Bigraphs versus Digraphs via Matrices

Richard A. Brualdi
UNIVERSITY OF WISCONSIN

Frank Harary
Zevi Miller
UNIVERSITY OF MICHIGAN

ABSTRACT

It was observed by Dulmage and Mendelsohn in their work on matrix reducibility that there is a one-to-one correspondence between bigraphs and digraphs determined by the utilization of the adjacency matrix. In this semiexpository paper we explore the interaction between this correspondence and a theory of matrix decomposability that is developed in several different articles. These results include: (a) a characterization of those bipartite graphs that can be labeled so that the resulting digraph is symmetric; (b) a criterion for the bigraph of a symmetric digraph to be connected; (c) a necessary and sufficient condition for a square binary matrix to be fully indecomposable in terms of its associated bigraph, and (d) matrix criteria for a digraph to be strongly, unilaterally, or weakly connected. We close with an unsolved extermal problem on the number of components of the bigraph of various orientations of a given graph. This leads to new amusing characterizations of trees and bigraphs.

Dedicated to the graph-theoretic partnership of Lloyd Dulmage and Nathan Mendelsohn.

1. INTRODUCTION

About 20 years ago Dulmage and Mendelsohn [8, 9] observed that there is a natural correspondence between bigraphs (bipartite graphs) and

Journal of Graph Theory, Vol. 4 (1980) 51–73

0364–9024/80/0004–0051$01.00

digraphs (directed graphs). This correspondence can be defined most easily through the intermediary use of matrices in the following way. Let $A=[a_{ij}]$ be an $n\times n$ binary matrix (of 0's and 1's). With A there is associated a directed graph $D(A)$ with n points $v_1,\ldots,v_n$ where v_i is adjacent to v_j [there is an arc (v_i, v_j) from v_i to v_j] iff $a_{ij}=1$ $(1\le i, j\le n)$. The matrix A is therefore the adjacency matrix [13, p. 150] of $D(A)$. With A there is also associated a bigraph $B(A)$. The points of $B(A)$ are the $2n$ points $s_1,\ldots,s_n$; $t_1,\ldots,t_n$ where s_i and t_j are adjacent (there is an edge joining s_i and t_j) iff $a_{ij}=1$ $(1\le i, j\le n)$. The adjacency matrix of $B(A)$ is the partitioned matrix

$$\begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix}, \tag{1.1}$$

where A^t is the transpose of A. It follows that through the use of the concept of an adjacency matrix there is associated with each digraph D a bipartite graph $B(D)$ and with each bigraph B a digraph $D(B)$. Throughout we reserve the symbols: (i) A; (ii) B; (iii) D, for, respectively,

(i) An $n\times n$ matrix of 0's and 1's.

(ii) An $n\times n$ bipartite graph with point set $V(B)$, partitioned as $S\cup T$ with $S=\{s_1,\ldots,s_n\}$ and $T=\{t_1,\ldots,t_n\}$, which is a spanning subgraph of $K_{n,n}$.

(iii) A directed graph with point set $V(D)=\{v_1,\ldots,v_n\}$.

The digraph D may contain (directed) loops corresponding to the facts that the matrix A may have main diagonal entries a_{ii} equal to 1 and the bigraph B may have edges of the form s_it_i. Thus in the sense of [13, 14], D is not exactly a digraph but it is a relation.

Let A be an $n\times n$ binary matrix, and let P and Q be $n\times n$ permutation matrices. The digraphs $D(A)$ and $D(PAQ)$ may have radically different structures. For example, let

$$A=\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and let

$$A_1=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\quad A_2=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad A_3=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

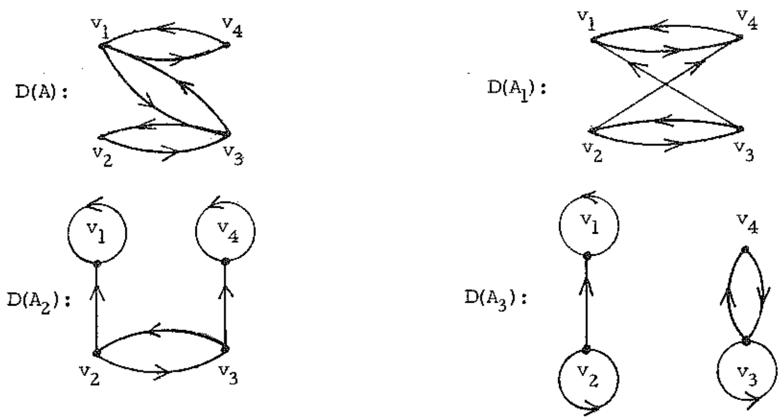

FIGURE 1. Digraphs corresponding to A, A_1, A_2, and A_3.

Then A_1, A_2, and A_3 are obtained from A by row and column permutations ($A_i = P_i A Q_i$ for some permutation matrices P_i and Q_i for $i = 1, 2, 3$). The digraphs $D(A)$, $D(A_1)$, $D(A_2)$, and $D(A_3)$ are shown in Figure 1. It is readily verified by inspection that $D(A)$ is strongly connected, $D(A_1)$ is unilaterally but not strongly connected, $D(A_2)$ is weakly but not unilaterally connected, and $D(A_4)$ is disconnected (not weakly connected). These and other concepts or notation not explicitly defined can be found in [13] and [14].

In contrast the bigraphs $B(A)$ and $B(PAQ)$ are isomorphic for each pair of permutation matrices P and Q. If σ (respectively, τ) is the permutation of $\{1, \ldots, n\}$ corresponding to P (respectively, Q), then $B(PAQ)$ is obtained from the bigraph $B(A)$ by relabeling the point s_i as $s_{\sigma(i)}$ $(i = 1, \ldots, n)$ and point t_j as $t_{\tau^{-1}(j)}$ $(j = 1, \ldots, n)$. In terms of the bigraph–digraph correspondence, the structure of the digraph $D(B)$ corresponding to the bigraph B depends on how the labels $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ are assigned to the point sets S and T of B. We call such an assignment a *bipartite labeling* of B. On the other hand, the structure of the bigraph $B(D)$ corresponding to a digraph D is independent of how the labels $v_1, \ldots, v_n$ are assigned to the point set of D. In Figure 2 we

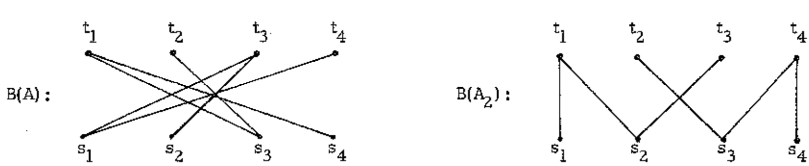

FIGURE 2. Bigraphs $B(A)$ and $B(A_2)$.

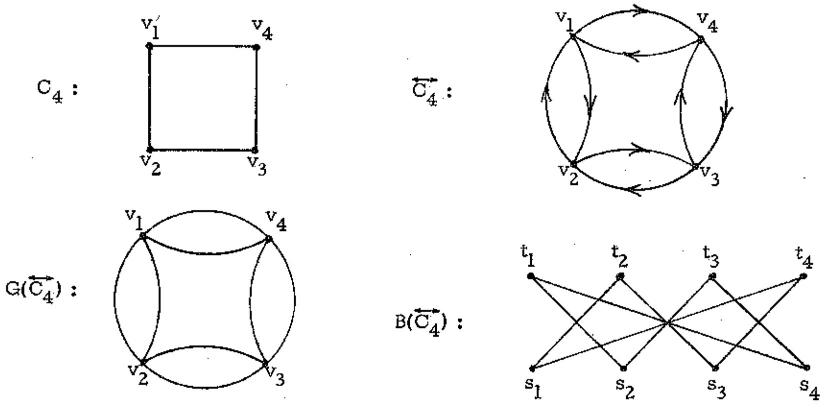

FIGURE 3. Graph C_4 and associated structures.

show the bigraphs corresponding to the digraphs $D(A)$ and $D(A_2)$ of Figure 1, i.e., bigraphs $B(A)$ and $B(A_2)$.

Now suppose that $A = [a_{ij}]$ is an $n \times n$ *symmetric* binary matrix. Then besides the bigraph $B(A)$ and digraph $D(A)$ corresponding to A, there is also a graph $G(A)$ whose point set is $\{v_1, \ldots, v_n\}$ where v_i and v_j are adjacent iff $a_{ij} = 1$ $(1 \le i, j \le n)$. Since a_{ii} may equal 1 $(i = 1, \ldots, n)$, $G(A)$ may have loops. The matrix A is the adjacency matrix of $G(A)$. Conversely, given a graph G we may compute the (symmetric) adjacency matrix $A(G)$ and then obtain the bigraph $B(A(G))$ and digraph $D(A(G))$. The symmetric digraph $D(A(G))$ is obtained from G by replacing each edge of G by a symmetric pair of arcs (a loop by a directed loop) and is denoted by $\overleftrightarrow{G}$. The bigraph $B(\overleftrightarrow{G})$ is readily seen to be the graph $K_2 \times G$, the *Kronecker product* [24] (also called *tensor product* [20] and *conjunction* [13]) of the graphs K_2 and G. Given a digraph D the *underlying graph* $G(D)$ of D is the graph (possibly a multigraph) obtained by disregarding the orientation of the arcs in D. Thus each arc (v_i, v_j) of D becomes an edge v_iv_j of $G(D)$. In Figure 3 we show the graph C_4 [the (undirected) cycle of length 4], the symmetric digraph $\overleftrightarrow{C}_4$, the underlying graph $G(\overleftrightarrow{C}_4)$ of $\overleftrightarrow{C}_4$, and the bigraph $B(\overleftrightarrow{C}_4)$.

2. SYMMETRIC DIGRAPHS FROM BIGRAPHS

Given a bigraph B, whether or not the digraph $D(B)$ is a symmetric digraph depends on the bipartite labeling of B. If A is the adjacency matrix of $D(B)$, then Eq. (1.1) is the adjacency matrix of B, and it follows that $D(B)$ is symmetric for some bipartite labeling of B iff there are

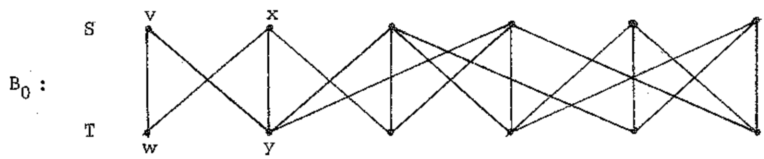

FIGURE 4. S and T have the same degree sequence but B_0 does not yield a symmetric digraph.

permutation matrices P and Q such that PAQ is a symmetric matrix. Equivalently, $D(B)$ is symmetric iff B can be represented in the form $K_2 \times G$ for some graph G. These graphs have been characterized by Sampathkumar [20]. An alternative characterization is the following:

Theorem 2.1. Let B be an $n \times n$ bipartite graph with $V(B) = S \cup T$. Then there exists a bipartite labeling of B for which $D(B)$ is a symmetric digraph iff there is an automorphism β of B of order 2 such that $\beta(S) = T$ and $\beta(T) = S$.

Proof. First suppose that $S = \{s_1, \ldots, s_n\}$ and $T = \{t_1, \ldots, t_n\}$ is a labeling of B for which $D(B)$ is symmetric. Then for each i and j with $1 \le i, j \le n$, $s_i t_j$ is an edge of B iff $s_j t_i$ is. Hence the function $\beta : V(B) \to V(B)$ defined by $\beta(s_i) = t_i$, $\beta(t_i) = s_i$ $(1 \le i \le n)$ is an automorphism of B with the properties specified in Theorem 2.1.

Now suppose that β is an automorphism of B of order 2 such that $\beta(S) = T$ and $\beta(T) = S$. Choose an arbitrary labeling $s_1, \ldots, s_n$ of the points of S and label the points of T by $t_1 = \beta(s_1), \ldots, t_n = \beta(s_n)$. Since β has order 2, $s_i t_j = s_i \beta(s_j)$ is an edge of B iff $s_j t_i = s_j \beta(s_i)$ is. It follows that with this labeling of B, $D(B)$ is symmetric. ∎

As an application of Theorem 2.1 we show that for the bigraph B_0 in Figure 4, no bipartite labeling yields a symmetric digraph. Note that S and T have the same degree sequence, a necessary condition for the existence of a bipartite labeling yielding a symmetric digraph that is not sufficient by Theorem 2.1.

Suppose to the contrary that such a bipartite labeling of B_0 exists. Then there is an automorphism β of B_0 with the properties stated in Theorem 2.1. Since v and w are the only points having degree 2, it follows that $\beta(v) = w$, $\beta(w) = v$. Since wx and vy are edges and v and w each have degree 2, it follows that $\beta(x) = y$, $\beta(y) = x$. This is a contradiction since x and y have different degrees.

3. CONNECTEDNESS OF BIGRAPHS

To each digraph D (or matrix A) we now have a corresponding bigraph $B(D)$ [or $B(A)$]. Likewise to each graph G (or symmetric matrix A) we have a corresponding bigraph $B(\overleftrightarrow{G})$ [or $B(A)$]. We now determine when the bigraph is connected in terms of properties of D, G, and A, respectively. First we require some definitions.

Let A be an $n \times n$ binary matrix. For $\alpha, \beta \subseteq \{1, \ldots, n\}$ let $A[\alpha, \beta]$ denote the submatrix of A formed by rows i with $i \in \alpha$ and columns j with $j \in \beta$, the rows and columns of $A[\alpha, \beta]$ being taken in the same order as in A. The matrix A is called *completely decomposable* if there exist partitions α_1, α_2 and β_1, β_2 of $\{1, \ldots, n\}$ where $\alpha_1 \cup \beta_1 \neq \varnothing \neq \alpha_2 \cup \beta_2$ such that both $A[\alpha_1, \beta_2]$ and $A[\alpha_2, \beta_1]$ are zero matrices. Thus the matrices $A[\alpha_1, \beta_1]$ and $A[\alpha_2, \beta_2]$ may be vacuous but must have either at least one row or at least one column; they can be vacuous only when A has a zero row or zero column. The definition of completely decomposable used here differs from that used in [3] where the matrices $A[\alpha_1, \beta_1]$ and $A[\alpha_2, \beta_2]$ are required to be nonvacuous. One easily shows the following:

Theorem 3.1. Let A be an $n \times n$ binary matrix. The bigraph $B(A)$ is connected iff A is not completely decomposable.

An *alternating path* in a digraph D is a sequence of distinct points $w_1, \ldots, w_m$ such that either (w_1, w_2), (w_3, w_2), $(w_3, w_4), \ldots$, are all arcs of D or (w_2, w_1), (w_2, w_3), $(w_4, w_3), \ldots$, are all arcs of D.

Theorem 3.2. Let D be a digraph. Then $B(D)$ is connected iff between each pair of distinct points of D there is an alternating path. Equivalently, a bigraph B is connected iff for any bipartite labeling of B there exists an alternating path between each pair of distinct points of $D(B)$.

This observation follows at once from the bigraph–digraph correspondence as an alternating path in a digraph gives a path in its bigraph and vice versa.

The most interesting relationship occurs for the correspondence between graphs and bigraphs. Let A be an $n \times n$ symmetric binary matrix, and consider the graph $G(A)$ with point set V whose adjacency matrix is A. The graph $G(A)$ is connected iff there does not exist a partition of V into nonempty sets V_1 and V_2 such that no edge joins a vertex of V_1 and a vertex of V_2. The matrix A is called *completely reducible* when there exists a partition of $\{1, \ldots, n\}$ into nonempty sets α_1 and α_2 such that both $A[\alpha_1, \alpha_2]$ and $A[\alpha_2, \alpha_1]$ are zero matrices. (The matrix A need not be symmetric in order to apply this definition.) Using the correspondence between A and $G(A)$, we immediately obtain the following:

Theorem 3.3. Let A be an $n \times n$ symmetric binary matrix. Then $G(A)$ is connected iff A is not completely reducible.

The graph $G(A)$ is bipartite provided there is a partition of its point set V into sets V_1 and V_2 such that each edge of $G(A)$ joins a point of V_1 and a point of V_2. Thus $G(A)$ is bipartite iff there is a partition of $\{1, \ldots, n\}$ into sets α_1 and α_2 such that both $A[\alpha_1, \alpha_1]$ and $A[\alpha_2, \alpha_2]$ are zero matrices.

Weichsel [24] has proved that the Kronecker product of two connected graphs is connected iff at least one of the two graphs has an odd cycle. Since for a graph G, $B(\overleftrightarrow{G}) = K_2 \times G$, we immediately obtain the following:

Theorem 3.4. Let G be a graph. Then $B(\overleftrightarrow{G})$ is connected iff G is connected and not bipartite.

4. CONNECTEDNESS OF DIGRAPHS

In this section we discuss the various types of connectedness for digraphs and how they depend on the matrices and bigraphs in the correspondences defined previously. We begin by recalling some definitions. Let D be a digraph with point set $V(D)$. Then D is

strongly connected (*or strong*) if for each pair of distinct points v, w there is a path from v to w and a path from w to v; (4.1)

unilaterally connected (*or unilateral*) if for each pair of distinct points v, w there is a path from v to w or a path from w to v; (4.2)

weakly connected (*or weak*) if the underlying graph of D is connected. (4.3)

The digraph D is *disconnected* if it is not weakly connected.

An $n \times n$ matrix A is called *reducible* if there exists a partition of $\{1, \ldots, n\}$ into nonempty sets α_1 and α_2 such that $A[\alpha_1, \alpha_2]$ is a zero matrix. Equivalently, A is reducible provided there exists a permutation matrix P such that

$$PAP^t = \begin{bmatrix} A_1 & 0 \\ X & A_2 \end{bmatrix}, \tag{4.4}$$

where A_1 and A_2 are square nonvacuous matrices. A matrix that is not reducible is called *irreducible*. Note that a 1×1 matrix is always irreducible. The following theorem is well known (see, e.g., [23]):

Theorem 4.1. Let A be an $n \times n$ binary matrix. Then $D(A)$ is strong iff A is irreducible.

A *directed cut* of a digraph D is a partition of its point set $V(D)=\{v_1, \ldots, v_n\}$ into two nonempty sets V_1 and V_2 such that there are no arcs from V_1 to V_2. A reformulation of Theorem 3.1 is then that D is strong iff it has no directed cuts. Referring to the definition of reducible, we observe that $V_1=\{v_i : i \in \alpha_1\}$ and $V_2=\{v_i : i \in \alpha_2\}$ is a directed cut.

We now consider two questions concerning the property of strong connectedness and the digraph–bigraph correspondence. Although we phrase them in terms of digraphs and matrices, the equivalent formulation can be obtained by replacing the matrix A by $B(A)$. Let A be an $n \times n$ binary matrix.

(i) When is $D(PA)$ strong for *some* permutation matrix P?

(ii) When is $D(PA)$ strong for *all* permutation matrices P?

We note that if Q is a permutation matrix, then $D(QAQ^t)$ is isomorphic to $D(A)$. Indeed $D(QAQ^t)$ is obtained from $D(A)$ by relabeling the points in the manner specified by the permutation of $\{1, \ldots, n\}$ corresponding to Q. Since for permutation matrices P and Q,

$$PAQ = Q^t((QP)A)Q\,,$$

questions (i) and (ii) are equivalent to those obtained by replacing PA by PAQ with the qualifications "for some permutation matrices P and Q," and "for all permutation matrices P and Q," respectively. The following theorem is proved in [4]:

Theorem 4.2. Let A be an $n \times n$ binary matrix. There is a permutation matrix P such that $D(PA)$ is strong iff A has no row or column all of whose entries are zero.

Thus if B is an $n \times n$ bipartite graph, there is some bipartite labeling of B that yields a strong digraph iff each point of B has degree at least one, i.e., B has no isolated points.

An $n \times n$ binary matrix A is called *partly decomposable* if $n=1$ and the unique entry of A is zero, or $n>1$ and there exist two partitions α_1, α_2 and β_1, β_2 of $\{1, \ldots, n\}$ into nonempty sets with $|\alpha_i|=|\beta_i|$ $(i=1, 2)$ such that $A[\alpha_1, \beta_2]$ is a zero matrix. Thus for $n>1$, A is partly decomposable iff there exist permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} A_1 & 0 \\ X & A_2 \end{bmatrix}, \tag{4.5}$$

where A_1 and A_2 are square nonvacuous matrices. The matrix A is *fully indecomposable* if it is not partly decomposable. It follows immediately that a reducible matrix is partly decomposable, and thus that a fully indecomposable matrix is irreducible. The converse does not hold but we do have the following result:

Theorem 4.3. Let A be an $n \times n$ binary matrix, and let $A^{(1)}$ be the matrix obtained from A by replacing each entry on the main diagonal of A with a 1. Then A is irreducible iff $A^{(1)}$ is fully indecomposable.

This theorem is proved in an equivalent form as Theorem 4 in [16]. A "one line proof" can be found in [4]. We refer also to Lemma 2.3 in [6].

Theorem 4.4. Let A be an $n \times n$ binary matrix with $n > 1$. Then $D(PA)$ is strong for every permutation matrix P iff A is fully indecomposable.

Proof. It follows from definitions that A is fully indecomposable iff PA is irreducible for all permutation matrices P. The result now follows from Theorem 4.1. ∎

We note that a digraph D with adjacency matrix A is called *ultrastrong* in [2] if $D(PA)$ is strongly connected for all permutation matrices P. Thus an ultrastrong digraph is a digraph D_0 with the property that $D(B(D_0))$ is strong for every bipartite labeling of $B(D_0)$.

We now consider what it means to $B(A)$ for A to be fully indecomposable. Let A be an $n \times n$ nonzero binary matrix. The matrix A is said to have *total support* [22] if there exist permutation matrices $P_1, \ldots, P_t$ such that $P_i \leq A$ $(i = 1, \ldots, t)$ and $A \leq P_1 + \cdots + P_t$. Since a permutation matrix P with $P \leq A$ corresponds to a 1-factor of $B(A)$, it follows that:

A has total support iff each edge of $B(A)$ belongs to a 1-factor of $B(A)$.

Using the definition of a fully indecomposable matrix and the Frobenius–König theorem [19, p. 97] (or the König–Egérvàry theorem [13, p. 96]) we see that:

A fully indecomposable matrix has total support.

Since a fully indecomposable matrix is clearly not completely decomposable, it follows from Theorem 3.1 that:

$B(A)$ is connected when A is fully indecomposable.

Now let A have total support. Suppose A is partly decomposable. Then there exist permutation matrices P and Q such that (3.5) holds. Since A has total support, it readily follows that $X = 0$ and that A_1 and A_2 have total support. Continuing with A_1 and A_2 (or using induction on n) we see that if A is a nonzero matrix with total support, then there exist fully indecomposable matrices $A_1, \ldots, A_t$ $(t \geq 1)$ and permutation matrices P and Q such that

$$PAQ = A_1 \oplus \cdots \oplus A_t,$$

the direct sum of $A_1, \ldots, A_t$. The matrices $A_1, \ldots, A_t$ are called the *fully indecomposable components* of A; we will see that they are uniquely determined apart from permutations of their rows and columns.

Theorem 4.5. An $n\times n$ binary matrix A is fully indecomposable iff $B(A)$ is connected and every edge of $B(A)$ belongs to a 1-factor.

Proof. We have proved above that for A fully indecomposable, $B(A)$ is connected and every edge belongs to a 1-factor. Moreover, it follows from the above discussion that if A has total support [every edge of $B(A)$ is on a 1-factor], then A is fully indecomposable iff $B(A)$ is connected. ∎

The fully indecomposable components can be achieved in another way and can be defined more generally. Let A be an $n\times n$ binary matrix, and let $A^{(1)}$ be the matrix obtained from A by replacing all entries on the main diagonal by 1's. Alternately, one can begin with a matrix $A^{(1)}$ for which there exists a permutation matrix $Q\leq A^{(1)}$; using row permutations we may then assume $I\leq A^{(1)}$ and define $A=A^{(1)}-I$. There exists a permutation matrix P such that

$$PAP^t=\begin{bmatrix} A_1 & 0 & \cdots & 0\\ A_{21} & A_2 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ A_{t1} & A_{t2} & \cdots & A_t \end{bmatrix},$$

where $A_1,\ldots,A_t$ are irreducible; as first shown in [11] and [12] the diagraphs $D(A_i)$ correspond to the strongly connected components of $D(A)$. Hence

$$PA^{(1)}P^t=\begin{bmatrix} A_1^{(1)} & 0 & \cdots & 0\\ A_{21} & A_2^{(1)} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ A_{t1} & A_{t2} & \cdots & A_t^{(1)} \end{bmatrix},$$

where $A_i^{(1)}$ is obtained from A_i by replacing all entries on the main diagonal by 1's. It follows from Theorem 4.3 that $A_1^{(1)},\ldots,A_t^{(1)}$ are fully indecomposable, and are the fully indecomposable components of $A^{(1)}$ (note that if $A^{(1)}$ has total support then $A_{ij}=0$ for all $i>j$). The correspondence above between fully indecomposable and irreducible components is given in Theorem 6 of [16] and occurs repeatedly in the literature.

A bigraph B with the property that the union of all its 1-factors form a connected spanning subgraph (hence every edge of the subgraph is in a 1-factor) has been called *elementary* by Lovász [17]. It follows from Theorem 3.5 that B is elementary iff the matrix A in its adjacency matrix (1.1) [thus $B=B(A)$] is fully indecomposable. Lovász and Plummer [17] have further investigated *minimal elementary* bigraphs, elementary bigraphs such that the removal of any edge results in a bigraph that is no longer elementary. Thus $B=B(A)$ is minimal elementary iff A is fully indecomposable but any matrix obtained from A by changing a 1 to a 0 is partly decomposable. Such matrices are called *nearly decomposable* [21] and have been studied in the literature especially for the role they play in the theory of doubly stochastic matrices and for their application to the study of the permanent function (see [5] and [21]).

The analogs of nearly decomposable matrices are the nearly reducible ones. An $n \times n$ binary matrix A is *nearly reducible* provided it is irreducible and each matrix obtained from A by replacing a 1 by a 0 is reducible. For a unified treatment of nearly reducible and nearly decomposable matrices see [7]. It follows from Theorem 4.1 that A is nearly reducible iff $D(A)$ is strong but no digraph obtained from $D(A)$ by removing an arc is. Digraphs with this property have been called *minimally strongly connected* (sometimes minimally connected) and their basic properties have been uncovered. It is well known (see [1, pp. 30–32]) that if D is a minimally strongly connected digraph with $n \geq 2$ points, then there exist points $u, v_1, \ldots, v_t, w$ $(t \geq 0)$ with the following properties:

(i) $u, v_1, \ldots, v_t, w$ are distinct except u may equal w.
(ii) $u, v_1, \ldots, v_t, w$ is a (directed) path of D.
(iii) The outdegree and indegree of v_i equals 1 $(1 \leq i \leq t)$.
(iv) The digraph D' obtained from D by removing $v_1, \ldots, v_t$ and all arcs of the path (ii) is minimally strongly connected.

In terms of matrices this translates as follows [15]. If A is an $n \times n$ nearly reducible matrix with $n \geq 2$, then there exists a permutation matrix P such that

$$PAP^t = \left[\begin{array}{cccccc|c}
0 & & & & & & 1 \\
1 & 0 & & & & & \\
 & 1 & 0 & & & & \\
 & & \ddots & \ddots & & & \\
 & & & & 0 & & \\
 & & & & 1 & 0 & \\
\hline
 & & & & & 1 & \\
 & & & & & & A' \\
 & & & & & &
\end{array}\right]$$

where A' is nearly reducible and all entries not specified are zero.

We now investigate the analogs of Theorems 4.1–4.3 for unilateral digraphs.

Theorem 4.6. Let A be an $n\times n$ binary matrix. Then $D(A)$ is unilaterally connected iff

$$\text{there exists an integer } t\geq 1 \text{ and a permutation matrix } P \text{ such that} \tag{4.6}$$

$$PAP^t=\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_{21} & A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{t1} & A_{t2} & \cdots & A_t \end{bmatrix},$$

where $A_1,\ldots,A_t$ are irreducible, and $A_{21},\ldots,A_{t,t-1}$ are not zero matrices.

Proof. This is precisely the statement in matrix form of Theorem 3.10 of [14, p. 66] that a digraph D is unilateral iff its condensed digraph D^* (with respect to strong components) has a unique spanning directed path. ∎

We shall need the following theorem to complete our investigation of unilateral digraphs:

Theorem 4.7. Let C be an $n\times n$ binary matrix having no row or column all of whose entries are zero. Then there exists a matrix $E\leq C$ and permutation matrices P and Q such that PEQ equals

$$\left[\begin{array}{cccc|cc} 1 & & & & & \\ & \ddots & & & & \\ & & & 1 & & \\ \hline & & & 1 & & \\ & & & & E' & \\ \hline & & & & & 1 \end{array}\right], \tag{4.7}$$

where E' is an irreducible matrix and where unspecified entries equal zero.

Proof. It follows from Theorems 4.1 and 4.2 that there exists a permutation matrix R such that RC is irreducible. Hence there exists a matrix $E\leq C$ such that RE is nearly reducible. It follows from the discussion preceding Theorem 4.6 that there exists a permutation matrix S such that $S(RE)S^t$ has the form (4.7). The theorem follows with $P = SR$ and $Q = S^t$. ∎

Theorem 4.8. Let A be an $n \times n$ binary matrix. Then there exists a permutation matrix P such that $D(PA)$ is unilaterally connected iff A has at most one row and at most one column all of whose entries are zero.

Proof. It follows readily from the definition that a unilateral digraph can have at most one point with indegree 0 and at most one point with outdegree 0. Hence if $D(PA)$ is unilaterally connected for some permutation matrix P, then A has at most one zero row and at most one zero column (this fact also follows from Theorem 4.6). We now investigate the converse.

First suppose that A has exactly one zero row and exactly one zero column. Then there exist permutation matrices R and S such that

$$RAS = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ & & & 0 \\ & & & \vdots \\ & C & & \vdots \\ & & & 0 \end{bmatrix},$$

where C has no zero row or zero column. It now follows from Theorem 4.7 that there exist permutation matrices P' and Q' and a matrix $A' \le A$ such that

$$P'A'Q' = \left[\begin{array}{cccc|c|c} 0 & 0 & \cdots & & 0 & 0 \\ \hline 1 & 0 & & & & 0 \\ & \ddots & \ddots & & & \vdots \\ & & 1 & 0 & & \vdots \\ \hline & & & 1 & & \vdots \\ & & & & E' & \\ & & & & & 0 \\ \hline & & & & 1 & 0 \end{array}\right],$$

where E' is an irreducible matrix and the unspecified entries are zero. Using Theorem 3.6 we conclude that $D(P'A'Q')$ is unilateral; hence $D(P'AQ')$ is unilateral. Since $P'AQ' = Q'^t(Q'P')AQ'$, $D(PA)$ is unilaterally connected for $P = Q'P'$.

Now suppose that A has exactly one nonzero column but no zero row. Then there exists a permutation matrix S such that

$$AS = \begin{bmatrix} & 0 \\ & 0 \\ & \cdot \\ F & \cdot \\ & \cdot \\ & 0 \end{bmatrix},$$

where F has no zero row or zero column. Let $F' \leq F$ be minimal (for its total number of 1's) with respect to the property of having no zero row or column. Hence each 1 of F' is either the only 1 in its row or the only 1 in its column. Since F' has n nonzero rows and $n-1$ nonzero columns, there must be a 1 of F' that is the only 1 in its row but not the only 1 in its column. If follows that there exists a permutation matrix R such that

$$RF'S = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & & 0 \\ & & & \cdot \\ & C & & \cdot \\ & & & \cdot \\ & & & 0 \end{bmatrix},$$

where C is an $(n-1)\times(n-1)$ matrix with no zero row or column. Applying Theorem 4.7 we conclude as in the preceding case that there is a permutation matrix P such that $D(PA)$ is unilaterally connected.

Finally if A has no zero row or column, then by Theorem 4.2 there is a permutation matrix P such that $D(PA)$ is strong and hence unilateral. This completes the proof of the theorem. ∎

As an example when

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

there is a permutation matrix P such that

$$PA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where $D(PA)$ is the unilateral digraph shown in Figure 5.

D(PA) :

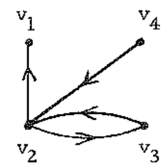

FIGURE 5. A unilateral digraph.

Let A be an $n \times n$ binary matrix. We say that A is *quasidecomposable* if there exist premutation matrices R and S such that

$$RAS = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ A_{21} & A_2 & 0 & 0 \\ A_{31} & 0 & A_3 & 0 \\ A_{41} & A_{42} & A_{43} & A_4 \end{bmatrix}, \qquad (4.8)$$

where A_1 and A_4 are possibly vacuous square matrices and A_2 and A_3 are nonvacuous square matrices. If A is not quasidecomposable, we call A *quasiindecomposable.*

Theorem 4.9. Let A be an $n \times n$ binary matrix. Then $D(PA)$ is unilaterally connected for every permutation matrix P iff A is quasi-indecomposable.

Proof. If A is quasidecomposable, and R and S are permutation matrices such that Eq. (4.8) holds, then it follows readily that $D(RAS)$ is not unilateral. Hence $D(PA)$ is not unilateral for $P = S^t R$. Now suppose that Q is a permutation matrix such that $D(QA)$ is not unilateral. Then it follows from Theorem 4.6 that there exist integers t and s with $t \geq s \geq 2$ and a permutation matrix P such that

$$P(QA)P^t = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ C_{21} & C_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ C_{t1} & C_{t2} & \cdots & C_t \end{bmatrix},$$

where $C_1, \ldots, C_t$ are square nonvacuous matrices and $C_{s,s-1} = 0$. If we define

$$A_1 = \begin{bmatrix} C_1 & \cdots & 0 \\ \vdots & & \vdots \\ C_{s-2,1} & \cdots & C_{s-2} \end{bmatrix}, \quad A_2 = C_{s-1}, \quad A_3 = C_s, \; A_4 = \begin{bmatrix} C_{s+1} & \cdots & 0 \\ \vdots & & \vdots \\ C'_{t1} & \cdots & C_t \end{bmatrix}$$

we see that A is quasidecomposable. ∎

We now investigate the property of a digraph being (weakly connected). The following theorem is an easy consequence of definitions:

Theorem 4.10. Let A be an $n \times n$ binary matrix. Then $D(A)$ is weakly connected iff A is not completely reducible.

Let A be an $n \times n$ binary matrix. According to Theorem 2.1, $B(A)$ is connected iff A is not completely decomposable. Suppose $B(A)$ has t connected components $B_1, \ldots, B_t$ of which k contain at least one edge. Then $1 \leq k \leq t \leq 2n$ and there exist permutation matrices P and Q such that PAQ has the form

$$\left[\begin{array}{cccc|c} A_1 & 0 & \cdots & 0 & \\ 0 & A_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & A_k & \\ \hline & & & & \\ & & 0 & & 0 \end{array}\right], \tag{4.9}$$

where A_i is a nonvacuous $r_i \times s_i$ matrix and $B(A_i)$ is connected $(1 \leq i \leq k)$. Note that here A_i is a rectangular matrix, but nonetheless there corresponds a bipartite graph whose adjacency matrix is

$$\begin{bmatrix} 0 & A_i \\ A_i^t & 0 \end{bmatrix}.$$

Lemma 4.11. If $S_1, \ldots, S_k$ is a collection of sets such that for each partition of $\{1, \ldots, k\}$ into nonempty sets I, J

$$\left(\bigcup_{i \in I} S_i\right) \cap \left(\bigcup_{j \in J} S_j\right) \neq \varnothing, \tag{4.10}$$

then

$$|S_1 \cup \cdots \cup S_k| \leq \sum_{i=1}^{k} |S_i| - k + 1.$$

Proof. We prove Lemma 4.11 by induction on k, noting first that it holds for $k=1$. Let $k>1$. Let G be the graph whose point set is $\{1, \ldots, k\}$ where there is an edge joining i and j iff $i \neq j$ and $S_i \cap S_j \neq \varnothing$. It follows from Eq. (4.10) that G is connected. Hence there is a point, say 1, such that the subgraph generated by $\{2, \ldots, k\}$ is connected. By the inductive assumption applied to $S_2, \ldots, S_k$,

$$|S_2 \cup \cdots \cup S_k| \leq \sum_{i=2}^{k} |S_i| - (k-1) + 1.$$

Since $S_1 \cap (S_2 \cup \cdots \cup S_k) \neq \emptyset$,

$$|S_1 \cup S_2 \cup \cdots \cup S_k| \leq \left(\sum_{i=2}^{k} |S_i| - (k-1) + 1 \right) + |S_1| - 1,$$

and Lemma 4.11 follows. ∎

Theorem 4.12. Let A be an $n \times n$ binary matrix. Then there exists a permutation matrix P such that $D(PA)$ is weakly connected iff $B(A)$ has at most $n+1$ connected components.

Proof. We continue with the notation introduced in the paragraph preceding Lemma 4.11. There is no loss of generality in assuming A has the form given in Eq. (4.9). The number of connected components of $B(A)$ equals

$$k + n - (r_1 + \cdots + r_k) + n - (s_1 + \cdots + s_k),$$

and hence the number of components of $B(A)$ is at most $n+1$ iff

$$(r_1 + \cdots + r_k) + (s_1 + \cdots + s_k) \geq n + k - 1. \tag{4.11}$$

Suppose first that there exists a permutation matrix P such that $D(PA)$ is weak. We define sets $\alpha_q, \beta_q \subseteq \{1, \ldots, n\}$ with $|\alpha_q| = r_q$ and $|\beta_q| = s_q$ by

$$(PA)[\alpha_q, \beta_q] = A_q \qquad (q = 1, \ldots, k).$$

Thus A_q occupies rows i with $i \in \alpha_q$ and columns j with $j \in \beta_q$ in A. Let $V = \{v_1, \ldots, v_n\}$ be the set of points of $D(PA)$ so that the 1's of A_q correspond to arcs which join a point in $U_q = \{v_i : i \in \alpha_q\}$ to a point in $W_q = \{v_i : i \in \beta_q\}$ $(1 \leq q \leq k)$. Let $V_q = U_q \cup W_q$ $(1 \leq q \leq k)$ and consider the graph G whose set of points is $\{1, \ldots, k\}$ where there is an edge joining i and j iff $i \neq j$ and $V_i \cap V_j \neq \emptyset$. Since $D(PA)$ is weak, it follows that G is connected and $n = |V_1 \cup \cdots \cup V_k|$. Now using Lemma 4.11 we obtain

$$\begin{aligned} n = |V_1 \cup \cdots \cup V_k| &\leq \sum_{i=1}^{k} |V_i| - k + 1 \\ &\leq \sum_{i=1}^{k} (r_i + s_i) - k + 1, \end{aligned}$$

which is Eq. (4.11). Hence $B(A)$ has at most $n+1$ connected components.

Now suppose $B(A)$ has at most $n+1$ components, so that Eq. (4.11) holds. We permute the rows and columns of A to obtain a matrix C so that

$$C[1,\ldots,r_1;r_1+1,\ldots,r_1+s_1]=A_1,$$

$$C[r_1+s_1+1,\ldots,r_1+s_1+r_2;r_1+s_1,r_1+s_1+r_2+1,\ldots,$$
$$r_1+s_1+r_2+s_2-1]=A_2,$$

$$C[r_1+s_1+r_2+s_2,\ldots,r_1+s_1+r_2+s_2+r_3-1;$$
$$r_1+s_1+r_2+s_2-1,r_1+s_1+r_2+s_2+r_3,$$
$$\ldots,r_1+s_1+r_2+s_2+r_3+s_3-2]=A_3,$$

etc.

It may happen that the indices used to define C above exceed n. When that happens, we replace them by any row index and column index, respectively, that has not already occurred. However, because of Eq. (4.11), each integer i with $1\le i\le n$ occurs either as a row index or column index in defining the positions of the matrices $A_1,\ldots,A_k$ in C above. Each of the matrices A_i determines in $D(C)$ a weak subdigraph D_i [since $B(A_i)$ is connected]. For $i=2,\ldots,k$, D_{i-1} and D_i have a point in common, and hence $D(C)$ is weak. The theorem now follows. ∎

As an example of the construction used in the proof, suppose $n=9$, $k=3$ and

$$A_1=A_2=A_3=\begin{bmatrix}1&1\\1&1\end{bmatrix}.$$

Then

$$C=\begin{bmatrix}0&0&1&1&0&0&0&0&0\\0&0&1&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&1&0&0\\0&0&0&1&0&0&1&0&0\\0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&1&0\\0&0&0&0&0&0&1&1&0\end{bmatrix},$$

and $D(C)$ is the weak digraph shown in Figure 6.

Finally we record the following observation:

Theorem 4.13. Let A be an $n\times n$ binary matrix. Then $D(PA)$ is weak for all permutation matrices P iff there are no permutation matrices R

D(C) :

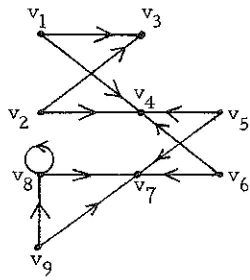

FIGURE 6. A weakly connected digraph.

and S such that

$$RAS = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where A_1 and A_2 are square, nonvacuous matrices.

Proof. The theorem follows readily from Theorem 4.10. ∎

5. CHARACTERIZATION OF BIPARTITE GRAPHS AND TREES

Using the digraph–bigraph correspondence, we give characterizations of bigraphs and trees.

Let G be any graph. We will use the symbol $\vec{G}$ to denote a digraph obtained by orienting the edges of G. Thus if $A = [a_{ij}]$ is the (symmetric) adjacency matrix of G, then $\vec{G} = D(A_0)$ where A_0 is obtained from A in the following manner: for each i, j with $i \neq j$ and $a_{ij} = a_{ji} = 1$, replace exactly one of a_{ij} and a_{ji} by 0. With an orientation $\vec{G}$ of G we may associate the bigraph $B(\vec{G}) = B(A_0)$. We then let $k(B(\vec{G}))$ denote the number of connected components of $B(\vec{G})$. In Figure 7, two different

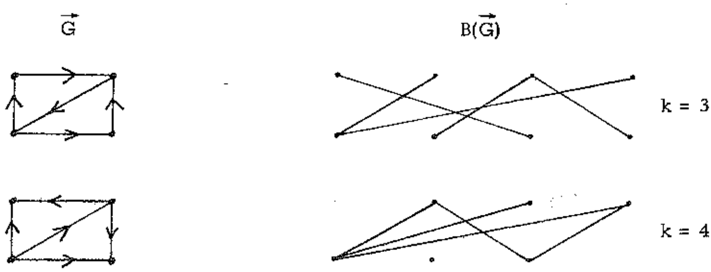

FIGURE 7. Two orientations of a graph with different values of k.

orientations of the graph G, obtained from the complete graph K_4 by removing an edge, with different values of $k(B(\vec{G}))$ are given. We let $m(G)$ and $M(G)$ denote the minimum and maximum values of $k(B(\vec{G}))$, respectively, over all possible orientations $\vec{G}$ of G.

We begin with a lemma which gives one direction of our characterization of trees.

Lemma 5.1. If T is a tree with n points, then $m(T)=M(T)=n+1$.

Proof. We prove the lemma by induction on n. When $n=1$, T has no edges and thus $\vec{T}$ is a digraph with 1 point and no arcs; hence $B(\vec{T})$ is the graph with 2 points and no edges, and has the required 2 components. Now let $n>1$. Let $\vec{T}$ be any orientation of T and let v be an endpoint of T. Then $T-v$ is a tree with $n-1$ points, and it follows from the inductive assumption that $m(T-v)=M(T-v)=n$, so that the subgraph $B(\overrightarrow{T-v})$ of $B(\vec{T})$ has n components. Since v is an endpoint of T, it follows that $B(\vec{T})$ is obtained from $B(\overrightarrow{T-v})$ by adjoining two new points and an edge from one of them to a point of $B(\overrightarrow{T-v})$. Hence $B(\vec{T})$ has exactly one more component than $B(\overrightarrow{T-v})$, and so $m(T)=M(T)=n+1$. ∎

The next theorem contains a characterization of connected bipartite graphs.

Theorem 5.2. Let G be a connected graph with n points. Then $M(G)\leq n+1$ with equality iff G is bipartite.

Proof. Let T be a spanning tree of G. It follows from Lemma 5.1 that

$$M(G)\leq M(T)=n+1.$$

We now consider the case of equality.

First suppose that G is bipartite with partitioned point set $V\bigcup W$ where $V=\{v_1,\ldots,v_p\}$ and $W=\{w_1,\ldots,w_q\}$. Let $\vec{G}$ denote the specific orientation of G obtained by orienting each edge from V to W. Since G is connected, it follows that $B(\vec{G})$ has one component of $n=p+q$ points and n components of one point each. Hence $k(B(\vec{G}))=n+1$, so that $M(G)=n+1$ in case G is a bipartite graph.

Now suppose that G has a cycle of odd length. Then there exists a spanning tree T of G and an edge e of G such that $G_e=T+e$ has an odd length cycle (necessarily e is an edge of this cycle). Let $\vec{G}$ by any orientation of G. Let the points of G (and thus those of $\vec{G}$) be $v_1,\ldots,v_n$ and let $\{s_1,\ldots,s_n\}$ and $\{t_1,\ldots,t_n\}$ be the partitioned point set of $B(\vec{G})$. Thus there is an edge joining s_i and t_j iff there is an arc from v_i to v_j. Suppose e joins v_p and v_q in G. Then v_p and v_q are joined by a path of even length in T. Let e be oriented from v_p to v_q in $\vec{G}$. Thus there exists an edge joining s_p and t_q in $B(\vec{G})$. Suppose s_p and t_q were in the same component of $B(\vec{T})$, where $\vec{T}$ is the orientation of T obtained by restricting the orientation $\vec{G}$ of G to the edges of T. Then it follows that v_p and v_q are joined by a path of odd length in T. But this contradicts the fact that v_p and v_q are joined by a path of even length in T. Hence s_p and t_q are in different components of $B(\vec{T})$. Since there is an edge joining s_p and t_q in $B(\vec{T})$, it now follows that

$$k(B(\vec{G})) \leq k(B(\vec{T})) - 1 = (n+1) - 1 = n.$$

Hence $M(G) \leq n$, and the proof is complete. ▮

We now give a characterization of trees in these terms.

Theorem 5.3. Let G be a connected graph with n points. Then G is a tree iff $m(G) = M(G) = n+1$.

Proof. If G is a tree, then it follows from Lemma 5.1 that $m(G) = M(G) = n+1$. Now suppose that G is not a tree. To complete the proof it suffices to show that there exists an orientation $\vec{G}$ of G such that $k(B(\vec{G})) \leq n$. If G is not bipartite this follows from Theorem 5.2. Thus we suppose G is bipartite and use the notation in the proof of Theorem 5.2. Since G is not a tree, there is an edge $e = v_i w_j$ such that $G-e$ is connected. Let $\overrightarrow{G-e}$ denote the orientation of $G-e$ obtained by orienting each edge from V to W and let $\vec{G}$ be the orientation of G obtained from $\overrightarrow{G-e}$ by orienting e from w_j to v_i. Since $\{v_i\}$ and $\{w_j\}$ are one-point components of $B(\overrightarrow{G-e})$, if follows that

$$k(B(\vec{G})) = k(B(\overrightarrow{G-e})) - 1 = (n+1) - 1 = n.$$

Hence the theorem follows. ▮

In closing we remark that several natural extremal problems are suggested in consideration of the m and M invariants defined above. For example, one could ask for the minimum and maximum number of edges in a graph with n points and given values of m and M. In addition, for a given graph G and integer t with $m(G) < t < M(G)$, does there exist an orientation $\vec{G}$ of G with $k(B(\vec{G})) = t$?

For a thorough study of the structure of bigraphs using matrices we refer the reader to Dulmage and Mendelsohn [8–10].

References

[1] C. Berge, *Graphs and Hypergraphs*. Elsevier-North Holland, New York (1973).

[2] R. A. Brualdi, Kronecker product of fully indecomposable matrices and of ultrastrong digraphs. *J. Combinatorial Theory* 2 (1967) 135–139.

[3] R. A. Brualdi, The DAD theorem for arbitrary row sums. *Proc. Amer. Math. Soc.* 45 (1974) 189–194.

[4] R. A. Brualdi, Matrices permutation equivalent to irreducible matrices and applications. *Linear and Multilinear Algebra.* 7 (1979) 1–12.

[5] R. A. Brualdi and P. M. Gibson, Convex polyhedra of doubly stochastic matrices: I. Applications of the permanent function. *J. Combinatorial Theory Ser. A* 22 (1977) 194–230.

[6] R. A. Brualdi, S. V. Parter, and H. Schneider, The diagonal equivalence of a nonnegative matrix to a stochastic matrix. *J. Math. Anal.* 16 (1966) 31–50.

[7] R. A. Brualdi and M. B. Hedrick, A unified treatment of nearly reducible and nearly decomposable matrices. *Linear Algebra and Appl.* 24 (1979) 51–73.

[8] A. L. Dulmage and N. S. Mendelsohn, Coverings of bipartite graphs. *Canad. J. Math.* 10 (1958) 517–534.

[9] A. L. Dulmage and N. S. Mendelsohn, A structure theory of bipartite graphs of finite exterior dimension. *Trans. Roy. Soc. Canad. Third Ser. Sec. III* 53 (1959) 1–13.

[10] A. L. Dulmage and N. S. Mendelsohn, Graphs and matrices. In *Graph Theory and Theoretical Physics.* Edited by F. Harary. Academic, London (1967) 167–227.

[11] F. Harary, A graph theoretic method for the complete reduction of a matrix with a view toward finding its eigenvalues. *J. Math. Phys.* 38 (1959) 104–111.

[12] F. Harary, A graph theoretic approach to matrix inversion by partitioning. *Numer. Math.* 4 (1962) 128–135.

[13] F. Harary, *Graph Theory.* Addison-Wesley, Reading, Mass. (1969).

[14] F. Harary, R. Z. Norman, and D. Cartwright, *Structural Models.* Wiley, New York (1965).

[15] D. J. Hartfiel, A simplified form for nearly reducible and nearly decomposable matrices. *Proc. Amer. Math. Soc.* 24 (1970) 388–393.

[16] D. M. Johnson, A. L. Dulmage, and N. S. Mendelsohn, Connectivity and reducibility of graphs. *Canad. J. Math.* 14 (1962) 529–539.

[17] L. Lovász, On the structure of factorizable graphs. *Acta Math. Acad. Sci. Hungar.* 23 (1972) 179–195.

[18] L. Lovász and M. D. Plummer, On minimal elementary bipartite graphs. *J. Combinatorial Theory Ser. B* 23 (1977) 127–138.

[19] M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities.* Allyn and Bacon, Boston (1964).

[20] E. Sampathkumar, On tensor product graphs. *J. Austral. Math. Soc. Ser. A* 20 (1975) 268–271.

[21] R. Sinkhorn and P. Knopp, Problems involving digonal products in nonnegative matrices. *Trans. Amer. Math. Soc.* 136 (1969) 67–75.

[22] R. Sinkhorn and P. Knopp, Concerning nonnegative matrices and doubly stochastic matrices. *Pac. J. Math.* 21 (1967) 343–348.

[23] R. S. Varga, *Matrix Iterative Analysis*. Prentice-Hall, Englewood Cliffs, N. J. (1962).

[24] P. M. Weichsel, The Kronecker product of graphs. *Proc. Amer. Math. Soc.* 13 (1963) 47–52.