On Graphs Containing a Given Graph as Center

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ABSTRACT

We examine the problem of embedding a graph H as the center of a supergraph G, and we consider what properties one can restrict G to have. Letting $A(H)$ denote the smallest difference $|V(G)| - |V(H)|$ over graphs G having center isomorphic to H it is demonstrated that $A(H) \leq 4$ for all H, and for $0 \leq i \leq 4$ we characterize the class of trees T with $A(T) = i$. For $n \geq 2$ and any graph H, we demonstrate a graph G with point and edge connectivity equal to n, with chromatic number $\chi(G) = n + \chi(H)$, and whose center is isomorphic to H. Finally, if $|V(H)| \geq 9$ and $k \geq |V(H)| + 1$, then for n sufficiently large (with n even when k is odd) we can construct a k-regular graph on n vertices whose center is isomorphic to H.

1. INTRODUCTION

In response to the statement by one of the authors that "for any graph H there is certainly some graph G whose center is isomorphic to H," Hedetniemi demonstrated the existence of such a G with only four more vertices than H using the construction illustrated in Figure 1. Each of two new points v and w

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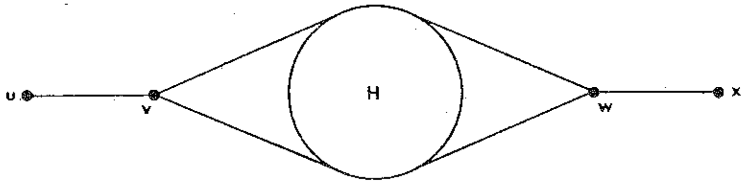

FIGURE 1. Graph G with $C(G) \cong H$ and $|V(G)| = |V(H)| + 4$.

is made adjacent to every point of H, and two other new points u and x are added with u adjacent only to v and x adjacent only to w. That for every graph H there exists a graph G the center of which (considered as an induced subgraph of G) is isomorphic to H is a result previously stated, without proof, by Kopylov and Timofeev [8]. In this paper we examine the problem of embedding H as the center of a graph G when G is restricted to having certain properties.

In general, our terminology is consistent with general usage, such as in Behzad, Chartrand, and Lesniak-Foster [1] and Harary [4]. The number of edges incident on a point v in a graph H will be denoted $d_H(v)$. Graphs H and G will be simple, undirected graphs. The distance between points u and v in H equals the smallest number of edges in a u-v path in H, and it is denoted $d_H(u, v)$. The *eccentricity* of a point u in H, denoted $e_H(u)$, is the distance from u to a point in H farthest from u, that is, $e_H(u) = \max_{v \in V(H)} d_H(u, v)$. The collection of points with minimum eccentricity is usually called the center, but in this paper by the *center* $C(G)$ of a graph G we mean the subgraph of G induced by the points of minimum eccentricity. With this alternative definition in mind, Jordan's well-known theorem takes the following form:

Theorem 1 (Jordan [7]). If T is a tree, then $C(T) \cong K_1$ or $C(T) \cong K_2$. That is, the point set of $C(T)$ consists of one point or two adjacent points in T.

Very recently Proskurowski proved a result which, like Jordan's theorem, identifies the set of centers in graphs belonging to a specified collection.

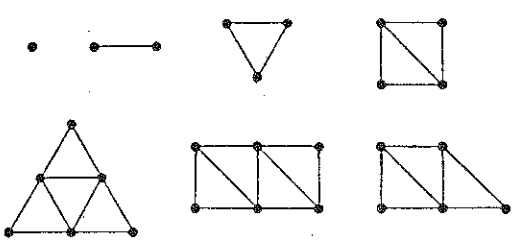

FIGURE 2. The seven possible centers of maximal outerplanar growth.

Theorem 2 (Proskurowski [9]). If H is a maximal outerplanar graph, then $C(H)$ is isomorphic to one of the seven graphs illustrated in Figure 2.

More recently Hedetniemi and Mitchell-Hedetniemi [5] defined $C_{(n)}$-trees (maximal outerplanar graphs are a subclass of the $C_{(3)}$-trees) and showed there to be only six possibilities for the center of a $C_{(4)}$-tree. Even more recently the centers and centroids of $C_{(n)}$-trees for $n \geq 4$ have been identified in [6]. Finally in [10] it is shown that the result for medians corresponding to the result of Kopylov and Timofeev for centers also holds. That is, for any graph H there is a graph G whose "median subgraph" is isomorphic to G.

In general one can ask the following question. For a specified graph H and a collection $\mathscr{S}$ of graphs, is there a graph $G \in \mathscr{S}$ whose center is isomorphic to H? In Section 3 we consider collections $\mathscr{S}$ such as the set of r-connected graphs or the set of k-regular graphs on n vertices.

Another possibility is to have the collection $\mathscr{S}$ depend on H. For example, as in Sec. 3, one might desire a graph G with $C(G) \cong H$ whose vertex independence number $\beta_0(G)$ satisfies $\beta_0(G) = \beta_0(H) + n$ for a given integer n. An intriguing problem, suggested by Hedetniemi's construction (Fig. 1), is to find the smallest G whose center is H. We therefore define $A(H)$ by

$$A(H) = \min_G\{\,|V(G)| - |V(H)| : C(G) \cong H\}.$$

By the construction in Figure 1 one has $A(H) \leq 4$ for every graph H. A graph H with $A(H) = 0$ is called self-centered, and such graphs have been studied in [2] and [3]. Clearly we have $A(H) \neq 1$ for every graph H. In Sec. 2 we observe that $A(T) \neq 3$ for every tree T, and we completely characterize the class of trees T with $A(T) = 0$, 2, and 4.

A determination of $A(H)$ for arbitrary H would include a characterization of the self-centered graphs and is quite difficult. The following simpler question is still unresolved by us. Does there exist a graph with $A(H) = 3$?

2. DETERMINING $A(T)$

We begin with some notation. The *diameter* $d(H)$ of a graph H is the maximum value among the eccentricities of the points in H. Similarly, the *radius* $r(H)$ is the minimum eccentricity among the points of H. We let $[s]$ and $\{s\}$ denote the maximum integer of size at most s and the minimum integer of size at least s, respectively. For $x \in V(H)$, we let $N_H(x)$ be the set of points in H adjacent to x.

As already mentioned, graphs H for which $A(H) = 0$ have been called self-centered graphs. Some familiar classes of graphs which are self-centered are complete graphs and cycles. For trees, Jordan's theorem demonstrates the next result.

Theorem 3. If T is a tree, then $A(T) = 0$ if and only if $T = K_1$ or K_2.

Although almost no tree is self-centered, there is a generous supply of connected self-centered graphs, as demonstrated by the next two theorems from [2].

Theorem 4. There exists a connected grpah H with $A(H) = 0$, p vertices, and q edges if and only if $q = \binom{p}{2}$ or $p \leq q \leq \frac{1}{2}p(p-2)$.

Theorem 5. If T is a tree, then $A(\bar{T}) = 0$ *if and only if* $d(T) \neq 3$.

Even if we restrict our attention to graphs with a given eccentricity, self-centered graphs still occur in abundance.

Theorem 6 [3]. Assume $p \geq 2k > 2$. With the exception of $p = 2k = 4$, there exists a connected graph H with $A(H) = 0$, $r(H) = k$, p vertices, and q edges if and only if

$$(pk - 2k - 1)/(k-1) \leq q \leq \tfrac{1}{2}(p^2 - 4pk + 5p + 4k^2 - 6k).$$

If $p = 2k = 4$, then $q = 4$.

As one would expect, there are more nonisomorphic graphs (satisfying the conditions of Theorem 4 or Theorem 6) near the middle of the described ranges than near the boundaries.

With the following notation, we may state our result concerning $A(T)$ for a tree T. Let S be a subgraph of a graph H. For each $v \in V(H)$, let $d_H(v, S) = \min_{w \in V(S)}\{d_H(v, w)\}$, and we call $d_H(v, S)$ the distance from v to S. For a tree T, let $\Omega(T)$ be the set of end points of T, and define $\Gamma(T)$ by $\Gamma(T) = \{x \in \Omega(T)\colon e_T(x) = d(T)\}$.

Theorem 7. Let T be a tree with $|V(T)| \geq 3$. Then $A(T) = 2$ if and only if any two end points of T are equidistant from the center of T.

Proof. We begin by proving sufficiency. Suppose then that $d_T(x, C(T)) = k$ for all end points x of T. Define the graph G by

$$V(G) = V(T) \cup \{v, w\},$$

$E(G) = E(T) \cup \{vy\colon y \in C(T)\} \cup \{wx\colon x$ and end point of $T\}$. It is then
It is then clear that $e_G(t) = k + 1$ for $t \in V(T)$ and $e_G(v) = e_G(w) = d_G(v, w) = k + 2$. It follows that $C(G) \cong T$, so $A(T) \leq 2$. Since $A(H) \neq 1$ for all graphs H, by Theorem 3 we get $A(T) = 2$ as desired.

For the converse, assume G is a supergraph of T with $V(G) = V(T) \cup \{v, w\}$ and $C(G) = T$. Hence we have $e_G(v) = e_G(w) = d_G(v, w) = d(G)$, while $e_G(u) = e_G(u') < d(G)$ for all $u, u' \in V(T)$.

First we claim that if $x \in N_G(v)$ and $y \in N_G(w)$, then any shortest x-y path $P(x, y)$ in G lies entirely in T (and hence is uniquely determined). For if not, let $P'(x, y)$ be a counterexample which without loss of generality contains v, so that $P'(x, y)$ may be decomposed $P'(x, y) = Q(x, v) + R(v, y)$ into paths Q and R each having v as an end point. But then we get

$$d_G(v, w) \leq |E(R(v, y))| + 1 \leq |E(P'(x, y))| = d_G(x, y),$$

a contradiction to $e_G(v) > e_G(u)$ for all $u \in V(T)$.

Next we show that $e_G(u) = d(G) - 1 = d_G(v, w) - 1$ for any $u \in V(T)$. Observe that if $u \in N_G(v)$, then $e_G(u) = d_G(u, w) \geq d_G(v, w) - 1$. which combined with $e_G(u) < e_G(v) = d_G(v, w)$ forces $e_G(u) = d_G(v, w) - 1$. Of course the same argument shows that if $u' \in N_G(w)$, then $e_G(u') = d_G(v, w) - 1$. But $e_G(x) = e_G(y)$ for any $x, y \in V(T)$, so we get $e_G(u) = d_G(v, w) - 1$ for all $u \in V(T)$ as desired.

We now claim that each end point of T is adjacent to exactly one of v or w. Now clearly $vw \notin E(G)$, and $\{v, w\} \not\subseteq N_G(u)$ for any $u \in V(T)$ since otherwise $d(G) = 2$ so $d(T) = 1$, a contradiction if $|V(T)| \geq 3$. If x is an end point of T and $N_G(x) \cap \{v, w\} = \emptyset$, then $N_G(x) = \{u\}$ for some $u \in V(T)$. This gives $e_G(x) = e_G(u) + 1$, contradicting $e_G(x) = e_G(u) = d(G) - 1$. Hence $N_G(x) \cap \{v, w\} \neq \emptyset$, which combined with $\{v, w\} \not\subseteq N_G(x)$ implies the desired claim.

Next we show that $N_G(v) \supseteq \Gamma(T)$ or $N_G(w) \supseteq \Gamma(T)$. To prove this, it would suffice to show that for any pair $r, s \in \Gamma(T)$ satisfying $e_T(r) = e_T(s) = d_T(r, s)$, we have $r, s \in N_G(v)$ or $r, s \in N_G(w)$. Suppose not, and let $r', s' \in \Gamma(T)$ satisfy $e_T(r') = e_T(s') = d_T(r', s')$ and $r'v, s'w \in E(G)$. If $c \in C(T)$, then using the observation $d_T(r', s') = d_G(r', s')$ derived above we have

$$\begin{aligned} e_G(c) \leq 1 + \tfrac{1}{2}[d_T(r', s') + 1] < 1 + d_T(r', s') &= 1 + d_G(r', s') \\ &= d_G(r', w) \\ &= e_G(r'). \end{aligned}$$

This contradicts $C(G) = T$, and our assertion follows. We therefore henceforth suppose that $N_G(v) \supseteq \Gamma(T)$.

We now show that $N_G(v) \supseteq \Omega(T)$. By the above we need only show $N_G(v) \supseteq \Omega(T) \setminus \Gamma(T)$. Suppose to the contrary that there exists $x \in \Omega(T) \setminus \Gamma(T)$ with $x \in N_G(w)$. Now there exist $y, y' \in \Gamma(T)$ such that $d_T(x, y) = e_T(x)$ and $d_T(y, y') = d(T)$. Since $x \in N_G(w)$ we have $e_G(x)$

$= d_G(x, v) = 1 + d_G(x, y)$, and by the above $d_G(x, y) = d_T(x, y)$ since $y \in N_G(v)$. Hence we get $e_G(x) = 1 + d_T(x, y)$, and similarly $e_G(x) = 1 + d_T(x, y')$. It follows that $d_T(x, y) = d_T(x, y')$. Now let $P(x, y)$ and $P(x, y')$ be the unique paths in T joining x to y and y' and let z be the vertex on $P(x, y) \cap P(x, y')$ for which $d_T(z, x) = \max\{d_T(u, x): u \in P(x, y) \cap P(x, y')\}$. Then $d_T(x, y) = d_T(x, y')$ implies $d_T(z, y') = d_T(z, y) = \frac{1}{2}d(T)$. It follows that $e_G(z) \leq 1 + \frac{1}{2}d(T) < 1 + d_T(x, y) = e_G(x)$, a contradiction to $e(s) = e_G(t)$ for all s, $t \in T$.

Next we show that $N_G(w) = C(T)$. First we prove $N_G(w) \subseteq C(T)$. Suppose the contrary, and let $u \in V(T)$ satisfy $u, w \in E(G)$ and $u \notin C(T)$. Let $y \in \Gamma(T)$ be such that the shortest u-y path in T contains $C(T)$. Since any shortest u-y path in G lies entirely in T, we have $d_G(u, y) = d_T(u, y) \geq r(T) + 1$. But now let $y' \in \Omega(T)$ be such that $d_T(y', u)$ is a minimum in the set $\{d_T(\bar{y}, u): \bar{y} \in \Omega(T)\}$. Thus y' and u lie on the same branch in T of some central point of T, so $d_T(y', u) < r(T)$. It follows that $d(G) = d_G(v, w) \leq 2 + d_G(y', u) \leq r(T) + 1 \leq d_G(u, y)$, a contradiction to $e_G(x) < d(G)$ for $x \in T$. Next we prove $N_G(w) \supseteq C(T)$. If $x \in \Omega(T)$, then we have $e_G(x) = d_G(x, w) = d(G) - 1$. In particular, if $r, s \in \Gamma(T)$ and $d_T(r, s) = d(T)$, then we get $d_G(r, w) = d_G(s, w) = d(G) - 1$. It follows that $N_G(w) \supseteq C(T)$.

The proof may now be completed. If $x \in \Omega(T)$, then $d_G(x, C(T)) = d_G(x, w) + 1 = d(G) - 2$, so all end points are equidistant from the center. ∎

We remark that Theorem 7 may also be stated in form "$A(T) = 2$ if and only if all end points have equal eccentricity."

Theorem 8. If T is a tree, then $A(T) \neq 3$.

The proof of Theorem 8 is similar to the previous proof, so it will be omitted.

Our characterization for trees is now complete. We have $A(T) = 0$ for K_1 and K_2, and for all other trees T we have $A(T) = 2$ or 4 depending upon whether or not all of the end points of T are equidistant from the center.

3. Graphs with Given Center and Prescribed Properties

In this section we show that graphs G having given center H may have various prescribed properties. We begin with connectivity and chromatic number. A graph G with center H having point and edge connectivity r and chromatic number $r + \chi(H)$, for $r \geq 2$, is illustrated in Figure 3. A double line between two subgraphs indicates that every point of one is adjacent to all points of the other, while a single line between two copies of K_r indicates that each vertex in one copy is adjacent to exactly one in the other (or, there is a

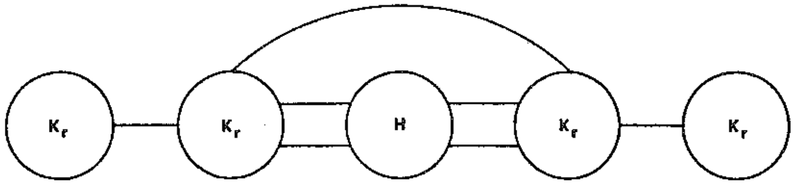

FIGURE 3. Graph having center H satisfying $\kappa = r$, $\lambda = r$, and $\chi = r + \chi(H)$.

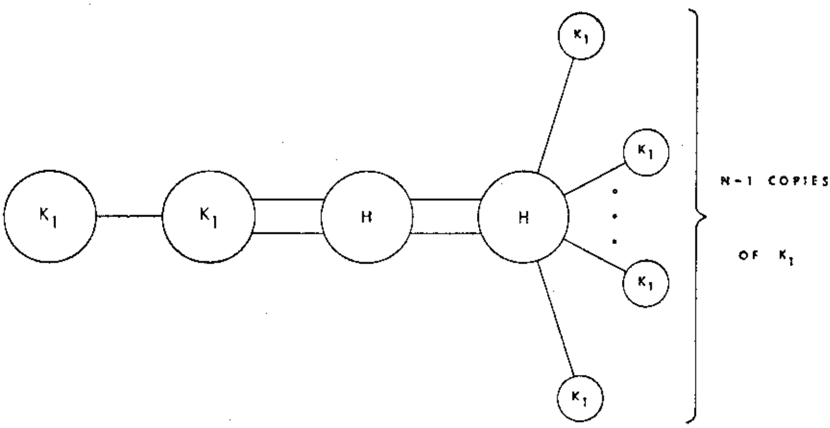

FIGURE 4. Graph with $\beta_0 = \beta_0(H) + n$ and center H.

matching between them). It is easily verified that $\chi(G) = r + \chi(H)$. A graph G satisfying $C(G) = H$ and $\beta_0(G) = \beta_0(H) + n$, for each $n \geq 2$ is illustrated in Figure 4. The graph H has n points, and the single lines between K_1 and H indicate a matching between the $n - 1$ copies of K_1 and $n - 1$ points of H. We summarize the above observations in the following proposition.

Theorem 9. Let H be a graph, and $n \geq 2$ a positive integer.

(i) There exists a graph G satisfying $C(G) \cong H$, $\kappa(G) = n$, $\lambda(G) = n$, and $\chi(G) = n + \chi(H)$.

(ii) There exists a graph G satisfying $C(G) \cong H$, and $\beta_0(G) = n + \beta_0(H)$.

We now turn to the problem of embedding a graph H as the center in a regular graph having a given number of points. The proof uses ideas discussed in [1, p. 10] for embedding a given grah H in a k-regular graph G. We omit this proof here for the sake of brevity.

Theorem 10. Let H be a graph with $|V(H)| = p \geq 9$. Then for each $k \geq$

$p + 1$ and for each n sufficiently large (where n is even weak k is odd) there exists a k-regular graph $G(k, n)$ on n points satisfying $C(G(k, n)) \cong H$.

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