Ultrafilter limits of asymptotic density are not universally measurable

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Given a nonprincipal ultrafilter $U$ on $\omega$ and a sequence $\bar{x} = \langle x_n : n \in \omega \rangle$ consisting of members of a compact Hausdorff space $X$, the $U$-limit of $\bar{x}$ (written $\lim_{n \to U} x_n$) is the unique $y \in X$ such that for every open set $O \subseteq X$ containing $y$, $\{ n \mid x_n \in O \} \in U$. Letting $X$ be the unit interval $[0, 1]$, this operation defines a finitely additive measure $\mu_U$ on $P(\omega)$ in terms of asymptotic density, letting $\mu_U(A) = \lim_{n \to U} |A \cap n|/n$, for each $A \subseteq \omega$. A medial limit is a finitely additive measure on $P(\omega)$, giving singletons measure 0 and $\omega$ itself measure 1, such that for each open set $O \subseteq [0, 1]$, the collection of $A \subseteq \omega$ given measure in $O$ is universally measurable, i.e., is measured by every complete finite Borel measure on $P(\omega)$ (see [1] for more on medial limits and universally measurable sets). If there could consistently be a nonprincipal ultrafilter $U$ such that measure given by the $U$-limit of asymptotic density were universally measurable, this would give a relatively simple example of a medial limit. We show here, however, that this cannot be the case.

**Theorem 0.1.** If $U$ is a nonprincipal ultrafilter on $\omega$, then the function

$$\mu_U : P(\omega) \to [0, 1]$$

defined by letting $\mu_U(A) = \lim_{n \to U} |A \cap n|/n$ is not universally measurable.

**Proof.** Let $I_0 = \{ 0 \}$, and for each positive $n \in \omega$ let

$$I_n = \{ 5^{n-1}, 5^{n-1} + 1, \ldots, 5^n - 1 \}.$$ 

Either the union of the $I_n$'s for $n$ even is in $U$, or the corresponding union for $n$ odd is. In the first case, let $J_0 = I_0 \cup I_1 \cup I_2$, and for each positive $n$, let $J_n = I_{2n+1} \cup I_{2n+2}$. In the second case, let $J_n = I_{2n} \cup I_{2n+1}$ for all $n \in \omega$. In either case, let $S$ be the set of $A \subseteq \omega$ such that $A \cap J_n \in \{ 0, J_n \}$ for all $n \in \omega$. Then $S$ is a perfect subset of $P(\omega)$, and the mapping $H : S \to P(\omega)$ sending $A \in S$ to $\{ n \mid A \cap J_n = J_n \}$ is a homeomorphism. Let $F_0$ be the set of $A \subseteq \omega$ such that $\mu_U(A) \in [0, 1/4)$, and let $F_1$ be the set of $A \subseteq \omega$ such that $\mu_U(A) \in (3/4, 1]$ (so $F_1$ is the set of complements of elements of $F_0$). It will be enough to show that $F_1 \cap S$ is not a universally measurable subset of $S$. 

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We claim for each $A \in S$, the set of $n$ such that $|A \cap n|/n \in [0, 1/4) \cup (3/4, 1]$ is in $U$. This follows from the fact that all (but possibly one) of the $J_n$'s are unions of two consecutive $I_m$'s, and that the union of the larger members of these pairs is in $U$. Each such consecutive pair (for $n > 0$) has the form $I_m = \{5^{m-1}, 5^{m-1}+1, \ldots, 5^m-1\}$ and $I_{m+1} = \{5^m, 5^m+1, \ldots, 5^{m+1}-1\}$, and if $A \in S$, then $A$ either contains or is disjoint from $I_m \cup I_{m+1}$. If it contains both, then for each $k \in I_{m+1}$,

$$|A \cap k|/k \geq (5^m - 5^{m-1})/5^m = 1 - 1/5 = 4/5 > 3/4,$$

and if it is disjoint from both then

$$|A \cap k|/k \leq 5^{m-1}/5^m = 1/5 < 1/4.$$

This establishes the claim. It follows that $S \subseteq F_0 \cup F_1$. Since $\mu_U$ is a finitely additive measure, the intersection of two sets of $\mu_U$-measure greater than $3/4$ cannot be less than $1/4$, so $F_1 \cap S$ is closed under finite intersections. It follows that $H$ maps $F_1 \cap S$ homeomorphically to a nonprincipal ultrafilter, and thus that $F_1 \cap S$ is not universally measurable.

A version of the proof just given, in the special case $\{5^n : n \in \omega\} \in U$, led to the proof in [1] that consistently there are no medial limits.

References


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