Ultrafilter limits of asymptotic density are not universally measurable

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Given a nonprincipal ultrafilter U on ω and a sequence $\bar{x} = \langle x_n : n \in \omega \rangle$ consisting of members of a compact Hausdorff space X, the U-*limit* of $\bar{x}$ (written $\lim_{n \to U} x_n$) is the unique $y \in X$ such that for every open set $O \subseteq X$ containing y, $\{n \mid x_n \in O\} \in U$. Letting X be the unit interval $[0,1]$, this operation defines a finitely additive measure μ_U on $\mathcal{P}(\omega)$ in terms of asymptotic density, letting $\mu_U(A) = \lim_{n \to U} |A \cap n|/n$, for each $A \subseteq \omega$. A *medial limit* is a finitely additive measure on $\mathcal{P}(\omega)$, giving singletons measure 0 and ω itself measure 1, such that for each open set $O \subseteq [0,1]$, the collection of $A \subseteq \omega$ given measure in O is universally measurable, i.e., is measured by every complete finite Borel measure on $\mathcal{P}(\omega)$ (see [1] for more on medial limits and universally measurable sets). If there could consistently be a nonprincipal ultrafilter U such that measure given by the U-limit of asymptotic density were universally measurable, this would give a relatively simple example of a medial limit. We show here, however, that this cannot be the case.

Theorem 0.1. *If U is a nonprincipal ultrafilter on ω, then the function*

$$\mu_U \colon \mathcal{P}(\omega) \to [0,1]$$

defined by letting $\mu_U(A) = \lim_{n \to U} |A \cap n|/n$ is not universally measurable.

Proof. Let $I_0 = \{0\}$, and for each positive $n \in \omega$ let

$$I_n = \{5^{n-1}, 5^{n-1}+1, \ldots, 5^n - 1\}.$$

Either the union of the I_n's for n even is in U, or the corresponding union for n odd is. In the first case, let $J_0 = I_0 \cup I_1 \cup I_2$, and for each positive n, let $J_n = I_{2n+1} \cup I_{2n+2}$. In the second case, let $J_n = I_{2n} \cup I_{2n+1}$ for all $n \in \omega$. In either case, let S be the set of $A \subseteq \omega$ such that $A \cap J_n \in \{\emptyset, J_n\}$ for all $n \in \omega$. Then S is a perfect subset of $\mathcal{P}(\omega)$, and the mapping $H \colon S \to \mathcal{P}(\omega)$ sending $A \in S$ to $\{n \mid A \cap J_n = J_n\}$ is a homeomorphism. Let F_0 be the set of $A \subseteq \omega$ such that $\mu_U(A) \in [0, 1/4)$, and let F_1 be the set of $A \subseteq \omega$ such that $\mu_U(A) \in (3/4, 1]$ (so F_1 is the set of complements of elements of F_0). It will be enough to show that $F_1 \cap S$ is not a universally measurable subset of S.

We claim for each $A \in S$, the set of n such that $|A \cap n|/n \in [0, 1/4) \cup (3/4, 1]$ is in U. This follows from the fact that all (but possibly one) of the J_n's are unions of two consecutive I_m's, and that the union of the larger members of these pairs is in U. Each such consecutive pair (for $n > 0$) has the form $I_m = \{5^{m-1}, 5^{m-1}+1, \ldots, 5^m - 1\}$ and $I_{m+1} = \{5^m, 5^m+1, \ldots, 5^{m+1}-1\}$, and if $A \in S$, then A either contains or is disjoint from $I_m \cup I_{m+1}$. If it contains both, then for each $k \in I_{m+1}$,

$$|A \cap k|/k \geq (5^m - 5^{m-1})/5^m = 1 - 1/5 = 4/5 > 3/4,$$

and if it is disjoint from both then

$$|A \cap k|/k \leq 5^{m-1}/5^m = 1/5 < 1/4.$$

This establishes the claim. It follows that $S \subseteq F_0 \cup F_1$. Since μ_U is a finitely additive measure, the intersection of two sets of μ_U-measure greater than 3/4 cannot be less than 1/4, so $F_1 \cap S$ is closed under finite intersections. It follows that H maps $F_1 \cap S$ homoemorphically to a nonprincipal ultrafilter, and thus that $F_1 \cap S$ is not universally measurable. $\square$

A version of the proof just given, in the special case $\{5^n : n \in \omega\} \in U$, led to the proof in [1] that consistently there are no medial limits.

References

[1] P.B. Larson, *The Filter Dichotomy and medial limits*, in preparation

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