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# EXTREMAL REGULAR GRAPHS FOR THE ACHROMATIC NUMBER

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The problem of constructing $(m, n)$ cages suggests the following class of problems. For a graph parameter $\theta$, determine the minimum or maximum value of $p$ for which there exists a $k$-regular graph on $p$ points having a given value of $\theta$. The minimization problem is solved here when $\theta$ is the achromatic number, denoted by $\psi$. This result follows from the following main theorem. Let $M(p, k)$ be the maximum value of $\psi(G)$ over all $k$-regular graphs $G$ with $p$ points, let $\{x\}$ be the least integer of size at least $x$, and let $\Omega(k) \subseteq \mathbb{Z}$ be given by $\Omega(k) = \{i(ik+1)+1 : 1 \leq i < \infty\}$. Define the function $f(p, k)$ by $f(p, k) = \max\{\lambda \in \mathbb{Z} : \lambda\{(\lambda-1)/k\} \geq p\}$. Then for fixed $k \geq 2$ we have $M(p, k) = f(p, k)$ if $p \notin \Omega(k)$ and $M(p, k) = f(p, k) - 1$ if $p \in \Omega(k)$ for all $p$ sufficiently large with respect to $k$.

## 1. Introduction

The achromatic number $\psi(G)$ of a graph $G$ has been studied by several authors. This invariant arises naturally from the concept of homomorphism, and as such it lies at the opposite extreme from the usual chromatic number $\chi(G)$. Motivated by the notion of the $(m, n)$ cages, we study in this paper the extremal behavior of $\psi(G)$ for regular graphs $G$.

We begin with some basic terminology and notation. First, all graphs in this paper are finite, with no loops or multiple edges. We denote by $V(G)$ and $E(G)$ the point set and edge set respectively of a graph $G$. If $S \subseteq V(G)$, then $\langle S\rangle_G$ (or just $\langle S\rangle$) is the graph having $S$ as point set and all edges of $G$ having both points in $S$ as its edge set. If $U \subseteq E(G)$, then $\langle U\rangle$ is the graph having $U$ as edge set and points incident on edges of $U$ as point set. We let $\{x\}$ denote the least integer of size at least $x$, while $nG$ denotes a disjoint union of $n$ copies of $G$.

Let $G$ and $H$ be graphs with $|V(H)| \leq |V(G)|$. A homomorphism $f: G \to H$ is a map from $V(G)$ onto $V(H)$ with the property that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$, and $f(u)f(v) \in E(H)$ implies the existence of $r \in f^{-1}(f(u))$ and $s \in f^{-1}(f(v))$ such that $rs \in E(G)$. Since loops are not allowed, adjacent points of $G$ cannot have the same image under $f$. Hence the inverse image $f^{-1}(u)$ for any $u \in V(H)$ is an independent set in $G$, and we get an induced partition of $V(G)$, $V(G) = \bigcup_{u \in V(H)} f^{-1}(u)$, such that for any $v, w \in V(H)$ there exist $r \in f^{-1}(v)$ and $s \in f^{-1}(w)$

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for which $rs \in E(G)$ if and only if $vw \in E(H)$. In case $H$ is the complete graph $K_\lambda$, we write the induced partition as $V(G) = \bigcup_{i=1}^{\lambda} V_i$ and we call the $V_i$ the *classes* of $G$ (the particular partition being fixed by context). A partition of $G$ as above induced by a homomorphism $f: G \to K_\lambda$ will be called a $\lambda$-*partition* of $G$. The minimum value of $\lambda$ for which an $f$ exists is well known as the chromatic number $\chi(G)$, where points have the same color if and only if they belong to the same class. The maximum possible value of $\lambda$ is called the *achromatic number* of $G$, and is denoted $\psi(G)$.

In [5] bounds for $\psi$ in terms of the independence and covering parameters $\alpha_0, \beta_0, \alpha_1$, and $\beta_1$ are obtained. It is also shown that $\chi(\bar{G}) + \psi(G) \leq p+1$, generalizing the bound $\chi(\bar{G}) + \chi(G) \leq p+1$ obtained by Nordhaus and Gaddum [6]. In [5] an irreducible graph is defined as one in which distinct points have distinct neighborhoods. It is then shown that there are only finitely many irreducible graphs with a given achromatic number. In [2] the values of $\psi(C_n)$ and $\psi(P_n)$ were found for the $n$ point cycle $C_n$ and path $P_n$. The result for cycles was the following.

**Theorem A** [2]. *If* $p = n\,\{\frac{1}{2}(n-1)\}$, $n \geq 3$, *then* $\psi(C_p) = n$. *For p between* $n\{\frac{1}{2}(n-1)\}$ *and* $(n+1)\{\frac{1}{2}n\}$, $\psi(C_p) = n$ *unless n is odd and* $p = n\{\frac{1}{2}(n-1)\}+1$, *in which case* $\psi(C_p) = n-1$.

An extremal problem to which much attention has been given is the determination of the smallest integer $h(m, n)$ for which there exists an $m$-regular graph having girth $n$ and $h(m, n)$ points. Such graphs have been called $(m, n)$ cages, and they have been constructed for certain values of $m$ and $n$. One can vary this problem by asking for the minimum or maximum value of the girth given the regularity degree and the number of points. More generally we may ask the following. Given a graph parameter $\theta$, determine the minimum or maximum value of $\theta$ over the set of $k$ regular graphs on $p$ points.

Turning our attention to the achromatic number, let $M(p, k)$ be the maximum value of $\psi(G)$ over all $k$ regular graphs on $p$ points, and let $h(\lambda, k)$ be the minimum $p$ for which there exists a $k$ regular graph on $p$ points having achromatic number $\lambda$. Our main result is the determination of $M(p, k)$ and then of $h(\lambda, k)$ as a corollary. Letting

$$f(p, k) = \max\{\lambda \in \mathbb{Z} : \lambda\{(\lambda-1)/k\} \leq p\},$$

it is easy to show that $M(p, k) \leq f(p, k)$. Combining this with Theorem A it follows that $M(p, 2) = f(p, 2)$ unless $p = n(\frac{1}{2}(n-1))+1$ for $n$ odd. It will be seen that the generalization $M(p, k) = f(p, k)$ for $k \geq 3$ holds when $p$ is sufficiently large (aside from a class of exceptional values of $p$).

We now introduce some notions that will be useful in constructions which we will later make. Suppose $G$ and $H$ are graphs such that $V(G) \supseteq V(H)$, $\psi(G) \geq \lambda$, $\psi(H) \geq \mu$, and $\lambda \geq \mu$. Let $V(G) = \bigcup_{i=1}^{\lambda} V_i$ and $V(H) = \bigcup_{i=1}^{\mu} W_i$ be the induced $\lambda$

and $\mu$ partitions of $G$ and $H$ respectively. We say that $G$ is a $(\lambda, \mu)$ *extension of* $H$ if (after possibly reindexing the $V_i$) we have $V_i \supseteq W_i$ for $1 \leq i \leq \mu$. We simply say $G$ is an extension of $H$ if $\lambda$ and $\mu$ are unspecified.

Let $G$ be a graph, and let $S \subseteq V(G) = V$. We define the *boundary of* $S$ *in* $G$, $\partial_G(S)$, by

$$\partial_G(S) = \{x \in V \setminus S: xy \in E(G) \text{ for some } y \in S\}.$$

Thus $\partial_G(S)$ is the set of points in $G$ not in $S$ adjacent to points of $S$. If $V \subseteq V(H)$ for some graph $H$, where possibly $\langle V \rangle_H \not\cong G$, then the set of points in $V$ adjacent to points of $S$ (in the graph $H$) will be denoted by $G \cap \partial_H(S)$, and not by $\partial_G(S)$. Thus the symbol $\partial_G(S)$ is only to be used when $S$ is viewed as a subset of $V$ in the original graph $\langle V \rangle_G \cong G$. When $S$ is a singleton, we make an exception and write $N_G(S)$ for $G \cap \partial_H(S)$.

We now define a transformation on graphs. Let $G$ be a graph, with $E \subseteq E(G)$ and $W \subseteq V(G)$ such that $W \cap V(\langle E \rangle) = \emptyset$. If $x = st \in E$, then an *elementary transfer* is a map $\tau: G \to G'$, where $G'$ is the graph defined by $V(G') = V(G)$, and $E(G') = (E(G) \setminus \{x\}) \cup \{sw_1, tw_2\colon \text{some } w_1, w_2 \in W, w_1 \text{ and } w_2 \text{ not necessarily distinct}\}$. Thus $G'$ is the graph obtained by deleting $x$ and joining each point incident on $x$ to some point in $W$. Notice that $\deg_{G'} s = \deg_G s$ and $\deg_{G'} t = \deg_G t$. When $G'$ is not a graph (if, say, multiple edges are introduced via $sw_1$ or $tw_2$), then $\tau$ is not defined. We sometimes write $G(\tau)$ for $G'$. A *transfer on* $G$ *with domain set* $E$ *and object set* $W$ is a map $\tau: G \to G'$ which can be factored as a composition $\tau = \tau_k \circ \tau_{k-1} \circ \cdots \circ \tau_2 \circ \tau_1$ of elementary transfers $\tau_i$. That is, the $\tau_i$ are maps $\tau_i: G_{i-1} \to G_i$, where $G_0 = G$ and $G_{i-1}(\tau_i) = G_i$, $1 \leq i \leq k$, and $G_k = G'$. We define the norm $\|\tau\|$ of $\tau$ by $\|\tau\| = 2|E|$. Of course a transfer $\tau: G \to G'$ so defined is not uniquely determined. We denote by $T_G(E, W)$ the set of transfers on $G$ with domain set $E$ and object set $W$. An example of a $\tau \in T_G(E, W)$ is illustrated in Fig. 1 for a particular choice of $G, E$, and $W$. The key property of transfers which we will use is that for each $v \in \langle E \rangle$, we have $\deg_G v = \deg_{G'} v$.

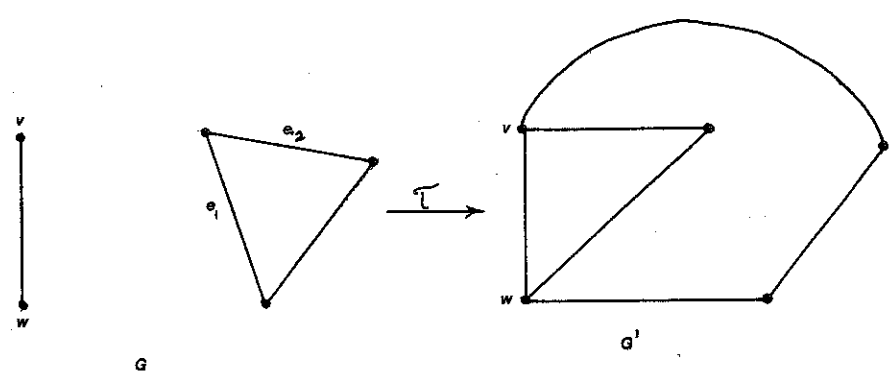


Fig. 1. A transfer $\tau \in T_G(E, W)$ with $E = \{e_1, e_2\}$ and $W = \{v, w\}$.

Finally, we call a graph $G$ *$\psi$-critical* if for any $x \in E(G)$ we have $\psi(G-x) < \psi(G)$. All definitions or notations not given explicitly here or below follow general usage, as may be found, for example, in [1] or [3].

## 2. The main result

Before proceeding with our theorem, we need the following lemma.

**Lemma.** *Let $G$ be a $k$-regular graph on $p$ points, $k$ even, satisfying $\psi(G)=\lambda$ with $\lambda$-partition $V(G)=\bigcup_{j=1}^{\lambda} V_j$. Suppose there is an integer $s$, $3 \leq s \leq \lambda$, such that each class $V_j$, $1 \leq j \leq s$, contains a set $B_j$ of cardinality $r$, where $r$ is a multiple of $\frac{1}{2}k$, such that $S=\bigcup_{j=1}^{s} B_j$ is an independent set in $G$ and $N_G(v) \cap N_G(w)=\emptyset$ for $v,w \in B_j$. Let $b$ be any integer satisfying $s \geq 2+\sqrt{2b+4}$ and $2 \leq b \leq sr$. Then there exists a $k$-regular graph $H$ satisfying:*

(i) $|H|=|G|+b$.

(ii) *$H$ is a $(\lambda, \lambda)$ extension of $G$.*

**Proof.** For convenience we think of $S$ as being arranged in an $s \times r$ rectangular array such that the points of $B_j$ form row $j$, $1 \leq j \leq s$, and are numbered $v_{j1}v_{j2}\cdots v_{jr}$. Write $b$ as $b=[b/s]s+t$, where $0 \leq t \leq s-1$, and $r=\frac{1}{2}dk$. Define a set $S'$ of $b$ points by

$$S'=\{y_{jm}: 1 \leq j \leq s, 1 \leq m \leq [b/s]\} \cup \{y_{l,([b/s]+1)}: 1 \leq l \leq t\}.$$

Thus $S'$ may be viewed as an array of $[b/s]+1$ columns, the first $[b/s]$ of which contain $s$ points while the last contains $t$ points. This array may be joined to $S$ to form a new array $A'$ in which the $j$th row, $j \leq s$, consists of the points

$$\{v_{jm}: 1 \leq m \leq r\} \cup \{y_{jm}: 1 \leq m \leq [b/s]+1\}$$

as in Fig. 2. Indeed, our object will be to form the required graph $H$ as $V(H)=V(G) \cup S'$ so that $H$ has a $\lambda$-partition $V(H)=\bigcup_{j=1}^{\lambda} V_j'$ in which $V_j'=V_j$ for $j>s$, and for $j \leq s$ $V_j'$ is obtained, modulo minor modifications, as the union of the $j$th row of $S'$ with $V_j$.

The graph $H$ will be constructed by a sequence of edge deletions and additions applied to the graph $G \cup \bar{K}_b = G \cup S'$. As notation, a *tower* $T$ in $S$ is a subset $T \subset S$ defined by

$$T=\{v_{jm}: 1+t(\tfrac{1}{2}k) \leq m \leq (t+1)\tfrac{1}{2}k,\ 1 \leq j \leq s\}$$

for some $t$, $0 \leq t \leq d-1$. Thus we may think of $S$ as being partitioned into $d$ towers, each tower consisting of the points in $\frac{1}{2}k$ consecutive columns of $S$. For a $T$ defined as above define $C(T)$ to be the set of column numbers in $T$, that is,

$$C(T)=\{x: 1+t(\tfrac{1}{2}k) \leq x \leq (t+1)\tfrac{1}{2}k\}.$$

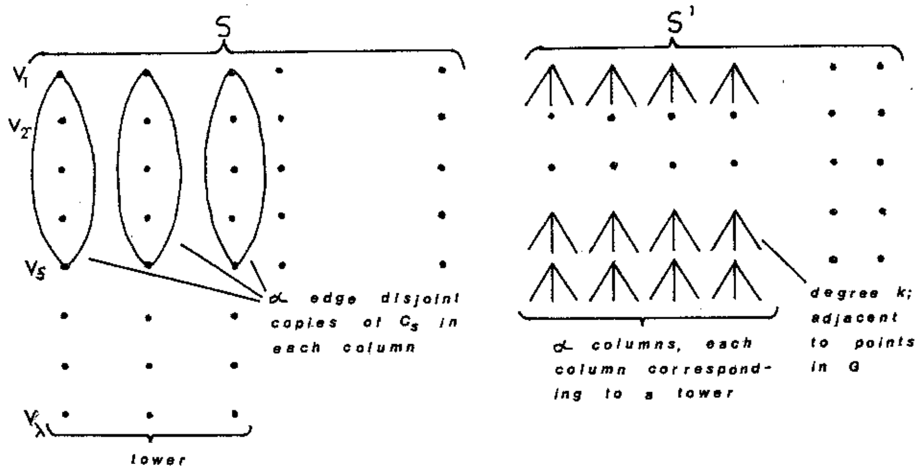


Fig. 2. The graph $G^{(\alpha)}$, an intermediate step in constructing $H$.

Now start with any tower $T_1$, and let $v_{jm} \in T_1$. Define $P_{jm}^{(1)}$ to be any pair of points in $N_G(v_{jm})$, and let $P_j^{(1)} = \bigcup_{m \in C(T_1)} P_{jm}^{(1)}$. Thus we have $|P_j^{(1)}| = k$ since the sets $P_{jm}^{(1)}$, $m \in C(T_1)$ have pairwise empty intersection by hypothesis. Now define the graph $G^{(1)}$ by

$$V(G^{(1)}) = V(G) \cup \{y_{j1}: 1 \leq j \leq s\},$$

$$E(G^{(1)}) = \left(E(G) \setminus \left[\bigcup_{j=1}^{s} \{v_{jm}x: x \in P_{jm}^{(1)}, m \in C(T_1)\}\right]\right)$$

$$\cup \{y_{j1}x: 1 \leq j \leq s, x \in P_j^{(1)}\} \cup \bigcup_{m \in C(T_1)} E(C_m^{(1)}),$$

where $C_m^{(1)}$ denotes any cycle having for its points the set of points $\{v_{jm}: 1 \leq j \leq s\}$ in the $m$th column of $S$. We have thus deleted two edges incident on each point of $T_1$ and for each $j$ we have joined $y_{j1}$ to the set of points in $G$ adjacent to points in $B_j$ via these deleted edges. This latter set is $P_j^{(1)}$, and since $|P_j^{(1)}| = k$ the process gives $y_{j1}$ degree $k$. To compensate the $v_{jm}$ for a decrease of 2 in their degree, the cycle $C_m^{(1)}$ is formed through the points in the $m$th column of $S$. Thus $G^{(1)}$ is $k$-regular, has $p+s$ points, and has $\lambda$-partition given by

$$G^{(1)} = \bigcup_{j=1}^{s} (V_j \cup \{y_{j1}\}) \cup \bigcup_{j=s+1}^{\lambda} V_j$$

since

$$(G \cap \partial_{G^{(1)}}(V_j \cup \{y_{j1}\})) = \partial_G(V_j) \quad \text{for} \quad 1 \leq j \leq s.$$

Proceeding inductively, let $\alpha$ be an integer satisfying $1 < \alpha < \{b/s\} \leq r$, and suppose the graphs $G^{(n)}$, $1 \leq n < \alpha$ have been defined, each definition being relative to a tower $T_n$ and sets $P_{jm}^{(n)}$ (for $m \in C(T_n)$) and $P_j^{(n)}$. We then define $G^{(\alpha)}$

as follows. Let $T_\alpha$ be a tower for which $\bigcup_{\beta=1}^{\alpha-1} P_{jm}^{(\beta)} \neq N_G(v_{jm})$ for $v_{jm} \in T_\alpha$ where by definition $P_{jm}^{(\beta)} = \emptyset$ for $\beta \leq \alpha - 1$ if $G^{(\beta)}$ is defined relative to a tower other than the one containing $V_{jm}$. For each $v_{jm} \in T_\alpha$ let $P_{jm}^{(\alpha)}$ be a pair of points in $N_G(v_{jm})$ satisfying $P_{jm}^{(\alpha)} \cap (\bigcup_{\beta=1}^{\alpha-1} P_{jm}^{(\beta)}) = \emptyset$. Again let $P_j^{(\alpha)} = \bigcup_{m \in C(T_\alpha)} P_{jm}^{(\alpha)}$, and observe that $|P_j^{(\alpha)}| = k$ and $P_j^{(\alpha)} \cap (\bigcup_{\beta=1}^{\alpha-1} P_j^{(\beta)}) = \emptyset$. We let

$$V(G^{(\alpha)}) = V(G^{(\alpha-1)}) \cup \{y_{j\alpha} : 1 \leq j \leq s\},$$

and

$$E(G^{(\alpha)}) = (E(G^{(\alpha-1)}) \setminus \bigcup_{j=1}^{s} \{v_{jm}x : x \in P_{jm}^{(\alpha)}, m \in C(T_\alpha)\})$$
$$\cup \{y_{j\alpha}x : 1 \leq j \leq s, x \in P_j^{(\alpha)}\} \cup \bigcup_{m \in C(T_\alpha)} E(C_m^{(\alpha)}),$$

where $C_m^{(\alpha)}$ denotes a cycle having for its points the set of points $\{v_{jm} : 1 \leq j \leq s\}$ in the $m$th column of $S$ satisfying $E(C_m^{(\alpha)}) \cap (\bigcup_{\beta-1}^{\alpha-1} E(C_m^{(\beta)})) = \emptyset$ where again $C_m^{(\beta)} = \emptyset$ by definition for $\beta \leq \alpha - 1$ if the definition of $G^{(\beta)}$ is not made relative to the tower containing column $m$. The existence of possibly $\alpha$ cycles $C_m^{(\beta)}$, $1 \leq \beta \leq \alpha$, each having the $m$th column of $S$ as point set, may be verified as follows. Each column of $S$ may be viewed as the point set of a copy of $K_s$. Since $\alpha \leq [b/s]$, it would suffice to show that $K_s$ contains at least $[b/s]$ hamiltonian cycles. But clearly $K_s$ contains at least $[\frac{1}{2}s] - 1$ hamiltonian cycles and the hypothesis $s \geq 2 + \sqrt{2b+4}$ implies $[\frac{1}{2}s] - 1 \geq 1 + [b/s] > [b/s]$, as required. The role of $C_m^{(\alpha)}$, as that of $C_1^{(\alpha)}$, is to add 2 to the degree of the $v_{jm}$, $1 \leq j \leq s$, to compensate for the loss of 2 in their degree resulting from the deletion of the edges $v_{jm}x$, $x \in P_{jm}^{(\alpha)}$, in the definition of $G^{(\alpha)}$.

We make some remarks concerning the relation of $G^{(\alpha)}$ to $G^{(\alpha-1)}$ and to $G$. First, $V(G^{(\alpha)})$ is obtained by adjoining the $\alpha$th column of $S'$ to $V(G^{(\alpha-1)})$. In forming the edges, we join $y_{j\alpha}$ to the points of the set $P_j^{(\alpha)}$, and $P_j^{(\alpha)}$ is precisely the set of points adjacent in $G^{(\alpha-1)}$ to points of the $j$th row $(B_j)$ of $S$ but not so adjacent in $G^{(\alpha)}$. That is, we have

$$N_{G^{(\alpha)}}(y_{j\alpha}) = \partial_{G^{(\alpha-1)}}(B_j) \setminus \partial_{G^{(\alpha)}}(B_j).$$

By definition we also have $N_{G^{(\alpha)}}(y_{j\alpha}) \subseteq G$, and hence

$$(G \cap \partial_{G^{(\alpha)}}(B_j)) \cup N_{G^{(\alpha)}}(y_{j\alpha}) = G \cap \partial_{G^{(\alpha-1)}}(B_j).$$

Now let $B_j'(\alpha) = \{y_{jx} : 1 \leq x \leq \alpha\}$, the first $\alpha$ points in the $j$th row of $S'$. It follows that

$$(G \cap \partial_{G^{(\alpha)}}(B_j)) \cup \partial_{G^{(\alpha)}}(B_j'(\alpha)) = \partial_G(B_j).$$

Hence if we adjoin $B_j'(\alpha)$ to the class $V_j$ of $G$ to obtain a new class $V_j(\alpha)$ in $G^{(\alpha)}$, then the points of $V_j(\alpha)$ are adjacent to the same points in $G$ as $V_j$ was (in the graph $G$), i.e. $G \cap \partial_{G^{(\alpha)}}(V_j(\alpha)) = \partial_G(V_j)$. Since $V_j(\alpha)$ is an independent set, it follows that $G^{(\alpha)}$ is a $(\lambda, \lambda)$ extension of $G$, with classes $V_j(\alpha)$ for $1 \leq j \leq s$ and $V_j$

for $j>s$. It is this property of $G^{(\alpha)}$ which allows us to obtain the extension $H$ of $G$ claimed in the lemma. We illustrate $G^{(\alpha)}$ in Fig. 2.

The graph $G^{(\alpha)}$, where $\alpha=[b/s]+1$, is similar, but certain changes are necessary since column $1+[b/s]$ of $S'$ has $t<s$ points. First we define a cycle $C$ with points in $S$ as follows. Let $T=T_{1+[b/s]}$ be a tower, say with

$$C(T_{1+[b/s]})=\{x: 1+y(\tfrac{1}{2}k)\leq x\leq (y+1)\tfrac{1}{2}k\}$$

for some $y$. For each $m$, $m\in C(T)$, let $C_m$ denote a cycle through the points $v_{jm}$, $1\leq j\leq s$, such that

$$E(C_m)\cap\left(\bigcup_{\beta=1}^{[b/s]}E(C_m^{(\beta)})\right)=\emptyset.$$

Starting with $v_m^{(1)}=v_{1m}$, let $v_m^{(1)}-v_m^{(2)}-v_m^{(3)}-\cdots-v_m^{(t)}$ be a cyclic traversal of the
8 first $t$ points of $C_m$. Thus the $v_m^{(n)}$, $1\leq n\leq t$, all lie in the $m$th column of $S$. Now join the points $v_m^{(t)}$ and $v_{m+1}^{(1)}$, $m\in C(T)$, where subscript $1+(y+1)\frac{1}{2}k$ is identified with subscript $1+y\frac{1}{2}k$ to obtain the cycle $C$. More formally, we can define $C$ by

$$V(C)=\bigcup_{m\in C(T)}\{v_m^{(n)}: 1\leq n\leq t\},$$

and

$$E(C)=\bigcup_{m\in C(T)}[\{v_m^{(n)}v_m^{(n+1)}: 1\leq n\leq t-1\}\cup\{v_m^{(t)}v_{m+1}^{(1)}\}],$$

identification of subscripts as above. Note that $C$ and the $C_m^{(\alpha)}$, $1\leq\alpha\leq[b/s]$, could be chosen in the following special way. Writing the points in a cyclic traversal of $C_m^{(\alpha)}$ as $v_m^1(\alpha)-v_m^2(\alpha)-\cdots-v_m^s(\alpha)$ we can arrange that the points $v_m^j(\alpha)$, $j$ and $\alpha$ fixed, all lie in the same row of $S$. Indeed, this can be achieved by requiring that for each $\alpha$ the $C_m^{(\alpha)}$, $m\in C(T)$, are exact replicas of each other obtained by translating one of them, say $C_1^{(\alpha)}$, horizontally to each of the other columns. Similarly, we can arrange all the points $v_m^{(n)}$ of $C$, $n$ fixed, to lie in the same row of $S$, and we let the index of this row be $\bar{n}$. This last condition on $C$ is used, as will be seen, in getting the required $\lambda$-partition of $H$. Now for each pair $(n, m)$, let $P_{nm}$ be a pair of points in $N_G(v_m^{(n)})$ such that

$$P_{nm}\cap\left(\bigcup_{\beta=1}^{[b/s]}P_{\bar{n}}^{(\beta)}\right)=\emptyset,$$

and let $P_n=\bigcup_{m=1}^{t}P_{nm}$. Now define $G^{(1+[b/s])}$ as in the inductive definition above, with $P_{nm}$, $P_n$, $E(C)$, and $t$ playing the roles of $P_{jm}^{(\alpha)}$, $P_j^{(\alpha)}$, $\bigcup E(C_m^{\alpha})$, and $s$ respectively. Observe that the role of $C$ in $G^{(1+[b/s])}$ is the same as that of the $C_m^{(\alpha)}$ in $G^{(\alpha)}$ for $\alpha\leq[b/s]$, namely, to restore $k$-regularity (by an increment of 2 to the degree) to the points $v_m^{(n)}$, $m\in C(T)$, $1\leq n\leq t$.

Our graph $H$ may now be defined by

$$H=\begin{cases}G^{(b/s)} & \text{if } s\mid b,\\ G^{(1+[b/s])} & \text{if } s\nmid b.\end{cases}$$

It is easily checked that $H$ has the required properties. Each of the subgraphs $G^{(\alpha)}$, $1 \leq \alpha \leq [b/s]+1$, is $k$-regular by essentially the same argument which showed that $G^{(1)}$ is $k$-regular. Hence $H$ is $k$-regular, and clearly $|H| = |G| + b$. As notation, denote the 1–1 map $n \to \bar{n}$, $n \in \{1, 2, \ldots, t\}$, $\bar{n} \in \{1, 2, \ldots, s\}$ by $\sigma$; so that $\sigma(n) = \bar{n}$ for $1 \leq n \leq t$. We also let $T = \{1, 2, \ldots, t\}$, so $\sigma(T) = \{\sigma(n): n \in T\}$. The $\lambda$-partition of $H$ may now be given by $H = \bigcup_{j=1}^{\lambda} V'_j$, where

$$V'_j = \begin{cases} V_j \cup \left\{y_{jm}: 1 \leq m \leq \dfrac{b}{s}\right\}, & j \leq s, \quad \text{if } s \mid b, \\ \left.\begin{array}{ll} V_j \cup \left\{y_{jm}: 1 \leq m \leq \left[\dfrac{b}{s}\right]\right\} \cup \{y_{\sigma^{-1}(j),\, 1+[b/s]}\}, & j \in \sigma(T) \\ V_j \cup \left\{y_{jm}: 1 \leq m \leq \left[\dfrac{b}{s}\right]\right\}, & j \notin \sigma(T) \end{array}\right\} & \text{if } s \nmid b, \\ V_j, \quad j > s. \end{cases}$$

Essentially the same proof as that given for $G^{(\alpha)}$, $\alpha \leq [b/s]$, shows that $G^{(1+[b/s])}$ is a $(\lambda, \lambda)$ extension of $G$ with $V'_j$ playing the role of $V_j(\alpha)$. The required graph $H$ is thereby constructed, and the lemma is proved. □

Our result may now be presented. We let $f(p, k) = \max\{\lambda \in \mathbb{Z}: \lambda\{(\lambda-1)/k\} \leq p\}$, and we define the subset $\Omega$ of the positive integers by $\Omega = \{i(ik+1)+1: 1 \leq i \leq \infty\}$.

**Theorem.** *Let $k \geq 2$ be an integer. Then*

(i) *$M(p, k) \leq f(p, k)$ for all $p$.*

(ii) *There is a function $Q(k)$ such that for all $p \geq Q(k)$ we have*

$$M(p, k) = \begin{cases} f(p, k) & \text{if } p \notin \Omega, \\ f(p, k) - 1 & \text{if } p \in \Omega. \end{cases}$$

**Proof.** First we prove (i). Suppose $G$ is a $k$-regular graph on $p$ points satisfying $\psi(G) = M(p, k)$. Hence there exists a homomorphism $h: G \to K_{M(p,k)}$, and we let $r = \min\{|f^{-1}(t)|: t \in K_{M(p,k)}\}$. Set $M = M(p, k)$ and let $C = f^{-1}(x)$ be a preimage under $h$ satisfying $|C| = r$ for some $x \in K_{M(p,k)}$. Since $|\partial C| \geq M-1$ we have $kr \geq M-1$. It follows that $r \geq \{(M-1)/k\}$. But since $r$ is the smallest cardinality of all preimages we also have $Mr \leq p$. Combining these two inequalities we get $M\{(M-1)/k\} \leq p$, and hence $M(p, k) \leq f(p, k)$ as desired.

The proof of (ii) is based on an inductive construction. Let us first suppose $p \notin \Omega$. Our task is then to prove that $M(p, k) \geq f(p, k)$, and hence we must produce for each possible pair $(p, k)$ ($p$ even when $k$ odd) a $k$-regular graph $G(p, k)$ on $p$ points satisfying $\psi(G(p, k)) = f(p, k)$. Our general method is as follows. Fixing $k$, consider the subsequence $\{n_i\}_{i=1}^{\infty}$ of the positive integers defined by $n_i = i(ki+1)$. First we will construct a set of graphs $\{G(n_i, k): 1 \leq i < \infty\}$ satisfying $\psi(G(n_i, k)) = f(n_i, k)$. It follows that $M(n_i, k) \geq f(n_i, k)$, so that our theorem will have been proved for the point-regularity pair $(n_i, k)$, $1 \leq i < \infty$. Our next step will be to use the lemma to construct graphs $G(p, k)$ satisfying $\psi(G(p, k)) = f(p, k)$, where $n_{i-1}+1 < p < n_i$, $1 \leq i < \infty$.

We begin with the construction of the $G(n_i, k)$. Since $f(n_i, k) = ik+1$, our graph $G(n_i, k)$ must satisfy $\psi(G(n_i, k)) = ik+1$. Proceeding inductively, let $G(n_1, k) = K_{k+1}$ so that $n_1 = k+1 = \psi(G(n_1, k))$ as required. For the purposes of the induction label the vertices of $G(n_1, k)$ with the labels $(t, 1)$, $1 \leq t \leq k+1$. Having constructed $G(n_{i-1}, k)$ we let

$$G(n_i, k) = G(n_{i-1}, k) \cup (i-1)K_{k,k} \cup K_{k+1}.$$

Each point $v$ in the graph $G(n_i, k)$ so defined will be given a label $(t(v), s(v))$, $1 \leq t(v) \leq ik+1$, $1 \leq s(v) \leq i$, so that the following two conditions hold:

(1) $t(v) = t(w) \Rightarrow vw \notin E(G(n_i, k))$.

(2) $1 \leq t_1 \neq t_2 \leq ik+1 \Rightarrow$ there exists an edge $vw \in E(G(n_i, k))$ such that $t(v) = t_1$ and $t(w) = t_2$.

Observe that this would imply $\psi(G(n_i, k)) \geq ik+1$ (and hence $\psi(G(n_i, k)) = ik+1$) since an $(ik+1)$-partition of $G(n_i, k)$ may then be defined by letting $V_d$, $1 \leq d \leq ik+1$, be the set of points $v \in G(n_i, k)$ satisfying $t(v) = d$. Suppose then by induction that we have labeled the points in the subgraph $G(n_{i-1}, k)$ of $G(n_i, k)$, $i \geq 2$, with labels $(t, s)$, $1 \leq t \leq (i-1)k+1$, $1 \leq s \leq i-1$, so that conditions (1) and (2) hold with $i$ replaced by $i-1$. Hence there is an $(i-1)k+1$-partition of $G(n_{i-1}, k)$. We then complete the induction by distributing labels to the points of

$$Q = G(n_i, k) \setminus G(n_{i-1}, k) = (i-1)K_{k,k} \cup K_{k+1}$$

as follows. Number the $i-1$ copies of $K_{k,k}$ in $Q$ by writing them as $K_{k,k}(r)$, $1 \leq r \leq i-1$, and let $K_{k,k}(r) = V(r) \cup W(r)$ be the bipartition of $K_{k,k}(r)$ into two color sets, with $V(r) = \{v_{1r}, v_{2r}, \ldots, v_{kr}\}$ and $W(r) = \{w_{1r}, w_{2r}, \ldots, w_{kr}\}$. We label the points in $\bigcup_{r=1}^{i-1} K_{k,k}(r) \subset Q$ by $t(w_{ab}) = (i-1)k+1+a$, $s(w_{ab}) = b$, $t(v_{ab}) = (b-1)k+a$, and $s(v_{ab}) = i$. The $k+1$ points of $K_{k+1}$ are given the $k+1$ labels $\{((i-1)k+x, i): 1 \leq x \leq k+1\}$. An illustration of this labeling for the graph $G(n_3, 3)$ is given in Fig. 3, where a point $v$ carries the symbol $xy$ if the above procedure labels it with $t(v) = x$ and $s(v) = y$. Under this labeling, the $t$ coordinate indicates the class of the point while the $s$ coordinate numbers the point among the other points in its class. The function of the subgraph $Q$ of $G(n_i, k)$ is to introduce (i.e., make adjacent) the $k$ new classes of points in $G(n_i, k)$ (but not in $G(n_{i-1}, k)$) to the old classes in $G(n_{i-1}, k)$. This introduction is done with $k$ classes of $G(n_{i-1}, k)$ at a time in the bigraphs $K_{k,k}(r)$, $1 \leq r \leq i-1$, and with one remaining class in the $K_{k+1}$.

With this description, it is then easy to see that our labeling of $G(n_i, k)$ satisfies conditions (1) and (2). Hence we get $\psi(G(n_i, k)) = ik+1 = f(n_i, k)$ as desired.

We have now constructed the graphs $G(n_i, k)$ such that $\psi(G(n_i, k)) = f(n_i, k)$ for $1 \leq i < \infty$, so that $M(n_i, k) = f(n_i, k)$ for the sequence $\{n_i\}_{i=1}^{\infty}$. It remains to prove that for all $i \geq 1$ there exist $k$-regular graphs $G(p, k)$ of order $p$ satisfying $\psi(G(p, k)) = f(p, k)$ for each $p$ satisfying $n_{i-1}+1 < p < n_i$. This will be done in the

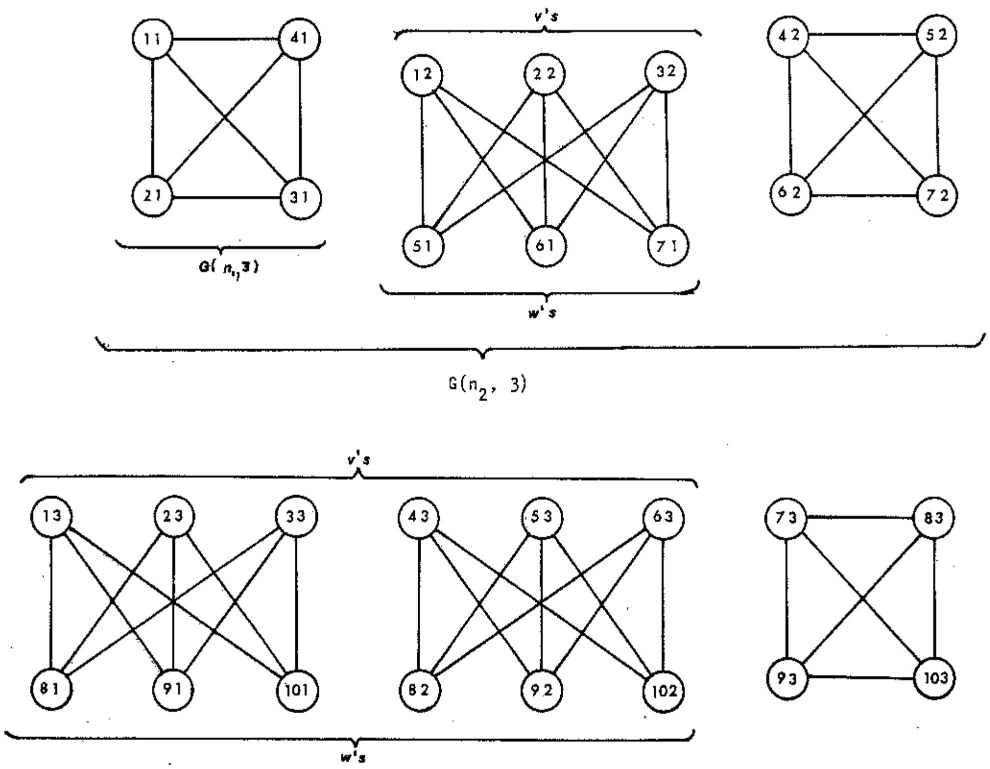


Fig. 3. The graph $G(n_3, 3)$ and its subgraphs $G(n_1, 3)$ and $G(n_2, 3)$.

following two stages. Fix the degree of regularity $k$. Suppose $k$ is even, with the case $k$ odd to be outlined later. For the first stage, let $p(\lambda)$ denote the minimum value of $p$ for which $f(p, k)=\lambda$. Viewing $f(p, k)$ as a function of $p$, note that it is a step function whose jumps occur at the integers $p(\lambda)$, $k+1\leqslant\lambda<\infty$. Now consider a fixed interval of integers $[n_{i-1}, n_i]$ for some $i\geqslant 2$, and let $\lambda_1=f(n_i, k)$ and $\lambda_0=f(n_{i-1}, k)$. Notice first that $\lambda_1-\lambda_0=k$, and that for each $\lambda$ satisfying $\lambda_0<\lambda<\lambda_1$ we have $\{(\lambda-1)/k\}=\{(\lambda_1-1)/k\}$. It follows that $p(\lambda)=p(\lambda+1)-\{(\lambda_1-1)/k\}$ for each $\lambda$, $\lambda_0<\lambda<\lambda_1$ and $p(\lambda_0)=n_{i-1}, p(\lambda_1)=n_i$, and $p(\lambda_1-r)=p(\lambda_1)-r\{(\lambda_1-1)/k\}$. We will use these observations to construct the graphs $G(p(\lambda), k), \lambda_0<\lambda<\lambda_1$, and this construction may be initially outlined as follows. Begin by removing $\{(\lambda_1-1)/k\}$ points from $G(n_i, k)$ and then adding certain edges to the resulting graph on $p(\lambda_1-1)$ points to obtain the graph $G(p(\lambda_1-1), k)$ satisfying $\psi(G(p(\lambda_1-1), k))=f(p(\lambda_1-1), k)$. We then proceed by descent. Having constructed the graph $G(p(\lambda_1-r), k)$ for $1\leqslant r<k-1$, we remove $\{(\lambda_1-1)/k\}$ points from $G(p(\lambda_1-r), k)$ and add certain edges to the resulting graph to obtain the graph $G(p(\lambda_1-r-1), k)$ satisfying

$$\psi(G(p(\lambda_1-r-1), k))=f(p(\lambda_1-r-1), k).$$

With the graphs $G(p(\lambda), k)$ thereby constructed, we will have shown that $f(p(\lambda), k) = M(p(\lambda), k)$ for $\lambda_0 < \lambda < \lambda_1$. In the second stage we will use a method of 'transfer' to develop an induction for showing that $f(p, k) = M(p, k)$ when $p$ satisfies $p(\lambda) < p < p(\lambda+1)$ for any $\lambda$. We thus first prove $M(n, k) = f(n, k)$ for integers $n$ of the form $p(\lambda)$, $\lambda_0 < \lambda < \lambda_1$, using descent, and second we use the lemma to prove that $M(p, k) = f(p, k)$ for integers $p$ in each subinterval $[p(\lambda), p(\lambda+1)]$, $\lambda_0 \leq \lambda < \lambda_1$. The result is $M(p, k) = f(p, k)$ for all $p \in [n_{i-1}, n_i]$, $p \notin \Omega$.

The method described above may be summarized in the graph illustrated in Fig. 4. This graph describes $f(p, k)$ as a function of $p$ with $k$ held constant. The phrase '$n$ achieved by' for an integer $n$ means '$G(n, k)$ is constructed by'.

Start with an interval $[n_{i-1}, n_i]$ for some $i \geq 2$. Since the construction of $G(p, k)$ is required only for $p$ sufficiently large, we may assume that $n_i$ is large enough so that $i-1 \geq k$. We begin with the construction of the graphs $G(p(\lambda), k)$ for $\lambda_0 < \lambda < \lambda_1$. The cases $i$ even or odd require similar constructions (differing only in minor details), and we restrict ourselves here to the case $i$ odd. Our first step is to define the graphs $G(p(\lambda_1 - r), k)$, $1 \leq r \leq k-1$. Fixing $r$, we begin by constructing a graph $Q$ from which $G(p(\lambda_1 - r), k)$ is obtained by the transfer operation. First we pair off the bigraphs $K_{k,k}(j)$, $1 \leq j \leq i-1$, of $G(n_i, k) \setminus G(n_{i-1}, k)$ into a pairing $\mathscr{P}$ having $\frac{1}{2}(i-1)$ disjoint pairs. For each pair $(K_{k,k}(l), K_{k,k}(m)) \in \mathscr{P}$, define the subgraph $G_{l,m}$ having point set

$$\{v_{1l}, v_{2l}, \ldots, v_{kl}, v_{1m}, v_{2m}, \ldots, v_{km}\}$$

(referring to the definition of $G(n_i, k)$) as follows. We make $G_{l,m}$ bipartite with bipartition $\{v_{1l}, v_{2l}, \ldots, v_{kl}\}$, $\{v_{1m}, v_{2m}, \ldots, v_{km}\}$, and we let $F_i$, $1 \leq i \leq r$, be any set of $r$ 1-factors that can be constructed in a bigraph with this bipartition. (Thus the

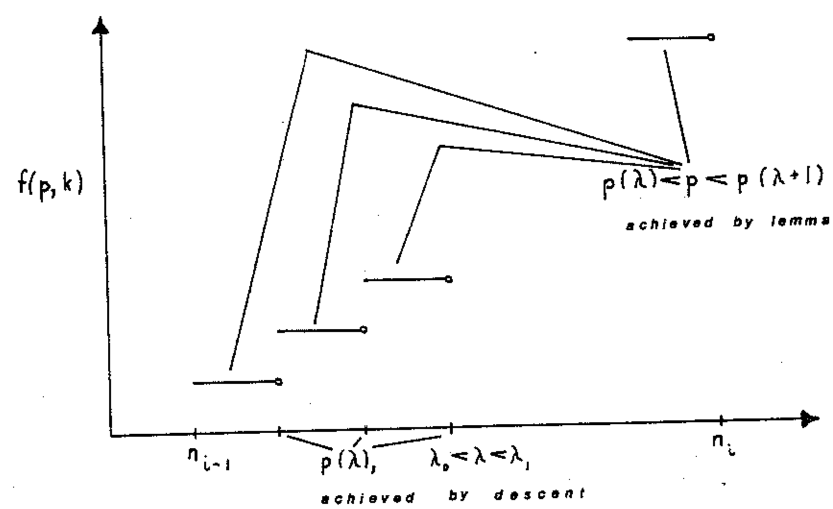


Fig. 4. The function $f(p, k)$, and construction of $G(p, k)$ in two stages.

$F_i$ can be thought of as coming from a copy of $K_{k,k}$ having color sets $\{v_{1l}, \ldots, v_{kl}\}$ and $\{v_{1m}, \ldots, v_{km}\}$.) Now define $G_{l,m}$ as the edge disjoint union of the $F_i$, that is, $G_{l,m} = F_1 + F_2 + \cdots + F_r$. Next define an edge set $F(\lambda_1 - r)$ by $F(\lambda_1 - r) = \bigcup_{\mathcal{P}} E(G_{l,m})$, where the union is over all pairs in $\mathcal{P}$.

We now define the graph $Q$ by

$$V(Q) = V(G(n_i, k)) \setminus \{(t, s) : \lambda_1 - r + 1 \leq t \leq \lambda_1\} = \bigcup_{j=1}^{t} V_j,$$

$$E(Q) = E\left(\left\langle \bigcup_{j=1}^{t} V_j \right\rangle_{G(n_i, k)}\right) \cup F(\lambda_1 - r),$$

where $t = \lambda_1 - r$. We illustrate the graph $Q$ in the case $i = 3$, $r = 2$, $k = 4$, and $\lambda_1 = f(n_3, 4)$. We therefore call it $Q(p(\lambda_1 - 2), 4)$. See Fig. 5.

We now make some observations concerning $Q$ which motivate the construction of $G(p(\lambda_1 - r), k)$. By definition, $V(Q)$ is obtained from $V(G(n_i, k))$ by removing all points in the classes $V_d$, $\lambda_1 - r + 1 \leq d \leq \lambda_1$, of the $(ik+1)$-partition $G(n_i, k) = \bigcup_{j=1}^{ik+1} V_j$ of $G(n_i, k)$. It follows that

$$|V(Q)| = n_i - r\{(\lambda_1 - 1)/k\} = (ik + 1 - r)i = p(\lambda_1 - r).$$

Now the removal of $\bigcup_{j=\lambda_1-r+1}^{ik+1} V_j$ from $G(n_i, k)$ obviously destroys $k$-regularity in the remaining points of $G(n_i, k) \setminus G(n_{i-1}, k)$, and the role of the edge set $F(\lambda_1 - r)$ is to restore $k$-regularity to the points in the bigraphs of $G(n_i, k) \setminus G(n_{i-1}, k)$. Hence the only points of $Q$ which do not have degree $k$ are those not incident with $F(\lambda_1 - r)$, namely, those in the component $K_{k+1-r}$ of $Q$.

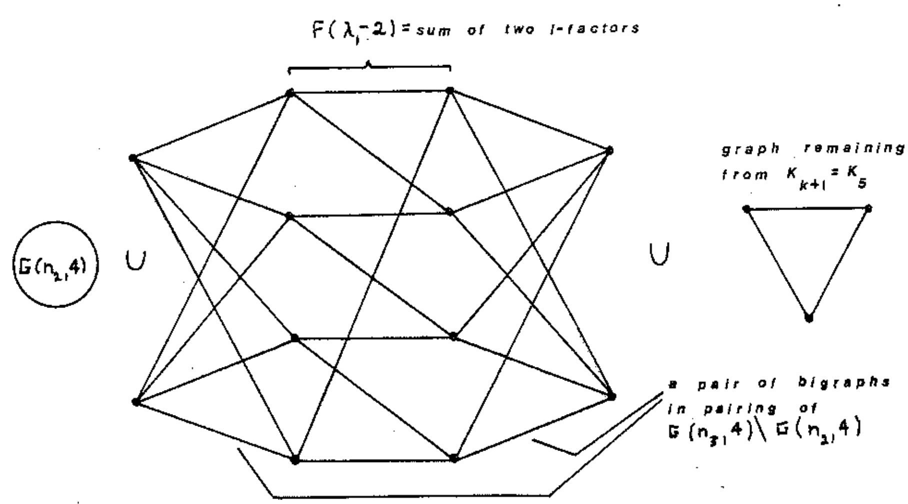


Fig. 5. The graph $Q(p(\lambda_1 - 2), 4)$, where $\lambda_1 = f(n_3, 4)$.

The graph $G(p(\lambda_1-r), k)$ will be obtained from $Q$ as the image $G(\tau)$ of some transfer $\tau$ on $Q$. The remarks above show that $\tau$ must restore $k$-regularity to the points of the component $K_{k+1-r}$ of $Q$. Indeed, the following two conditions would be sufficient for $G(\tau)$ to fulfill the role of $G(p(\lambda_1-r), k)$.

(A) $G(\tau)$ has the same $(\lambda_1-r)$-partition that $Q$ does, namely, $G(\tau)=\bigcup_{j=1}^{\lambda_1-r} V_j$, and

(B) $G(\tau)$ is $k$-regular.

For then $G(\tau)$ would be a $k$-regular graph on $p(\lambda_1-r)$ points satisfying $\psi(G(\tau))=\lambda_1-r$, precisely what we required in the definition of $G(p(\lambda_1-r), k)$.

The key property of $Q$ which makes the transfer possible is its subgraph $\langle F(\lambda_1-r)\rangle$. Now the points in $G_{l,m}$ (for any pair $l, m$) are (in the language of the definition of $V(Q)\subseteq V(G(n_i, k))$) $v$'s, and hence under the labeling $(t, s)$ of $G(n_i, k)$ they belong to the classes $V_t$, $1\leq t\leq (i-1)k$. But any pair of these classes are already joined in $G(n_{i-1}, k)$ since $G(n_{i-1}, k)$ has the $(i-1)k+1$ partition $G(n_{i-1}, k)=\bigcup_{t=1}^{(i-1)k+1} V_t$. Hence each edge of $G_{l,m}$, and therefore each edge of $F(\lambda_1-r)$ by the arbitrariness of $l, m$, joins a pair of classes already joined in $G(n_{i-1}, k)$. This fact makes possible our transfer since each edge $e\in F(\lambda_1-r)$ 'frees' some edge $e'\in E(G(n_{i-1}, k))\subseteq E(Q)$. That is, the graph $Q-e'$ retains the same $(\lambda_1-r)$-partition that $Q$ does. If we defined $\tau$ so that its domain set is a collection of such $e'$'s then condition (A) would be automatically satisfied. In order to satisfy (B), we will use the component $K=K_{k+1-r}$ of $Q$ as the object set of $\tau$.

Our transfer $\tau$ may now be constructed as follows. First we find an appropriate domain set for $\tau$. For each edge $e\in F(\lambda_1-r)$, define (as above) $e'$ to be the unique edge in $G(n_{i-1}, k)$ which joins the same pair of classes that $e$ does. Now if $e_1, e_2\in F(\lambda_1-r)$ are distinct, then the two pairs of classes joined by $e_1$ and $e_2$ are distinct, and hence the two pairs of classes joined by $e_1'$ and $e_2'$ are distinct. In particular, $e_1'$ and $e_2'$ are distinct. It follows that there is a one-one correspondence between the edge sets $F(\lambda_1-r)$ and $E'=\{e': e\in F(\lambda_1-r)\}$, so $|E'|=|F(\lambda_1-r)|=\frac{1}{2}(i-1)kr$. We show that $E'$ is large enough to contain a domain set for $\tau$. Now the proposed object set $K$ of $\tau$ has $k+1-r$ points each having degree $k-r$. Therefore, we must require that $\tau$ have norm $\|\tau\|=(k+1-r)r$ (and note that $k$ even implies $(k+1-r)r=\|\tau\|$ is even, as required in the definition of transfer). Now since $2|E'|=kr(i-1)\geq(k+1-r)r$ there exists a set $E''\subset E'$ such that $2|E''|=(k+1-r)r$, and we will choose such an $E''$ as the domain set of $\tau$. By assumption we have $i-1\geq k$, so that $(i-1)k>kr\geq(k+1-r)r$. Thus if $F'$ is any subset of $E'$ satisfying $2|F'|=(i-1)k$, then $F'$ too contains a subset $E''$ such that $2|E''|=(k+1-r)r$. We will choose $F'$ and subsequently $E''$ as follows. In each graph $G_{l,m}$ let $F_{l,m}$ be the set of edges of a 1-factor. Now let $F=\bigcup F_{l,m}$ as $(l, m)$ varies over the pairs of $\mathcal{P}$. Observe that each class $V_t$, $1\leq t\leq(i-1)k$, is represented exactly once among the points of $\langle F\rangle$, and $|F|=\frac{1}{2}(i-1)k$. Letting $F'=\{e': e\in F\}$, it follows that $(i-1)k$ classes are represented in $\langle F'\rangle$. Hence we have $|V(\langle F'\rangle)|\geq(i-1)k$, which combined with $|F'|=\frac{1}{2}(i-1)k$ forces $\langle F'\rangle$ to be 1-regular.

Now let $E''$ be any subset of $F'$ satisfying $2|E''|=(k+1-r)r$, and since $E''\subset F$ it follows that $\langle E''\rangle$ is 1-regular. Choose this $E''$ for the domain set of $\tau$. As already observed, the containments $E''\subset F'\subset E'$ guarantee that $G(\tau)$ satisfies (A).

The 1-regularity of $\langle E''\rangle$ may be used to show that a $\tau$ with domain set $E''$ exists which satisfies (B). Observe that the evenness of $\|\tau\|$ allows us to construct $\tau$ one edge of $E''$ at a time. That is, we may view $\tau$ as a composition $\tau_x\circ\tau_{x-1}\cdots\circ\tau_1$ of $x=\frac{1}{2}\|\tau\|$ transfers, each having a single edge of $E''$ as domain set and one or two points of $K$ as object set. With each successive $\tau_i$, we increase the degree of some point of $K$ by 2 or the degrees of two such points by 1 each until the degree of each point in $K$ reaches $k$. This procedure yields a $k$-regular graph $G(\tau)$ provided that no multiple edges are introduced by $\tau$. But this is guaranteed by the 1-regularity of $\langle E''\rangle$. For each edge introduced by $\tau_i$, $1\leq i\leq\frac{1}{2}\|\tau\|$, is incident with some point of $\langle E''\rangle$ and 1-regularity implies that no point of $\langle E''\rangle$ can be incident with more than one edge introduced in any of the $G(\tau_i)$. Hence the $G(\tau)$ so constructed has no multiple edges, and is therefore a $k$-regular graph. Condition (B) is thereby satisfied.

We have thus found a transfer $\tau\in T_Q(E'',K)$ with the property that $G(\tau)$ satisfies (A) and (B). Letting $G(\tau)=G(p(\lambda_1-r),k)$, we have thereby constructed the graphs $G(p(\lambda_1-r),k)$, $1\leq r\leq k-1$, and it follows that $f(p(\lambda_1-r),k)=M(p(\lambda_1-r),k)$ or equivalently $f(p(\lambda),k)=M(p(\lambda),k)$, $\lambda_0<\lambda<\lambda_1$.

To complete our construction and proof, we must fill in the gaps $p(\lambda)<p<p(\lambda+1)$ by showing that $M(p,k)=f(p,k)$ for all such $p$ and all $\lambda$, $\lambda_0\leq\lambda<\lambda_1$, $\lambda_0=f(n_{i-1},k)$, $\lambda_1=f(n_i,k)$, and $p\notin\Omega$. Our goal is therefore to construct, for each $p(\lambda)<p<p(\lambda+1)$ and $p\notin\Omega$ a $k$-regular graph $G(p,k)$ satisfying $|G|=p$ and $\psi(G)=f(p,k)$. Our method will be to apply the lemma with the above constructed graph $G(p(\lambda),k)$ playing the role of $G$, and the required graph $G(p,k)$ the role of $H$, with $b=p-p(\lambda)$. For if this can be done, then the lemma guarantees that $\psi(G(p,k))\geq\psi(G(p(\lambda)),k)$, while

$$\psi(G(p,k))\leq M(p,k)\leq f(p,k)=\lambda=f(p(\lambda),k)=\psi(G(p(\lambda),k))$$

by definition of $p(\lambda)$ and property (A) of the graph $G(p(\lambda),k)$. Hence we would have $\psi(G(p,k))=f(p,k)$, and the desired conclusion $M(p,k)=f(p,k)$ would follow.

It remains to verify that the hypotheses of the lemma hold for our $G=G(p(\lambda),k)$ and our desired $H=G(p,k)$ with $b=p-p(\lambda)$. It is in meeting these hypotheses that we will require freedom in taking $p(\lambda)$ and $p$ large enough, and this freedom is given in the conditions of our theorem.

The conditions on $G(p(\lambda),k)$ required by the lemma can be stated explicitly. As observed previously, we have $p(\lambda+1)-p(\lambda)=\{(\lambda_1-1)/k\}$ for $\lambda>\lambda_0$ and

$$p(\lambda_0+1)-p(\lambda_0)=1+\lambda_0+(\lambda_0-1)/k.$$

Since $(\lambda_0-1)/k+1=\{(\lambda_1-1)/k\}$, it follows that

$$b=p-p(\lambda)\leq p(\lambda+1)-p(\lambda)\leq 1+\lambda_0+(\lambda_0-1)/k.$$

Hence it suffices to find $l$ classes $V_1, V_2, \ldots, V_l$ in a $\lambda$-partition of $G(p(\lambda), k)$, $l$ playing the role of $s$ in the lemma and satisfying the conditions of the lemma with

$$l \geq 2+\sqrt{6+2(\lambda_0+(\lambda_0-1)/k)} \geq 2+\sqrt{2b+4}.$$

In fact, these classes will be the extensions to $G(p(\lambda), k)$ of the first $l$ classes (where $l$ will be given below), call them $\bar{V}_1, \bar{V}_2, \ldots, \bar{V}_l$, in the $\lambda_0$-partition of $G(n_{i-1}, k)$ (recall that the latter classes are the ones whose points have $t$ coordinate satisfying $1 \leq t \leq l$ in the $(t, s)$ labeling of $G(n_{i-1}, k)$). We will show that the required sets $B_j \subseteq V_j$, $1 \leq j \leq l$, are already present in the subsets $\bar{V}_j$ of $V_j$.

For concreteness, let $l=\{3\sqrt{\lambda_0}\}$ so that $l \geq 2+(6+2(\lambda_0+(\lambda_0-1)/k))^{1/2}$ for $i$ (and hence $\lambda_0$) sufficiently large. We now show that for $i$ sufficiently large the $l \times r$ array of $G(p(\lambda), k)$ required in the lemma can be found in the subgraph $G(n_{i-1}, k) \subseteq G(p(\lambda), k)$. First observe that for any integer $l$ the first $l$ classes $V_1, V_2, \ldots, V_l$ in a $\lambda_0$ partition of $G(n_{i-1}, k)$ are already pairwise adjacent in the subgraph $G(n_x, k)$ where $x=\{(l-1)/k\}$. Hence adjacencies between these classes all occur among a set of $xl$ points (the ones in $(\bigcup_{j=1}^{l} V_j) \cap G(n_x, k)$) which may be partitioned into $l$ subsets, each contained in a different $V_j$, $1 \leq j \leq l$, and each of cardinality $x$. Since $G(n_{i-1}, k)$ is $\psi$-critical and an extension of $G(n_x, k)$ (for $i$ sufficiently large), it follows that there are no further adjacencies among these classes besides those just mentioned. Since each class in the $\lambda_0$ partition of $G(n_{i-1}, k)$ has $i-1$ points, it follows that each of the classes $V_j$, $1 \leq j \leq l$, contains a set $B_j$ of cardinality $i-\{(l-1)/k\}-1$ such that $S=\bigcup_{j=1}^{l} B_j$ is an independent set in $G(n_{i-1}, k)$. Now recall that the construction of $G(p(\lambda), k)$ for $\lambda > \lambda_0$ introduces edges joining certain points of $G(n_{i-1}, k)$ to points in $G(n_i, k) \setminus G(n_{i-1}, k)$. But the construction guaranteed, by the 1-regularity of $\langle E'' \rangle$, that at most one point from each class $V_j$ (in the $\lambda_0$ partition of $G(n_{i-1}, k)$) is incident with an edge from this set. Therefore let $t_j$ be the single point of $B_j$ (if there is one) incident on one of these introduced edges. We will form our array using the sets $\bar{B}_j = B_j \setminus \{t_j\}$. Since $S$ is independent in $G(n_{i-1}, k)$, so is $\bar{S}=\bigcup_{j=1}^{l} \bar{B}_j$. Also we have $\bar{B}_j = |B_j|-1=i-\{(l-1)/k\}-2$ and $\bar{B}_j \subset B_j \subset V_j$, $1 \leq j \leq l$. Note also that since $G(p(\lambda), k)$ is an extension of $G(n_{i-1}, k)$, there are classes $W_j$, $1 \leq j \leq l$, in a $\lambda$-partition of $G(p(\lambda), k)$ such that $W_j \supset \bar{B}_j$. We have thus produced an $l \times (i-\{(l-1)/k\}-2)$ array $\bar{S}$ and classes $W_j$ of $G(p(\lambda), k)$ as analogues of $S$, $V_j$, and $G$ respectively in the lemma.

We must now verify that the $\bar{B}_j$ and $l$ have the required properties. Set $K=G(n_{i-1}, k)$ and $L=G(p(\lambda), k)$ for brevity. Let us verify first that $N_L(v) \cap N_L(w)=\emptyset$ for $v, w \in \bar{B}_j$, $1 \leq j \leq l$. The $\psi$-criticality of $K$ implies that $N_K(v) \cap N_K(w)=\emptyset$, while $v, w \in \bar{B}_j$ implies that $N_L(v)=N_K(v)$ and $N_L(w)=N_K(w)$. It follows that $N_L(v) \cap N_L(w)=\emptyset$ as desired. As observed already, we have

$$l=\{3\sqrt{\lambda_0}\}>2+\sqrt{6+2(\lambda_0+(\lambda_0-1)/k)} \geq 2+\sqrt{2b+4}.$$

In case the number of columns in our array, $i=\{(l-1)/k\}-2$, is not a multiple of $\frac{1}{2}k$, we delete the minimum number of columns necessary so that we are left with a multiple of $\frac{1}{2}k$ number of columns. We will be left with at least $i-\{(l-1)/k\}-\frac{1}{2}k-2$ columns, and hence our new set of rows, which we continue to call $\bar{B}_j$, $1 \leq j \leq l$, satisfy $|\bar{B}_j| = i-\{(l-1)/k\}-\frac{1}{2}k-2$. It remains to show that

$$l(i-\{(l-1)/k\}-\tfrac{1}{2}k-2) \geq b,$$

and since $b \leq 1+\lambda_0+(\lambda_0-1)/k$ it suffices to verify the inequality

$$l(i-\{(l-1)/k\}) \geq \lambda_0+(\lambda_0-1)/k+1$$

for $i$ sufficiently large. Since $l=\{3\sqrt{\lambda_0}\}$ and $\lambda_0 = f(n_{i-1}, k) = (i-1)k+1$, the left side is $O(i^{3/2})$ while the right is $O(i)$. Hence the inequality holds for large enough $i$. The lemma may therefore be applied to yield our graph $G(p, k)$, and the construction is complete. The function $Q(k)$ may be defined by $Q(k) = n_t = t(tk+1)$, where $t$ is the smallest integer such that $i \geq t$ insures that the inequality holds.

We now outline the construction of the graphs $G(p, k)$, $n_{i-1}+1<p<n_i$ (i.e. $p \notin \Omega$), for $k$ odd. Certain details will be omitted as they are similar to ones given in the case $k$ even.

First we consider changes in the lemma. The main alteration occurs in constructing the graphs $G^{(\alpha)}$. With $k$ even the definition of $G^{(\alpha)}$ employs cycles of length $s$, $C_m^{(\alpha)}$, constructed on the set $C=\{v_{jm}: 1 \leq j \leq s\}$ of points forming a column of the array $S$. The role of $C_m^{(\alpha)}$ is to add 2 to the degree of each $v_{jm}$ to compensate for a loss of 2 resulting from the deletion of edges joining $v_{jm}$ to points in $G$. The degree of $v_{jm}$ is thereby maintained at $k$. By the evenness of $k$ we can iterate this process, using the same column $C$ (as the point set of successive $C_m^{(\alpha)}$) at most $\frac{1}{2}k$ times if necessary, and be sure that the loss of $k$ in the degree of $v_{jm}$ is compensated for by a gain of $k$ accumulating from $\frac{1}{2}k$ increments of 2 by the successive $C_m^{(\alpha)}$. In the case $k$ is odd we can perform $\frac{1}{2}(k-1)$ iterations on a fixed column $C$ in the same way. In the final iteration we delete from each $v_{jm}$ one edge going to $G$, and we compensate by constructing a 1-factor on $C$. Thus when $k$ is odd we alter the construction by using, if the number of iterations on some column $C$ is large enough, a 1-factor in the final iteration instead of the cycle $C_m^{(\alpha)}$. Naturally there are corresponding changes in the definition of the sets $P_{jm}^{(\alpha)}$ and $P_j^{(\alpha)}$. Finally, in the conclusion of the lemma we take $b$ even.

The case $k$ odd also forces a change in the proof of the theorem itself. The change occurs in the construction of the graph $G(p(\lambda_1-r), k)$. For convenience set $G(p(\lambda_1-r)) = G(p(\lambda_1-r, k)$. Referring to the integer $i$ defined by $p(\lambda_1)=n_i$ and $p(\lambda_0)=n_{i-1}$, we consider separately the case $i$ even and $i$ odd.

Suppose first $i$ is odd. If $r$ is even, then we may define $G(p(\lambda_1-r))$ and the transfer $\tau$ as in the case $k$ even since then $\|\tau\| = (k+1-r)r$ is even, as required. Assume then $r$ is odd. Since a $k$-regular graph ($k$ odd) has an even number of points and since

$$p(\lambda_1-r) \geq p(\lambda_1) - r\{(\lambda_1-1)/k\} = p(\lambda_1) - ri,$$

it follows that

$$p(\lambda_1 - r) = p(\lambda_1) - ri + \delta(i, r),$$

where $\delta(i, r) = \frac{1}{2}(1 - (-1)^{ri})$. Hence we must have

$$|G(p(\lambda_1 - r + 1))| - |G(p(\lambda_1 - r))| = i - 1 \quad \text{if } r \text{ is odd.}$$

We therefore begin the construction of $G(p(\lambda_1 - r))$ by first forming $G(p(\lambda_1 - r + 1))$ as in the case $k$ even, recalling that $V(G(p(\lambda_1 - r + 1))) = \bigcup_{j=1}^{\lambda_1 - r + 1} V_j$ (the $V_j$ coming from the $(ik+1)$-partition $G(n_i, k) = \bigcup_{j=1}^{ik+1} V_j$ of $G(n_i, k)$). We now form $V = V(G(p(\lambda_1 - r)))$ by deleting all points in $G(p(\lambda_1 - r + 1))$ belonging to the class $V_{\lambda_1 - r + 1}$ except for the one (call it $w$) in the component $K_{k+2-r}$ ($w$ had the $(t, s)$ label $(\lambda_1 - r + 1, i)$ in $G(n_i, k)$). Since each class of $G(n_i, k)$ has $i$ points, we have thus removed the required $i - 1$ points, and we have the partition $V = \bigcup_{j=1}^{\lambda_1 - r + 1} V_j \cup \{w\}$, while

$$\langle V \rangle_{G(n_i, k)} \cong G(n_{i-1}, k) \cup (i-1)K_{k, k-r} \cup K_{k+2-r}.$$

Now construct $G(p(\lambda_1 - r))$ in two steps. First build the edge set $F(\lambda_1 - r)$ on the points of $(i-1)K_{k, k-r}$ in $V$ using some pairing $\mathcal{P}$ as in the case $k$ even, and call the resulting graph $G_1$. Note that all points of $G_1$ have degree $k$ except for those in the component $K = K_{k+2-r}$. In the second step we define a transfer $\tau$ on $G_1$ with object set $K$ such that $G_1(\tau)$ is $k$-regular and $\psi(G_1(\tau)) = \lambda_1 - r$. We then let $G(p(\lambda_1 - r)) = G_1(\tau)$. Observe that $\|\tau\| = (k + 2 - r)(r - 1)$ is even as required by definition. The existence of $\tau$ is proved as in the case $k$ even.

The case $i$ even is treated with some changes. We let $V = V(G(p(\lambda_1 - r))) = \bigcup_{j=1}^{\lambda_1 - r} V_j$, and hence

$$\langle V \rangle_{G(n_i, k)} \cong G(n_{i-1}, k) \cup (i-1)K_{k, k-r} \cup K_{k+1-r}.$$

The graph $G_1$ is obtained by building a set $F'(\lambda_1 - r)$ of edges (analogous to $F(\lambda_1 - r)$) based on $i - 2$ of the $i - 1$ copies of $K_{k, k-r}$. Thus in $G_1$ all points have degree $k$ except for those in $B \cup K$, where $B$ is the exceptional copy of $K_{k, k-r}$ and $K \cong K_{k+1-r}$. These points have degree $k - r$, and we define a transfer $\tau$ on $G_1$ with object set $B \cup K$ and norm $\|\tau\| = |B \cup K|\, r = (2k + 1 - r)r$ such that $G_1(\tau)$ is $k$-regular with $\psi(G_1(\tau)) = \lambda_1 - r$. We let $G(p(\lambda_1 - r)) = G_1(\tau)$. The norm of $\tau$ is even as required, and $\tau$ is found as in the case $k$ even.

This completes an outline of the proof when $k$ is odd.

We now prove that $M(p, k) = f(p, k) - 1$ for $p \in \Omega$ sufficiently large. Let $p = i(ik+1) + 1$ for some $i$.

For the inequality $M(p, k) \geq f(p, k) - 1$, a $k$-regular graph $G(p, k)$ may be found which satisfies $|G| = p$ and $\psi(G(p, k)) = f(p, k) - 1$. Let $\lambda_1 = ik + 1 = f(n_i, k)$, and notice that $\lambda_1 - 1 = f(n_i, k) - 1 = f(p, k) - 1$. We simply apply the lemma using $G(p(\lambda_1 - 1), k)$ as the graph $G$, and we obtain $G(p, k)$ as $H$. The value of $b$ we require is

$$b = p - p(\lambda_1 - 1) = 1 + \{(\lambda_1 - 1)/k\}.$$

But the proof above shows that for $p$ sufficiently large the lemma may be applied for all $b$ satisfying $2 \leq b \leq 1+\lambda_0+(\lambda_0-1)/k$, where $\lambda_0=(i-1)k+1=f(n_{i-1}, k)$. Since $(\lambda_1-1)/k=(\lambda_0-1)/k+1$, we have

$$1+\{(\lambda_1-1)/k \leq 1+\lambda_0+(\lambda_0-1)/k$$

and hence the lemma may be used with $b=1+(\lambda_1-1)/k$. The graph $G(p, k)$ is thereby constructed.

We now show that $M(p, k)<f(p, k)$. Suppose the contrary, and let $G(p, k)$ be a $k$-regular graph of order $p$ satisfying $\psi(G(p, k)) \geq f=f(p, k)=f(i(ik+1)+1, k)=ik+1$. In any $f$-partition $\bigcup_{j=1}^{f} V_j$ of $G(p, k)$ we must have $|V_j| \geq i$. Hence $|V_t|=i+1$ for some $1 \leq t \leq f$ and $|V_j|=i$ for $j \neq t$. Without loss of generality take $t=f$. Now since $|\partial V_j|=ik$ for $1 \leq j \leq f-1$ and $\psi(G)=ik+1$, it follows that $G$ is $\psi$-critical. Hence any two classes $V_j$, $V_m$, $1 \leq j, m \leq f$, are joined by exactly one edge. It follows that $\partial V_j=ik$ for $1 \leq j \leq f$. But $V_f$ has $i+1$ points, so there are two points $v, w \in V_f$ such that $N(v) \cap N(w) \neq \emptyset$. Let $z \in N(v) \cap N(w)$. Then one of the edges $zv$ or $zw$ could be deleted while maintaining the $f$-partition of $G(p, k)$. Hence $G$ is not $\psi$-critical, a contradiction. The theorem is thus proved. □

Let us now turn our attention to the analogue of the classical problem involving cages. In this problem, we are asked to determine the minimum $p$ for which there exists a $k$-regular graph on $p$ points with given girth $b$. By analogy, define $p(\lambda, k)$ as the minimum $p$ for which there exists a $k$-regular graph on $p$ points having achromatic number $\lambda$. Letting $h(\lambda, k)=\min\{p: p \geq \lambda\{(\lambda-1)/k\}$, and $p$ even if $k$ odd$\}$, our theorem implies the following.

**Corollary.** *For any $k \geq 2$ and $\lambda$ sufficiently large (depending on $k$), we have $p(\lambda, k)=h(\lambda, k)$.*

## 3. Open problems

Natural outcomes of our result are the following two types of problems.

(1) Given a parameter $\theta$ in graph theory, determine its extremal behavior for graphs with a given regularity degree and number of points. Specifically, find

$$M_\theta(p, k)=\max\{\theta(G): G \text{ is } k\text{-regular and } |G|=p\},$$

or

$$m_\theta(p, k)=\min\{\theta(G): G \text{ is } k\text{-regular and } |G|=p\}.$$

(2) Solve the analogue of the cage problem by finding $p(\alpha, k)=\min\{p$: there exists a $k$-regular $G$ satisfying $|G|=p$ and $\theta(G)=\alpha\}$.

The author believes that there are interesting parameters for which problems 1 and 2 are open. Their solutions should lead to interesting constructions for the extremal graphs involved.

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