Compressing Grids into Small Hypercubes

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Abstract

Let G be a graph, and denote by $Q(G)/2^t$ the hypercube of dimension $\lceil \log_2|G| \rceil - t$. Motivated by the problem of simulating large grids by small hypercubes, we construct maps $f:G \rightarrow Q(G)/2^t$, $t \geq 1$, when G is any two or three dimensional grid, with a view to minimizing communication delay and optimizing distribution of G-processors in $Q(G)/2^t$.

Let dilation(f) = $\max\{dist(f(x),f(y)): xy \in E(G)\}$, where "dist" denotes distance in the image network $Q(G)/2^t$, and let the load factor of f be the maximum value of $|f^{-1}(h)|$ over all vertices $h \in Q(G)/2^t$. Our main results are the following.

(1) Let G be any two dimensional grid. Then for any $t \geq 1$ there is a map $f:G \rightarrow Q(G)/2^t$ having dilation 1 and load factor at most $1+2^t$.

(2) Given certain upper bounds on the "densities" $\frac{|G|}{|Q(G)/4|}$ or $\frac{|G|}{|Q(G)/8|}$ of G in Q(G)/4 or Q(G)/8 respectively, we get dilation 1 maps $f:G \rightarrow Q(G)/4$ or $f:G \rightarrow Q(G)/8$ with improved (i.e. smaller) load factor over that given in (1).

(3) Let G be any three dimensional grid. Then there is a map $f:G \rightarrow Q(G)/2$ of dilation at most 2 and load factor at most 3, and a map $f:G \rightarrow Q(G)/4$ of dilation at most 3 and load factor at most 5.

I. Introduction

The analysis of how effectively one network can simulate another and the resulting implications for optimal design of parallel computation networks have become in recent years important topics in graph theory and computer science. One of the measures of the effectiveness of a simulation is the dilation of the corresponding map of networks, defined as follows. Let G and H be two graphs and $f:V(G)\to V(H)$ a map from the vertices of G to those of H (we often just write $f:G\to H$ for such a map). We let $\text{dilation}(f) = \max\{\text{dist}_H(f(x),f(y)): xy\in E(G)\}$, where $\text{dist}_H(p,q)$ is the smallest number of edges on any path joining vertices p an q in H. The map f indicates how the vertices of H play the roles of the vertices of G, and dilation(f) is a measure of the communication delay in this roleplaying, since a message between vertices x and y joined by an edge in G would take unit time while the same message between corresponding vertices f(x) and f(y) in H takes time proportional to at least $\text{dist}_H(f(x),f(y))$ which in the worst case is dilation(f). Indeed the delay may turn out to be worse when one defines the full simulation, requiring in addition to f a routing of each edge xy in G along along some path in H joining f(x) and f(y). The edge congestion in the routing (i.e. the maximum number of routing paths in H passing over any edge in H) is then an additional contribution to the delay of the simulation.

In this paper we minimize dilation in simulations of 2 and 3 dimensional grid structures by small hypercubes. To fix notation, the k-dimensional grid $\underline{[a_1 x a_2 x ... x a_k]}$ has vertex and edge sets given by $V([a_1 x\ a_2 x ... x a_k]) = \{(x_1, x_2,..., x_k): 1 \le x_i \le a_i\}$ and $E([a_1 x\ a_2 x ... x a_k]) = \{xy: \sum_{i=1}^{k} |x_i - y_i| = 1\}$. The n-dimensional hypercube $\underline{Q(n)}$ has vertex set consisting of all n digit words over the alphabet {0,1}, with two vertices joined by an edge if and only if they disagree in exactly one coordinate (so that Q(n) is isomorphic to the n-dimensional grid [2x2x...x2]). For any graph G, denote by $\underline{Q(G)}$ the smallest hypercube having at least as many vertices as G; so that $Q(G) = Q(\lceil \log_2|G| \rceil)$. More generally, for $t \ge 1$ denote by $\underline{Q(G)/2^t}$ the hypercube $Q(\lceil \log_2|G| \rceil - t)$.

There is already a substantial literature on the simulation of various networks by hypercubes and their related networks the butterfly, shuffle exchange, and DeBruijn networks; see [Lei] for an excellent exposition. In particular, one result of [BCLR] shows that for any complete binary tree T there is a one to one map $f:T\to Q(T)$ having dilation 2, and in fact satisfying $\text{dist}_{Q(T)}(f(x),f(y)) = 1$ for all but one of the edges xy of T. A second result shows that any binary tree T can be embedded into Q(T) with O(1) dilation and O(1) edge congestion. It was also proved in [MS] that any binary tree T can be embedded into Q(T) with dilation 5, and into some hypercube Q satisfying $\frac{|Q|}{|Q(T)|} = O(1)$ with dilation 3. There are a number of results concerning the embedding of grids into hypercubes. To state these, for a grid G (of any dimension) let d(G) be the minimum possible value of dilation(f) over all one to one embeddings $f:G \to Q(G)$, and let $\alpha(k)$ be the maximum of d(G) over all k-dimensional grids G. In [C1] it was shown that $\alpha(2) = 2$, and in [C2] that $\alpha(3) \leq 7$ and $\alpha(k) \leq 4k+1$. It was independently shown in [Lee] that $\alpha(3) \leq 8$ and $\alpha(k) \leq 4k-1$, the latter upper bound being realized by a parallel algorithm on the hypercube. In [BMS] it was shown that $\alpha(k) \leq k$ provided that we only allow k-dimensional grids whose sides $a_1, a_2,...,a_k$ satisfy certain inequality constraints. It is still unknown whether $\alpha(k)$ is bounded by a constant as $k \to \infty$.

The specific goal of the work reported here is to construct maps $f:G \to Q(G)/2^t$, $t \geq 1$, for 2 and 3 dimensional grids (which we abbreviate by "2D" and "3D grids")which are optimal or nearly optimal with respect to dilation and distribution of guest processors from G. Here dilation(f) for such many to one maps is defined in exactly the same way as above. As for discussing the distribution of guest processors, when $f:G \to H$ is a (possibly many to one) map of graphs, we let <u>load-factor(f)</u> (which we also refer to as the "load factor of f") be the maximum value of $|f^{-1}(h)|$ over all $h \in V(H)$, and we say that f has <u>uniform load factor</u> L if $|f^{-1}(h)| = L$ for all $h \in H$. Since in a simulation of G by H (using the map f) the processors of G assigned to a given processor h will be simulated sequentially, an unbalanced distribution of G-processors in H will degrade the simulation time. That is, lightly used H-processors must wait for heavily used ones to finish their tasks. Thus the object is to construct a map f which in addition to having small dilation also has load factor as small as possible. When f is of the form $f:G \to Q(G)/2^t$, the best possible upper bound one could hope to prove for the load factor of f, independent of the grid G, is of course 2^t.

This work was at least partly motivated by earlier work of Fiat and Shamir [FS] describing an architecture, called a "polymorphic array", into which arbitrarily large grids could be embedded with small dilation and fairly uniform distribution of grid points in the host array. An additional motivation is that most results on simulations of grids by hypercubes assume that the host hypercube can grow arbitrarily large with the guest grid. This does not correspond well to the real world, where typically one would have a hypercube machine with a fixed number of processors. Seen in this light, the problem of efficiently embedding a large grid in a small hypercube becomes important.

We follow standard graph theoretic notation, and we write log(m) and ln(n) for the logarithm to the base 2 and the natural logarithm respectively.

II. Arbitrarily High Compression of 2D Grids

It will be convenient to construct our embeddings so that their domains and ranges are certain infinite graphs. An appropriate restriction then yields the actual embedding of a finite grid into a finite hypercube.

Let G(m) be the infinite graph with vertices and edges given by

$V(G(m)) = \{(i,j): 1 \le i \le m, 1 \le j < \infty\}$, and $E(G(m)) = \{(i,j)(i',j'): |i-i'| + |j-j'| = 1\}$. Obviously G(m) is just an infinite grid with m rows. We call the subset $\{(i,j): 1 \le j < \infty\}$ of G(m) chain i, and sometimes denote it by C_i, and the subset $\{(i,j): 1 \le i < m\}$ column j.

Next define the graph $H(2^r)$, $r \ge 1$, inductively as follows. Starting with H(2), let $V(H(2)) = \{(i,t): i = 0,1; 1 \le j < \infty\}$ and $E(H(2)) = \{(i,j)(i',j'): |i-i'| + |j-j'| = 1\}$, so that H(2) is isomorphic to G(2) with its rows being indexed by 0 and 1 (instead of 1 and 2 as in G(2)). Given $H(2^{r-1})$, we define the vertices and edges of $H(2^r)$ by

$V(H(2^r)) = \{(i,t): 0 \le i \le 2^r-1, 1 \le t < \infty\}$, and

$E(H(2^r)) = E(H(2^{r-1})) \cup \{(i+2^r,t)(i'+2^r,t'): (i,t)(i',t') \in E(H(2^{r-1}))\} \cup \{(i,t)(2^r-1-i,t): (i,t) \in V(H(2^{r-1}))\}$.

We call the subset $\{(i,t): 1 \le t < \infty\}$ of $H(2^r)$ rod i, and continue to call the subset $\{(i,t): 0 \le i \le 2^r-1\}$ column t. Now the subgraph A of G(m) induced by the first k columns is isomorphic to [mxk], while the subgraph B of $H(2^r)$ induced by the first 2^s columns is a spanning subgraph of Q(r+s). Hence if we construct a map $f:G(m) \to H(2^r)$ of dilation d for which $f(A) \subset B$, then we can interpret f as a map $f:[mxk] \to Q(r+s)$ satisfying dilation(f) $\le$ d. Typically we do not make this interpretation explicit, and instead continue to regard f as a map $f:G(m) \to H(2^r)$, writing $f([mxk]) \subset Q(r+s)$ if we have proved that $f(A) \subset B$.

We will say that a map $f:G(m) \to H(2^r)$ is periodic if there are positive integer constants p, K, and C for which the following holds.

Let $d \ge 0$ and s be nonnegative integers with $1 \le s \le K$. Let $f(i,dK+s) = (x,(m-1)p+u)$ for integers $m \ge 1$ and $0 \le u < p$. Then either

(i) $f(i,(d+1)K+s) = (x+C, mp+u)$ for all i,d, and s with $1 \le s \le K$, or

(ii) $f(i,(d+1)K+K-s+1) = (x+C, (m+1)p-u+1)$ for all i,d, and s with $1 \le s \le K$,

where the addition x+C is taken modulo 2^r. Note that up to a translation by C in the rod index, the image of f either repeats itself every p columns or reflects itself every p columns, and in so doing maps columns tK+1 through (t+1)K of G(m) to columns tp+1 through (t+1)p of $H(2^r)$ for any integer $t \ge 0$. If either either (i) or (ii) hold, then the smallest p for which one of them holds is called the period of f. If (i) holds then we say f is forward periodic, and if (ii) holds then f is reflection periodic (note that f may be both forward and reflection periodic, of course for different integers p). If p is the period of f, then any set of columns of the form tp+1 through (t+1)p of $H(2^r)$, with $t \ge 0$, will be called a frame of f.

A map $f:G(m) \to H(2^r)$ will be called a braiding if it has the following properties.

(B1) dilation(f) = 1.

(B2) If $f(i,t) = (x_1,y_1)$ and $f(i,t') = (x_2,y_2)$ with $t < t'$, then $y_1 \le y_2$.

(B3) If for some i and j the point f(i,j) is in column t of $H(2^r)$ for some t, then for all i' and j' satisfying j' > j the points f(i',j') lie in columns t or bigger. (Note that (B3) implies (B2), but we list (B2) separately for convenience of referencing.)

(B4) f is periodic.

If in addition to these properties, f also has uniform load factor L, then we sometimes denote f by the symbol $m/2^r/L$. Given a map $f:G(m)\rightarrow H(2^r)$, we say that chain i of G(m) switches in column c of $H(2^r)$ if for some b we have f(i,b) and f(i,b+1) being distinct and both lying in column c of $H(2^r)$. If the points f(i,b), f(i,b+1),...,f(i,b+e) all lie in column c with $f(i,b+t)\neq f(i,b+t+1)$ for $0\leq t\leq e-1$, then we say that (i,b), (i,b+1), ...,(i,b+e) (resp.) is the first, second,...(e+1)'st point (resp.), of chain i in column c, and that chain i switches e times in column c. We say that chain i walks at column c if it does not switch in column c. Suppose now that f is a braiding. Then if chain i switches in column c under f, and (i,t) is the last point of chain i mapped to column c, say with f(i,t) = (p,c), then properties (B1) and (B2) imply that f(i,t+1) = (p,c+1). That is, after a chain switches its required number of times in some column c, its next point proceeds along the current rod to column c+1.

For a braiding $f:G(m)\rightarrow H(2^r)$ we make the following conventions concerning the index set $S_t \subset \{1,2,...,m\}$ of chains of G(m) switching in column t .

(1) S_t is a consecutive set of integers, and will be listed as such in the form $S_t = \{x_t, x_t+1, ..., x_t+h_t\}$.

(2) Any integer x > m listed in S_t is to be interpreted as x', where x' is the residue of x mod m if x is not a multiple of m, and x'= m if x is a multiple of m.

(3) The number of times that an integer x appears in S_t is the number of times that chain x' switches in column t.

We let S_i(final) (resp. S_i(final)') be the final segment {1,2,..,z} (resp. {k,k+1,...,u}) if it exists in the listing of S_i such that no element appears more than once in this segment. If such a segment does not exist, then we let S_i(final) (resp. S_i(final)') = ϕ. As an example, for a braiding $f:G(5)\rightarrow H(4)$, the list S_1 = {1,2,3,4,5,6,7} indicates that chains 1 and 2 switch twice, while chains 3, 4, and 5 switch once in column 1. We have S_1(final) = {1,2} while S_1(final)' = ϕ after interpreting 6 and 7 as 1 and 2 respectively modulo 5. If for some i we had S_i = {3,4,5}, then S_i(final)=ϕ while S_1(final)' = {3,4,5}.

Let f be a periodic map. We say that f is consecutive on switches if for all $t \geq 1$

(a) $|S_t|$ is a constant for all t, so that $S_t = \{x_t, x_t+1, ..., x_t+h\}$ with h uniform for all t.

(b) For any two successive columns t and t+1 in the same frame of f, we have either $x_{t+1} = x_t+h+1$, or $x_{t+1}+h = x_t -1$, the choice among the two possibilities being uniform through the frame. That is, if column t+2 is in the same frame as column t+1, then $x_{t+2} = x_{t+1}+h+1$ (resp. $x_{t+2}+h = x_{t+1}-1$) if $x_{t+1} = x_t+h+1$ (resp. $x_{t+1}+h = x_t -1$).

When for any two successive columns t and t+1 of a frame F we have $x_{t+1} = x_t+h+1$ (resp. $x_{t+1}+h = x_t -1$), then we say F is a forward (resp. backward) frame.

The first class of maps we consider in detail are $f_{d<r}:G(r) \rightarrow H(2)$, $r \geq 2$, defined by;

(1) chains begin alternately on rod 0 and rod 1 in column 1. That is,

$f_{d,r}(i,1) = (0,1)$ if i is odd

$= (1,1)$ if i is even.

(2) $f_{d,r}$ makes the chains {1+d(c-1), 2+d(c-1), 3+d(c-1),..., d+d(c-1)} switch at column c, where all integers are read modulo r. All other chains walk at column c.

(3) If (i,t) is the last point of chain i mapped to column c, say with f(i,t) = (p,c), then f(i,t+1) = (p,c+1).

We also define $f_{\underline{0<r}}$:G(r) → H(2) to be the map satisfying (1), but in which every chain walks at every column.

Assume f:G(m)→ H(2^r) is consecutive on switches and for simplicity that $|S_t| < m$ for all t. Let s_1 be the smallest integer such that every chain in G(m) has switched in at least one column from among columns 1 through s_1. We call the set of columns 1 through s_1 the first cycle of f, and we denote it by C_1. Suppose inductively that the k'th cycle C_k of f has been defined and that s_k is the largest column in C_k. Now let s_{k+1} be the smallest integer larger than s_k such that every chain in G(m) has switched in at least k+1 columns from among columns 1 through s_{k+1}. We then let the (k+1)'st cycle C_{k+1} be the set of columns $\{s_k,s_k+1, ...,s_{k+1}\}$ if some chain has switched for the (k+1)'st time (starting from column 1) at column s_k, and alternatively the set of columns $\{s_k+1,s_k+2, ...,s_{k+1}\}$ otherwise. If c is the first column in some cycle C, then we call S_c(final) (resp. S_c(final)') the defining segment of C provided the frame F containing C is forward (resp. backward). Observe that the defining segment of C contains chain 1 (resp. chain m) if F is forward (resp. backward).

Now let d≥c be any column of C, where again c is the first column of C and F is the frame containing C. Assume first that F is forward. Then we define the set (C-chains)$_d$ to be $S_c(\text{final})\cup(\cup_{c<t\leq d}S_t)$ (resp. $S_c(\text{final})\cup(\cup_{c<t<d}S_t)\cup\{S_d\backslash S_d(\text{final})\}$) if d is not the last column (resp. is the last column) of C. Now suppose that F is backward. Then (C-chains)$_d$ is defined as above with S_c(final)' and S_d(final)' taking the place of S_c(final) and S_d(final) repectively.

We may view any chain i∈(C-chains)$_d$ as having switched "during" the cycle C (as opposed to either the cycle preceding or following C) at some column up to d. Accordingly we will say in such a case that i has a second point (resp. has a second point up to column d) in C iff i∈(C-chains)$_t$ for some t (resp. for some t≤d).

Clearly an analogous, but more complicated, definition of cycle and related notions can be made without the assumption $|S_t| < m$. We omit this for brevity. Observe that any frame of a periodic braiding is a (not necessarily disjoint) union of cycles.

As an example, the first five columns in the image of the map $f_{3,5}$ are illustrated in Figure 1. In Figure 2 we show the same map in the more abbreviated form that we shall adopt for the rest of our illustrations. We show 5 arrays, where the j'th array (reading from left to right), 1 ≤ j ≤ 5, shows how the points of G(5) are mapped in the j'th column of H(2). A vertical line in rod p (p = 0 or 1 in these two figures) of an array indicates that any entry i of rod p to the left (resp. right) of the line is the first (resp. second) point of chain i in the column represented by that array. In general there would of course be 2^r many rods in each array when we illustrate a braiding $f:G(m)\to H(2^r)$. The map $f_{3,5}$ forward periodic with period 5 and constants p=5, K=5, and C=1 (in the definition of periodicity). The first few cycles, with integers standing for column numbers, are $C_1=\{1,2\}$, $C_2=\{2,3,4\}$, and $C_3=\{4,5\}$. The frame (consisting of columns 1 through 5) containing these cycles is forward, so the defining segments of these cycles are respectively $\{1,2,3\}$, $\{1\}$, and ϕ. In Figure 6a we have the first eight columns of a reflection periodic map f = 7/4/3 with period 7 and constants p=7, K=12, and C=0. The first few cycles are $C_1=\{1,2\}$, $C_2=\{2,3\}$, $C_3=\{3,4,5\}$, $C_4=\{5,6\}$, and $C_5=\{6,7\}$, and the frame containing them (consisting of columns 1 through 7) is forward. Noting that $S_5=\{1,2,3,4\}$, we have $(C_3\text{-chains})_4=\{1,2,3,4,5,6\}$ and $(C_3\text{-chains})_5=\{1,2,3,4,5,6,7\}$. The second frame, which begins with column 8, is backward. Noting that $S_9=\{5,6,7,1,2\}$, it follows that the defining segments of C_6 and C_7 are $\{3,4,5,6,7\}$ and $\{5,6,7\}$ respectively, and $(C_6\text{-chains})_9$ $=\{1,2,3,4,5,6,7\}$.

[Put Figure 1 followed by Figure 2 here.]

Lemma 1: The map $f_{d,r}:G(r) \to H(2)$ has the following properties.
(a) $f_{d,r}$ is a braiding, and
(b) For any $t \geq 1$, we have $\text{dist}(f_{d,r}(i,t),f_{d,r}(j,t)) \leq 2$ for all i and j, $1\leq i,j\leq r$.

Proof: As an abbreviation we let $f = f_{d,r}$ since d and r will be fixed throughout the proof. To avoid unimportant complications, we will make the simplifying assumption that $d\leq r$. As a consequence each chain of G(r) has either one or two of its points mapped to any column, depending on whether the chain walks or switches (respectively) in that column. The proof when $d>r$ is essentially the same, requiring the extra detail related to keeping track of how many points a given chain has mapped to a column. We start with (a) by verifying the properties (B1)-(B4) in the definition of braiding. The proof of (B3) (and hence of (B2)) is similar to that of (B1), so we give (B1) in detail and mention briefly how it applies to proving (B3).

Property (B1): Consider two successive points on chain i, say (i,b) and (i,b+1), and assume wlog that f(i,b) = (0,t) for some t. If chain i walks at column t, then f(i,b+1) = (0,t+1) showing that f(i.b) and f(i,b+1) are at distance 1. Suppose then that chain i switches in column t. If (i,b) is the first point of chain i in column t then f(i,b+1) = (1,t), while if (i,b) is the second point of chain i in column t then f(i,b+1) = (1,t) by condition (3) in the definition of the maps $f_{d,r}$. In either case we again see that f(i.b) and f(i,b+1) are at distance 1, as desired.

Now we prove that for any i and b the points f(i,b) and f(i+1,b) are at distance 1 in H(2). Fixing i, we prove this by induction on b. For the base of the induction, we verify that this is true when $1\leq b\leq 3$. Indeed for such b one can check that either f(i,b) and f(i+1,b) are in the same column t of H(2) with $t\leq 3$, or for some $p\in\{0,1\}$ we have f(i,b) = (p,t) and f(i+1,b) = (p,t+1) with $t\leq 2$. We take as our inductive hypothesis that for all $b \leq m$, for some $m\geq 3$, the points f(i,b) and f(i+1,b) are related in one of the following three ways. We let $p \varepsilon \{0,1\}$, and we read p and p+1 as integers modulo 2.

(a) f(i,b) = (p,t), f(i+1,b) = (p+1,t), and {f(i,b), f(i+1,b)} are either both the first or both the second points of their chains mapped to column t .

(b) f(i,b) = (p,t), f(i+1,b) = (p,t+1), and (i,b) is the second point of chain i mapped to column t while (i+1,b) is the first point of chain i+1 mapped to column t+1.

(c) f(i,b) = (p,t), f(i+1,b) = (p+1,t), and (i,b) is the first point of chain i mapped to column t while (i+1,b) is the second point of chain i+1 mapped to column t .

As our inductive step we will show that f(i,m+1) and f(i+1,m+1) are related to each other in one of the three ways described above, and the lemma will follow. We argue by cases according to which relation holds between f(i,m) and f(i+1,m). For brevity, we use the phrase "in column x" to mean "mapped to column x". We also note that f(s,m-1) and f(s,m-2) are defined for all s since $m \geq 3$.

Assume case (a) holds with b = m. If {(i,m), (i+1,m)} are both second points of their chains in column t, then f(i,m+1) = (p,t+1) and f(i+1,m+1)} = (p+1,t+1) so case (a) is preserved. We can therefore assume that {(i,m), (i+1,m)} are both first points of their chains in column t.

Suppose first that chain i switches at column t. If chain i+1 also switches at column t, then {f(i,m+1), f(i+1,m+1)} = {(p+1,t), (p,t)} so relation (a) holds between f(i,m+1) and f(i+1,m+1). Now suppose that chain i+1 walks at column t. Then f(i+1,m+1) = (p+1,t+1) while f(i,m+1) = (p+1, t), so relation (b) holds between f(i,m+1) and f(i+1,m+1).

Now suppose that chain i walks at column t. We claim that chain i+1 also walks at column t. This would suffice since then relation (a) would continue to hold in column t+1. Suppose to the contrary that chain i+1 switches at column t. Then by the definition of our maps it follows that chain i must have switched at column t-1 while chain i+1 walked at column t-1. This forces f(i+1,m-2) = (p+1,t-2), while f(i,m-1) = (p,t-1) and f(i,m-2) = (p+1,t-1). The result is that f(i,m-2) and f(i+1,m-2) are related in none of the three ways, contrary to the the inductive hypothesis.

Now assume case (b) holds with b = m. Thus chain i switched at column t, and f(i,m+1) = (p,t+1). We claim that chain i+1 must have walked at column t. For if chain i+1 switched at column t, then we have f(i+1,m-2) = (p+1,t) and f(i,m-2) = (p+1,t-1). Clearly (i+1,m-2) is the first point of chain i+1 in column t. If (i,m-2) is the first point of chain i in column t-1, then (i,m-2) and (i+1,m-2) are related in none of the three ways, a contradiction to the inductive hypothesis. If (i,m-2) is the second point of chain i in column t-1, then chain i switched at columns t and t-1. This implies that every chain switches at every column, and then a simple induction shows that relation (a) must hold between f(i,b) and f(i+1,b) for all b, contrary to the assumption that case (b) holds. Having shown then that chain i+1 walked at column t, we see from the fact that chain i switched at column t and from the definition of our maps $f_{d,r}$ that chain i+1 must switch at column t+1.

Therefore we have $f(i+1,m+1) = (p+1,t+1)$, which together with $f(i,m+1) = (p,t+1)$, shows that that we are in case (c) as desired.

Finally assume that case (c) holds with $b = m$. Thus $f(i+1,m+1) = (p+1,t+1)$. If chain i walks in column t, then $f(i,m+1) = (p,t+1)$, so that $f(i,m+1)$ and $f(i+1,m+1)$ are related as in case a). If chain i switches in column t, then $f(i,m+1) = (p+1,t)$, so that $f(i,m+1)$ and $f(i+1,m+1)$ are related as in case b). This completes the inductive step, and hence the proof of B1.

Properties (B2)-(B4): Property B3 is a tedious but straightforward induction on j, using the relations (a)-(c) as in the proof of B1. Property (B2) follows from (B3) (or alternatively from the fact that by definition every chain of G(m) either walks or switches at every column). For B4, note that f is consecutive on switches with d chains switching in each column. Letting L be the least common multiple of d and r, we see that every chain switches a total of L/r times in columns 1 through L/d. The same is of course true about columns iL/d+1 through (i+1)L/d for any $i \geq 1$. Hence B4 follows with $p = L$ and $K = L(1/r + 1/d)$. Thus $f_{d,r}$ is a braiding.

Consider property (b) in the statement of Lemma 1. We in fact prove the stronger assertion that $f_{d,r}(i,t)$ and $f_{d,r}(j,t)$ are either in the same column of H(2) or in successive columns of H(2). We proceed by induction, the case $t = 1$ being obvious. Assume the statement is true for $t \leq T$. As notation, let $f(i,T) = p = (p_1,p_2)$ and $f(j,T) = q = (q_1,q_2)$, while $f(i,T+1) = p' = (p_1',p_2')$ and $f(j,T+1) = q'= (q_1',q_2')$. Also let C_i (resp.C_j) denote chain i (resp. chain j). It suffices to show that $|p_2'-q_2'| \leq 1$; that is, images are in adjacent columns.

The inductive hypothesis implies that $|p_2-q_2| \leq 1$. We may suppose that $i < j$. Since f is consecutive on switches, we know that for any k, the column at which C_i switches for the k'th time is the same as or comes before the corresponding column for C_j. Thus $p_2 \leq q_2$.

Assume first that $p_2 = q_2$. Then whichever (if any) of the chains C_i or C_j switches at column p_2, the condition dilation(f) = 1 forces $|p_2'-q_2'| \leq 1$, as required.

So assume $p_2 = q_2-1$. If p is the second point of C_i in column p_2, then p' must be the first point of C_i in column p_2+1, while q' is either in column q_2 or q_2+1 depending on whether C_j switches in column q_2 or not, respectively. It follows that $|p_2'-q_2'| \leq 1$. Now assume that p is the first point of C_i in column p_2. Then C_i has switched one more time than C_j in columns 1 through p_2-1. Let c be the last column $\leq p_2-1$ at which C_i switched, and let c_j (resp. c_i) be the first column after c at which C_j (resp. C_i) switches. We must have $c_j < c_i$ and $c_j \geq q_2$. Hence we have again $p_2' = p_2+1$ and $q_2 \leq q_2' \leq q_2+1$, implying $|p_2'-q_2'| \leq 1$ and completing the proof of the lemma. ∎

Our next lemma treats the load factor of these maps $f_{d,r}$. As notation, for any point (p,t) of H(2) in the image of a map $f=f_{d,r}$, $p \in \{0,1\}$, we let <u>L(p,t)</u> $= |f^{-1}(p,t)|$.

<u>Lemma 2</u>: For any point $(p,t) \in H(2)$ in the image of a map $f_{d,r}$, where d+r is even,

we have $L(p,t) = \frac{r+d}{2}$.

Proof: To avoid routine details we make the simplifying assumption d≤r, so that now each chain switches at most once in any given column.As notation, let S(p,t) be the set of chains of G(r) switching in column t of H(2) and having their first point of column t in rod p (and hence their second point in rod p+1 mod 2). Let W(p,t) be the set of chains of G(r) walking in column t and having their (only) point of column t in rod p. Also in this proof we refer to chains just by their indices, i.e. chain C_i will be denoted by i. From the definition of $f_{d,r}$ in column 1, we have $S(0,1) = \lceil \frac{d}{2} \rceil$ and $S(1,1) = \lfloor \frac{d}{2} \rfloor$, while $S(0,1) \cup W(0,1) = \lceil \frac{r}{2} \rceil$ and $S(1,1) \cup W(1,1) = \lfloor \frac{r}{2} \rfloor$. It follows that

$L(0,1) = \lceil \frac{r}{2} \rceil + \lfloor \frac{d}{2} \rfloor = \frac{r+d}{2} = \lfloor \frac{r}{2} \rfloor + \lceil \frac{d}{2} \rceil = L(1,1)$, proving the lemma for t = 1.

We proceed by induction on t. First observe that $L(0,t) = |W(0,t-1)| + |S(1,t)| = L(0,t-1) - |S(0,t-1)| + |S(1,t)|$. The same reasoning applies to L(1,t), and we get the recurrence

$$L(p,t) = L(p,t-1) - |S(p,t-1)| + |S(p+1,t)|$$

where again p and p+1 are read modulo 2. It therefore suffices to show that

(1) $|S(0,t)| = \lceil \frac{d}{2} \rceil$ for t odd and $|S(0,t)| = \lfloor \frac{d}{2} \rfloor$ for t even, and

(2) $|S(1,t)| = \lceil \frac{d}{2} \rceil$ for t even and $|S(1,t)| = \lfloor \frac{d}{2} \rfloor$ for t odd.

For then we get $L(p,t) = L(p,t-1)$ for all $t \geq 2$, and the lemma would follow.

Toward proving (1) and (2), we need the following claim. For $t \geq 1$, let $z(t) = 1+d(t-1)$ and $z'(t) = d+d(t-1)$, so that z(t) and z'(t) are the first and last chains switching in column t under the map $f_{d,r}$. Also let S be any increasing sequence of integers drawn from the set {1,2,...,r}. When we write ... x, x+2 for the initial segment of S we mean to include in that segment precisely those integers between 1 and x+2 which are translates of x+2 by a multiple of 2. Similarly when we write y, y+2, ... for the final segment of S we mean to include in that segment precisely those integers between y and r which are translates of y by a multiple of 2. This notation for initial and final segments will be combined.

Claim: Let $t \geq 1$, and set $z = z(t)$ and $z' = z'(t)$. Then for some rod $p = p(t) \in \{0,1\}$ we have

$S(p,t) \cup W(p,t) = \{..., z-3, z-1, z, z+2, z+4, ...\}$ and

$S(p+1,t) \cup W(p+1,t) = \{..., z-4, z-2, z+1, z+3, ...\}$, all integers being taken modulo r.

Proof of claim: We proceed by induction on t. For t = 1 the claim just says that $S(0,t) \cup W(0,t)$ consists of the odd integers between 1 and r, while $S(1,t) \cup W(1,t)$ consists of the even integers between 1 and r. This follows immediately from the definition of $f_{d,r}$ in column 1.

For the inductive step, suppose the claim is true for some t. From the defintion of $f_{d,r}$ the chains switching in column t are z through z', that is, $S(p,t)\cup S(p+1,t) = \{z,z+1,...,z'\}$. So using the inductive hypothesis on $S(p,t)\cup W(p,t)$ we can deduce that $S(p+1,t) = \{z+1, z+3, ..., z'-1\}$ and $W(p+1,t) = \{..., z-4, z-2, z'+1, z'+3, ...\}$. It follows that

$S(p+1,t+1)\cup W(p+1,t+1) = W(p+1,t)\cup S(p,t) = \{..., z-4, z-2, z'+1, z'+3, ...\}\cup\{z, z+2, ...,z'\}$
$= \{..., z-4, z-2, z, z+2, ...,z', z'+1, z'+3,...\}$.

Similarly we can use the definition of $f_{d,r}$ and induction to get $W(p,t) = \{..., z-3, z-1, z'+2, z'+4, ...\}$, from which it follows that

$S(p,t+1)\cup W(p,t+1) = W(p,t)\cup S(p+1,t) = \{..., z-3, z-1, z'+2, z'+4, ...\}\cup\{z+1, z+3, ..., z'-1\} =\{..., z-3, z-1, z+1, z+3, ..., z'-1, z'+2, z'+4, ...\}$.

Thus we see that by letting p(t+1) play the role of p+1 and z'+1 the role of z, the claim holds for t+1, completing the inductive step and proving the claim.

Returning to the proof of (1) and (2), we have (again using the fact that the chains switching in column t are z through z') that $|S(p,t)| = \lceil \frac{d}{2} \rceil$ and $|S(p+1,t)| = \lfloor \frac{d}{2} \rfloor$. Also since the chains switching in column t+1 are z'+1 through z'+d, we see from the description of $S(p+1,t+1)\cup W(p+1,t+1)$ and $S(p,t+1)\cup W(p,t+1)$ that $|S(p+1,t+1)| = \lceil \frac{d}{2} \rceil$ and $|S(p,t+1)| = \lfloor \frac{d}{2} \rfloor$. It follows from this and $|S(0,1)| = \lceil \frac{d}{2} \rceil$ with $|S(1,1)| = \lfloor \frac{d}{2} \rfloor$ that (1) and (2) hold. ∎

We now establish some simple ways of combining braidings to get new braidings. Starting with some map f, we will define the maps k·f, $f\oplus 2^t$, and $f_{m;q,c}$.

The map k·f: Let $f:G(m)\rightarrow H(2^r)$ be a braiding. For a positive integer k, we define the map <u>k·f</u>$:G(km)\rightarrow H(2^r)$ chainwise by

(1) $k\cdot f(C_i) = f(C_i)$ for $1 \le i \le m$,

(2) $k\cdot f(C_{pm+i}) = f(C_i)$ for $1 \le i \le m$, p even

(3) $k\cdot f(C_{pm+i}) = f(C_{m-i+1})$ for $1 \le i \le m$, p odd.

Notice that k·f maps successive groups of m chains by successive reflections of chain indices; so that chains m+1, m+2,...,2m are mapped the same way as chains m, m-1, ...,1 respectively, and chains 2m+1, 2m+2,...,3m are mapped the same way as chains 1, 2, ...,m respectively, etc.. This is illustrated in Figure 3.

[Put Figure 3 here.]

The map $f\oplus 2^t$: Let m,k, and r be nonnegative integers with $0\le k\le m-1$. Consider the subgraph G(k) of $H(m2^r)$ induced by rods $k2^r, k2^r+1, ..., (k+1)2^r -1$ for some k, so that obviously $G(k)\cong H(2^r)$. Given a set $A\subset V(G(k))$, the reflection $Ref_{2^r}(A)$ of A is defined by <u>$Ref_{2^r}(A)$</u> $= \{(k2^{r+1}+2^r-1-x,y): (x,y)\in A\}$. We retain 2^r in the symbol for reflection to emphasize that for a given $A\subset H(m2^r)$, the set $\mathrm{Ref}_{2^r}(A)$ lies within the same set of 2^r rods $k2^r, k2^r+1, ..., (k+1)2^r -1$ in $H(m2^r)$ as does A, for the appropriate integer k. Observe that $\mathrm{Ref}_{2^r}(A)$ is obtained from A by reflecting in the rod index (the first coordinate) within these set of rods, that is, within the subgraph G(k) of $H(m2^r)$. When (x,y) and (x',y') are two points in $H(m2^r)$ with $x+x' \le m2^r-1$, we write $(x,y)\oplus(x',y')$ for the point (x+x',y+y'). When S is a set of vertices in $H(2^t)$, we write $S\oplus(x',y')$ for the set $\{(x,y)\oplus(x',y'):(x,y)\varepsilon S\}$ provided the elements described in the set are defined.

Starting with a map $f:G(u)\rightarrow H(2^r)$, we indicate how to in some sense replicate f a total of 2^t times to get a map $g:G(2^t u)\rightarrow H(2^{r+t})$. Again the definition is recursive on chains:

(1) $g(C_i) = f(C_i)$ for $1 \le i \le u$

(2) $g(C_{ku+i}) = \mathrm{Ref}_{2^r}(g(C_{ku-i+1}))\oplus(2^r,0)$ for $1 \le i \le u, k \ge 1$.

We denote the map g just defined by <u>$f\oplus 2^t$</u>.

Note that $f\oplus 2^t$ is similar to $2^t\cdot f$ in that it reflects successive groups of u chains, the difference of course being that $f\oplus 2^t$ then translates the rod index in the image by a successive multiples of 2^r for successive groups. Thus the domain $G(2^t u)$ of $f\oplus 2^t$ can be partitioned into 2^t groups, the i'th group ($1\le i\le 2^t$), call it Z_i, consisting of chains (i-1)u+1 through iu, while the range $H(2^{r+t})$ can also be partitioned into 2^t groups, the i'th group ($1\le i\le 2^t$) consisting of rods $(i-1)2^t$ through $i2^t -1$. The map sends the i'th group of chains to the i'th group of rods. A pictorial representation of the process is given in Figure 4 (where we abbreviate Ref_{2^r} by Ref and use the notation $S\oplus(x',y')$ defined above), and an example is given in Figure 5a and 5b below.

Note that the frame and cycle structure of f carries over in a natural way to the restriction of f to any group Z_i. Indeed the corresponding cycle structure for the restriction is identical, and a frame of f is forward iff the corresponding frame of the restriction of f to Z_i (i.e. consisting of the same columns) is forward (resp. backwards) when i is odd (resp. even). Let {1,2,..,z} (resp. {u,u-1,..,k} be the defining segment of some cycle σ of f. Then we let the <u>defining segment of σ for Z_i</u> be defined by {(i-1)u+1,...,(i-1)u+z} (resp. {(i-1)u+u,(i-1)u+u-1,.., (i-1)u+k} if i is odd, and alternatively {(i-1)u+u,...,(i-1)u+u-z+1} (resp. {(i-1)u+1,..., (i-1)u+u-k+1} if i is even.

[Put Figure 4 here.]

For finite graphs G and H we let <u>dens(G,H)</u> $= \frac{|G|}{|H|}$. For a braiding $f:G(u)\rightarrow H(2^r)$ we let $(f)_{[t]}$ be the number of points of G(u) mapped into the first t columns of $H(2^r)$ by f. We then let <u>$\mathrm{dens}_t(f)$</u> $= \frac{(f)_{[t]}}{t\cdot 2^r}$, which we can view as the "density" of the map f.

The map $f_{m;q,c}$: Suppose $f:G(u)\rightarrow H(2^r)$ is a periodic braiding of load factor L which is consecutive on switches. Let $c\ge 1$, $m\ge 1$,and $q = 2^e$ for some $e\ge 0$ be positive integers, and let s be the residue class of m modulo u, so that m = du+s with s<u for some integer d. We will build a map $f_{m;q,c}:G(m)\rightarrow H(2^{r+e})$ by restricting the domain of $f\oplus 2^e$ to a certain set of m chains.

Let σ be the last cycle of f containing column c. Suppose first that the defining segment of σ includes chain 1. Then we let we let $f_{m;q,c}$:G(m)→ H(2^{r+e}) be the map obtained from f⊕2^e by restricting the domain to the m chains 1 through s together with chains u+1 through (d+1)u of G(u·2^e). Formally we must view these m chains as the chains of G(m) by reindexing them appropriately with the integers 1 through m. This can be done by a correspondence $\kappa:\left\{\{1,2,...s\}\cup\{u+1,u+2,...(d+1)u\}\right\}\to\{1,2,...m\}$, where for chain i of G(u), with 1≤i≤s or u+1≤i≤(d+1)u, we take κ(i) to be the index of that chain in G(m) given by κ(i) = s-i+1 if 1≤i≤s and κ(nu+i) = nu-i+s+1 if 1≤n≤d and 1≤i≤u.

Now suppose that the defining segment of σ includes chain u. Then we let $f_{m;q,c}$:G(m)→ H(2^{r+e}) be the map obtained from f⊕2^e by restricting the domain to the m chains u-s+1 through (d+1)u of G(u·2^e). The reindexing $\kappa:\left\{u\text{-}s+1, u\text{-}s+2,..., (d+1)u\right\}\to\{1,2,...m\}$ in this case is given by κ(i) = i-(u-s) for u-s+1≤i≤(d+1)u.

When the defining segment of σ includes both 1 and u, then we take either definition.

Examples of these reindexing schemes, applied to the appropriate values of c, are given in Figures 5 and 6.

[Put Figure 5 followed by Figure 6 here.]

Recall the partition of the domain G(qu) and range H(q2^r) into q groups each, the i'th group of the chains in the domain being mapped by f⊕q to the i'th group of rods in the range. We can view $f_{m;q,c}$ in a similar way, only understanding that the first group of chains has size s (because of the u-s chains that were removed) and the last q-d+1 groups are empty. We call these removed u-s chains that are in the first group of the domain of f⊕q but not in the domain of $f_{m;q,c}$ the <u>absent chains</u> of $f_{m;q,c}$. In the case where the chains switching in the first column of σ include chain 1, the absent chains are s+1 through u. In the case where these chains include chain u, the absent chains are 1 through u-s. Note that the set of absent chains is so chosen that it has smallest possible intersection, given its size and consecutiveness of chain numbers, with the set of chains switching under f in the first column of σ. In this sense we minimize the number of points removed from the image of f to form the image of $f_{m;q,c}$ in the first group. We will describe this property of the absent chains more precisely in the next lemma, and it will be the key to proving a useful lower bound on $\mathrm{dens}_c(f_{m;q,c})$. A pictorial representation is given in Figure 7 of the map $f_{m;q,c}$ together with the indexing scheme in the case where chain 1 switches in the first column of σ.

[Put Figure 7 here.]

We can also define maps $(f\oplus 2^t)_{m;q,c}$ and and $(k\cdot f)_{m;q,c}$, despite f⊕2^t and k·f being in general not consecutive on switches, in analogous ways. The details for $(f\oplus 2^t)_{m;q,c}$ will be now be given, but are omitted for $(k\cdot f)_{m;q,c}$ as they are similar.

Write m = du2^t+s, where s is the residue class of m modulo u2^t. Also write s = eu+h where h is the residue class of s modulo u. For an overview, we form $(f\oplus 2^t)_{m;q,c}$ as the restriction of

$(f\oplus 2^t)\oplus q$ to m chains found in a collection of successive groups of u chains each (from the decomposition of $G(qu2^t)$ into $q2^t$ such groups). As above, we will take for the domain (of $(f\oplus 2^t)_{m;q,c}$) h chains from the first group, and all chains in the remaining groups of the collection. The parameters show that we need $d2^t+e+1$ groups, and the first group in our collection will consist of chains $(2^t-(e+1))u+1$ through $(2^t-e)u$. Denote this group by S.

More precisely, let σ as above be the last cycle of f containing column c. Assume first that defining segment of σ for S includes chain $(2^t-(e+1))u+1$. We define the map $(f\oplus 2^t)_{m;q,c}:G(m)\rightarrow H(2^{t+q}\cdot u)$ as the map obtained from $(f\oplus 2^t)\oplus q:G(qu2^t)\rightarrow H(2^{t+q}\cdot u)$ by restricting the domain to the m chains $(2^t-(e+1))u+1$ through $(2^t-(e+1))u+h$ together with the chains $(2^t-e)u+1$ through $(d+1)u2^t$. The reindexing of these chains (in order to identify their chain numbers in G(m)) is given by $\kappa((2^t-(e+1))u+i) = h-i+1$ for $1\le i\le h$, and $\kappa(nu+i) = u-i+1+h+(n-2^t-e)u$ for $1\le i\le h$ and $2^t-e \le n \le (d+1)2^t$.

Now suppose that the defining segment of σ for S includes chain $(2^t-e)u$. Then $(f\oplus 2^t)_{m;q,c}$ is defined as the restriction of $(f\oplus 2^t)\oplus q$ to the chains m chains $(2^t-e)u-h+1$ through $(d+1)u2^t$. The indexing is given by $\kappa(i) = i-u+s$ for $(2^t-e)u-h+1 \le i \le (d+1)u2^t$.

The next lemma implies that the average, over all absent chains, of the number of points mapped to columns 1 through c is at most the same average, taken over all u chains of G(u).

<u>Lemma 3</u>: Let $f:G(u)\rightarrow H(2^r)$ be a periodic braiding which is consecutive on switches, and σ the last cycle of f containing column c. Let $A\subset G(u)$ be the set of absent chains of $f_{m;q,c}$ (for integers m and q). Then

$$\frac{|A\cap(\sigma\text{-chains})_c|}{|A|} \le \frac{|(\sigma\text{-chains})_c|}{|u|}$$

<u>Proof</u>: We assume that $|S_t| < u$ for all t (i.e. each chain switches at most once in a column), as an analogous but more cumbersome proof applies without this assumption. It suffices show that $\frac{|A\cap(\sigma\text{-chains})_c|}{|(\sigma\text{-chains})_c|} \le \frac{|A|}{u}$. But observe that $|A| - |A\cap(\sigma\text{-chains})_c| = u - |(\sigma\text{-chains})_c| = z$ for some integer $z \ge 0$ by consecutiveness of f and the definition of A. Hence the last inequality follows from the validity of the inequality $\frac{x}{y} \le \frac{x+z}{y+z}$ for any integers $0\le x\le y$ and $z\ge 0$, as desired. ∎

In the next lemma we collect some facts about the maps defined above.

<u>Lemma 4</u>: Let $f:G(u)\rightarrow H(2^r)$ be a peiodic braiding with period p and uniform load factor L which is consecutive on switches. Then

(a) $k\cdot f$ is a braiding with uniform load factor kL and period p.
(b) $f\oplus 2^t$ is a braiding with uniform load factor L and period p for any $t \ge 1$.

(c) For any $c \geq 1$ and q a power of 2, $f_{m;q,c}$, $(f\oplus 2^t)_{m;q,c}$, and $(k\cdot f)_{m;q,c}$ are braidings of load factor at most L, L, and kL respectively, satisfying $\mathrm{dens}_c(f_{m;q,c}) \geq \frac{m}{qu} L$, $\mathrm{dens}_c((f\oplus 2^t)_{m;q,c}) \geq \frac{m}{qu_1} L$, and $\mathrm{dens}_c((k\cdot f)_{m;q,c}) \geq \frac{m}{qu_2} L$, where $u_1 = 2^t u$ and $u_2 = ku$. .

Proof: Items (a) and (b) follow directly from the definitions. In particular, the fact that we have dilation 1 for these maps comes from f having dilation 1 and from considering the "vertical" edges in $H(2^r)$ of the form $(i,t)(2^x\text{-}1\text{-}i,t)$.

Consider the statements in (c) concerning $f_{m;q,c}$. The construction of the map $f_{m;q,c}$ guarantees that the images of the q groups (in the discussion following the definiton of $f_{m;q,c}$) are pairwise disjoint. Hence the load factor of $f_{m;q,c}$ is at most the maximum, over all $1 \leq i \leq q$, of the load factor of the restriction of $f_{m;q,c}$ to the i'th group. This maximum is load factor(f) = L, proving the statement on load factor. To check property (B1) for $f_{m;q,c}$, let y and z be adjacent points of G(m). If y and z are successive points on the same chain of G(m), then $\mathrm{dist}(f_{m;q,c}(y),f_{m;q,c}(z))=1$ because $f_{m;q,c}$ is a restriction of $f\oplus q$. If y and z are corresponding points on successive chains opf G(m), then $\mathrm{dist}(f_{m;q,c}(y),f_{m;q,c}(z))=1$ as a consequence of the reindexing κ of chain numbers in the definition of $f_{m;q,c}$. It is easy to see that the remaining properties (B2)-(B4) are inherited by $f_{m;q,c}$ from f.

Next we show that $\mathrm{dens}_c(f_{m;q,c}) \geq \frac{m}{qu} L$. Since f is consecutive, it suffices to consider just the case in which the number k of chains in G(u) switching under f in any column satisfies k<u, that is, each chain switches at most once in any column. Let A be the set of absent chains of f. As above, let s be the residue class of m modulo u and let x=u-s, so that x = |A|. Let w be the number of points on the chains in A which are mapped by f to columns 1 through c. We begin by showing that

$$\frac{w}{x} \leq c\left(\frac{u+k}{u}\right) . \qquad (1)$$

Let E be the number of complete cycles of f among the columns 1 through c, so that E is the maximum integer such that every of chain of G(u) has switched at least E times under f in columns 1 through c. As in the definition of $f_{m;q,c}$ let σ be last, possibly partial, cycle of f containing column c. Then we may write w = cx + Ex + t, where cx counts the number of first points of chains in A lying in columns 1 through c, Ex counts the number of second points of these chains among the E complete cycles (each chain having exactly one second point per cycle), and t counts the number of second points of chains in A up to column c in σ (unless σ is itself complete in which case we take t=0). Hence $t = |A \cap (\sigma\text{-chains})_c|$, and (1) is reduced to showing that

$$E + \frac{t}{x} \leq \frac{ck}{u} . \qquad (2)$$

To see (2), note that $\frac{ck}{u}$ is the average, over all u chains in a group, of the number of second points of a chain among columns 1 through c under the map f. Thus $\frac{ck}{u}$ - E is the average, over all u chains, of the number of second points of a chain up to column c in σ. Thus $\frac{ck}{u} - E = \frac{|(\sigma\text{-chains})_c|}{|u|}$. But $\frac{t}{x}$ is the average, over the the chains in A, of the number of second points of a chain up to column c in σ. By Lemma 3 we have $\frac{t}{x} \leq \frac{ck}{u} - E$. Hence (2) and therefore also (1) follows.

Let $\mu(c) = L - dens_c(f_{m;q,c})$. Recall that the map $f_{m;q,c}:G(m)\rightarrow H(q\cdot 2^r)$ is obtained from the map $f\oplus q:G(qu)\rightarrow H(q\cdot 2^r)$ by restriction to the nonabsent chains in the first group, and all the chains in groups 2 through d+1. Thus we may view $\mu(c)$ as the percentage, from among the points in G(qu) mapped to columns 1 through c under $f\oplus q$, of points lying in the absent chains or in chains (d+1)u+1 through qu. It suffices to prove that

$$\mu(c) \leq \left(\frac{qu - m}{qu} \right) L. \qquad (3)$$

Let us write m = du + s, where as above s is the residue class of m modulo u, and x = u-s is the number of absent chains. Thus we have q = d+1+b for some b≥0. We can view the $q\cdot 2^r$ rods of $H(q\cdot 2^r)$ as coming in q groups, the i'th group, $1\leq i\leq q$, consisting of rods $(i-1)2^r$ through $i2^r - 1$. Similarly we can partition the chains of G(qu) into q groups of consecutively indexed chains, each of size u, the i'th group of u chains being mapped by $f\oplus q$ into the i'th group of 2^r rods. In particular the last b groups of rods are the images under $f\oplus q$ of chains (d+1)u+1 through qu, and the first group are the images of chains 1 through u. By the symmetry in the way $f\oplus q$ maps these groups, it follows that the contribution of chains (d+1)u+1 through qu to $\mu(c)$ is $\frac{b}{q}L$. As for the contribution of the absent chains to $\mu(c)$, inequality (1) implies $w \leq xc\left(\frac{u+k}{u}\right)$. But also we have $L = \frac{u+k}{2^r}$ since f maps u+k points to any column, has load factor L, and there are 2^r points in a column (of $H(2^r)$). Thus the contribution of the absent chains is $\frac{w}{q2^r c} \leq \frac{xc}{uq}L$. Combining these two contributions we obtain the bound

$$\mu(c) \leq L\left(\frac{b}{q} + \frac{xc}{uq}\right) = \frac{bu + xc}{uq} = \left(\frac{qu - m}{qu}\right)L$$

as desired.

We outline the lower bound on $dens_c((f\oplus 2^t)_{m;q,c})$ as it is similar to the lower bound just proved. The lower bound on $dens_c((k\cdot f)_{m;q,c})$ is again similar and we omit the argument for it. Let A be the set of absent chains of $(f\oplus 2^t)_{m;q,c}$ and S the group of u chains (chains $(2^t - (e+1)u)+1$ through $(2^t - e)u$) in $G(qu2^t)$ containing A. By the obvious analogues of Lemma 3 and inequality (1) above, the average, over chains of A, of the number of points mapped by $(f\oplus 2^t)\oplus q$ to columns 1 through c is at most the same average over the chains of S. But by the symmetry in the way the different groups of chains are mapped, the latter average is equal to the same average over all chains in the domain $G(qu2^t)$ of $(f\oplus 2^t)\oplus q$. This implies the analogue of inequality of (3) above, with u replaced by u_1, q by $q2^t$, and f by $f\oplus 2^t$ in the definition of $\mu(c)$. The statement on $dens_c((f\oplus 2^t)_{m;q,c})$ follows. ∎

Corollary 4.1: Let $f:G(u)\rightarrow H(2^r)$ be a periodic braiding of load factor L which is consecutive on switches. Let m, b, and x be nonnegative integers satisfying $x<\lfloor \log(u)\rfloor$ and $b+x \le \lfloor \log(m)\rfloor$. Let $R = \frac{m}{2^{\lfloor \log(m)\rfloor - b}}$, and $q = 2^{\lfloor \log(m)\rfloor - b - x}$. If $\frac{u-s}{2^x} \le R \le \frac{u}{2^x}$ for some positive integer s, then the map $f_{m;q,c}:G(m)\rightarrow H(q\cdot 2^r)$ satisfies $dens_c(f_{m;q,c}) \ge L(1 - \frac{s}{u})$ for any integer $c \ge 1$. Similarly we have $dens_c((f\oplus 2^t)_{m;q,c}) \ge L(1 - \frac{s}{u_1})$ and $dens_c((k\cdot f)_{m;q,c}) \ge L(1 - \frac{s}{u_2})$ respectively, if in the above hypothesis u is replaced by $u_1 = u2^t$ and $u_2 = uk$ respectively.

Proof: We prove the statement just for $dens_c(f_{m;q,c})$ as the other proofs are essentially the same. From the hypothesis on R we have $m \ge (u-s)q$. Hence by part c of lemma 4 we have $dens_c(f_{m;q,c}) \ge \frac{(u-s)q}{uq} L = L(1 - \frac{s}{u})$. ∎

Lemma 5: Let G be a two dimensional grid with m chains (i.e. G=[mxk] for some k≥1), and H a hypercube "smaller" than G (i.e. $H=Q(G)/2^t$ for some $t \ge 0$). Let $r \ge 1$ and let $c = \frac{|H|}{2^r}$. Suppose that the map $f:G(m)\rightarrow H(2^r)$ has property B3 (and hence also (B2)) and $dens_c(f) \ge \frac{|G|}{|H|}$. Then $f(G)\subset H$.

Proof: Since $dens_c(f)\ge\frac{|G|}{|H|}$, f has mapped at least |G| points of G(m) into columns 1 through c of $H(2^r)$. It suffices to show that all points of G have been mapped into columns 1 through c. Suppose not. Then some point of G has been mapped to a column c+i, i ≥1. But then property B3 implies that all points mapped to columns 1 through c must be from G. It follows that G has more than |G| points, a contradiction. ∎

We can now prove our theorem on compressions of two dimensional grids into hypercubes.

Theorem 1: Let G = [mxk] be a two dimensional grid. Then for any $i \ge 1$ there is a map $F: G\rightarrow Q(G)/2^i$ having dilation 1 and load factor at most $1+ 2^i$.

Proof: Consider the ratio $R = \frac{m}{2^{\lfloor \log(m)\rfloor - i}}$. Clearly $2^i \le R < 2^{i+1}$ for any $i \ge 0$, and we let u be the integer such that $u-1 \le R \le u$, and $2^i + 1 \le u$. Let $L = 1+2^i$.

Assume first that $\lfloor \log(m)\rfloor - i \ge 0$. Set $f = f_{2L-u,u}$, so that $f:G(u)\rightarrow H(2)$ is a braiding of uniform load factor L by Lemma 2. Now apply corollary 4.1 with x = 0, b = i, s = 1, r = 1, and c =

$\frac{|Q(G)/2^i|}{2^{\lfloor \log(m) \rfloor - i + 1}}$. We then obtain the dilation 1 and load factor at most L map $f_{m;q,c}$:G(m)→H(q·2), where $q = 2^{\lfloor \log(m) \rfloor - i}$, satisfying $dens_c(f_{m;q,c}) \geq L(1 - \frac{1}{u}) \geq$ $(1+2^i)(1 - \frac{1}{1+2^i}) = 2^i = \frac{|G|}{|Q(G)/2^i|}$. By Lemma 5 we have $f_{m;q,c}(G) \subset Q(G)/2^i$, as desired.

Now assume $\lfloor \log(m) \rfloor - i < 0$. Then form the map f:G(m)→H(2), where $f = f_{2P-m,m}$ and $P = 2^i$. By Lemma 2, f has load factor 2^i. Hence by Lemma 5 we have $f(G) \subset Q(G)/2^i$. ∎

III. Compressions of a 3D Grid M into Q(M)/2

In this section we take up the subject of mapping 3D grids M into their "half" size hypercubes Q(M)/2. As a first step it will be necessary to analyze further embeddings of 2D grids.

We now introduce a set of braidings of type $(m2^s +1)/ 2^s / L$, with $L \geq m$, which are natural generalizations of the $f_{d,r}$ braidings. Define the map <u>cyclic $m2^s+1/ 2^s / L$</u>, abbreviated here by f, as follows:

(1) The intial points of chains are mapped by f(i,1) = ((i-1)',1) where x' is the residue class of x mod 2^s.

(2) Write S_t for the set of chains switching in column t. Then $S_1 = \{i: 1 \leq i \leq (L-m)2^s -1\}$, and for t ≥1 we have $S_{t+1} = \{i+(L-m)2^s -1: i \in S_t\}$. (Thus f is consecutive on switches.)

(3) Suppose chain i switches in column t, and let (i,b) and (i,b+1) be successive points of chain i mapped to column t. Then for some p, $0 \leq p \leq 2^s -1$, we have f(i,b) = (p,t) and f(i,b+1) = (p+1,t) where p and p+1 are read mod 2^s.

(4) If (i,b) is the last point of chain i mapped to column t, say with f(i,b) = (p,t), then f(i,b+1) = (p,t+1).

An illustration of the cyclic 9/4/3 map is given in Figure 8. An induction along the lines of Lemma 1 shows that the maps cyclic $m2^s+1/ 2^s / L$ are braidings of uniform load factor L.

[Put Figure 8 here.]

We now begin on a series of theorems giving the existence of maps G→Q(G)/4, where G is a 2D grid, having dilation 1 and various load factors. These maps will ultimately be used in constructing dilation 2 maps M→Q(M)/2, with M a 3D grid.

The main point of the following theorem is that for G a 2D grid, a dilation 1 map G→Q(G)/4 of load factor 5 (the existence of which is already guaranteed by Theorem 1) can be pasted together using certain special braidings as building blocks. The significance of these braidings will be explained later in the context of constructing compressions of 3D grids.

<u>Theorem 2</u>: Let G = [mxk] be a 2D grid with dens(G, Q(G)/4) ≥ $\frac{10}{3}$. Then one can construct a dilation 1 map g:G→Q(G)/4 of load factor at most 5, where g is of the form $f_{m;q,c}$, and f is one of the braidings $f_{2,8}$, $f_{4,6}$, or $f_{5,5}$.

Proof:Let $R = R(m) = \frac{m}{2^{\lfloor \log(m) \rfloor -1}}$, so that $2 \le R \le 4$. We partition the interval [2,4] into the subintervals $(\frac{7}{2},4]$, $(3,\frac{7}{2}]$, $(\frac{5}{2},3]$, and $(2,\frac{5}{2}]$. The proof proceeds by cases, based on which of the four intervals contains R.

We begin by proving that not both R(m) and R(k) can belong to the interval $I = [3, \frac{7}{2}]$. Suppose not. Then $m \ge 3 \cdot 2^{\lfloor \log(m) \rfloor -1}$, from which it follows that

$$\log(m) \ge \log(3) + \lfloor \log(m) \rfloor - 1 = \frac{\ln(3)}{\ln(2)} + \lfloor \log(m) \rfloor - 1 > .58 + \lfloor \log(m) \rfloor.$$

The same inequalities hold for k in place of m by assumption, so it follows that

$$\lfloor \log(m) + \log(k) \rfloor = \lfloor \log(m) \rfloor + \lfloor \log(k) \rfloor + 1.$$

We can now apply this fact to get

$$\mathrm{dens}(G, Q(G)/4) = \frac{mk}{2^{\lfloor \log(mk) \rfloor -1}} = \frac{mk}{2^{\lfloor \log(m) \rfloor + \lfloor \log(k) \rfloor}} = \left(\frac{m}{2^{\lfloor \log(m) \rfloor}}\right)\left(\frac{k}{2^{\lfloor \log(k) \rfloor}}\right) \le \left(\frac{7}{4}\right)^2 < \frac{10}{3},$$

a contradiction to the density assumption of the theorem.

We can now eliminate the possibility $R(m) \in I$ by observing that in that case $R(k) \notin I$, and hence we can let R(k) play the role of R = R(m) in one of the cases below.

We treat each of the three remaining cases as follows. Suppose we are in a case defined by $\frac{u-1}{2} \le R \le \frac{u}{2}$ and that a braiding $f:G(u) \to H(2)$ has been given which has load factor 5 and is consecutive on switches. We then apply Corollary 4.1 with x=1, b=1, s=1, and r=1, so that $q = \frac{m}{2^{\lfloor \log(m) \rfloor -1}}$. The result is that for those m satisfying the inequality hypotheses of Corollary 4.1, i.e. $\lfloor \log(m) \rfloor \ge b+x = 2$ and hence $m \ge (u-1)q \ge u-1$, we get for any $c \ge 1$ a map $f_{m;q,c}:G(m) \to H(2^{\lfloor \log(m) \rfloor -1})$ satisfying $\mathrm{dens}_c(f_{m;q,c}) \ge 5(1 - \frac{1}{u})$. If also $u \ge 5$, then we get $\mathrm{dens}_c(f_{m;q,c}) \ge 4 \ge \frac{|G|}{|Q(G)/4|}$. Hence by Lemma 5 (with $f_{m;q,c}$, $\lfloor \log(m) \rfloor - 1$, and Q(G)/4 playing the roles of f, r, and H respectively) we get $f_{m;q,c}(G) \subset Q(G)/4$.

To summarize, in each case defined by $\frac{u-1}{2} \le R \le \frac{u}{2}$, it suffices to produce an appropriate braiding $f:G(u) \to H(2)$ of load factor 5 with $u \ge 5$. The reasoning above then yields the required map $g:G \to Q(G)/4$ for those grids G=[mxk] satisfying $\lfloor \log(m) \rfloor \ge 2$ and $m \ge (u-1)q$. The specification of f in each case is done in tabular form below.

The theorem is thereby proved in each case $\frac{u-1}{2} \le R \le \frac{u}{2}$ except for the at most finitely many grids G=[mxk] for which m is covered by the case but fails to satisfy the inequalities $\lfloor \log(m) \rfloor \ge 2$ and $m \ge (u-1)q$; for example m=3 in the case $\frac{5}{2} \le R \le 3$. For these m one can verify that the theorem holds (for the finitely many grids [mxk] involved) occasionaly using braidings different than the ones specified in the theorem. In the later applications of this theorem to constructing compressions of 3D grids, the dependence on the braidings specified in Theorem 2 is eliminated for the finitely many exceptional cases (arising from the parameters m, k and p in [mxkxp] being too small) by using ad hoc constructions for these cases. These constructions, as well as the current ones showing that Theorem 2 holds for "small" m and k are omitted for brevity.

case $\frac{u-1}{2} < R \le \frac{u}{2}$	u	braiding f:G(u)→H(2) of load factor 5
$\frac{7}{2} < R \le \frac{8}{2}$	8	$f_{2,8}$
$\frac{6}{2} < R \le \frac{7}{2}$	This case is eliminated above.	
$\frac{5}{2} < R \le \frac{6}{2}$	6	$f_{4,6}$
$\frac{4}{2} < R \le \frac{5}{2}$	5	$f_{5,5}$

∎

While Theorem 1 applies uniformly to all 2D grids, one might ask whether for some grids a dilation 1 embedding (into the appropriate hypercube) is possible with a smaller load factor. We will see below that this can indeed be done, and that the smaller load factor makes possible our later results on embeddings of 3D grids. We will confine these reduced load factor results to maps G→ Q(G)/b where b =4, or 8, at least partly because the latter maps are sufficient for obtaining our results on the embeddings of 3D grids. The main idea behind reducing the load factor in the Theorem 1 is to take advantage of the density of the 2D grid. In fact we will show that as dens(G, Q(G)/b) decreases, one can obtain a progressively smaller load factor map G→Q(G)/b.

The first of these results follows, giving a map G→Q(G)/4 of load factor 4 under a density assumption, and giving our first improvement on the load factor 5 map from Theorem 1.

Theorem 3: Let a 2D grid G=[mxk] satisfy dens(G, Q(G)/4) $\le \frac{10}{3}$. Then there is an embedding g:G→Q(G)/4 having dilation 1 and load factor at most 4.

Proof: Let $R = \frac{m}{2^{\lfloor \log(m) \rfloor -1}}$, so that $2 < R \le 4$. We partition the interval [2,4] into the subintervals $(\frac{7}{2},4]$, $(3,\frac{7}{2}]$, $(\frac{5}{2},3]$, $(\frac{9}{4},\frac{5}{2}]$, and $(2,\frac{9}{4}]$. The proof proceeds by cases, based on which of the five intervals contains R.

The five cases are treated as were the cases in the previous theorem, only with some variation in parameter values from case to case. Each case is defined by a condition of the form $\frac{u-1}{2^x} < R \le \frac{u}{2^x}$, where now (in the language of corollary 4.1) x and s may change with the case, but r=1 and b=1 in all cases, and thus $q=2^{\lfloor \log(m) \rfloor -1-x}$ for each case. We give a braiding

$f:G(u) \to H(2)$ of load factor 4 for the given case. An application of Corollary 4.1 then gives a map $f_{m;q,c}:G(m) \to H(2q)$ of load factor at most 4 satisfying $dens_c(f_{m;q,c}) \geq 4(1 - \frac{s}{u})$. The map f and the values x,s,u,r and b necessary for the application of the corollary in each case are summarized in the table below, together with the resulting lower bound $4(1 - \frac{s}{u})$ for the density of the map $f_{m;q,c}$. In this table and similar ones which follow, we use for brevity the single parameter u to head the second column even though in the language of the corollary we are actually using u, u_1 (for a braiding of the form $f \oplus 2^t$), or u_2 (for a braiding of the form $k \cdot f$). We have $4(1 - \frac{s}{u}) \geq \frac{10}{3} \geq \frac{|G|}{|Q(G)/4|}$ in each case, so by Lemma 5 we get $f_{m;q,c}(G) \subset Q(G)/4$, proving the theorem.

Again we implicitly assume that the conditions $\lfloor \log(m) \rfloor \geq b+x = 1+x$ and $m \geq (u-1)q$ necessary for the application of Corollary 4.1 hold in each case. When m and k are too small for this to hold in a given case, one can verify that the theorem holds for the finitely many cases not covered by using other braidings f, which we omit here. In future applications of Corollary 4.1, we will make this implicit assumption without again mentioning it, having verified in the finitely many cases not covered by this assumption that the theorem in question holds through the use of different braidings than the ones given under this assumption in the the text.

case defined by $\frac{u-s}{2^x} < R \leq \frac{u}{2^x}$	u	x	s	r	b	braiding $f:G(u) \to H(2)$ of load factor 4	$4(1 - \frac{s}{u})$ (Lower bound for $dens_c(f_{m;q,c})$ for all $c \geq 1$)
$\frac{7}{2} < R \leq \frac{8}{2}$	8	1	1	1	1	$f_{0,8}$	$\frac{7}{2}$
$\frac{6}{2} < R \leq \frac{7}{2}$	7	1	1	1	1	$f_{1,7}$	$\frac{24}{7}$
$\frac{5}{2} < R \leq \frac{6}{2}$	6	1	1	1	1	$f_{2,6}$	$\frac{10}{3}$
$\frac{9}{4} < R \leq \frac{10}{4}$	10	2	1	1	1	2·(cyclic 5/4/2)	$\frac{18}{5}$
$\frac{16}{8} < R \leq \frac{18}{8}$	18	3	2	1	1	2·(cyclic 9/8/2)	$\frac{32}{9}$

∎

The next theorem shows that with a yet stronger density assumption than that of Theorem 3, one can obtain obtain maps $g:G \to Q(G)/4$ of smaller load factor. The price one pays for this improvement is a lengthier proof, consisting of a greater number of cases and using more involved "nonstandard" braidings (i.e. ones which are neither $f_{d,r}$ nor cyclic r/s/t) in some cases.

Theorem 4: Let a 2D grid G=[mxk] satisfy $dens(G, Q(G)/4) \leq \frac{8}{3}$. Then there is an embedding $g:G \to Q(G)/4$ having dilation 1 and load factor at most 3.

Proof: In outline the proof is exactly the same as that of theorem 3. We will again have a number of cases defined by the interval in which the number $R = \frac{m}{2^{\lfloor \log(m) \rfloor -1}}$ can be found. As above, in each case we give the parameters x,s,u, r, with b=1 in all cases, and a braiding $f:G(u) \rightarrow H(2^r)$ of uniform load factor 3. We can then form the map $f_{m;q,c}$, and the last column lists the lower bound $3(1 - \frac{s}{u})$ for $dens_c(f_{m;q,c})$ given by Corollary 4.1. In each case this lower bound is at least $\frac{8}{3} \geq$ dens(G, Q(G)/4), so that by Lemma 5 we have $f_{m;q,c}(G) \subset Q(G)/4$. Thus $f_{m;q,c}$ can be taken to be the map g required by the theorem.

case defined by $\frac{u-s}{2^x} < R \leq \frac{u}{2^x}$	u	x	s	r	b	braiding $f:G(u) \rightarrow H(2^r)$ of load factor 3	$3(1 - \frac{s}{u})$ (Lower bound for $dens_c(f_{m;q,c})$ for all c≥1)
$\frac{15}{4} < R \leq \frac{16}{4}$	16	2	1	3	1	$f_{2,4} \oplus 4$	$\frac{45}{16}$
$\frac{14}{4} < R \leq \frac{15}{4}$	15	2	1	3	1	15/8/3 (Appendix 1)	$\frac{14}{5}$
$\frac{13}{4} < R \leq \frac{14}{4}$	14	2	1	3	1	(7/4/3)⊕2 (Appendix 2)	$\frac{39}{14}$
$\frac{12}{4} < R \leq \frac{13}{4}$	13	2	1	3	1	13/8/3 (Appendix 3)	$\frac{36}{13}$
$\frac{11}{4} < R \leq \frac{12}{4}$	12	2	1	2	1	$f_{0,6} \oplus 2$	$\frac{11}{4}$
$\frac{10}{4} < R \leq \frac{11}{4}$	11	2	1	3	1	11/8/3 (Appendix 4)	$\frac{30}{11}$
$\frac{9}{4} < R \leq \frac{10}{4}$	10	2	1	2	1	$f_{1,5} \oplus 2$	$\frac{27}{10}$
$\frac{8}{4} < R \leq \frac{9}{4}$	9	2	1	2	1	cyclic 9/4/3	$\frac{8}{3}$

∎

We will use these results to obtain embeddings of 3-D grids into their half size hypercubes with dilation 2 and (optimum) load factor 2. Before describing the method precisely, let us first give an informal outline of the approach. Let M = [mxkxp] be a 3D grid. Let $Q = Q(\lfloor \log(mk) \rfloor - s - 1)$ for some s≥1, and consider the infinite graph $Q \times H(2)$ (where "×" refers to the usual Cartesian product of graphs). Then Q(M)/2 is spanned by the subgraph $Q \times (H(2)_{[s]})$ of $Q \times H(2)$, where $H(2)_{[s]}$ is the subgraph of H(2) induced by vertices lying in the first 2^s columns of H(2). If we viewed Q as a 2D object (say as the first 2^d columns of $H(2^r)$ for appropriate r and d) then Q(M)/2 can be viewed as a 3D object having Q as a hypercube "face". We can then proceed in two phases.

Phase 1: Construct a map $f:[mxk]\rightarrow[2^r x 2^s]$ with dilation 1 of a 2D face [mxk] of M onto a hypercube face Q of Q(M)/2. We observe that f induces a natural partition $\cup_{h\in Q}D_h$, indexed by Q, of the mk chains of length p in M having their "first" point in the [mxk] face, i.e., a set D_h consists of precisely those chains whose first points are mapped by f to the same point h of Q.

Phase 2: Construct a braiding g of the the chains in each set D_h along the copy of H(2) paired with h in $Q\times H(2)$.

The combination of these phases in sequence gives our desired map of a 3D grid M into Q(M)/2. We will now make these constructions more precise, with an eye to facilitating the proof that the map has the desired load factor and dilation.

For each $(x,y)\in[mxk]$, we define the subset C(x,y) of M (which we can regard as a chain) by $C(x,y) = \{(x,y,z)\text{: } 1\le z\le p\}$. Now for any $h\in Q$ let $D_h = \{C(x,y)\text{: } f(x,y) = h\}$. Notice that since f has dilation 1, adjacent chains C(x,y) and C(x',y') (i.e. chains for which $|x-x'| + |y-y'| = 1$) belong to sets D_h and $D_{h'}$ respectively such that either h = h', or h and h' are adjacent in Q.

Let $q:G(m)\rightarrow H(2^r)$ be a dilation 1 map of some finite load factor. Suppose that for each $h\in H(2^r)$ there is a one to one map $N_h:\{q^{-1}(h)\}\rightarrow\{1,2,...,|q^{-1}(h)|\}$. We then call the collection fo maps $\{N_h:h\in H(2^r)\}$ a numbering of q, and we denote the numbering by N. Consider the following property.

Let $p_1 = (x_1,y_1)$ and $p_2 = (x_2,y_2)$ be points of G(m) satisfying $dist(p_1,p_2) = 1$.
Then $|N_{q(p_1)}(p_1) - N_{q(p_2)}(p_2)| \le 1$.

Any numbering N of q having this property will be called a consistent numbering for q. We will typically abbreviate the notation $N_h(p)$ by writing simply N(p), the identity of h being either irrelevant or fixed by context.

Given a map $f:[mxk]\rightarrow Q$, where $Q = Q(\lfloor\log(mk)\rfloor-s-1)$ for some $s\ge1$, and a map $g:G(Y)\rightarrow H(2)$, where Y is the load factor of f, we can then define the map

$$f\otimes_N g\text{: } M \rightarrow Q\times H(2)$$

by defining it on chains C(x,y) as follows. For any chain C(x,y) of M let

$$(f\otimes_N g)(C(x,y)) = (f(x,y), g(C_{N(x,y)})).$$

Thus we see that $f\otimes_N g$ can be viewed as mapping the chain C(x,y) along the third dimension (the one orthogonal to Q) along a pair of rods (the copy of H(2) paired with f(x,y) in $Q\times H(2)$) in the same manner that g would braid the chain $C_{N(x,y)}$ of G(Y). The map $f\otimes_N g$ is illustrated in terms of its two phases in Figure 9, and some of its basic properties are summarized in the lemma below.

[Put Figure 9 here.]

Lemma 6: Let M = [mxkxp] be a 3D grid, and Q = Q([mxk])/2^s. Suppose f:[mxk] $\rightarrow$ Q is a map of dilation 1 and load factor at most Y, and g:G(Y) $\rightarrow$ H(2) is a braiding of the form g = $f_{d,r}$ (for some d and r) having load factor L. Assume dens([mxkl,Q([mxk])/2^s)) $\geq \frac{Y}{L} 2^u$. Then given any numbering N for f, the map $f \otimes_N g$: M $\rightarrow$ Q$\times$H(2) has the following properties.

(1) $f \otimes_N g$ has load factor L.

(2) dilation($f \otimes_N g$) $\leq$ 3; and if N is a consistent numbering for f, then dilation($f \otimes_N g$) $\leq$ 2.

(3) ($f \otimes_N g$)(M) $\subset$ Q(M)/2^u.

Proof: Observe that when h and h' are distinct points of Q, we have ($f \otimes_N g$)(D_h)$\cap$($f \otimes_N g$)($D_{h'}$) = ϕ. Hence, load-factor($f \otimes_N g$) = $\max_h${load factor of the restriction of $f \otimes_N g$ to D_h}

= load-factor(g)

= L.

For the dilation statement, let p = (x,y,z) and p' = (x',y',z') be adjacent points of M, i.e. points satisfying |x-x'|+|y-y'|+|z-z'| = 1. If (x,y) = (x',y'), then p and p' are successive points of chain C(x,y), so since dilation(g) = 1 we have dist(($f \otimes_N g$)(p),($f \otimes_N g$)(p')) = 1.

Suppose then that |x-x'|+|y-y'| = 1 so that z = z'. Then since dilation(f) = 1, we have either f(x,y) = f(x',y') = h for some hεQ, or f(x,y) = h and f(x,y) = h' for a pair of adjacent points h and h' in Q.

Case 1: f(x,y) = f(x',y') = h for some hεQ:

Applying lemma 1(b) with i = N(x,y), j = N(x',y'), and t = z, we find that dist(($f \otimes_N g$)(p),($f \otimes_N g$)(p')) = $dist_{G(Y)}$(g(N(x,y),z), g(N(x',y'),z)) $\leq$ 2, as desired.

Case 2: f(x,y) = h and f(x',y') = h' for a pair of adjacent points h and h' in Q:

Let N_1 = N(x,y) and N_2 = N(x',y') and assume wlog that $N_1 \leq N_2$. Let (x",y")ε[mxk] be such that C(x",y")$\varepsilon D_{h'}$ and N(x",y") = N_1. Then since C(x,y) and C(x",y") both behave as chain C_{N_1} (of G(Y)) under the braiding g, we see that g(N(x,y),z)) = g(N(x",y"),z)) for any z. Thus dist(($f \otimes_N g$)(x,y,z), ($f \otimes_N g$)(x",y",z)) = $dist_Q$(f(x,y),f(x",y")) = $dist_Q$(h,h') = 1. But since f(x',y') = f(x",y"), we can apply case 1 to get

$$dist((f \otimes_N g)(x",y",z), (f \otimes_N g)(x',y',z)) \leq 2.$$

Hence we have

$$dist((f \otimes_N g)(x,y,z), (f \otimes_N g)(x',y',z)) \leq 3,$$

completing the proof of the first dilation bound.

Now suppose that f has a consistent numbering, so that continuing with the above notation we have $|N_1-N_2| \leq 1$. Then since g is a dilation 1 map we know that $dist_{G(Y)}$(g(N_1,z), g(N_2,z)) = 1. Hence the next to last inequality in the previous paragraph (obtained by reducing to case 1) becomes

$$dist((f \otimes_N g)(x",y",z), (f \otimes_N g)(x',y',z)) \leq 1.$$

The triangle inequality for graph distance now gives

$$\text{dist}((f\otimes_N g)(x,y,z), (f\otimes_N g)(x',y',z)) \le 2.$$

We now consider (3). Recall that $f\otimes_N g$ maps each chain C(x,y) along the pair of rods (f(x,y),H(2)) in $Q\times H(2)$ in the same way as g braids $C_{N(x,y)}$, and observe that there is an obvious bijection between the set {C(x,y): (x,y)ε[mxk]} and the chains of G(mk). Since the total number of rods into which the chains are mapped is 2|Q|, we can view $f\otimes_N g$ as the restriction to the first p columns of G(mk) of a map F:G(mk)→H(2|Q|) defined in the obvious way. This F has properties B2 and B3 in the definition of a braiding, essentially by inheriting them from g. Let us then identify the vertices of M with the vertices in the subgraph H of G(mk) induced by the first p columns of G(mk). It suffices to show that $F(H)\subset Q(H)/2^u$.

As a convenience we may index the sets D_h, hεQ, by $D_0, D_1, .., D_{|Q|-1}$ in such a way that the chains in D_i are mapped by F (using the braiding g) along rods 2i and 2i+1 of H(2|Q|). Let $|D_i| = Y - d_i$, with $0 \le d_i \le Y$, so that in phase 2 we may regard the restriction g_i of g to D_i as a braiding g_i:G(Y - d_i)→H(2) (with rods 2i and 2i+1 playing the role of H(2)). For any $t \ge 1$, let P_t be the set of vertices of H(2|Q|) lying in the first t columns of H(2|Q|), and let $p_{i,t} = |g_i^{-1}(P_t)|$. Finally, set A=dens([mxkl,Q([mxk])/2^s)).

We now prove that for any t, $t \ge 1$, we have

$$|F^{-1}(P_t)| \ge 2^u|P_t|. \qquad (*)$$

By Corollary 4.1 we have $p_{i,t} \ge 2tL\left(1 - \frac{d_i}{Y}\right)$. Hence we have

$$|F^{-1}(P_t)| = \sum_{i=0}^{|Q|-1} p_{i,t} \ge 2tL \sum_{i=0}^{|Q|-1} \left(1 - \frac{d_i}{Y}\right).$$

Now since $|P_t| = 2|Q|t$, we are reduced to showng that

$$L \sum_{i=0}^{|Q|-1} \left(1 - \frac{d_i}{Y}\right) \ge 2^u|Q|.$$

Using the fact that $A = \frac{1}{|Q|}\left(\sum_{i=0}^{|Q|-1} Y\text{-}d_i\right)$, the last inequality is equivalent to our assumption $A \ge \frac{Y}{L}2^u$. Thus (*) is proved.

To complete the argument, we note that (*) implies dens(F) $\ge 2^u \ge$ dens(H,Q(H)/2^u). It follows by Lemma 5 that $F(H)\subset Q(H)/2^u$. ∎

We will need to know some braidings for which consistent numberings exist, and also the fact that consistent numberings respect the ways of combining braidings we have previously introduced. This will be the subject of the next two lemmas.

Lemma 7: The following braidings have consistent numberings; $f_{0,m}$, $f_{1,m}$ for m odd, and $f_{2,m}$ for m even.

Proof: Let $f{:}G(m)\to H(2)$ be a braiding, with $f = f_{0,m}$, $f_{1,m}$, or $f_{2,m}$. For any $h\varepsilon H(2)$, let $f^{-1}(h) = \{(x_1,y_1), (x_2,y_2), ..., (x_k,y_k)\}$, where the indexing is such that $x_1 \le x_2 \le ... \le x_k$. The intuition behind our construction of the desired numbering N is that N orders the elements of $f^{-1}(h)$ by chain number in G(m), that is, $N((x_i,y_i)) = i$.

For the case $f = f_{0,m}$ we have the simple formula $f_{0,m}(x,y) = (p(x),y)$ where $p(x)=0$ if x is odd and $p(x)=1$ if x is even. Hence for any $h=(p,t)\varepsilon H(2)$, p=0 or 1, and $t \ge 1$, we have $f^{-1}(h) = \{(x,t)\text{: x odd, } 1\le x\le m\}$ if p=0, and $f^{-1}(h) = \{(x,t)\text{: x even, } 1\le x\le m\}$ if p=1. It follows directly from this that the ordering by chain number is a consistent numbering of f.

To treat the remaining two cases, we make the following notation. Since we are concerned only with maps $f_{1,m}$ or $f_{2,m}$, we may suppose that each chain of G(m) switches at most once in a column. Let us then write $i^{(1,t)}$ (resp. $i^{(2,t)}$) for the first (resp. second, if it exists) point of chain i mapped to column t of H(2) under the map f, where as before the first point of the chain is the one with smaller second coordinate (i.e. column number) in G(m). It now suffices to define $N(i^{(s,t)})$, s=0 or 1, just for $t \le p$ where p is the period of the map f, with the understanding that $N(i^{(s,t+kp)}) = N(i^{(s,t)})$ for $1\le t \le p$, and k = 1,2,... .

First for $f = f_{1,r}$, we define the numbering N as follows.

a) For i even, let

$$N(i^{(1,t)}) = \frac{i}{2} + 1 \quad \text{for } t \le i$$
$$= \frac{i}{2} \quad \text{for } t \ge i+1$$
$$N(i^{(2,t)}) = \frac{i}{2} \quad \text{(noting that chain i of G(m) switches at column i of H(2)).}$$

b) For i odd, let

$$N(i^{(1,t)}) = \left\lceil \frac{i}{2} \right\rceil \text{ for all t, and}$$
$$N(i^{(2,t)}) = \left\lceil \frac{i}{2} \right\rceil \text{ for all t.}$$

To verify that the numbering N is consistent, let ~ be the relation on integers for which $x\sim y$ iff $|x-y| \le 1$. Then we need to verify the following relations for all i, $1 \le i \le m$.

(A) $N(i^{(1,t)})\sim N(i+1^{(1,t)})$ for $t \ne i+1$
(B) $N(i^{(2,i)})\sim N(i+1^{(1,i+1)})$
(C) $N(i^{(2,i)})\sim N(i-1^{(1,i)})$
(D) $N(i^{(1,t)})\sim N(i^{(1,t+1)})$ for $t \ne i$
(E) $N(i^{(1,i)})\sim N(i^{(2,i)})$
(F) $N(i^{(2,i)})\sim N(i+1^{(1,i)})$.

The relations (A)-(C) ensure that pairs of corresponding points on successive chains are numbered with a difference of at most 1, these relations describing the possible ways in which such pairs could be mapped by f (e.g. in (B) the pair consists of {(i,d),(i+1,d)} for some d, where (i,d) happens to be the second point of chain i in column i, while (i+1,d) is the first point of chain i+1 in column i+1). Similarly the realtions (D)-(F) ensure this for pairs of successive points on the same chain. The straightforward verification that the numbering N defined above satisfies these relations is omitted.

Consider next the map $f = f_{2,m}$. The period of this map is m/2 and for any i, $1 \le i \le m$, chain i switches once (in columns 1,2,...,m/2) at column $\lceil \frac{i}{2} \rceil$. The numbering N is then defined as follows.

a) For i even, let

$N(i^{(1,t)}) = \frac{i}{2} + 1$ for all t , and

$N(i^{(2,i/2)}) = \frac{i}{2} + 1$.

b) For i odd, let

$N(i^{(1,t)}) = \lceil \frac{i}{2} \rceil + 1$ for $t < \lceil \frac{i}{2} \rceil$,

$N(i^{(1,t)}) = \lceil \frac{i}{2} \rceil$ for $t \ge \lceil \frac{i}{2} \rceil$, and

$N(i^{(2,\lceil i/2 \rceil)}) = \lceil \frac{i}{2} \rceil$.

To check that N is consistent, we must verify that certain relations analogous to (A) through (F) above hold. The four relations which follow check that corresponding points on successive chains of G(m) receive numbers that differ by at most one. These are;

(1) $N(i^{(1,t)}) \sim N(i+1^{(1,t)})$, $t \ne \lceil \frac{i}{2} \rceil, \lceil \frac{i+1}{2} \rceil$ (neither chain i nor chain i+1 switches at column t),

(2) $N(i^{(2,i/2)}) \sim N(i+1^{(1,i/2+1)})$, i even (chain i switches at column i/2 and chain i+1 does not)

(3) $N(i^{(1,\lceil (i+1)/2 \rceil)}) \sim N(i+1^{(2,\lceil (i+1)/2 \rceil)})$, i even (chain i does not switch at column $\lceil (i+1)/2 \rceil$ and chain i+1 does)

(4) $N(i^{(2,\lceil i/2 \rceil)}) \sim N(i+1^{(2,\lceil i/2 \rceil)})$, i odd (both chains i and i+1 switch at column $\lceil i/2 \rceil$).

Then there are the three relations (D) through (F) above which again check that successive points on the same chain receive numbers which differ by at most one. Again we omit the obvious calculations. ∎

Lemma 8: If a braiding f has a consistent numbering, then so do the braidings $k \cdot f$, $f \oplus 2^t$, and $f_{m;q,c}$. We omit the obvious proof.

Armed with the above results we can now compress 3D grids into their half size hypercubes with nearly optimal load factor and dilation.

Theorem 5: For any 3D grid M = [mxkxp], there is a map $M \to Q(M)/2$ of dilation ≤ 2 and load factor 3.

Proof: Let G = [mxk] be a 2D face of M, and let Q = Q([mxk])/4. By lemma 6 it suffices to produce a braiding $f: G \to Q$ of some load factor Y, a braiding $g: G(Y) \to H(2)$ of load factor 3, and a consistent numbering N for f, such that $\text{dens}(G,Q) \ge \frac{2Y}{3}$. For then by that lemma (applied with s = 2 and u = 1) the map $f \otimes_N g: M \to Q \times H(2)$ would be the desired map.

Let D = dens(G,Q(G)/4). Since $2 \le D \le 4$, we know that D must lie in one of the three intervals $[2, \frac{8}{3}]$, $[\frac{8}{3}, \frac{10}{3}]$, and $[\frac{10}{3}, 4]$. We argue by cases, based on which interval contains D. In each case, the existence of a braiding f of load factor Y, $3 \le Y \le 5$, is guaranteed by one of our previous theorems, where of course f depends on the case. The braiding g (also appropriate to the case) will then be specified. So in the table below for each case we will identify Y (together with a reference to the appropriate theorem guaranteeing f), the braiding g, and verify that the left endpoint E of the interval defining the case satisfies $E \ge \frac{2Y}{3}$ (in fact, with equality).

case $\alpha \le D \le \beta$	Y (= load factor of f:G→Q/4)	braiding g:G(Y)→H(2) of load factor 3	$\frac{2Y}{3}$
$\frac{10}{3} \le D \le 4$	5 (Theorem 2)	$f_{1,5}$	$\frac{10}{3}$
$\frac{8}{3} \le D \le \frac{10}{3}$	4 (Theorem 3)	$f_{2,4}$	$\frac{8}{3}$
$2 \le D \le \frac{8}{3}$	3 (Theorem 4)	$f_{0,6}$	2

Next we check that for every map f used (from the referenced theorems) there is a numbering N for which the corresponding g in the third column is such that dilation($f \otimes_N g$) ≤ 2, proving the dilation 2 statment. For those f coming from Theorem 2 or 3, this will be done by showing that they have consistent numberings. Since in these theorems f is of the form $h_{m;q,c}$, $h \oplus 2^t$, or k·h for some braiding h, it suffices by lemma 8 to check that there is a consistent numbering for each of the braidings h listed listed in the tables for these two theorems. As for those f coming from Theorem 4, we will see that dilation($f \otimes_N g$) ≤ 2 regardless of the numbering N used. We now check all this, considering each theorem in turn.

(1) Theorem 2 - A consistent numbering for $f_{2,8}$ exists by lemma 7, and for $f_{4,6}$ and $f_{5,5}$ by inspection, letting N be the ordering by chain number (defined in the first paragraph of the proof of lemma 7).

(2) Theorem 3 - A consistent numbering for $f_{1,7}$, $f_{2,6}$, and $f_{0,8}$ exists by lemma 7, and for 2·(cyclic 5/4/2) and 2·(cyclic 9/8/2) using lemma 8 and the fact that any numbering is consistent for a load factor 2 map, in particular for cyclic 5/4/2 and cyclic 9/8/2.

(3) Theorem 4 - Any numbering N for any of the braidings listed will suffice for dilation 2, based on the fact that for such f we use the map $g = f_{0,6}$ in forming $f \otimes_N g$, and this g makes every chain walk at every column. Thus take points $p_1=(x_1,y_1, z_1)$ and $p_2=(x_2,y_2, z_2)$ of M satisfying $|x_1-x_2| + |y_1-y_2| = 1$ (so $z_1=z_2$). Since under g every chain walks at every column, corresponding points on chains $C(x_1,y_1)$ and $C(x_2,y_2)$ will have the same column number in H(2) during phase 2. Hence the distance between $g(N(x_1,y_1),z_1)$ and $g(N(x_2,y_2),z_2)$ is either 0 or 1 depending on the relative

parities of their rod numbers in H(2), so since $\mathrm{dist}(f(x_1,y_1),f(x_2,y_2)) = 1$, it follows that $\mathrm{dist}((f\otimes_N g)(p_1),(f\otimes_N g)(p_2)) \leq 2$. ∎

IV. Compressions of a 3D Grid M into Q(M)/4

We will now extend the above results by obtaining compressions of 3D grids into their quarter size hypercubes following an approach similar to the one used above. We retain the two phase procedure analogous to the one just employed.

(1) For any 2D grid G we construct a dilation 1 map $f:G\rightarrow Q(M)/8$ of progressively smaller load factor as dens(G,Q(G)/8) decreases.

(2) Now let M be a 3D grid, and G some 2D face of M. We apply Lemma 6, using f from phase 1, a braiding g that depends on dens(G,Q(G)/8), and any numbering N, to obtain a map $f\otimes_N g:M\rightarrow Q\times H(2)$ with appropriate dilation and load factor for which $(f\otimes_N g)(M) \subset Q(M)/4$.

Let us start on this plan by observing that by Theorem 1 we already know that for any 2D grid G there is a braiding $G\rightarrow Q(G)/8$ of load factor ≤ 9. By analogy with Theorems 3 and 4, we must now find dilation 1 maps of smaller load factor (8 down to 5) as dens(G,Q(G)/8) decreases. Our first and easiest such result treats load factor 8, and its proof requires only the $f_{d,r}$ braidings and their generalizations, the cyclic r/s/t braidings.

Theorem 6: Let G = [mxk] be a 2D grid satsifying dens(G,Q(G)/8) ≤ 7.2. Then there is a braidng g: $G\rightarrow Q(G)/8$ with load factor at most 8.

Proof: Let $R = \frac{m}{2^{\lfloor \log(m) \rfloor -2}}$, so that $4 < R \leq 8$. The proof is along the same lines as that given for Theorems 3 and 4, being based on treating cases defined by which interval contains R. For each case we give the values of u, x, s, r, with b=2 in all cases, and braiding $f:G(u)\rightarrow H(2^r)$ of load factor 8 necessary for applying Corollary 4.1. Again the last column verifies that $\mathrm{dens}_c(f_{m;q,c}) \geq 7.2 =$ dens(G,Q(G)/8) for all c≥1, and hence by Lemma 5 shows that $f_{m;q,c}(G) \subset Q(G)/8$ and thus provides $f_{m;q,c}$ as the map g required by the Theorem.

case defined by $\frac{u-s}{2^x} < R \leq \frac{u}{2^x}$	u	x	s	r	b	braiding $f:G(u)\rightarrow H(2^r)$ of load factor 8	$8(1-\frac{s}{u})$ (Lower bound for $\mathrm{dens}_c(f_{m;q,c})$ for all c≥1)
$\frac{15}{2} < R \leq \frac{16}{2}$	16	1	1	1	2	$f_{0,16}$	$\frac{15}{2}$
$\frac{14}{2} < R \leq \frac{15}{2}$	15	1	1	1	2	$f_{1,15}$	$\frac{112}{15} > 7.4$
$\frac{13}{2} < R \leq \frac{14}{2}$	14	1	1	1	2	$2\cdot f_{1,7}$	$\frac{52}{7} > 7.42$

$\frac{12}{2} < R \le \frac{13}{2}$	13	1	1	1	2	$f_{3,13}$	$\frac{96}{13} > 7.3$
$\frac{11}{2} < R \le \frac{12}{2}$	12	1	1	1	2	$2 \cdot f_{2,6}$	$\frac{22}{3} > 7.3$
$\frac{10}{2} < R \le \frac{11}{2}$	11	1	1	1	2	$f_{5,11}$	$\frac{80}{11} > 7.25$
$\frac{9}{2} < R \le \frac{10}{2}$	10	1	1	1	2	$2 \cdot f_{3,5}$	7.2
$\frac{17}{4} \le R \le \frac{18}{4}$	18	2	1	2	2	2·(cyclic 9/4/4)	$\frac{68}{9} > 7.5$
$\frac{16}{4} < R \le \frac{17}{4}$	17	2	1	2	2	cyclic 17/4/8	$\frac{128}{17} > 7.5$

∎

Our next theorem provides load factor 7 braidings G→Q(G)/8 under a stronger density assumption than the one of the previous theorem. In this theorem and the ones which follow on braidings G→Q(G)/8, the maps $f_{d,r}$ and cyclic r/s/t are not sufficient for constructing the required maps in all cases. We return to the use of "nonstandard" braidings in the necessary cases.

Theorem 7: Let G = [mxk] be a 2D grid satisfying dens(G,Q(G)/8) ≤ 6.4. Then there is a braiding g: G→Q(G)/8 with load factor at most 7.

Proof: Again the proof proceeds along the same lines as the ones before, only with more intricacy involved in the construction of the necessary braidings.

case defined by $\frac{u-s}{2^x} < R \le \frac{u}{2^x}$	u	x	s	r	b	braiding $f{:}G(u) \rightarrow H(2^r)$ of load factor 7	$7(1 - \frac{1}{u})$ (Lower bound for $dens_c(f_{m;q,c})$ for all $c \ge 1$)
$\frac{15}{2} < R \le \frac{16}{2}$	16	1	1	2	2	$f_{6,8} \oplus 2$	$\frac{105}{16} > 6.5$
$\frac{14}{2} < R \le \frac{15}{2}$	15	1	1	2	2	15/4/7 (Appendix 5)	$\frac{98}{15} > 6.5$
$\frac{13}{2} < R \le \frac{14}{2}$	14	1	1	2	2	$f_{0,7} \oplus 2$	6.5
$\frac{12}{2} < R \le \frac{13}{2}$	13	1	1	1	2	$f_{1,13}$	$\frac{84}{13} > 6.46$
$\frac{11}{2} < R \le \frac{12}{2}$	12	1	1	1	2	$f_{2,12}$	$\frac{77}{12} > 6.41$
$\frac{21}{4} < R \le \frac{22}{4}$	22	2	1	2	2	$f_{3,11} \oplus 2$	$\frac{147}{22} > 6.6$
$\frac{20}{4} < R \le \frac{21}{4}$	21	2	1	2	2	cyclic 21/4/7	$\frac{20}{3} > 6.6$
$\frac{19}{4} < R \le \frac{20}{4}$	20	2	1	2	2	$f_{4,10} \oplus 2$	$\frac{133}{20} > 6.6$
$\frac{18}{4} < R \le \frac{19}{4}$	19	2	1	2	2	19/4/7 (Appendix 6)	$\frac{126}{19} > 6.6$

$\frac{17}{4} < R \le \frac{18}{4}$	18	2	1	2	2	$f_{5,9}\oplus 2$	$\frac{119}{18} > 6.6$
$\frac{16}{4} < R \le \frac{17}{4}$	17	2	1	2	2	cyclic 17/4/7	$\frac{112}{17} > 6.5$

∎

The next theorem continues the series by providing braidings of load factor 6 assuming a smaller density. As in the hypothesis of Theorem 2 we also impose a lower bound in the density assumption, as a convenience that will allow us to eliminate certain cases (as was done in Theorem 2).

Theorem 8: Let G = [mxk] be a 2D grid satsifying $4.8 \le \text{dens}(G,Q(G)/8) \le 5.6$. Then there is a braiding g: $G \rightarrow Q(G)/8$ with load factor at most 6.

Proof: The method of proof is the same as in the previous theorem, only now we will eliminate in advance the need to consider certain cases (motivated by the difficulties involved in constructing the associated braidings). For an integer t, let $R(t) = \frac{t}{2^{\lfloor \log(t) \rfloor -2}}$.

Specifically, we show that if $\frac{19}{4} \le R(m) \le \frac{24}{4}$, then under our assumptions we must have either $R(k) > \frac{24}{4}$ or $R(k) < \frac{19}{4}$. It would then follow that in the table below we need not consider the case $\frac{19}{4} \le R(m) \le \frac{24}{4}$. This is because we could view G as [kxm], and consider the cases which treat the value of R(k) instead of R(m).

So suppose to the contrary that $\frac{19}{4} \le R(m),R(k) \le \frac{24}{4}$. Assume first that $\lfloor \log(mk) \rfloor = \lfloor \log(m) \rfloor + \lfloor \log(k) \rfloor$. Then
$\text{dens}(G,Q(G)/8) = \frac{mk}{2^{\lfloor \log(mk) \rfloor -2}} = \left(\frac{m}{2^{\lfloor \log(m) \rfloor -2}}\right)\left(\frac{k}{2^{\lfloor \log(k) \rfloor}}\right) \ge \frac{19}{4} \cdot \frac{19}{16} > 5.6$, a contradiction. So assume that $\lfloor \log(mk) \rfloor = 1 + \lfloor \log(m) \rfloor + \lfloor \log(k) \rfloor$. Then
$\text{dens}(G,Q(G)/8) = \frac{mk}{2^{\lfloor \log(mk) \rfloor -2}} = \left(\frac{m}{2^{\lfloor \log(m) \rfloor -1}}\right)\left(\frac{k}{2^{\lfloor \log(k) \rfloor}}\right) \le \frac{24}{8} \cdot \frac{24}{16} < 4.8$, again a contradiction.

We can now present the rest of the proof in the table below, giving the appropriate parameter values necessary for the application of Corollary 4.1 and Lemma 5.

case defined by $\frac{u-s}{2^x} < R \le \frac{u}{2^x}$	u	x	s	r	b	braiding $f:G(u) \rightarrow H(2^r)$ of load factor 6	$6(1 - \frac{s}{u})$ (Lower bound for $\text{dens}_c(f_{m;q,c})$ for all $c \ge 1$)
$\frac{15}{2} < R \le \frac{16}{2}$	16	1	1	2	2	$f_{4,8}\oplus 2$	$\frac{45}{8} > 5.6$
$\frac{28}{4} < R \le \frac{30}{4}$	30	2	2	3	2	2·(15/8/3) (Appendix 1)	$\text{dens}(f_{(-2)}) = 5.6$
$\frac{27}{4} < R \le \frac{28}{4}$	28	2	1	3	2	$f_{5,7}\oplus 4$	$\frac{81}{14} > 5.7$

$\frac{26}{4} < R \le \frac{27}{4}$	27	2	1	3	2	27/8/6 (Appendix 7)	$\frac{52}{9} > 5.7$
$\frac{25}{4} < R \le \frac{26}{4}$	26	2	1	3	2	2·(13/8/3) (Appendix 3)	$\frac{75}{13} > 5.7$
$\frac{24}{4} < R \le \frac{25}{4}$	25	2	1	3	2	cyclic 25/8/6	$\frac{144}{25} > 5.7$
$\frac{19}{4} < R \le \frac{24}{4}$	This case is eliminated above.						
$\frac{18}{4} < R \le \frac{19}{4}$	19	2	1	2	2	19/4/6 (Appendix 8)	$\frac{108}{19} > 5.6$
$\frac{17}{4} < R \le \frac{18}{4}$	18	2	1	2	2	$f_{3,9} \oplus 2$	$\frac{17}{3} > 5.6$
$4 < R \le \frac{17}{4}$	17	2	1	2	2	cyclic 17/4/6	$\frac{96}{17} > 5.6$

∎

Our last, and most complicated, theorem in this series is the next one giving load factor 5 under the strongest of our density assumptions. The small load factor and the small gap between the allowed density and this load factor require the greatest number and the most intricate of the nonstandard (i.e not cyclic) braidings we have seen so far.

Theorem 9: Let G = [mxk] be a 2D grid satisfying dens(G,Q(G)/8) ≤ 4.8. Then there is a braiding g: G→Q(G)/8 with load factor at most 5.

Proof: As in the last theorem we will begin by eliminating certain cases in advance. The remaining cases are then handled by the usual method, with the parameters used in the application of Corollary 4.1 and Lemma 5 for each case summarized in a table. We retain the notation $R(t) = \frac{t}{2^{\lfloor \log(t) \rfloor -2}}$.

We claim that under our assumptions not both of R(m) and R(k) can lie in the interval $(\frac{9}{2}, \frac{45}{8}] = (\frac{36}{8}, \frac{45}{8}]$. As above this would imply (by interchanging the roles of m and k) that we need not consider the case $\frac{9}{2} < R(m) \le \frac{45}{8}$. So suppose to the contrary that both R(m) and R(k) lie in this interval.

By the upper bound we have

$\log(m) \le \log(\frac{45}{8}) + \lfloor \log(m) \rfloor - 2 < .49 + \lfloor \log(m) \rfloor$. It follows that $\lfloor \log(mk) \rfloor = \lfloor \log(m) \rfloor + \lfloor \log(k) \rfloor$.

Hence $\text{dens}(G,Q(G)/8) = \frac{mk}{2^{\lfloor \log(mk) \rfloor -2}} = \left(\frac{m}{2^{\lfloor \log(m) \rfloor -2}}\right)\left(\frac{k}{2^{\lfloor \log(k) \rfloor}}\right) \ge \frac{9}{2} \cdot \frac{9}{8} > 4.8$, a contradiction to the assumed upper bound for dens(G,Q(G)/8) .

Proceeding now to the remaining cases we have the following table. Three of the cases have been asterisked since the braidings $f{:}G(u) \to H(2^r)$ involved do not have a uniform load factor, so that the lower bound $L(1 - \frac{s}{u})$ for $\text{dens}_c(f_{m;q,c})$ is not applicable. Instead we use the fact that each of these braidings is consecutive on switches. It can then be shown almost exactly as in the proof of Lemma 4c that $dens_c(f_{m;q,c}) \geq L'(1 - \frac{s}{u})$, where $L' = \frac{(f)[p]}{2^r p}$ and p is the period of f. (Note that if f had uniform load factor L, then $\frac{(f)[p]}{2^r p} = L$.) Thus for the asterisked cases we list $L'(1 - \frac{s}{u})$ in the last column, and since $L'(1 - \frac{s}{u}) \geq 4.8$ in each case, we get by the usual appeal to Lemma 5 that $f_{m;q,c}(G) \subset Q(G)/8$, proving the theorem.

case defined by $\frac{u-s}{2^x} < R \leq \frac{u}{2^x}$	u	x	s	r	b	braiding $f:G(u) \to H(2^r)$ of load factor 5	$5(1 - \frac{s}{u})$ (Lower bound for $dens_c(f_{m;q,c})$ for all $c \geq 1$), except for asterisked cases
$\frac{31}{4} < R \leq \frac{32}{4}$	32	2	1	3	2	$f_{2,8} \oplus 4$	$\frac{155}{32} > 4.84$
$\frac{30}{4} < R \leq \frac{31}{4}$	31	2	1	3	2	31/8/5* (Appendix 9)	$\frac{135}{28} > 4.82$
$\frac{29}{4} < R \leq \frac{30}{4}$	30	2	1	3	2	(15/4/5)⊕2 (Appendix 14)	$\frac{29}{6} > 4.83$
$\frac{28}{4} < R \leq \frac{29}{4}$	29	2	1	3	2	29/8/5* (Appendix 10)	$\frac{77}{16} > 4.81$
$\frac{27}{4} < R \leq \frac{28}{4}$	28	2	1	3	2	$f_{3,7} \oplus 4$	$\frac{135}{28} > 4.82$
$\frac{26}{4} < R \leq \frac{27}{4}$	27	2	1	3	2	27/8/5 (Appendix 11)	$\frac{130}{27} > 4.81$
$\frac{25}{4} < R \leq \frac{26}{4}$	26	2	1	3	2	(cyclic 13/4/5)⊕2	$\frac{125}{26} > 4.807$
$\frac{24}{4} < R \leq \frac{25}{4}$	25	2	1	3	2	cyclic 25/8/5	4.8
$\frac{47}{8} < R \leq \frac{48}{8}$	48	3	1	4	2	$f_{4,6} \oplus 8$	$\frac{235}{48} > 4.89$
$\frac{46}{8} < R \leq \frac{47}{8}$	47	3	1	4	2	47/16/5* (Appendix 12)	$\frac{391}{80} > 4.88$
$\frac{45}{8} < R \leq \frac{46}{8}$	46	3	1	4	2	(23/8/5)⊕2 (Appendix 15)	$\frac{225}{46} > 4.89$
$\frac{36}{8} < R \leq \frac{45}{8}$	This case is eliminated above.						
$\frac{35}{8} < R \leq \frac{36}{8}$	36	3	1	4	2	$f_{1,9} \oplus 4$	$\frac{175}{36} > 4.86$
$\frac{34}{8} < R \leq \frac{35}{8}$	35	3	1	4	2	35/8/5 (Appendix 13)	$\frac{34}{7} > 4.85$
$\frac{33}{8} < R \leq \frac{34}{8}$	34	3	1	4	2	(cyclic 17/4/5)⊕2	$\frac{165}{34} > 4.85$
$\frac{32}{8} < R \leq \frac{33}{8}$	33	3	1	3	2	cyclic 33/8/5	$\frac{160}{33} > 4.84$

∎

We are now ready for the analogue of theorem 5 for maps M →Q(M)/4, where M=[mxkxp] is a 3D grid. The method is the same except for an obvious change in parameters. There is a first phase consisting of a map f:[mxk] → Q, where Q = Q([mxk])/8 is regarded as a face of Q(M)/4 and f is provided by one of theorems 6 - 9. Then there is a second phase consisting of the construction of a braiding g of chains into the "third" dimension orthogonal to Q. The required map M →Q(M)/4 is then $f \otimes_N g$ where N is any numbering for f. By not producing an N which is consistent for f we settle for a dilation 3 (rather than a dilation 2) result using Lemma 6, part (2).

Theorem 10: For any 3D grid M = [mxkxp], there is a map M → Q(M)/4 of dilation ≤ 3 and load factor 5.

Proof: We proceed as in the proof of theorem 5, the only difference being that we give up the requirement that the numbering N is consistent for f. Again let G = [mxk] be a 2D face of M, and let Q = Q(G)/8. We produce a braiding f:G → Q of some load factor Y, and a braiding g:G(Y) → H(2) of load factor 5 such that $\text{dens}(G,Q) \geq \frac{4Y}{5}$. Then by lemma 6 (applied with s = 3 and u = 2) the map $f \otimes_N g: M \to Q \times H(2)$ would be the desired map.

Noting that D = dens(G,Q(G)/8) satisfies 4 ≤ D ≤ 8, we partition the interval [4,8] into subintervals, and argue by cases based on which subinterval contains D. In each case, a braiding f of load factor Y, 5 ≤ Y ≤ 9, is guaranteed by one of theorems 6 - 9. So in each case it suffices to specify the braiding g together with the integer Y, and to verify that the left endpoint E of the interval defining the case satisfies $E = \frac{4Y}{5}$. The results are summarized in this table.

case $\alpha \leq D \leq \beta$	Y (= load factor of f:G→Q/8)	braiding g:G(Y)→H(2) of load factor 5	$\frac{4Y}{5}$
7.2 ≤ D ≤ 8	9 (Theorem 1)	$f_{1,9}$	7.2
6.4 ≤ D ≤ 7.2	8 (Theorem 6)	$f_{2,8}$	6.4
5.6 ≤ D ≤ 6.4	7 (Theorem 7)	$f_{3,7}$	5.6
4.8 ≤ D ≤ 5.6	6 (Theorem 8)	$f_{4,6}$	4.8
4 ≤ D ≤ 4.8	5 (Theorem 9)	$f_{5,5}$	4

∎

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	Column 1	Column 2	Column 3
Rod 0	(1,1)-(3,1)-(5,1)-(2,2)	(2,3)-(5,2)-(4,3)-(1,4)	(1,5)-(4,4)-(2,4)-(3,5)
Rod 1	(2,1)-(4,1)-(1,2)-(3,2)	(3,3)-(1,3)-(4,2)-(5,3)	(5,4)-(3,4)-(2,5)-(4,5)
	Chains {1,2,3} switch	Chains {4,5,1} switch	Chains {2,3,4} switch

	Column 4	Column 5
Rod 0	(3,6)-(1,6)-(5,6)-(2,7)	(2,8)-(5,7)-(3,7)-(4,8)
Rod 1	(4,6)-(2,6)-(5,5)-(1,7)	(1,8)-(4,7)-(3,8)-(5,8)

Chains {5,1,2} switch   Chains {3,4,5} switch

Figure 1: The first 5 columns in the image of the braiding $f_{3,5}$:G(5)→H(2) of uniform load factor 4

1-3-5|2   2-5|4-1   1-4-2|3   1-3|5-2   5-2-3|4
2-4|1-3   1-3-4|5   5-3|2-4   4-2-5|1   1-4|3-5

Figure 2: The braiding $f_{3,5}$ in abbreviated form - an entry i to the left (resp. right) of a vertical line represents the first (resp. second) point of chain i in the column

1-10-3-8-5-6|2-9   2-9-5-6-|4-7-1-10   1-10-4-7-2-9|3-8
2-9-4-7|1-10-3-8   1-10-3-8-4-7|5-6   5-6-3-8|2-9-4-7

1-10-3-8-|5-6-2-9   5-6-2-9-3-8|4-7
4-7-2-9-5-6|1-10   1-10-4-7-|3-8-5-6

Figure 3: The first 5 columns in the image of the braiding $2\cdot f_{3,5}$

1-5|4	4|6-3	4-6-3
2-6|5	6-5|2	5-2|1
3-7|2	7-2|1	1|5-7
4|1-3	1-3|7	7|4-6

a) First three columns of a braiding f:G(7)→H(4) of load factor 3

1-5|4	4|6-3	4-6-3
2-6|5	6-5|2	5-2|1
3-7|2	7-2|1	1|5-7
4|1-3	1-3|7	7|4-6
11|14-12	14-12|8	8|11-9
12-8-|13	8-13|14	14|10-8
13-9-|10	9-10|13	10-13|14
14-10|11	11|9-12	11-9-12

b) First three columns of f⊕2:G(14)→H(8)

1|4-*	4|3-*	4-3-*
2-*-*	*|2-*	2|1-*
3|2-*	2|1-*	1-*-*
4|1-3	1-3-*	*|4-*
11|14-12	14-12|8	8|11-9
12-8-|13	8-13|14	14|10-8
13-9-|10	9-10|13	10-13|14
14-10|11	11|9-12	11-9-12

c) Restriction of f⊕2:G(14)→H(8) to m = 11 chains; the absent chains are 5,6, and 7 indicated by * (disregarding the relative order of * and the vertical line on a row).

4|1-*	1|2-*	1-2-*
3-*-*	*|3-*	3|4-*
2|3-*	3|4-*	4-*-*
1|4-2	4-2-*	*|1-*
8|5-7	5-7|11	11|8-10
7-11|6	11-6|5	5|9-11
6-10|9	10-9|6	9-6|5
5-9|8	8|10-7	8-10-7

d) Reindexing of chain numbers using κ(i) = s-i+1 if 1≤i≤s, and κ(nu+i) = nu-i+s+1 for 1≤n≤d and 1≤i≤u (with u=7 and s=4), thereby obtaining the first three columns of the map $f_{11;2,3}$:G(11)→H(8)

Figure 5: The first three columns of the map $f_{11;2,3}$:G(11)→H(8) obtained from the braiding f:G(7)→H(4) of appendix 2.

```
1-5|4      4|6-3      4-6-3      3|2-4      2-4|1      1|5-7      5-7|4      4|7-5
2-6|5      6-5|2      5-2|1      2-1|3      1-3|2      2|6-1      6-1|5      5-1|6
3-7|2      7-2|1      1|5-7      5-7|6      7-6|3      6-3|2      3-2|6      6-2|3
4|1-3      1-3|7      7|4-6      4-6|5      5|7-4      5-7-4      4|3-7      7-3|4
```

a) First eight columns of a braiding f:G(7)→H(4) of load factor 3

```
5|4-*      4|6-*      4-6-*      *|4-*      4-*-*      *|-5-7     5-7|4      4|7-5
6|5-*      6-5-*      5-*-*      *-*-*      *-*-*      *|-6-*     6|5-*      5|6-*
7-*-*      7-*-*      *|5-7      5-7|6      7-6-*      6-*-*      *|6-*      6-*-
*
4-*-*      *|7-*      7|4-6      4-6|5      5|7-4      5-7-4      4|7-*      7|4-*
```

```
11|14-12   14-12|8    8|11-9     11-9|10    10|8-11    10-8-11    11|12-8    8-12-|11
12-8-|13   8-13|14    14|10-8    10-8|9     8-9|12     9-12-|13   12-13|9    9-13-|12
13-9-|10   9-10|13    10-13|14   13-14|12   14-12|13   13|9-14    9-14-|10   10-14|9
14-10|-11  11|9-12    11-9-12    12|13-11   13-11|14   14|10-8    10-8-|11   11|8-10
```

b) Restriction of f⊕2:G(14)→H(8) to m = 11 chains; the absent chains are 1,2, and 3, again indicated by (disregarding the relative order of * and the vertical line on a row) .

```
2|1-*      1|3-*      1-3-*      *|1-*      1-*-*      *|2-4      2-4|1      1|2-4
3|2-*      3-2-*      2-*-*      *-*-*      *-*-*      *|3-*      3|2-*      2|3-*
4-*-*      4-*-*      *|2-4      2-4|3      4-3-*      3-*-*      *|3-*      3-*-
*
1-*-*      *|4-*      4|1-3      1-3|2      2|4-1      2-4-1      1|4-*      4|1-*
```

```
8|11-9     11-9|5     5|8-6      8-6|7      7|5-8      7- 5-8     8|9-5      5-9|8
9-5|10     5-10|11    11|7-5     7-5|6      5-6|9      6-9|10     9-10-|6    6-10|9
10-6-|7    6-7|10     7-10-|11   10-11|9    11-9|10    10|6-11    6-11-|7    7-11|6
11-7-|8    8|6-9      8-6-9      9|10-8     10-8|11    11|7-5     7-5|8      8|5-7
```

c) Reindexing of chain numbers using κ(i) = s-i+1 for 1≤i≤s, and κ(i) = i-u+s for u+1≤i≤(d+1)u (with u=7 and s=4), thereby obtaining the map $f_{11;2,3}$:G(11)→H(8)

<u>Figure 6</u>: The first eight columns of the map $f_{11;2,8}$:G(11)→H(8) obtained from the braiding f:G(7)→H(4) of appendix 2.

```
1-5-9      9-5|4      4-9|8
2-6|1      1-6|5      5-1|9
3-7|2      2-7|6      6-2-7
4-8|3      3-4-8      8-3|7
```

<u>Figure 8</u>: The first 3 columns in the image of the braiding cyclic 9/4/3

Appendix

Each of the braidings which follow is understood to be reflection periodic, and we illustrate its first p columns, where p is the period of the braiding.

Appendix-1: 15/8/3

1-9|2	2|10-1	1-10|11	1-11|2	2-11|12
2-10|9	9-10|2	2-9|8	2-8|1	1-8|9
3-11|4	4-11|3	3-4|9	3-9|15	9-15|10
4-12|3	3-12|11	11|4-10	4-10|3	3-10|11
5-13|6	6-13|12	6-12|5	5|13'-4	4-13|14
6-14|5	5-14|15	5-15|6	6-15|14	14|7-15
7-15|8	8-15|14	8-14|7	7-14|6	6-7|8
8|1-7	1-7|13	7-13|12	12-13|5	5-12|13

Appendix-2: 7/4/3

1-5|4	4|6-3	4-6-3	3|2-4	2-4|1	1|5-7	5-7|4
2-6|5	6-5|2	5-2|1	2-1|3	1-3|2	2|6-1	6-1|5
3-7|2	7-2|1	1|5-7	5-7|6	7-6|3	6-3|2	3-2|6
4|1-3	1-3|7	7|4-6	4-6|5	5|7-4	5-7-4	4|3-7

Appendix-3: 13/8/3

1-9|8	8|1-7	1-7|6	6|10-5	5-6-10	5|6-4	4-6|7
2-10|9	9|2-8	2-8|7	7-8|9	7-9|8	8|7-9	7-9|8
3-11|10	10|3-9	3-9-10	9|8-12	8-12|9	9|8-10	8-10|9
4-12|11	11-12|13	11-13|10	10|11-13	11-13|10	10|5-11	5-11|10
5-13|4	4-13|12	12|11-13	11-13|1	1|13-11	11-13|12	12|5-11
6|3-5	3-5|4	4|12-3	3-12|2	2|12-1	1-12|13	13|2-12
7|2-6	2-6|5	5|2-4	2-4|3	3|7-2	2-7|1	1-2|3
8|1-7	1-7|6	6|1-5	1-5|4	4|6-3	3-4-6	3|4-6

7|4-6	4-6|3	3|9-2	2-9|10	10|11-9	9-11|8
8|1-3	1-3|2	2|12-1	1-12|11	11|12-1	1-12|7
9|2-8	2-8|1	1|11-13	11-13|12	12|7-13	7-13|6
10|7-9	7-9|8	8-9|10	8-10|13	13|8-6	6-8|9
5|11-10	10-11|7	7-10|6	6-7|8	6-8|5	5|4-10
2-12-13	12|11-13	11-13|5	5|7-4	4-5-7	4|5-13
1-3|13	13|12-5	5-12|4	4|1-3	1-3|2	2-3|12
4-6|5	5|6-4	4-6|3	3|9-2	2-9|10	10|3-11

Appendix-4: 11/8/3 (A prime indicates the second point, and a double prime the third point in a column.)

1-9-10'	10-5'-4"	4-5-10'	10-9'-8"	8-9-7'	7-2'1"	1-2-7'
2-10-7'	7-6'-3"	3-6-4'	4-10'-3'	3-10-4'	4-3'-11"	3-11-1'
3-11-6'	6-7'-2'	2-7-3'	3-11'-2'	2-11-3'	3-4'-10'	4-10-11'
4-5'-11'	5-11-1'	1-11'-2'	2-11-1'	1-2'-8'	2-8-9'	9-8'-10'
5-4'-1"	1-4-11'	11-8'-1'	1-8-6'	6-1'-9"	1-9-8'	8-5'-9'
6-3'-2"	2-3-8'	8-7'-6"	6-7-5'	5-11'-10"	10-11-5'	5-4'-3"
7-2'-8'	8-3'-9'	9-6'-5"	5-7'-4'	4-10'-5'	5-11'-6'	6-3'-2"
8-1'-9'	9-4'-10'	10-5'-9'	9-8'-7"	7-9'-6'	6-1'-7'	7-2'-6'

7-6'-5"	5-6-4'	4-10'-9"	9-10-10"
1-7'-11'	7-11-1'	1-11'-8"	8-11-11"
11-8'-10'	8-10-11'	11-1'-7'	1-7-8'
8-10-9'	9-10'-5'	5-10-6'	6-7'-9'
5-9-3'	3-9'-6"	6-9-5'	5-2'-6'

3-4-2'   2-8'-7"   7-8-2'   2-1'-5'
2-4'-1'   1-7'-2'   2-8'-3'   3-11'-4'
6-5'-4"   4-6'-3'   3-9'-4'   4-10'-3'

<u>Appendix 5</u>: 15/4/7

1-5-9-13|4-8-12   4-8-12-|14-3-7-11   12-14-3-7-11|-2-6   11-2-6|-10-12-1-5   10-12-1-5|11-15-
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32-1-17-33-47	17-33-47-16-32	47-16-32-1-15	32-1-15-31-47	31-47-32-16-*
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31-47-14-32-46	14-32-46-15-29	46-15-29-47-14	47-14-30-46-13	30-46-13-31-47

<u>Appendix 13</u>: 35/8/5

1-9-11-23-35	9-11-23-35-10	11-23-35-10-14	23-35-10-14-16	23-35-10-14-16
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26-28-11-15-25	11-15-25-31-35
29-31-12-24-30	31-12-24-30-32
5-13-17-23-29	5-13-17-23-29
33-4-6-18-22	33-4-6-18-22
30-32-3-7-21	32-3-7-21-33
27-2-8-20-26	2-8-20-26-34
34-1-9-19-27	34-1-9-19-27

<u>Appendix 14</u>: 15/4/5

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<u>Appendix 15</u>: 23/8/5

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