JOURNAL OF ALGORITHMS 6, 10–16 (1985)

# NP-Completeness for Minimizing Maximum Edge Length in Grid Embeddings

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Received February 10, 1983

Given an embedding $f\colon G \to \mathbb{Z}^2$ of a graph $G$ in the two-dimensional lattice, let $|f|$ be the maximum $L_1$ distance between points $f(x)$ and $f(y)$ where $xy$ is an edge of $G$. Let $B_2(G)$ be the minimum $|f|$ over all embeddings $f$. It is shown that the determination of $B_2(G)$ for arbitrary $G$ is NP-complete. Essentially the same proof can be used in showing the NP-completeness of minimizing $|f|$ over all embeddings $f\colon G \to \mathbb{Z}^n$ of $G$ into the $n$-dimensional integer lattice for any fixed $n \geq 2$. © 1985 Academic Press, Inc.

## 1. INTRODUCTION

Many combinatorial optimization problems, especially in the theory of VLSI, are really problems about graph embeddings. Specifically, let $G$ and $H$ be graphs with $H$ connected and $|H| \geqslant |G|$. Given a one–one map $f$: $V(G) \to V(H)$, let $|f| = \max\{d_H(f(x), f(y))\colon xy \in E(G)\}$, where $d_H$ is distance in $H$. Thus $|f|$ measures the maximum stretching in $H$ experienced by an edge of $G$. We let $C(G, H) = \min|f|$, taken over all maps $f$ above.

As an example, take the parameter $B(G) = C(G, P_{|G|})$, where $P_n$ denotes the path graph on $n$ points. In the literature, $B(G)$ is known as the *bandwidth* of $G$. Consider this problem.

(BANDWIDTH) Input: Graph $G$, integer $k$

Question: Is $B(G) \leqslant k$?

*Partially supported by a Miami University Summer Research Award.



0196-6774/85 $3.00

BANDWIDTH was shown to be NP-complete by Papadimitriou [12]. Garey *et al.* [6] later showed among other things that BANDWIDTH is NP-complete even when $G$ is restricted to trees of maximum degree 3. Saxe [15], and later Gurari and Sudborough [7] with an improvement, have shown that BANDWIDTH is polynomial time solvable for any fixed $k$. For further results on bandwidth see [2, 5].

Other possibilities for the host graph $H$ are $k$-ary trees and lattices or arrays (i.e., grids of arbitrary dimension). The examination of these possibilities is motivated by the attempt to minimize various costs associated with communication between points of $G$ when $G$ is embedded in $H$. Thus $G$ might represent a data structure and $H$, computer storage. The time delay in communication between neighboring points of $G$ is proportional to the graph theoretic distance between them in $H$. In order to minimize the worst time delay over all pairs of neighbors in $G$, we would like to find an embedding $f\colon G \to H$ for which $|f|$ is as small as possible, that is, where $|f|$ is as close as possible to $C(G, H)$. Results on $C(G, H)$ and on the averaging parameter $\min_f\{\sum_{xy \in E(G)} d_H(f(X), f(y))\}$ for the $H$ above appear in work of Rosenberg [13, 14], for example. Also, a survey of recent work on $C(G, H)$ is given in Chung [18]. For a good description of applications of costs of graph embeddings see these same references.

For a summary of research on graph embeddings in VLSI, see Johnson's column. In that column [9, p. 95] the question of finding complexity results for problems of optimal embeddings of graphs is mentioned. Results on $C(G, H)$, particularly when $H$ is the graph of the two-dimensional integer lattice, have appeared in several recent papers. In [4], for example, an "$f(n)$-separator" technique, based on the separator theorem of Lipton and Tarjan [11], is used to produce layouts $f$ of trees or outerplanar graphs satisfying $|f| = O(\sqrt{n}/\log n)$. In [10], arguments involving crossing number and wire area are used in showing that there are planar graphs for which $|f| = O((n/\log n)^{1/2})$.

Let $\mathbb{Z}^2$ be the two-dimensional integer lattice; i.e., $\mathbb{Z}^2 = \{(x, y)\colon x, y \in \mathbb{Z}\}$. Let $\langle \mathbb{Z}^2 \rangle$ be the graph of the two-dimensional integer lattice; i.e., $V(\langle \mathbb{Z}^2 \rangle) = \{(x, y) \in \mathbb{Z}^2\}$, $E(\langle \mathbb{Z}^2 \rangle) = \{\{(x_1, y_1), (x_2, y_2)\}\colon |x_1 - x_2| + |y_1 - y_2| = 1\}$. For convenience let $B_2(G) = C(G, \langle \mathbb{Z}^2 \rangle)$. In this paper we show that the problem

(EDGE LENGTH) Input: Graph $G$, integer $k$
Question: Is $B_2(G) \leqslant k$?

is NP-complete.

## 2. Some Notation

Points in $\langle \mathbb{Z}^2 \rangle$ will be denoted $(i, j)$, $i, j \in \mathbb{Z}$, retaining their usual names as lattice points. If $z \in \mathbb{Z}^2$, let $x(z)$ and $y(z)$ be the $x$ and $y$

coordinates of $z$, respectively. Given $z$ and $w \in \mathbb{Z}^2$, denote the $L_1$ distance between $z$ and $w$ (which is $d(z,w)$ in $\langle \mathbb{Z}^2 \rangle$) by $|z - w|$. Thus $|z - w| = |x(z) - x(w)| + |y(z) - y(w)|$. If $S \subseteq \mathbb{Z}^2$, then $[S]_k = (V^k(S), E^k(S))$ is the graph defined as follows:

(1) For each $(i, j) \in S$, there is a corresponding vertex $\langle i, j \rangle \in V^k(S)$.

(2) Two vertices in $V^k(S)$ (say $\langle i_1, j_1 \rangle$ and $\langle i_2, j_2 \rangle$) are adjacent in $[S]_k$ iff $|(i_1, j_1) - (i_2, j_2)| \leqslant k$.

A one–one map $f$: $V(G) \to \mathbb{Z}^2$ will be called a *layout*, and we let $|f| = \max\{|f(v) - f(w)|: vw \in E(G)\}$. Note that the routing of edges of $G$ along grid lines plays no role. We say $f$ is *k-feasible* if $|f| \leqslant k$.

Two $k$-feasible layouts $f, f'$ of some $G$ are *equivalent* if one can be obtained from the other by one or more of the following operations:

(i) translation in $\mathbb{Z}^2$, i.e., $g(i, j) = (i + a, j + b)$, all $(i, j) \in \mathbb{Z}^2$, some $a, b \in \mathbb{Z}$,

(ii) a 90° rotation of $\mathbb{Z}^2$ around the origin, i.e., $g(i, j) = (-j, i)$,

(iii) a reflection along the $x$-axis, i.e., $g(i, j) = (i, -j)$.

Thus $f' = g \circ f$, where $g$ is a composition of the elementary isometries of $\mathbb{Z}^2$ given in (i), (ii), and (iii). We say that there is a *unique* $k$-feasible layout of the graph $G$ if there is some $k$-feasible layout of $G$ and any two $k$-feasible layouts of $G$ are equivalent.

We let $D^t$ denote the set of all points in $\mathbb{Z}^2$ having $L_1$ distance at most $t$ from the origin; i.e., $D^t = \{z: |z - (0,0)| \leqslant t\}$.

Given a graph $G = (V, E)$ and $S \subseteq V$, let $\langle S \rangle_G$ be the subgraph induced by $S$, defined by

$$V(\langle S \rangle_G) = S \qquad \text{and} \qquad E(\langle S \rangle_G) = \{xy: x \in S,\ y \in S,\ xy \in E(G)\}.$$

For any graph theoretic terminology not given here, see [1] or [8].

## 3. NP-Completeness

Our proof of NP-completeness will use the reduction BANDWIDTH $\to$ EDGE LENGTH. That reduction relies on the uniqueness of the $\mathbb{Z}^2$ embedding of a certain graph.

As notation, let $k, p \in \mathbb{Z}$, $k, p \geqslant 2$, and $t = 4k + p$. Also let $S(k, p) = D^t - \{(0, j): |j| \leqslant p\}$. The set $S(1,1)$ is illustrated in Fig. 1.

Lemma 1.1. *The graph $[S(k, p)]_k$ has a unique k-feasible embedding.*

The proof of the lemma will follow the proof of the reduction.

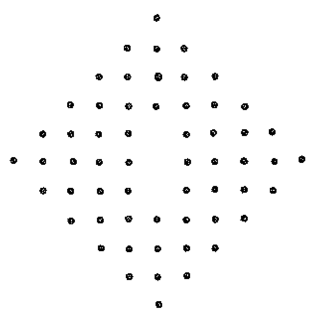

FIG. 1. The set $S(1,1)$.

THEOREM 1. *EDGE LENGTH is* NP-*complete.*

*Proof.* Let $G = (V, E)$ and $k \in \mathbb{Z}^+$ be an instance of BANDWIDTH. The corresponding instance of $\mathbb{Z}^2$-COST will be a graph $G^*$ and $k$.

We define $G^*$ as follows. First let $G' = [S(k, |V|)]_k$ where $S(k, |V|)$ is defined as above. Now let $G^*$ be defined by $V(G^*) = V(G) \cup V(G')$, and

$$E(G^*) = E(G) \cup E(G') \cup \{(\langle 1, 0\rangle, v^*)\}, \qquad \text{some } v^* \in V.$$

Thus the vertex set of $G^*$ is the union of the vertex sets of $G$ and $G'$, while the edge set of $G^*$ is the union of the edge sets of $G$ and $G'$ plus one additional edge from some designated vertex of $G$ to the vertex $\langle 1, 0\rangle$ of $G'$.

We claim that $B_2(G^*) \leqslant k$ if and only if $B(G) \leqslant k$.

Suppose first that $B(G) \leqslant k$. We will build a layout of $G^*$ by "tucking in" a bandwidth mapping of $G$ in a canonical layout of $G'$.

Define the layout $\beta$: $G^* \to \mathbb{Z}^2$ as follows. First let $\beta$: $G' \to \mathbb{Z}^2$ be the canonical embedding; that is, $\beta(\langle i, j\rangle) = (i, j)$ for all $\langle i, j\rangle \in G'$. To define $\beta$ on the $G$, let $\alpha$: $G \to \mathbb{Z}$ be a layout of $G$ such that $|\alpha| = B(G)$. We may assume $\alpha$ has range $\{1, 2, \ldots, |V|\}$. Now let $\beta$: $G \to \mathbb{Z}^2$ be defined by $\beta(w) = (0, \alpha(w) + b)$, $w \in V$, where $b$ is chosen so that $-(1 + |V|) \leqslant b \leqslant 0$ and $|\beta(v^*) - (1, 0)| \leqslant k$. Letting $T = \{(0, j): |j| \leqslant |V|\}$, we see that $\beta(G) \subseteq T$, and hence $\beta$ is a layout of all of $G^*$ satisfying $|\beta| \leqslant k$.

Conversely, suppose to the contrary that $B(G) > k$, and let $\gamma$: $G^* \to \mathbb{Z}^2$ be a layout satisfying $|\gamma| \leqslant k$. Since $G'$ has a unique $k$-feasible embedding, we may assume without loss of generality that the embedding of $G'$ is canonical. Consider now where $v^*$ is mapped. Since $(\langle 1, 0\rangle, v^*) \in E(G^*)$, we have $|\beta(v^*) - (1, 0)| \leqslant k$. Now all points $z$ outside of $D^{4k+|V|}$ satisfy $|z - (1, 0)| > k$. Hence $\beta(v^*) \in D^{4k+|V|}$, and hence $\beta(v^*) \in T$ since $\beta$ maps $G'$ canonically. Since $B(G) > k$, we have $\beta(G) \not\subset T$ so that at least one of the vertices $u$ of $G$ is mapped to a point outside of $T$ and thus outside of $D^{4k+|V|}$. But since there is a path from $u$ to $v^*$ in $G$, there are adjacent vertices of $V$ with one mapped inside $T$ and the other outside $D^{4k+|V|}$. Since $|a_1 - a_2| > k$ for any $a_1 \in T$ and $a_2$ outside $D^{4K+|V|}$, we have contradicted $B_2(G^*) \leqslant k$. □

The proof of Lemma 1.1 will follow a proof of some elementary results covering $k$-feasible embeddings of $D^k$.

Consider any graph $G = (V, E)$, a subset $V' \subseteq V$, and a collection $\mathscr{F}$ of layouts of $G$. We say that $V'$ is *uniquely embeddable under* $\mathscr{F}$ if given any two $f_1, f_2 \in \mathscr{F}$, the restriction layouts $f_i\colon V' \to \mathbb{Z}^2$, $i = 1, 2$, are equivalent. Let $[D^k]_k = (V^k, E^k)$, and let $V' \subseteq V^k$ be defined by $V' = \{\langle i, j\rangle \in V^k\colon |i| + |j| = k\} \cup \{\langle i, j\rangle \in V^k\colon |i| + |j| \leqslant 1\}$. Let $\mathscr{F}_k$ be the set of all $k$-feasible embeddings of $[D^k]_k$.

LEMMA 1.2. *The subset $V' \subseteq V^k$ is uniquely embeddable under $\mathscr{F}_k$.*

*Proof.* Let $f$ be a layout of $[D^k]_k$, $f \in \mathscr{F}_k$. Without loss of generality we may assume that the vertex $\langle 0, 0\rangle$ is mapped under $f$ to the point $(0,0) \in \mathbb{Z}^2$. It follows that all other vertices of $V^k$ are mapped into $D^k$ because $\langle 0, 0\rangle$ is adjacent to all of the points.

If vertices $\langle i_1, i_2\rangle$ and $\langle j_1, j_2\rangle$ are *not* vertices corresponding to opposite corners in $D^k$ (e.g., $\langle 0, k\rangle$ and $\langle 0, -k\rangle$) then these two vertices have more than one neighbor in common and cannot be mapped to opposite corners of $D^k$ under $f$. Thus we know that the pair $\langle 0, k\rangle$ and $\langle 0, -k\rangle$ must be mapped to opposite corners of $D^k$ as must the pair $\langle k, 0\rangle$ and $\langle -k, 0\rangle$. By using rotations and reflections we may assume without loss of generality that the vertices $\langle 0, k\rangle$, $\langle 0, -k\rangle$, $\langle k, 0\rangle$, and $\langle -k, 0\rangle$ are mapped to the lattice points $(0, k)$, $(0, -k)$, $(k, 0)$, and $(-k, 0)$, respectively.

The vertex $\langle 1, 0\rangle$ is the unique remaining vertex of $V^k$ adjacent to a maximum number of vertices of $V^k$ and not adjacent to $\langle 0, -k\rangle$, $\langle 0, k\rangle$, and $\langle -k, 0\rangle$. Thus $\langle 1, 0\rangle$ must be mapped to $(1, 0)$. Similarly $\langle 0, 1\rangle$, $\langle 0, -1\rangle$, and $\langle -1, 0\rangle$ must be mapped to $(0,1)$, $(0,-1)$, and $(-1,0)$, respectively.

The vertex set $R = \{\langle i, j\rangle\colon |i| + |j| = k,\ i > 0,\ j > 0\}$ is the unique set of vertices not adjacent to both $\langle -1, 0\rangle$ and $\langle 0, -1\rangle$ and thus this set must be mapped to the northeast border of $D^k$, i.e., the set $\{(i, j)\colon |i| + |j| = k,\ i > 0,\ j > 0\}$. The vertices $\langle k-1, 1\rangle, \langle k-2, 2\rangle, \ldots, \langle [k/2], [k/2]\rangle$ are such that $n(\langle k-1, 1\rangle) < n(\langle k-2, 2\rangle) < \cdots < n(\langle \lceil k/2\rceil, \lfloor k/2\rfloor\rangle)$ where $n(v)$ denotes the number of neighbors of $v$ in $[D^k]_k$. Using this along with the information that these are the vertices in $R$ that are adjacent to $\langle k, 0\rangle$, it is easy to see that vertex $\langle k-j, j\rangle$ must get mapped to lattice point $(k-j, j)$. Similarly, all vertices $\langle i, j\rangle$ with $|i| + |j| = k$ must get mapped to their corresponding points. □

The set of lattice points $z$ satisfying $|z - (i, j)| = k$ will be called the *periphery* of $(i, j)$.

*Proof of Lemma 1.1.* Consider the set $S(k, p)$. Let us call a vertex $\langle i, j\rangle$ of $G = [S(k, p)]_k$ a *diamond center* if it is adjacent to $2k^2 + 2k$ other vertices of $G$, and thus is the center of an induced diamond of "radius" $k$. Two distinct vertices $\langle i_1, j_1\rangle$ and $\langle i_2, j_2\rangle$ are *immediate neighbors* $|i_1 - i_2| + |j_1 - j_2| = 1$.

Start at any diamond center $v$ and embed the periphery of the induced diamond and the immediate neighbors of $v$ in a canonical way. By Lemma 1.2, if $u$ is an immediate neighbor of $v$ and if $u$ is a diamond center, then the periphery of its induced diamond must be embedded in the corresponding lattice points in any $k$-feasible embedding. It is easy to see that it is possible to find a path in $G$ from any diamond center to any other diamond center such that every two adjacent vertices in the path are immediate neighbors, and thus all of these centers must be mapped to their corresponding lattice points (as proved inductively). Every non-diamond center of $G$ is on the periphery of some induced diamond of $G$ and thus must also be embedded to its corresponding lattice point, completing the proof. □

## 4. Remarks

(a) Our NP-completeness proof generalizes in a straightforward way to layouts of graphs in $n$-dimensional grids. That is, let $\mathbb{Z}^n = \{(a_1, a_2, \dots, a_n): a_i \in \mathbb{Z}\}$, and let $\mathbb{Z}^n$ be the graph defined by $V(\langle \mathbb{Z}^n\rangle) = \{w \in \mathbb{Z}^n\} E(\langle \mathbb{Z}^n\rangle) = \{(w, s): \sum_{i=1}^{n} |w_i - s_i| = 1\}$. Then the following problem can be proved NP-complete by a similar argument. Let $n \geqslant 2$ be a fixed integer. Given a graph $G$ and integer $k$, is $C(G, \langle \mathbb{Z}^n\rangle) \leqslant k$? In the proof, the role of $S(k, p)$ is played by $D^t - \{(0, 0, \dots, 0, j): |j| \leqslant p\}$, where $D^t$ is the "disk" of radius $t$ (under the $L_1$ distance) for appropriate $t$ depending on $n$.

(b) We have obtained a polynomial time algorithm for computing $C(T, \langle \mathbb{Z}^n\rangle)$ for any $n$, where $T$ is a "caterpillar" graph; i.e., $T$ is a tree such that the tree obtained by removing all endpoints of $T$ is a path. This generalizes the results of [16], where a polynomial time algorithm for $C(T, \langle \mathbb{Z}\rangle) = B(T)$ is found.

(c) We have just learned that Bhatt and Cosmadakis [3] have recently obtained the result of this paper (for two-dimensional grids) independently. Their proof is quite different; it uses a reduction from NOT-ALL-EQUAL 3CNFSAT. (Given a Boolean formula $\varnothing$ in conjunctive normal form with three literals per clause, does there exist a truth assignment which satisfies $\varnothing$ such that each clause contains at least one false literal?)

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