ is a V-generic filter such that $G \cap \mathbb{P}_{<\alpha}$ is V-generic for $\mathbb{P}_{<\alpha}$. Let $j_\alpha \colon V \to \mathrm{Ult}(V, G \cap \mathbb{P}_\alpha)$ and $j_\beta \colon V \to \mathrm{Ult}(V, G)$ be the corresponding elementary embeddings. Then the embedding $k \colon \mathrm{Ult}(V, G \cap \mathbb{P}_{<\alpha}) \to \mathrm{Ult}(V, G)$ defined by setting $k([f]_{G \cap \mathbb{P}_{<\alpha}}) = [f]_G$ is elementary, and has critical point at least α. A similar fact holds for partial orders of the form $\mathbb{Q}^\kappa_{<\beta}$.

6 Absoluteness for the Chang Model

The *Chang Model* is $L(\mathrm{Ord}^\omega)$, the constructible closure of the class of all countable sequences of ordinals. It is a model of ZF + DC, but not necessarily a model of the Axiom of Choice, as shown by Kunen. Solovay's theorem on collapsing a strongly inaccessible cardinal applies to the Chang Model, showing the following.

Theorem 6.1 (Solovay)**.** *If κ is a strongly inaccessible cardinal, then after forcing with $\mathrm{Col}(\omega, {<}\kappa)$, every set of reals in the Chang Model is Lebesgue measurable, and satisfies the perfect set property and the property of Baire.*

Theorem 6.2. *Suppose that δ is a Woodin limit of Woodin cardinals. Then in a forcing extension there is an elementary embedding from the Chang Model of V to the Chang Model of a forcing extension of V by $\mathrm{Col}(\omega, {<}\delta)$.*

Proof. Let $G \subseteq \mathbb{Q}_{<\delta}$ be a V-generic filter, and let $j \colon V \to M$ be the corresponding elementary embedding. Then M and $V[G]$ have the same Chang Model, and j maps the Chang Model of V to the Chang Model of M. Working in $V[G]$, let P be the partial order consisting of V-generic filters for partial orders of the form $\mathrm{Col}(\omega, {<}\alpha)$, for some $\alpha < \delta$, ordered by extension. Since $\delta = \omega_1^{V[G]}$ and δ is strongly inaccessible in V, there exist such generic filters for each such α.

Let H be $V[G]$-generic for P. It suffices to see that H is V-generic for $\mathrm{Col}(\omega, {<}\delta)$, and that $V[G]$ and $V[H]$ have the same Chang Model. That H is V-generic for $\mathrm{Col}(\omega, {<}\delta)$ follows from the fact that if $D \subseteq \mathrm{Col}(\omega, {<}\delta)$ is predense and $h \subseteq \mathrm{Col}(\omega, {<}\alpha)$ is V-generic, for some $\alpha < \delta$, then $p \cap \mathrm{Col}(\omega, {<}\alpha) \in h$ for some $p \in D$. Fixing $\beta < \delta$ such that $p \in \mathrm{Col}(\omega, {<}\beta)$, we have that $\mathrm{Col}(\omega, {<}\beta)$ is isomorphic to $\mathrm{Col}(\omega, {<}\alpha) \times \mathrm{Col}(\omega, {<}[\alpha, \beta))$, which means that there is a V-generic filter $h' \subseteq \mathrm{Col}(\omega, {<}\beta)$ extending h with $p \in h'$. By genericity, then, $H \cap D$ is nonempty.

To see that $V[G]$ and $V[H]$ have the same Chang Model, note first of all that $\delta = \omega_1^{V[H]}$, which means that every countable set of ordinals in $V[H]$ is in $V[H \cap \mathrm{Col}(\omega, \alpha)]$ for some $\alpha < \delta$, and thus in $V[G]$. For the reverse inclusion, suppose that x is a countable set of ordinals in $V[G]$, and that $h \in V[G]$ is V-generic for $\mathrm{Col}(\omega, {<}\alpha)$, for some $\alpha < \delta$. By Lemma 5.1, and the fact that $\delta = \omega_1^{V[G]}$, we may fix a $\beta < \delta$ such that h and x are in $V[G \cap \mathbb{Q}_{<\beta}]$. By Remark 4.1, there is a $\mathrm{Col}(\omega, {<}\alpha)$-name τ such that $\mathrm{Col}(\omega, {<}\alpha) * \tau$ is forcing equivalent to $\mathbb{Q}_{<\beta}$. Moreover, using a fixed $\mathbb{Q}_{<\beta}$-name ρ for which $h = \rho_{G \cap \mathbb{Q}_{<\beta}}$, we may (applying the proof of Remark 4.1) choose τ so that $V[G \cap \mathbb{Q}_{<\beta}]$ is a generic extension of $V[h]$ via τ. Furthermore, there exist an ordinal $\gamma < \delta$ and a τ-name $\sigma \in V[h]$ such that, in $V[h]$, $\tau * \sigma$ is forcing equivalent to $\mathrm{Col}(\omega, <[\alpha, \gamma))$. As $(2^\gamma)^{V[h]}$ is countable in $V[G]$, there exists in $V[G]$ a $V[h]$-generic filter $h' \subseteq \mathrm{Col}(\omega, <[\alpha, \gamma))$ such that $G \cap \mathbb{Q}_{<\beta} \in V[h, h']$. $\square$

The following classical theorem is due to McAloon.

6.3 Exercise. Show that if P is a partial order such that forcing with P makes P countable, then P is forcing-equivalent to $\mathrm{Col}(\omega, P)$. (Hint : Fix a P-name τ for a bijection between ω and the generic filter. Recursively build a function $\pi\colon \mathrm{Col}(\omega, P) \to P$ in such a way that $\pi(\emptyset) = 1_P$, and, for all $a \in \mathrm{Col}(\omega, P)$, $\pi[\{a \cup (|a|, p) : p \in P\}]$ is a maximal antichain below $\pi(a)$ and $\pi(a)$ decides $\tau \restriction |a|$. To see that the range of π is dense in P, fix $p \in P$ and a condition $p' \leq p$ forcing that $\tau(p) = n$, for some $n \in \omega$. Fix $a \in \mathrm{Col}(\omega, P)$ of length greater than n such that $\pi(a)$ and p' are compatible, and show that $\pi(a) \leq p$.)

6.4 Remark. The previous exercise shows that for any cardinal κ, and any partial order P of cardinality less than κ, $P \times \mathrm{Col}(\omega, <\kappa)$ is forcing-equivalent to $\mathrm{Col}(\omega, <\kappa)$ (as for any $\gamma > |P|$, $\mathrm{Col}(\omega, <\{\gamma\})$ can be replaced with $P \times \mathrm{Col}(\omega, <\{\gamma\})$). Combined with Theorem 6.2, this shows that if δ is a Woodin limit of Woodin cardinals, than no forcing of cardinality less than δ can change the theory of the Chang Model. The large cardinal hypothesis required for this result is much weaker. It suffices to assume that δ is a limit of Woodin cardinals, and that there is a measurable cardinal above δ, and even this can be weakened.

6.5 Exercise. Suppose that $\langle \delta_\alpha : \alpha \leq \omega \rangle$ are Woodin cardinals, listed in increasing order, and let $\lambda = \sup\{\delta_i : i < \omega\}$. Show that the $L(\mathbb{R})$ of V is elementarily equivalent to a model of the form $L(\mathbb{R}^*)$ in a forcing extension of V by $\mathrm{Col}(\omega, <\lambda)$, where $\mathbb{R}^*$ is the set of reals existing in models of the form $V[G \cap \mathrm{Col}(\omega, <\gamma)$ for some $\gamma < \lambda$, where G is the generic filter for $\mathrm{Col}(\omega, <\lambda)$. (Hint : Force with $\mathbb{Q}_{<\delta_\omega}$ below the condition a from Exercise 5.3. For each $i < \omega$, let $j_i\colon V \to M_i$ be the embedding induced by $G \cap \mathbb{Q}_{<\delta_i}$. Let N be the direct limit of the models M_i ($i \in \omega$) via the factor embeddings defined in Remark 5.4. Show that N embeds into M_ω, and is therefore wellfounded. Now mimic the proof of Theorem 6.2 to show that the $L(\mathbb{R})$ of N is the same as a model of the form $L(\mathbb{R}^*)$ of a forcing extension of V by $\mathrm{Col}(\omega, <\lambda)$.)

7 $\mathbb{R}^\#$

Our next goal is to sketch a proof of the fact that, assuming sufficiently many Woodin cardinals, every set of reals is weakly homogeneously Suslin. This will be useful in the development of $\mathbb{P}_{\max}$. We will sketch the relationship between this fact and Woodin's theorem that the existence of infinitely many Woodin cardinals below a measurable implies that the Axiom of Determinacy holds in $L(\mathbb{R})$. This will involve some black boxes, however. The first of these is the set $\mathbb{R}^\#$.

Very briefly, given a set of reals A such that $L(A)\cap\mathbb{R}=A$, $A^{\#}$ is a complete, consistent theory in the language of set theory expanded by adding constant symbols c_x for each $x\in A$ and constant symbols i_n $(n\in\omega)$ which represent ordinal indiscernibles. This theory in effect gives a recipe which for any ordinal α builds a model $\Gamma(A^{\#},\alpha)$ of ZF + V=$L(\mathbb{R})$ whose reals are exactly A, where we have α many ordinals playing the role of the indiscernibles. Note that in $L(\mathbb{R})$, every set is definable from a finite set of reals and ordinals, and for each fixed finite set of reals a the class of sets ordinal definable from a has a definable wellordering. In the models $\Gamma(A^{\#},\alpha)$, every set is definable from a finite set indiscernibles and elements of A, and the theory $A^{\#}$ explicitly defines the relations $\in$ and $=$ on these terms.

Let us black box the following facts about $A^{\#}$.

- If U is a normal uniform measure on a cardinal κ, then there is a set $X\in U$ such that for each $n\in\omega$, any two increasing n-tuples from X satisfy the same formulas in $L_\kappa(\mathbb{R})$, allowing constants for real numbers. The theory satisfied by the finite tuples from X is $\mathbb{R}^{\#}$ (in fact, a completely Jónsson cardinal is enough).

- There is a Δ_0 formula ϕ such that, for any set of reals A for which $A=\mathbb{R}\cap L(A)$, $A^{\#}$ is the unique set $B\subseteq A$ such that
  - $\langle A,\{B\},\in\rangle\models\phi(B)$;
  - for all countable ordinals α, $\Gamma(B,\alpha)$ is wellfounded.

- In each model $\Gamma(A^{\#},\alpha)$, the indiscernibles form a a club of ordertype α, and for all cardinals $\kappa>|A|$, the ordinal height of $\Gamma(\mathbb{R}^{\#},\kappa)$ is κ, and κ is an indiscernible of $\Gamma(A^{\#},\alpha)$ for all $\alpha>\kappa$.

- Every set of real in $L(A)$ is definable from $A^{\#}$ and a finite sequence of elements of A.

Since $\mathbb{R}^{\#}$ is a definable element of the Chang model, we have that if δ is a Woodin limit of Woodin cardinals, if $V[G]$ is any generic extension via a partial order of cardinality less than δ, then $(\mathbb{R}^{\#})^V\subseteq(\mathbb{R}^{\#})^{V[G]}$.

Now suppose that λ is the limit of an increasing sequence of Woodin cardinals $\langle\delta_i:i<\omega\rangle$, and that $\alpha>\lambda$ is a limit ordinal. Let a be the set of countable $X\prec V_{\lambda+1}$ such that for all $i<\omega$, and every predense $D\subset\mathbb{Q}_{<\delta_i}$ in X, there is a $d\in X\cap\bigcup D$ such that $X\cap\bigcup d\in d$. Then $a\in\mathbb{P}_{<\alpha}$, and a forces in $\mathbb{P}_{<\alpha}$ that $G\cap\mathbb{Q}_{<\delta_i}$ is V-generic for each $i<\omega$.

The generic filter G then gives a sequence of embeddings $j_i:V\to M_i$, each induced by $G\cap\mathbb{Q}_{<\delta_i}$, with factor embeddings and limit model N. Adapting the argument above for the Chang Model, one can force over $V[G]$ to find a V-generic $H\subset\mathrm{Col}(\omega,\lambda)$ such that $\mathbb{R}^N=\bigcup\{\mathbb{R}^{V[H\cap\mathrm{Col}(\omega,\alpha)]}:\alpha<\lambda\}$. Assuming enough wellfoundedness for N (which holds if α is Woodin or even completely Jónsson or less), we get that $(\mathbb{R}^{\#})^N=(\mathbb{R}^N)^{\#}$. Again, this gives us that whenever $V[G]$ is any generic extension via a partial order of cardinality less than λ, then $(\mathbb{R}^{\#})^V\subseteq(\mathbb{R}^{\#})^{V[G]}$.

8 Towers of measures

8.1 Definition. A *tower of measures* is a sequence $\langle \mu_i : i \in \omega \rangle$ such that, for some fixed set Z,

- each μ_i is an ultrafilter on Z^i;
- for all $i < j < \omega$ and all $A \subseteq Z^i$, $A \in \mu_i$ if and only if

$$\{\sigma \in Z^j \mid \sigma \restriction i \in A\} \in \mu_i\}.$$

Note that μ_0 is always the trivial ultrafilter on $\emptyset$.

A tower of measures $\bar{\mu} = \langle \mu_i : i < \omega \rangle$ gives rise to a sequence of elementary embeddings $j_i \colon V \to \mathrm{Ult}(V, \mu_i)$, and for each $i_0 < i_1 < \omega$ there is an elementary factor embedding $k_{i_0,i_1} : \mathrm{Ult}(V, \mu_{i_0}) \to \mathrm{Ult}(V, \mu_{i_1})$ defined by setting $k_{i_0,i_1}([f]_{\mu_0}) = [f']_{\mu_1}$, where $f'(\sigma) = f(\sigma \restriction i_0)$ for all $\sigma \in Z^{i_1}$ (and Z is such that each μ_i concentrates on Z^i).

The direct limit of the embeddings k_{i_0,i_1} gives rise to a limit model $\mathrm{Ult}(V, \bar{\mu})$, whose elements are represented by functions $f \colon Z^i \to V$, for some $i < \omega$. For a relation $R \in \{\in, =\}$, we set $[f]_{\bar{\mu}} R [g]_{\bar{\mu}}$ if and only if

$$\{\sigma \in Z^{i_2} \mid f(\sigma \restriction i_0) R g(\sigma \restriction i_1)\} \in \mu_{i_2},$$

for $f \colon Z^{i_0} \to V$, $g \colon Z^{i_1} \to V$ and $i_2 = \max\{i_0, i_1\}$.

8.2 Definition. A tower of measures $\langle \mu_i : i < \omega \rangle$ is said to be *countably complete* if, for each sequence $\langle A_i : i \in \omega \rangle$ such that each $A_i \in \mu_i$, there is a sequence σ of length ω such that $\sigma \restriction i \in A_i$ for all $i \in \omega$. We call such a σ a *thread* through $\langle A_i : i < \omega \rangle$.

8.3 Exercise. Suppose that $\langle \mu_i : i < \omega \rangle$ is a countably complete tower of measures. Show that each μ_i is countably complete, which implies that $\mathrm{Ult}(V, \mu_i)$ is wellfounded. Show furthermore that $\mathrm{Ult}(V, \bar{\mu})$ is wellfounded.

Lemma 8.4. *If $\langle \mu_i : i < \omega \rangle$ is a tower of measures which is not countably complete, then* $\mathrm{Ult}(V, \bar{\mu})$ *is not wellfounded*

Proof. By Exercise 1.9 from the first set of notes, a failure of countable completeness for any μ_i gives that $\mathrm{Ult}(V, \mu_i)$ is illfounded, which implies that $\mathrm{Ult}(V, \bar{\mu})$ is illfounded since $\mathrm{Ult}(V, \mu_i)$ embeds into it. So assume that each μ_i is countably complete and fix $\langle A_i : i\omega \rangle$ such that each $A_i \in \mu_i$ but there is not thread through $\langle A_i : i < \omega \rangle$. Replacing each A_i with $\bigcap_{n \in \omega \setminus i} \{\sigma \restriction i \mid \sigma \in A_n\}$ be may assume that for all $i_0 < i_1 < \omega$ and all $\sigma \in A_{i_1}$, $\sigma \restriction i_0 \in A_{i_0}$. Then $T = \bigcup_{i \in \omega} A_i$ is a wellfounded tree, and there is a function f on T defined by letting each $f(\sigma)$ be the least ordinal greater than $f(\tau)$ for all $\tau \in T$ properly extending σ (i.e., a *rank function* on T). For each $i < \omega$, let $g_i \colon Z^i \to V$ (where each μ_i is an ultrafilter on Z^i) be such that $g_i(\sigma) = f(\sigma)$ for all $\sigma \in A_i$. Then the functions g_i $(i \in \omega)$ represent a descending ω-sequence in $\mathrm{Ult}(V, \bar{\mu})$. $\square$

8.5 Definition. Given a cardinal κ, a set $A \subset \omega^\omega$ is *κ-homogeneously Suslin* if, for some set Z, there is a set $\{\mu_s \mid s \in \omega^{<\omega}\}$ such that

- each μ_s is a κ-complete ultrafilter on $Z^{|s|}$;
- A is the set of $x \in \omega^\omega$ for which $\langle \mu_{x\restriction i} : i < \omega\rangle$ is a countably complete tower.

We say that A is *homogeneously Suslin* if it is $\aleph_1$-homogeneously Suslin. We say that A is *$<\kappa$-homogeneously Suslin* if it is γ-homogeneously Suslin for all $\gamma < \kappa$.

8.6 Definition. Given a cardinal κ, a set $A \subset \omega^\omega$ is *κ-weakly homogeneously Suslin* if, for some set Z, there is a set $\{\mu_{s,t} \mid s, t \in \omega^{<\omega}\}$ such that

- each $\mu_{s,t}$ is a κ-complete ultrafilter on $Z^{|s|}$;
- A is the set of $x \in \omega^\omega$ for which there exists a $y \in \omega^\omega$ such that

  $$\langle \mu_{x\restriction i, y\restriction i} : i < \omega\rangle$$

  is a countably complete tower.

We say that A is *weakly homogeneously Suslin* if it is $\aleph_1$-homogeneously Suslin. We say that A is *$<\kappa$-weakly homogeneously Suslin* if it is γ-homogeneously Suslin for all $\gamma < \kappa$.

We will black box the following facts, which together give the relationship between Woodin cardinals and determinacy in the projective hierarchy (i.e., that the existence of n Woodin cardinals below a measurable cardinal implies that all $\undertilde{\Pi}^1_{n+1}$ sets are determined). The Martin-Steel theorem also shows that if λ is a limit of Woodin cardinals, then the $<\lambda$-weakly homogeneously Suslin sets are exactly the $<\lambda$-homogeneously Suslin sets.

Theorem 8.7 (Martin)**.** *If κ is a measurable cardinal, then coanalytic sets are κ-homogeneously Suslin.*

Theorem 8.8 (Martin)**.** *Homogeneously Suslin subsets of ω^ω are determined.*

Theorem 8.9 (Martin-Steel)**.** *If δ is a Woodin cardinal and $A \subseteq \omega^\omega$ is weakly homogeneously Suslin, then $\omega^\omega \setminus A$ is $<\delta$-homogeneously Suslin.*

8.10 Definition. For our purposes, a *tree* on a set X is a set of finite sequences from X, closed under initial segments. Given a set X and a tree T on $\omega \times X$, the *projection* of T, $\mathrm{p}[T]$, is the set of $x \in \omega^\omega$ such that for some $y \in X^\omega$, $(x\restriction n, y\restriction n) \in T$ for all $n \in \omega$.

8.11 Definition. Given a cardinal κ, a set $A \subseteq \omega^\omega$ is *κ-universally Baire* if for some ordinal γ there are trees S and T on $\omega \times \gamma$ such that $A = \mathrm{p}[S]$ and, in all forcing extensions by partial orders of cardinality at most κ, $\mathrm{p}[S] = \omega^\omega \setminus \mathrm{p}[T]$. We say that A is *$<\kappa$-universally Baire* if it is λ-universally Baire for all $\lambda < \kappa$, and *universally Baire* if it is κ-universally Baire for all cardinals κ.

8.12 Remark. By McAloon's result, the definition of κ-universally Baire does not change if one replaces "by partial orders of cardinality at most κ" with $\mathrm{Col}(\omega, \kappa)$.

The next two theorems show that when λ is a limit of Woodin cardinals, the $<\lambda$-weakly homogeneously Suslin subsets of ω^ω are exactly the $<\lambda$-universally Baire sets. The second of these theorems uses the stationary tower.

Theorem 8.13 (Martin-Solovay)**.** *If κ is a cardinal, and $A \subseteq \omega^\omega$ is κ-weakly homogeneously Suslin, then A is $<\kappa$-universally Baire.*

Theorem 8.14 (Woodin)**.** *Suppose that δ is a Woodin cardinal and S, T are trees on $\omega \times \gamma$, for some ordinal γ such that S and T project to complements in all generic extensions via $\mathbb{Q}_{<\delta}$. Then $\mathrm{p}[S]$ is $<\delta$-weakly homogeneously Suslin.*

The following theorem is known as the *Tree Production Lemma*. It is our primary means of showing that sets of reals are universally Baire. Note that whenever r is subset of V which exists in a set generic extension of V, $V[r]$ is also a set-generic extension of V (via the complete Boolean subalgebra generated by the terms $[\![x \in \tau]\!]$, for each x in some superset of r in V, and a name τ giving rise to r). The proof of the Lemma uses the stationary tower.

Theorem 8.15 (Woodin)**.** *Suppose that δ is a Woodin cardinal. Let ϕ and ψ be binary formulas, let x and y be sets, and assume that the empty condition in $\mathbb{Q}_{<\delta}$ forces that for each real number r,*

$$M \models \phi(r, j(y)) \Leftrightarrow V[r] \models \psi(r, x),$$

where $j \colon V \to M$ is the induced embedding. Then there exist trees S and T on $\omega \times \gamma$, for some ordinal γ such that $p[S] = \{r \in R \mid \phi(r, y)\}$ and S and T project to complements in any forcing extension via $\mathrm{Col}(\omega, <\delta)$.

Let us consider the Tree Production Lemma in the context where there exist $\omega + 1$ many Woodin cardinals above δ (or just ω many plus a measurable cardinal). Let λ be the limit of the first ω Woodin cardinals above δ. Let $\phi(y)$ be the formula "$r \in \mathbb{R}^{\#}$", and let ψ be the formula "$r \in (\mathbb{R}^*)^{\#}$ after forcing with $\mathrm{Col}(\omega, \lambda)$". The arguments given above show that the hypotheses of the Tree Production Lemma are satisfied, and thus that $(\mathbb{R}^{\#})^V$ is $<\delta$-universally Baire. Since we could apply the same argument for each Woodin cardinal below λ, we have that $(\mathbb{R}^{\#})^V$ is $<\delta$-universally Baire. Since every set of reals is a continuous preimage (in some sense just a projection) of $\mathbb{R}^{\#}$, we have that each such set is $<\delta$-universally Baire, and thus $<\delta$-homogeneously Suslin.

The Tree Production Lemma is also used to show that if λ is a limit of Woodin cardinals, then the $\mathrm{Col}(\omega, <\lambda)$-extension contains inner models satisfying determinacy.