Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints

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Abstract

We consider a canonical Ramsey type problem. An edge-coloring of a graph is called *m-good* if each color appears at most m times at each vertex. Fixing a graph G and a positive integer m, let $f(m, G)$ denote the smallest n such that every m-good edge-coloring of K_n yields a properly edge-colored copy of G, and let $g(m, G)$ denote the smallest n such that every m-good edge-coloring of K_n yields a rainbow copy of G. We give bounds on $f(m, G)$ and $g(m, G)$. For complete graphs $G = K_t$, we have $c_1 mt^2/\ln t \leq f(m, K_t) \leq c_2 mt^2$, and $c'_1 mt^3/\ln t \leq g(m, K_t) \leq c'_2 mt^3/\ln t$, where c_1, c_2, c'_1, c'_2 are absolute constants. We also give bounds on $f(m, G)$ and $g(m, G)$ for general graphs G in terms of degrees in G. In particular, we show that for fixed m and d, and all sufficiently large n compared to m and d, $f(m, G) = n$ for all graphs G with n vertices and maximum degree at most d.

1 Introduction

In Ramsey Theory we study, for fixed k, monochromatic subgraphs that are forced to appear in every k-coloring of the edges of K_n. If we allow arbitrarily many colors to be used, we can still ask what types of subgraphs are forced if we replace *monochromatic* by some other condition on the subgraph's coloring. Erdős and Rado [11] were among the first to study problems of this type. In 1950 they proved a counterpart of Ramsey's theorem on colorings of finite sets using arbitrarily many colors. Their theorem is often known as the *Canonical Ramsey Theorem.* We paraphrase that theorem, in terms of the following notation. When the end vertices x and y of an edge e are integers, we call $max(x, y)$ the *higher endpoint* and call $min(x, y)$ the *lower endpoint.*

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Theorem A (Erdős-Rado [11]) *Let p be a positive integer. Then there exists a least positive integer $N = er(p)$ such that if the edges of the complete graph K_N with vertex set $\{1, \cdots, N\}$ are colored using an arbitrary number of colors, then there exists a complete subgraph with p vertices on which the coloring is of one of four canonical types.*

1) monochromatic — all edges have the same color;

2) rainbow — no two edges have the same color;

3) upper lexical — two edges have the same color if and only if they have the same higher endpoint;

4) lower lexical — two edges have the same color if and only if they have the same lower endpoint.

The best known estimates of $er(p)$ are due to Lefmann and Rödl [15] who showed that there exist constants c, c' such that for every positive integer p, $2^{\,cp^2} \leq er(p) \leq 2^{\,c'p^2 \log p}$.

Motivated by the Canonical Ramsey Theorem, Jamison, Jiang and Ling [14], and independently, Chen, Schelp and Wei [8], introduced the following notion. Given two graphs G and H, let $R^*(G, H)$ denote the smallest n such that every coloring of $E(K_n)$ using an unlimited number of colors yields either a monochromatic copy of G or a rainbow copy of H. It follows from the Canonical Ramsey Theorem that $R^*(G, H)$ exists iff either G is a star or H is acyclic.

This points toward two natural directions for study of R^*.

(1) Study $R^*(G, H)$ when H is a fixed acyclic graph and G is any graph.

(2) Study $R^*(G, H)$ when G is a fixed star and H is any graph.

In particular, along direction (1), consider the choice $H = K_{1,k+1}$. Then $R^*(G, H)$ is exactly the smallest n such that every coloring of $E(K_n)$ with at most k different colors appearing at each vertex contains a a monochromatic copy of G. This was earlier introduced and studied (see [12, 13, 17, 19]) as the *k-local Ramsey number* of G, which can be considered a local version of the k-color Ramsey number $R_k(G)$.

The purpose of this paper is to study $R^*(G, H)$ along direction (2). In this case, $G = K_{1,m+1}$ for some positive integer m, and H is any specific graph. For convenience, we rephrase the problem as follows. Given a positive integer m, we define an edge-coloring of a (host) graph to be *m-good* if each color appears at most m times at each vertex. Given any graph H, let $g(m, H)$ denote the smallest n such that every m-good coloring of $E(K_n)$ yields a rainbow copy of H. It is not hard to see that $R^*(K_{1,m+1}, H)$ is exactly $g(m, H)$. So, study of R^* along direction (2) amounts to study of the function $g(m, H)$. In addition, we consider a related function $f(m, H)$, defined to be the smallest n such that every m-good coloring of $E(K_n)$ yields a properly edge-colored copy of H. We emphasize that the important definitions in this paragraph involve no upper limit on the number of colors used in an edge-coloring. Since the colorings in this paper are never vertex-colorings and are usually edge-colorings, the phrases *coloring* and *proper coloring* are to be understood to refer to edge-colorings except where indicated otherwise.

One of our motivations in studying the function f comes from previous work by various authors (see [4, 9, 18]) on the conjecture of Bollobás and Erdős [7] that every $\lfloor n/2 \rfloor$-good edge-coloring of K_n has a properly colored Hamiltonian cycle, which in terms of our notations says that $f(m, C_n) = n$ if $m \leq \lfloor n/2 \rfloor$. In section 4, we extend results in [4, 9, 18] from Hamiltonian cycles to sparse spanning subgraphs. It is also worth noting that a key ingredient in Lefmann and Rödl's proof [15] of their upper bound on $er(p)$ dealt with what amounts to an analysis of $g(m, K_n)$. In fact, our bounds here enable us to slightly improve their upper bound.

This paper is organized as follows. We first make some observations in Section 2. We give bounds on $f(m, K_t)$ in Sections 3. In Section 4 we study $f(m, G)$ for graphs G with bounded maximum degree. In Section 5 we derive bounds on $g(m, K_t)$, and in Section 6 we bound $g(m, G)$ for general graphs G in terms of degrees in G. Section 7 contains some concluding remarks. Throughout the paper, we make no attempt to optimize the absolute constants involved.

For any notation or definition not given here, we follow West [20]. In particular, for a graph G and a vertex subset S of $V(G)$, we let $G[S]$ denote the subgraph of G induced by S. As usual, $E(X)$ denotes the expectation of a random variable X (and although for a graph G we let $E(G)$ denote its set of edges, it will always be clear from the context if we refer to edges or expectations).

2 Preliminaries

In this section we give a couple of lemmas useful in deriving lower bounds on $f(m,G)$ and $g(m,G)$. For a positive integer n, let $[n]$ denote the set $\{1,2,\cdots,n\}$.

Lemma 2.1 *Let n,p,s,t be positive integers. Suppose there exists a p-good coloring of $E(K_n)$ containing no properly colored K_t. Then there exists a ps-good coloring of K_{ns} containing no properly colored K_t. That is, we have*

$$f(ps,K_t) > s[f(p,K_t)-1].$$

Proof. Let $x_1,x_2,\cdots,x_n$ denote the n vertices of $K=K_n$. By our assumption, there exists a p-good coloring c of $E(K)$ containing no properly colored K_t (i.e., $n \le f(p,K_t)-1$). From this assumption, note that $t \ge 3$. We use the coloring c to obtain a ps-good edge-coloring c^* of K_{ns} that yields no properly colored K_t, as follows.

Let $N = ns$ and $K^* = K_N$. Partition $V(K^*)$ into n subsets $A_1,A_2,\cdots,A_n$, each of order s. We define the coloring c^* of $E(K^*)$ as follows. For each pair $i,j \in [n], i \ne j$, assign color $c(x_ix_j)$ to all the edges joining A_i and A_j. For the remaining edges, which are edges with both endpoints in the same subset A_i, we assign them a new color α not used in c. Clearly, c^* is a ps-good coloring of $E(K^*)$.

Consider a properly colored complete subgraph H^* in c^* of order $p \ge 3$. It suffices to prove that $p \le t$. It follows easily from our definition of c^* and the fact that H^* is properly colored that $|V(H^*)\cap A_i| \le 1$ for all $i \in [n]$. Let H be the subgraph of K induced by $\{x_i : |V(H^*)\cap A_i| = 1\}$. Then H is a complete subgraph of K of order p which is properly colored under coloring c. By our assumption about c, we have $p \le t$. ∎

The proof of Lemma 2.1 can be easily modified to show the following.

Lemma 2.2 *Let n,p,s,t be positive integers, where $n,t \ge 3$. Suppose there exists a p-good coloring of $E(K_n)$ containing no rainbow K_t. Then there exists a ps-good coloring of K_{ns} containing no rainbow K_t. That is, we have*

$$g(ps,K_t) > s[g(p,K_t)-1].$$

Next we give some trivial lower bounds for $f(m,G)$ and $g(m,G)$.

Lemma 2.3 *Let G be a graph with q edges, and maximum degree Δ. Then $g(m,G) > m(q-1)$ and $f(m,G) > m(\Delta-1)$.*

Proof. Based on its edge-chromatic number, $K_{m(q-1)}$ has a proper edge-coloring c using colors from $[m(q-1)]$. Define c' by letting $c'(e) = \lceil c(e)/m \rceil$ for each edge $e \in E(K_{m(q-1)})$. The resulting c' is clearly m-good, and it uses at most $q-1$ colors, hence it has no rainbow copy of G. Thus, $g(m, G) > m(q-1)$. A similar argument shows $f(m, G) > m(\Delta - 1)$. ■

Toward developing Lemma 2.5, which is useful in establishing probabilistic lower bounds on $f(n, G)$ and $g(n, G)$ for dense graphs G, first we need the following simple lemma. Given positive integers n, j with $n \geq j$, let $[n]_j = n(n-1)\cdots(n-j+1)$.

Lemma 2.4 *Let a, b, j be positive integers such that $a \geq b \geq j$. Then*

$$\frac{[b]_j}{[a]_j} \leq \left(\frac{b - \frac{j-1}{2}}{a - \frac{j-1}{2}} \right)^j .$$

Proof. For $a = b$ this is trivial, hence assume $a > b$. Let $f(x) = \frac{b-x}{a-x}$, where $x < b$, and let $g(x) = \ln f(x) = \ln(b-x) - \ln(a-x)$. It is easy to check that $g''(x) = \frac{1}{(a-x)^2} - \frac{1}{(b-x)^2} < 0$. Hence $g(x)$ is concave down for $x < b$. So in particular we have $\ln \Pi_{i=0}^{j-1} f(i) = \Sigma_{i=0}^{j-1} g(i) \leq j \cdot g(\frac{j-1}{2})$ [by convexity] $= \ln[f(\frac{j-1}{2})]^j$. Thus $\Pi_{i=0}^{j-1} f(j) \leq [f(\frac{j-1}{2})]^j$, which is the same as the claim. ■

Let N, m, q be positive integers with $N = mq$. Given a set S of N elements, a coloring of the elements of S is *m-perfect* if exactly q colors are used, and each color is used on exactly m elements.

Lemma 2.5 *Let N, m, q be positive integers with $N = mq$. Let S be a set of N elements. Let $\mathcal{C}$ denote the probability space consisting of all m-perfect colorings of S using colors from $[q]$, with each m-perfect coloring of S being equally likely. Let T be a fixed subset of S of size d, where $d \leq q$. Let $c \in \mathcal{C}$ be a randomly chosen m-perfect coloring of S. Let A_T denote the event that c assigns distinct colors to the elements of T. Then*

$$Prob\ (A_T) \leq \left[\ exp(-\frac{m-1}{2N})\ \right]^{d(d-1)} .$$

Proof. The assertion trivially holds for $d = 0$, hence assume that $d > 0$. Since there are $(mq)!/(m!)^q$ many m-perfect colorings of S in the space $\mathcal{C}$, of which $[q]_d(qm-d)!/\left([(m-1)!]^d(m!)^{q-d}\right)$ assign different colors to the d elements of T, we have

$$\begin{aligned}
\text{Prob } (A_T) &= \frac{[q]_d(mq-d)!/\left([(m-1)!]^d(m!)^{q-d}\right)}{(mq)!/(m!)^q} \\
&= m^d \cdot \frac{[q]_d}{[mq]_d} \leq m^d \left(\frac{q - \frac{d-1}{2}}{mq - \frac{d-1}{2}} \right)^d \\
&= \left(\frac{mq - \frac{m(d-1)}{2}}{mq - \frac{d-1}{2}} \right)^d = \left(1 - \frac{\frac{(m-1)(d-1)}{2}}{mq - \frac{d-1}{2}} \right)^d \\
&\leq \left(1 - \frac{(m-1)(d-1)}{2mq} \right)^d = \left(1 - \frac{(m-1)(d-1)}{2N} \right)^d \\
&\leq \left[\ exp(-\frac{(m-1)(d-1)}{2N})\ \right]^d = \left[\ exp(-\frac{m-1}{2N})\ \right]^{d(d-1)} .
\end{aligned}$$

■

3 Bounds on $f(m, K_t)$

In this section we determine $f(m, G)$ to within a factor of $\ln t$ when G is a complete graph K_t. The upper and lower bounds are both established by probabilistic arguments. A first step toward a lower bound on $f(m, K_t)$ is given in the following lemma, its proof relying on Lemma 2.5. Then we will use the special case of $m = 2$, combined with Lemma 2.1, to obtain our final lower bound in Theorem 3.2 on $f(m, K_t)$. Note concerning our bounds for $f(m, K_t)$ and $g(m, K_t)$ that we ignore the trivial case $t = 2$, since $f(m, K_2) = 2$.

Lemma 3.1 *There exists an absolute constant $c_0 > 0$ such that for all integers $m \geq 2$ and $t \geq 3$, $f(m, K_t) > c_0 mt^2/(\ln t + \ln m)$.*

Proof. We may assume that $t \geq 4$ without loss of generality. For a choice of c_0 to be determined later, let $q = \lceil (1/2m)(c_0 mt^2/(\ln t + \ln m) - 1) \rceil$, and $n = 2mq + 1$. We prove that for some c_0 there always exists an m-good edge-coloring of $K = K_n$ containing no properly colored complete subgraph of order t. It follows for that c_0 that whenever $N \leq c_0 mt^2/(\ln t + \ln m)$, some m-good coloring of $K = K_N$ contains no properly colored complete subgraph of order t, completing the proof.

First, we take an orientation D of K such that each vertex has outdegree exactly $(n-1)/2 = mq$. (This is easily accomplished by numbering the vertices 1 through n, and joining each vertex to the $(n-1)/2$ vertices which follow it *mod* n in the ordering.) Let $\{\alpha_{i,j} : i \in [n], j \in [q]\}$ be a fixed set of qn distinct colors. For each $i \in [n]$, let $E^+(v_i)$ denote the set of out-edges at v_i; we have $|E^+(v_i)| = mq$.

Let $\mathcal{C}$ denote the set of all possible colorings of $E(D)$ obtained by assigning for each $i \in [n]$ an m-perfect coloring to $E^+(v_i)$, using each of the colors $\alpha_{i,1}, \cdots, \alpha_{i,q}$ on exactly m edges of $E^+(v_i)$. We make $\mathcal{C}$ into a probability space by letting the choices of $c \in \mathcal{C}$ be equally likely. We estimate the probability that c yields a properly colored copy of K_t.

Let $X = \{x_1, \cdots, x_t\}$ be a fixed set of t vertices in D. Let A denote the event that $K[X]$, the complete subgraph of K induced by X, is properly colored. For $j = 1, \cdots, t$, let A_j denote the event that the edges in $E^+(x_j) \cap E(K[X])$ are all colored differently. Clearly, the events $A_1, \cdots, A_t$ are mutually independent. Letting $A = A_1 \wedge A_2 \cdots \wedge A_t$, we then have

$$\text{Prob } (A) = \Pi_{j=1}^{t} \text{ Prob } (A_j).$$

For $j = 1, \cdots, t$, let $d_j = |E^+(x_j) \cap E(K[X])|$. Note that $\Sigma_{j=1}^{t} d_j = |E(K[X])| = \binom{t}{2}$. For each $j = 1, \cdots, t$, we estimate Prob (A_j). First, assume $d_j > 0$. Since we independently color each $E^+(v_i)$ for $i \in [n]$, we see that Prob (A_j) equals the probability that a randomly chosen m-perfect coloring of $E^+(x_j)$ assigns different colors to the d_j edges of $E^+(x_j) \cap E(K[X])$. By Lemma 2.5 we have

$$\text{Prob } (A_j) \leq \left[exp(-\frac{m-1}{2mq}) \right]^{d_j(d_j-1)} \leq \left[exp(-\frac{m-1}{n}) \right]^{d_j(d_j-1)}.$$

Note that Prob $(A_j) \leq [exp(-\frac{m-1}{n})]^{d_j(d_j-1)}$ still holds when $d_j = 0$.
Therefore we have

$$\text{Prob } (A) \quad = \quad \Pi_{j=1}^{t} \text{ Prob } (A_j)$$

$$\begin{aligned}
&\le \Pi_{j=1}^{t}\left[exp(-\frac{m-1}{n}) \right]^{d_j(d_j-1)} \\
&= \left[exp(-\frac{m-1}{n}) \right]^{\Sigma_{j=1}^{t} d_j(d_j-1)} \\
&\le \left[exp(-\frac{m-1}{n}) \right]^{t(\frac{t-1}{2})^2-\binom{t}{2}} \qquad \text{(by convexity of } x^2\text{)} \\
&= exp\left[-\frac{m-1}{4n} t(t-1)(t-3) \right].
\end{aligned}$$

Since there are $\binom{n}{t}$ t-subsets of $V(K)$, we have

$$\begin{aligned}
&\text{Prob (Some } t\text{-subset is properly colored)} \\
\le\ & \binom{n}{t} exp(-\frac{m-1}{4n} t(t-1)(t-3)) \\
\le\ & \left(\frac{ne}{t}\right)^t exp(-\frac{m-1}{4n} t(t-1)(t-3)).
\end{aligned}$$

By choosing $c_0 > 0$ sufficiently small, for our choice $n = 1 + 2m\lceil(1/2m)(c_0 mt^2/(\ln t + \ln m) - 1)\rceil$, the expression above is less than 1, since the logarithm of the expression is negative. Thus we have

$$\text{Prob (Some } t\text{-subset is properly colored)} < 1.$$

Hence

$$\text{Prob (no } t\text{-subset is properly colored)} > 0.$$

Thus there exists a coloring c in $\mathcal{C}$ of K_n that yields no properly colored complete subgraph of order t. Hence by definition, $f(m, K_t) > n$ as required. ■

Theorem 3.2 *There exists an absolute constant* $c_1 > 0$ *such that for all positive integers* m *and* t*, we have*

$$f(m, K_t) \ge \frac{c_1 mt^2}{\ln t}.$$

Proof. Applying Lemma 3.1 with $m = 2$, we have $f(2, K_t) > 2c_0t^2/(\ln t + \ln 2) \ge c_0t^2/\ln t$ for some positive constant c_0. By Lemma 2.1, $f(m, K_t) \ge f(2\lfloor m/2\rfloor, K_t) > \lfloor m/2\rfloor[f(2, K_t) - 1] \ge c_1mt^2/\ln t$, where c_1 is a positive constant appropriately chosen. ■

Next, we prove an upper bound on $f(m, K_t)$ which differs from the lower bound by a factor of $\ln t$. Define a path subgraph L of length 2 in an edge colored graph to be a *bad 2-claw* if the two edges of L have the same color.

Theorem 3.3 *Given positive integers* m, t*, we have* $f(m, K_t) \le 4mt^2$*.*

Proof. Let $n = 4mt^2$, and let c be an m-good coloring of the edges of $K = K_n$. We show that c yields a properly colored complete subgraph of order at least t.

It is not hard to see by convexity of $\binom{x}{2}$ that each vertex is the middle point of at most $(n/m)\binom{m}{2} < (1/2)mn$ bad 2-claws. So in particular there are at most $(1/2)mn^2$ bad 2-claws in K.

Let the set of all $2t$-subsets of $V(K)$ be the probability space. With equal probability, randomly choose a $2t$-subset T. For each bad 2-claw L, let A_L denote the event that T contains all three vertices of L. Clearly for any L,

$$\text{Prob }(A_L) = \frac{\binom{n-3}{2t-3}}{\binom{n}{2t}} < (2t/n)^3.$$

Let X denote the number of bad 2-claws in K contained in T. Then (where the summation is over all bad 2-claws L in K)

$$\begin{aligned} E(X) &= \Sigma \text{ Prob }(A_L) \\ &\leq (1/2)mn^2(2t/n)^3 \\ &= 4mt^3/n = t. \end{aligned}$$

Thus, there exists a $2t$-subset T in K that contains at most t bad 2-claws. Now by deleting t vertices of K we can delete at least one vertex from each bad 2-claw in T to obtain a subset T' of size at least t that contains no bad 2-claws. Clearly, T' induces a properly colored complete subgraph of order at least t. ■

4 Bounds on $f(m, G)$ for sparse graphs G

In this section, we give bounds on $f(m, G)$ when G is sparse. Intuitively, when G is sparse and $n = n(G)$ is sufficiently large compared to m, it is expected that every m-good edge-coloring of K_n yields a properly colored copy of G. Indeed when G is a cycle, this intuition is true even in the stronger sense that n need not be very large at all relative to m, as indicated by the following result of Alon and Gutin [4].

Theorem 4.1 [4] *Given $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n > n_0$ and $m \leq (1 - 1/\sqrt{2} - \varepsilon)n$, every m-good edge-coloring of K_n has a properly colored Hamiltonian cycle, i.e. $f(m, C_n) = n$.*

This result is the best known result toward the following conjecture of Bollobás and Erdős.

Conjecture 4.2 [7] *Every $\lfloor n/2 \rfloor$-good edge-coloring of K_n has a properly colored Hamiltonian cycle.*

In this section, we generalize Theorem 4.1 by showing that for all positive integers n, m, d, where d is fixed and n is sufficiently large compared to m, $f(m, G) = n$ holds for all graphs G on n vertices of maximum degree at most d. Furthermore, for certain classes of graphs G, it suffices for n to grow as slowly as a linear function in m to ensure $f(m, G) = n(G)$. We emphasize that the results of this section therefore represent a drastic departure from results in other sections of the paper, in that now we make the extremal requirement that a SPANNING properly colored copy of G is sought within an arbitrary m-good edge-coloring of $K_{n(G)}$. Thus the issue is no longer about how large we must take n to be as a function of m and $|V(G)|$ in order to ensure the existence of a properly colored copy of G in an m-good edge-coloring of K_n; we have instead constrained the choice of n to be exactly $|V(G)| = n(G)$, and we seek reasonable conditions on m and G for which $f(m, G)$ must equal its minimum conceivable value, namely $n(G)$. Our first main result in this direction is the following.

Theorem 4.3 *Let $G = (V, E)$ be a graph with n vertices, maximum degree at most d, and suppose that $n > 216(3m + 2d)^7(d + 1)^{20}m$. Then every m-good coloring of K_n contains a properly colored copy of G, i.e. $f(m, G) = n(G)$.*

Toward proving the above theorem, we start with a result which essentially says that $f(m, G)$ cannot be too much larger than $n(G)$ for a graph G having bounded maximum degree.

Theorem 4.4 *Suppose $\varepsilon \in (0, 1)$, with d, n any positive integers. If G is a graph with $n(G) < (1 - \varepsilon)n - (2d^2\sqrt{m/\varepsilon})\sqrt{n} - md^2$ and maximum degree at most d, then $f(m, G) < n$.*

Proof. Let c be an m-good edge-coloring of $K = K_n$. We show that c has a properly colored copy of G. Set $x = \sqrt{mn/\varepsilon}$. Define a pair $u, v \in V(K)$ to be a *bad pair* if there are more than x monochromatic (u, v)-paths of length 2. Using convexity, the number of monochromatic paths of length 2 centered at any one vertex is at most $(n/m)\binom{m}{2}$. Hence there are at most $(n/m)\binom{m}{2}(n) \leq (1/2)mn^2$ monochromatic paths of length 2 in K, so there are at most $(1/2)mn^2/x$ bad pairs in K. Let H be the graph with vertex set $V(K)$ in which $uv \in E(H)$ iff u, v is a bad pair in K. We have $e(H) \leq (1/2)mn^2/x$, and thus $\overline{d}(H) \leq mn/x$, where $\overline{d}(H)$ denotes the average degree in H.

Let $L = \{u : d_H(u) > (1/\varepsilon)(mn/x)\}$. Clearly we have $|L| \leq \varepsilon n$. Let $F = H - L$. Then $n(F) \geq (1 - \varepsilon)n$, and $\Delta(F) \leq (1/\varepsilon)(mn/x)$. Let $V = V(F)$, and let $K' = K[V]$. Note that $n(K') = n(F) \geq (1 - \varepsilon)n$. Consider the coloring c restricted to K'. Note that if $u, v \in V$ and $uv \notin E(F)$, then u, v are connected by at most x monochromatic paths of length 2 in K'.

We now embed G in K' greedily as follows. Let $t = n(G)$ and denote the vertices of G by $v_1, v_2, \cdots, v_t$. We embed them one by one in that order, eventually assigning each vertex $v_i \in V(G)$ to some vertex in $V(K')$. We say that this associated vertex *plays the role* of v_i within $V(K')$, sometimes slightly abusing notation by referring to that associated vertex by the name v_i (within $V(K')$) once having settled on which vertex is associated with v_i. For each i, let G_i denote the subgraph of G induced by $v_1, \cdots, v_i$. We insist that the following two conditions are met for all i:
(1) The embedded copy of G_i is properly colored, and
(2) If $v_j, v_k \in G_i$ have a common neighbor in G then $v_jv_k \notin E(F)$.
The embedding of G subject to these two conditions then guarantees that K' contains a properly colored copy of G.

We show that all of $v_1, \cdots, v_t$ can be embedded satisfying (1) and (2). Suppose we have embedded $v_1, \cdots, v_{j-1}$ and are now to embed v_j. In a graph G, for a vertex y (resp. a subset Y of $V(G)$) let $N_G(y)$ (resp. $N_G(Y)$) denote the set of neighbors of y (resp. the set of vertices having at least one neighbor in Y). Now let $N_j = N_G(v_j) \cap V(G_{j-1})$ and $N_j' = N_G(N_G(v_j)) \cap V(G_{j-1})$. We have $|N_j| \leq d$ and $|N_j'| \leq d(d-1) \leq d^2$. Among all vertices outside G_{j-1}, only the following three types of vertices cannot play the role of v_j.

Type 1. Vertices w for which there exists $u \in N_j'$ such that $uw \in E(F)$. Note that such u, w violate condition (2) above. Since there are at most d^2 vertices in N_j' and each is incident to at most $\Delta(F) \leq (1/\varepsilon)(mn/x)$ edges of F, there are at most $d^2(1/\varepsilon)(mn/x)$ such vertices w.

Type 2. Vertices w for which there exist $u, v \in N_j$ such that uw and vw have the same color. Since u, v have a common neighbor in G and $u, v \in V(G_{j-1})$, by condition (2), $uv \notin E(F)$. So there are at most x monochromatic (u, v)-paths of length 2 in K'. Thus there are at most $\binom{d}{2}x \leq d^2x$ such w.

Type 3. Vertices w such that there exists $u \in N_j$ for which uw has a color already used on an edge of G_{j-1} incident to u. There are at most $dm(d-1) \leq md^2$ such w.

Clearly any vertex w not of one of the three types above can play the role of v_j. Recalling that $x = \sqrt{mn/\varepsilon}$, there are at most $\phi = (d^2/\varepsilon)(mn/x) + d^2x + md^2 = 2d^2\sqrt{mn/\varepsilon} + d^2m$ vertices of those three types. So, if our assumed condition $n(G) < (1-\varepsilon)n - 2d^2\sqrt{mn/\varepsilon} - d^2m$ holds, then $n(K') \geq (1-\varepsilon)n > n(G) + \phi$, and one can always find a vertex w in K' to play the role of v_j, completing the proof. ■

Remark 4.5 The proof of Theorem 4.4 can be slightly modified to yield an upper bound on $g(m, G)$, concerning rainbow copies of G in an m-good edge-coloring of K_n. Namely, one defines Type 3 vertices to be vertices w such that there exists $u \in N_j$ for which uw has a color already used on any edge of G_{j-1}. This observation, later called Lemma 6.4, will be used in Section 6.

Corollary 4.6 *Let n, m, d be positive integers with $n > 8md^4$. If G has maximum degree d and $n(G) \leq n - 6d^{4/3}m^{1/3}n^{2/3}$, then $f(m, G) \leq n$.*

Proof. Apply Theorem 4.4 with $\epsilon = \frac{2d^{4/3}m^{1/3}}{n^{1/3}}$, and check that indeed $\epsilon < 1$, that $\epsilon n > 2d^2\sqrt{mn/\epsilon}$, and that $\epsilon n > md^2$. Thus $n(G) \leq n - 6d^{4/3}m^{1/3}n^{2/3} = n - 3\epsilon n$ satisfies the assumptions in Theorem 4.4. ■

We are ready to prove the main result of the section.

Proof of Theorem 4.3:

Let G be as in the theorem. Let $p = 3md + 2d^2$, $D = p(d+1) = (3md + 2d^2)(d+1)$, and $q = 6D^{4/3}m^{1/3}n^{2/3}$. Because n is sufficiently large, we can (as indicated below) select a set U of q vertices, together with q pairwise disjoint subsets V_u in $V(G) - U$, one for each $u \in U$, satisfying the following conditions.
(i) U is an independent set in G.
(ii) $|V_u| = p$ for all $u \in U$.
(iii) For each $u \in U$, V_u contains all the neighbors of u and all their neighbors in G.
(iv) There are no edges of G with ends in two distinct sets V_u.

We construct the set U and the sets V_u for each $u \in U$ iteratively as follows. To select the next vertex u' of U, we select any vertex not within distance 3 of any vertex of any V_u or any u previously selected. Note that this ensures that each vertex v' within distance 2 of u' is nonadjacent with each vertex of each previously selected V_u, so that by next including each such v' in the set $V_{u'}$ we satisfy condition (iv). It is then a simple matter to fill out the rest of the set $V_{u'}$ using any remaining vertices that are not adjacent to any vertices of any $\{u\} \cup V_u$ previously selected. This construction approach succeeds as follows. From the hypothesis $n > 216(3m+2d)^7(d+1)^{20}m$ we have $n > 216(3md+2d^2+1)^7(d+1)^{13}m$, so $n^{1/3} > 6(3md+2d^2+1)^{7/3}(d+1)^{13/3}m^{1/3} > 6(p(d+1))^{4/3}m^{1/3}(3md+2d^2+1)d^3$, so upon multiplying by $n^{2/3}$ we have $n > 6(p(d+1))^{4/3}m^{1/3}n^{2/3}(3md+2d^2+1)d^3 = q(p+1)d^3$, i.e. $n > q(p+1)d^3$. But in selecting u' there are at most $q(p+1)$ previously selected vertices among all the u and V_u vertices combined, each of them with at most d^3 vertices within distance 3 of them (since G has maximum degree at most d). Thus, a choice of u' exists among the other $n - q(p+1)d^3$ remaining vertices. Filling out the rest of the set $V_{u'}$ to size p, after including vertices within distance 2 of u', is possible since $n > q(p+1)d^3 \geq p + q(p+1)(d+1)$. Hence, the set U and the V_u's exist, as desired.

For each $u \in U$, let W_u denote the set of vertices in $V(G) - U - V_u$ adjacent to at least one vertex in V_u. Note that condition (iv) above ensures that W_u is disjoint from $\bigcup_{u \in U} V_u$.

Now, we obtain a graph H from G as follows. First, we delete all vertices of U from G. Then for each $u \in U$ we replace the subgraph of G induced by V_u by a complete graph on V_u and replace the bipartite subgraph of G between V_u and W_u by a complete bipartite graph between V_u and W_u. One can easily check that H has $n(H) = n - |U| = n - q = n - 6D^{4/3}m^{1/3}n^{2/3}$ vertices, and has maximum degree at most $|V_u|(d+1) = D$.

Given an m-good edge-coloring c of $K = K_n$, we show that there exists a properly colored copy of G. First, by Corollary 4.6 we can find a properly colored copy H' of H in K. For each $u \in U$ let V'_u denote the copy of V_u in H'. Let $U' = V(K) - V(H')$; note that $|U'| = |U|$. Consider an arbitrary bijection from U to U', letting $u' \in U'$ be the image of any $u \in U$. Our aim now is to show that by joining U' to H' using appropriate edges and deleting certain edges from H' if necessary, one can find a properly colored copy of G.

For each $u \in U$, we first look for a set of d_u many vertices in V'_u to play the role of $N_G(u)$, where $d_u = |N_G(u)| \leq d$. To achieve that, it suffices to find a subset S of V'_u with cardinality d_u such that the complete subgraph induced by $\{u\} \cup S$ is properly colored. This can be accomplished as follows. First, we obtain a subgraph F'_u of $H'[V'_u]$ by the following operation: for each $x' \in V'_u$, if there exists an edge in $H'[V'_u]$ incident to x' having the same color as $u'x'$, we remove this edge. Note that since H' is properly colored there exists at most one such edge for x'. Clearly F'_u obtained this way has $p = 3md + 2d^2$ vertices and at least $\binom{p}{2} - p$ edges. By Turán's Theorem, F'_u contains a complete subgraph L'_u of order at least $p/3 \geq md$. Note by our definition of F'_u and L'_u that no color used on an edge between u' and L'_u is used on an edge in L'_u. Now, since $n(L'_u) > md \geq md_u$, and c is m-good, it is easy to see that there exist d_u vertices in L'_u that are connected to u' by different colors. Denote by S the set of these d_u vertices; S plays the role of $N_G(u)$.

To finish the embedding, we embed $V_u - N_G(u)$ into $V'_u - S$ as follows. Let us now return to the graph $H'[V'_u]$, setting aside the graph F'_u previously formed. First, for each $x' \in S$, if there is an edge incident to x' in $H'[V'_u]$ having the same color as $u'x'$ then we remove this edge (note: this repeats the operation of the above paragraph, this time restricted to edges incident on vertices x' of S). Note that altogether at most $|S| = d_u \leq d$ edges are removed; let R'_u denote the remaining subgraph. Let $T = \bigcup_{x \in N_G(u)} N_G(x)$. Since G has maximum degree at most d, $|T| \leq d^2$. But R'_u has at least $p - d$ vertices outside S, and at most d vertices of R'_u are not adjacent to all of S. Thus R'_u has at least $p - 2d = 3md + 2d^2 - 2d \geq d^2 \geq |T|$ vertices outside S which are adjacent to all of S. Thus one can embed T. Finally we map the rest of V_u to the remaining vertices of V'_u by an arbitrary bijection and let the rest of the vertices in H' keep their identities. It is straightforward to check that the mapping described above provides a properly colored copy of G. ■

Roughly speaking, Theorem 4.3 asserts that if G is a graph on n vertices and maximum degree at most d, and n is sufficiently large relative to m and d, then every m-good coloring of $E(K_n)$ yields a proper copy of G. However, for fixed d, we require n to be at least on the order of m^8. Even though it is not very hard to improve the exponent of m in Theorem 4.3 , we have been unable reduce it down to 1 (although we suspect that a corresponding claim holds in which n grows linearly in m). For special classes of graphs, however, we are able to improve the exponent of m to 1. Specifically, we will lay out the proof for powers of cycles, although the same proof can be adapted for somewhat larger classes of graphs.

We need some preparations. Mainly, we need a couple of splitting lemmas, which are modified from the one used in [3]. We also use the Chernoff inequality and the Lovász Local Lemma in our proofs. We cite here a somewhat conservative version of the Chernoff inequality which can be found, for example, in [2] or [16].

Lemma 4.7 (The Chernoff Inequality) *Let X_i be independent, identically distributed random variables, where $X_i = 1$ with probability p and $X_i = 0$ with probability $1-p$. Consider the random variable $X = \Sigma_{i=1}^{n} X_i$. Then whenever $0 \le t \le np$,*

$$Prob\ (X - np > t) < exp(-\frac{t^2}{3np}).$$

Lemma 4.8 (The Local Lemma) *Let $A_1, \cdots, A_n$ be events in an arbitrary probability space. Suppose each A_i is mutually independent of all but at most b other events A_j, and suppose the probability of each A_i is at most p. If $ep(b+1) < 1$, then with positive probability none of the events A_i holds.*

As with our notation $G[A]$ for the subgraph of G induced by A, we also use the notation $G[A, B]$ for the subgraph of G with vertex set being the union of A and B, and with edge set $\{ab : a \in A, b \in B\}$.

Lemma 4.9 *Let n, m be positive even integers, where $m > 125(\ln n)^3$. Let c be an m-good edge-coloring of $K = K_n$. Then there exists a partition of $V = V(K)$ into subsets A, B such that $|A| = |B| = n/2$ and the colorings obtained from c by restricting it to $K[A], K[B]$, and $K[A, B]$ are all $(m/2 + m^{2/3})$-good.*

Proof. Let $q = n/2$. First, pair up the vertices of V arbitrarily into q pairs, say $\{x_1, y_1\}, \cdots, \{x_q, y_q\}$. Next, we construct a random partition of V into two disjoint subsets A and B of cardinality q as follows. For each $i \in [q]$, place one element of the pair $\{x_i, y_i\}$ in A and the other in B, each choice made independently with equal likelihood $1/2$.

Fix a vertex w and a color used in c, say red. Let $N_A(w)$ (resp. $N_B(w)$) denote the number of neighbors of w in A (resp. B) joined to w by a red edge. Thus $N_A(w)$ can be written as the sum of q independent indicator random variables $\sigma_1, \cdots, \sigma_q$, where σ_i is the number of neighbors of w in A among x_i, y_i such that the edge from w to this neighbor is red. Thus each σ_i has one of the following three simple distributions: either its value is 1 with probability 1 (if wx_i, wy_i are both red), or its value is 0 with probability 1 (if neither of wx_i, wy_i is red), or its value is 1 with probability 1/2 (if exactly one of wx_i, wy_i is red) and 0 otherwise. Let t denote the number of i's such that exactly one of wx_i, wy_i is red. Let r denote the number of red edges incident to w, so that $r \le m$. Our discussion above indicates that

Prob $(N_A(w) > (r/2) + \mu)$ equals the probability that more than $(t/2) + \mu$ flips among t independent flips of a fair coin yield "heads". By the Chernoff bound, this probability is at most $e^{-\mu^2/(3t/2)} \le e^{-2\mu^2/3m}$.

Clearly, the same argument also applies to $N_B(w)$. Now, since there are n choices for a vertex w, at most $n^2/2$ choices for a color used in c, and 2 choices for a partite set (A or B), we conclude that the probability that there exists a vertex with more than

$$m/2 + m^{2/3}$$

neighbors of the same color in either A or B is at most

$$n^3 e^{-2m^{4/3}/3m} = n^3 e^{-2m^{1/3}/3},$$

which is less than 1 for $m > 125(\ln n)^3$. Therefore, there exists a choice of A and B so that the above does not happen. Clearly, for this choice of A and B, $K[A], K[B], K[A, B]$ are all $(m/2 + m^{1/3})$-good. ■

Note that Lemma 4.9 holds even when m is not an integer, as long as we understand that an m-good coloring is meant to be an $\lfloor m \rfloor$-good coloring.

Lemma 4.10 *Suppose $n = 2^q \cdot N$. Let $m_0, m_1, \cdots, m_q$ be a sequence of positive numbers such that $m_{i+1} = m_i/2 + m_i^{2/3}$ for each $i \in [q-1]$, and $m_i > (3\ln N + 12)^3$ all for $i \in [q]$. Let K be an m-good edge-colored K_n, where $m = m_0$. Then there exists a partition of $V(K)$ into 2^q subsets of cardinality N such that for each $i \in [2^q]$, $K[V_i]$ and $K[V_i, V_{i+1}]$ are both m_q-good, where indices are taken modulo 2^q.*

Proof. We prove the claim by induction on q. For $q = 1$, the claim follows from Lemma 4.9. For the induction step, we suppose that $V(K)$ can be split into P subsets $U_1, U_2, \cdots, U_P$ of cardinality $2N$ where $P = 2^{q-1}$, such that for each $i \in [P]$, $K[V_i]$ and $K[V_i, V_{i+1}]$ are both m_{q-1}-good, where indices are taken modulo P.

Independently for each $i \in [P]$, we obtain a random splitting of U_i into two subsets $U_{i,1}, U_{i,2}$ of cardinality N like in the proof of Lemma 4.9: first pairing up vertices of U_i into N pairs, then randomly and independently putting one vertex of each pair in $U_{i,1}$ and the other in $U_{i,2}$. We show that there exists a splitting (for all i simultaneously) that yields a desired partition.

First, let us fix a color, say red. Consider an arbitrary vertex $u \in U_i$ for some i. For $s = i-1, i, i+1, l = 1, 2$, let $X_{s,l}^u$ denote the event that the number of red neighbors of u in $U_{s,l}$ exceeds $m_q = m_{q-1} + m_{q-1}^{2/3}$. Since $K[U_i], K[U_{i-1}, U_i]$ and $K[U_i, U_{i+1}]$ are all m_{q-1}-good, by the same calculations as in the proof of Lemma 4.9, we have for each $s \in \{i-1, i, i+1\}, l \in \{1, 2\}$, Prob $(X_{s,l}^u) \leq e^{-\frac{2}{3}m_{q-1}^{1/3}}$. Let $Z_u = \bigcup_{s=i-1}^{i+1} \bigcup_{l=1}^{2} X_{s,l}^u$. We have Prob $(Z_u) \leq 6e^{-\frac{2}{3}m_{q-1}^{1/3}}$. Let D_u denote the union of Z_u over all possible colors that are incident to u in $U_i, [U_{i-1}, U_i]$, or $[U_i, U_{i+1}]$; there are at most $3(2N) = 6N$ such colors. We have

$$\text{Prob } (D_u) \leq (6N)6e^{-\frac{2}{3}m_{q-1}^{1/3}} = 36Ne^{-\frac{2}{3}m_{q-1}^{1/3}}.$$

Note that if we can show that with positive probability none of the events D_u's holds, then there exists a choice of $U_{i,1}, U_{i,2}$ for all i such that $K[U_{i,1}], K[U_{i,2}], K[U_{i,1}, U_{i,2}], K[U_{i,1}, U_{i+1,2}]$ are all m_q-good (indices taken modulo P). We can then obtain our splitting by defining $V_{2j} = U_{j,1}$ and $V_{2j-1} = U_{j,2}$ for $j = 1, \cdots, P = 2^{q-1}$.

To show that with positive probability none of the events D_u's holds, we apply the Local Lemma. Note that D_u is mutually independent of all D_v's except for those v that lie in U_j, where $|j - i| \leq 2$; there are $5(2N) = 10N$ such v. Now, we have

$$e \cdot (36Ne^{-\frac{2}{3}m_{q-1}^{1/3}}) \cdot (10N) < 1,$$

since $m_{q-1} > (3\ln N + 12)^3$ by our condition. By the Local Lemma, Prob $(\bigcap_u \overline{D_u}) > 0$, completing the proof. ∎

Corollary 4.11 *For every $\varepsilon > 0$ there exists a constant $c_2 = c_2(\varepsilon)$ such that the following holds. Let n, m, N, q be positive integers such that $n = 2^q \cdot N$ and $m/2^q > \max\{c_2, (3\ln N + 12)^3\}$. Let K be an m-good edge-colored K_n. There exists a partition of $V(K)$ into 2^q subsets of cardinality N such that for each $i \in [2^q]$, $K[V_i]$ and $K[V_i, V_{i+1}]$ are both $(1+\varepsilon)m/2^q$-good, where indices are taken modulo 2^q.*

Proof. Given ε, let $c_2 = c_2(\varepsilon)$ satisfy

$$c_2 \geq \frac{512}{((1+\varepsilon)^{1/3}-1)^3}. \qquad (1)$$

Let $m_0 = m$, and let $m_{i+1} = m_i/2 + m_i^{2/3}$ for $i = 0, \cdots, q-1$. Note that for all i, $m_i > m/2^q > (3\ln N + 12)^3)$. By Lemma 4.10, it suffices to show that $m_q \leq (1+\varepsilon)m/2^q$. Clearly, we have $m_{i+1} = m_i/2 + m_i^{2/3} \leq 1/2(m_i^{1/3}+2)^3$. Hence, by taking cube roots and subtracting $\frac{2}{2^{1/3}-1}$ from both sides, we have

$$m_{i+1}^{1/3} - \frac{2}{2^{1/3}-1} \leq \frac{1}{2^{1/3}}(m_i^{1/3}+2) - \frac{2}{2^{1/3}-1} = \frac{1}{2^{1/3}}\left(m_i^{1/3} - \frac{2}{2^{1/3}-1}\right).$$

Therefore

$$m_q^{1/3} - \frac{2}{2^{1/3}-1} \leq \frac{1}{2^{q/3}}\left(m_0 - \frac{2}{2^{1/3}-1}\right),$$

and since $m_0 = m$ and $2^{1/3} - 1 > 1/4$,

$$m_q^{1/3} \leq \frac{m^{1/3}}{2^{q/3}} + 8 \leq (1+\varepsilon)^{1/3}\frac{m^{1/3}}{2^{q/3}}.$$

The last inequality follows from (1) and the assumption that $m/2^q \geq c_2$. Thus, $m_q \leq (1+\varepsilon)m/2^q$, completing the proof. ∎

Now, we are ready to prove the result for cycle powers. For positive integers n, d, C_n^d denotes the d-th power of a cycle of length n.

Theorem 4.12 *There exist constants c_3, c_4 such that $f(m, C_n^d) = n$ provided $n \geq c_3 m d^{c_4}$.*

Proof. Let $D = 2d^5$. Choose c_3, c_4 to be large enough such that

$$\frac{c_3}{2}d^{c_4+1} > 216(3d+2D)^7(D+1)^{20}d. \qquad (2)$$

For convenience we make a few assumptions, each of which is not essential, but is convenient. Assume first that $m = 2^{q-1}\cdot d$ for some positive integer q. Assume also that $n = c_3 m d^{c_4}$. Let $N = n/2^q$. (We thus assume that n is divisible by 2^q. This is convenient, but not really essential, as if this is not the case we can split the vertex set into nearly equal parts, instead of splitting it, in Corollary 4.11, into equal classes. As this does not change the arguments, besides making them a bit cumbersome, we assume n is indeed divisible by 2^q). Note that by our assumption $m/2^q = \frac{d}{2}$ and $N = \frac{c_3}{2}d^{c_4+1}$. Let c_2 be the constant in Corollary 4.11 corresponding to $\epsilon = 0.5$. Assume d is large enough such that $d > \max\{c_2, (3\ln N + 12)^3\} = \max\{c_2, [3\ln(\frac{c_3}{2}d^{c_4+1}) + 12]^3\}$. Clearly, we lose no generality by making this assumption, as for small d we can compensate by adjusting c_3.

Let K be an m-good colored K_n. Note that n, m, N, q satisfy the conditions of Corollary 4.11 for $\epsilon = 0.5$. Applying Corollary 4.11, we obtain a partition of $V = V(K)$ into $V_1, V_2, \cdots, V_{2^q}$, each of size $N = n/2^q$ such that for each $i \in [2^q]$, both $K[V_i]$ and $K[V_i, V_{i+1}]$ are s-good, where $s = (1+0.5)m/2^q = .75d \leq d$ (indices taken modulo 2^q). In particular, for each $i \in [2^q]$, both $K[V_i]$ and $K[V_i, V_{i+1}]$ are d-good.

By our discussion above, for each i, $K[V_i]$ is d-good and $|V_i| = N = \frac{c_3}{2}d^{c_4+1} > 216(3d+2D)^7(D+1)^{20}d$, where the last inequality follows from (2). By Theorem 4.3, $K[V_i]$ contains a properly colored copy P_i of the d^5-th power of a spanning path of $K[V_i]$ (note that such a spanning subgraph has maximum degree $2d^5$). Our idea is, roughly speaking, to link up the P_i's to get a properly colored C_n^d. Let $U_{i,1}$ and $U_{i,2}$ denote the set of the first d^5 and the last d^5 vertices on P_i. Since $|V_i| = N > 2d^5$, $U_{i,1}$ and $U_{i,2}$ are disjoint from each other.

Claim. For each $i \in [2^q]$, there exist d vertices $x_{i,1}, \cdots, x_{i,d}$ of $U_{i,2}$ and d vertices $y_{i+1,1}, \cdots, y_{i+1,d}$ of $U_{i+1,1}$ such that the subgraph induced by these $2d$ vertices is properly colored.

Proof of Claim. Randomly select d elements $x_{i,1}, \cdots, x_{i,d}$ from $U_{i,2}$ and d elements $y_{i+1,1}, \cdots, y_{i+1,d}$ from $U_{i+1,1}$. Recall that $K[V_i], K[V_{i+1}]$, and $K[V_i, V_{i+1}]$ are all d-good. Thus the probability for a fixed pair of incident edges in the subgraph L induced by $x_{i,1}, \cdots, x_{i,d}, y_{i+1,1}, \cdots, y_{i+1,d}$ to have the same color is less than (roughly) d/d^5. Hence, the probability that L has a pair of incident edges of the same color is less than $2d \cdot \binom{2d-1}{2} \cdot d/d^5 < 1$. Therefore there exists a choice of $x_{i,1}, \cdots, x_{i,d} \in U_{i,2}$ and $y_{i+1,1}, \cdots, y_{i+1,d} \in U_{i+1,1}$ such that the subgraph L induced by these $2d$ vertices is properly colored, proving the claim.

Note that our choices of the $x_{i,j}$'s and $y_{i+1,j}$'s are independent for different i's. For convenience we let $X_i = \{x_{i,1}, \cdots, x_{i,d}\}$ and $Y_i = \{y_{i,1}, \cdots, y_{i,d}\}$. Now, if we can find as a subgraph in each P_i a properly colored copy P_i' of the d-th power of a spanning path of P_i satisfying

(1) $y_{i,1}, \cdots, y_{i,d}$ are the first d vertices on P_i' and $x_{i,1}, \cdots, x_{i,d}$ are the last d vertices on P_i',
(2) the color on an edge of P_i' with exactly one endpoint in Y_i is not used in $K[Y_i \cup X_{i-1}]$,
(3) the color on an edge of P_i' with exactly one endpoint in X_i is not used in $K[X_i \cup Y_{i+1}]$,

then $\bigcup_{i=1}^{2^q} P_i'$ together with the edges in $K[X_i, Y_{i+1}]$ for $i = 1, \cdots, 2^q$ (indices modulo 2^q) clearly contains a properly colored copy of C_n^d, and we are done.

Thus, it remains to show that one can find such P_i' inside P_i. Let S denote the set of vertices on P_i outside Y_i that are connected to vertices of Y_i by edges of P_i using colors used in $K[Y_i \cup X_{i-1}]$. Since there are fewer than $(2d)^2/2 = 2d^2$ colors used in $K[Y_i \cup X_{i-1}]$ and P_i is properly colored, it is easy to see that $|S| \leq 2d^2$. Recall that $U_{i,1}$ denotes the first d^5 vertices on P_i. Let S_i' be a subset of $U_{i,1} - S_i \cup Y_i$ of cardinality d. Since $d^5 - 2d^2 - d \geq d$, S_i' exists. Similarly, let T_i denote the set of vertices on P_i outside X_i that are connected to vertices of X_i by edges of P_i using colors used in $K[X_i \cup Y_{i+1}]$. We have $|T_i| \leq 2d^2$. Let T_i' be a subset of $U_{i,2} - T_i \cup X_i$ of cardinality d; such T_i' exists. Furthermore, S_i' and T_i' are disjoint since $U_{i,1}$ and $U_{i,2}$ are disjoint.

For convenience, we let π denote the the ordering of vertices of P_i along P_i. We now define a re-ordering π' of the vertices of P_i as follows. We let Y_i occupy the first d spots and S_i' occupy the next d spots. At the other end, we let X_i occupy the last d spots and T_i' the d spots immediately preceding the last d spots. Finally, we fill the remaining vertices of P_i into the remaining spots in accordance to their relative order in π. Note that the first d^5 vertices in π still occupy the first d^5 spots in π' and the last d^5 vertices in π still occupy the last d^5 spots in π'. Each of the remaining vertices occupies the same spot in π as in π'. In particular, two vertices at most d apart in π' are certainly at most d^5 apart in π.

Let P_i' be a spanning subgraph of $K[V_i]$ consisting of all edges connecting vertices at most d apart in π'. Clearly P_i' is the d-th power of a spanning path of $K[V_i]$ that satisfies condition (1). Since two vertices in π' at most d apart are at most d^5 apart in π, all edges of P_i' are present in P_i. Hence P_i' is a (spanning) subgraph of P_i and thus is properly colored. It remains to verify that P_i' satisfies conditions (2) and (3). For that, note that an edge of P_i' with exactly one endpoint in Y_i has the other endpoint in one of the first $2d$ spots. Since no element of S_i occupies any of the first $2d$ spots, condition (2) is satisfied. Condition (3) is satisfied by a similar argument. ■

5 Bounds on $g(m, K_t)$

Recall that $g(m, G)$ denotes the smallest n such that every m-good coloring of $E(K_n)$ yields a rainbow copy of G. We first establish a lower bound on $g(m, K_t)$ using the following result of Babai.

Lemma 5.1 [6] *Let n be a positive integer. Every proper coloring of $E(K_n)$ yields a rainbow complete subgraph of order at least $(2n)^{1/3}$. Furthermore, there exists a coloring of $E(K_n)$ whose largest rainbow complete subgraph has order less than $8(n \ln n)^{1/3}$.*

The lower bound in Lemma 5.1 was later improved by Alon, Lefmann, and Rödl [5] to $c'(n \ln n)^{1/3}$, where c' is a positive constant. Building on Lemma 5.1, we obtain a lower bound on $g(m, K_t)$.

Theorem 5.2 *There exists a positive constant c'_1 such that for all positive integers m, t, where $t \geq 3$, we have*

$$g(m, K_t) > \frac{c'_1 m t^3}{\ln t}.$$

Proof. For $t \leq 20$, say, the result trivially follows from Lemma 2.3 (for an appropriate choice of c'_1). We thus assume $t > 20$. Choose c'_1 to be small enough such that $c'_1 m t^3 / \ln t \leq m \lfloor (1/1536) t^3 / \ln t \rfloor$ for all positive integers m, t, where $t > 20$; such c'_1 clearly exists. Let $N = \lfloor (1/1536) t^3 / \ln t \rfloor$. By definition it suffices to show that there exists an m-good edge-coloring of K_{mN} containing no rainbow copy of K_t. By Lemma 5.1 there exists a proper coloring c of $E(K_N)$ such that the largest rainbow complete subgraph has order less than

$$\begin{aligned} 8(N \ln N)^{1/3} &\leq 8[\frac{1}{1536} \cdot \frac{t^3}{\ln t} \cdot \ln(\frac{1}{1536} \cdot \frac{t^3}{\ln t})]^{1/3} \\ &< 8[\frac{1}{1536} \cdot \frac{t^3}{\ln t} \cdot \ln t^3]^{1/3} = t \end{aligned}$$

In other words, c is a 1-good coloring of $E(K_N)$ containing no rainbow copy of K_t. By Lemma 2.2, there exists an m-good coloring of $E(K_{mN})$ containing no rainbow copy of K_t. ∎

Next, we give upper bounds on $g(m, K_t)$. First, we give a simple, general upper bound on $g(m, K_t)$ which is within a factor of $\ln t$ from the lower bound given in Theorem 5.2. Then we improve it to within a constant factor of the lower bound. This is done by first obtaining such a bound when m grows modestly slowly as a function of t, and then by observing that using the splitting argument in Corollary 4.11 we can reduce the general case to this one.

Theorem 5.3 *For all positive integers m, t we have*

$$g(m, K_t) \leq 2mt^3 + 4mt^2.$$

Proof. Let $n = 2mt^3 + 4mt^2$, and let c be an arbitrary m-good coloring of $E(K_n)$. We show that c yields a rainbow complete subgraph of order t. Similar to the definition of a bad 2-claw from Section 3, we define a *bad* $2K_2$ to be a set of two independent edges of the same color. Since c is m-good, the number of edges of each fixed color does not exceed $nm/2$. Thus, one can easily deduce that

$$\begin{aligned} \#\text{bad 2-claws} &\leq n \cdot (n/m) \cdot \binom{m}{2} < \frac{1}{2} m \cdot n^2 \\ \#\text{bad } 2K_2\text{'s} &\leq \frac{1}{2}\binom{n}{2} \cdot n \cdot m/2 < \frac{1}{8} m \cdot n^3 \end{aligned}$$

Pick randomly and uniformly a $2t$-subset T of $V(K_n)$. For any fixed bad 2-claw L let A_L denote the event that T contains all three vertices of L. For any fixed bad $2K_2$ graph Q let B_Q denote the event that T contains all four vertices of Q. Clearly we have

$$\begin{aligned}\text{Prob }(A_L) &= \frac{\binom{n-3}{2t-3}}{\binom{n}{2t}} \leq \left(\frac{2t}{n}\right)^3 \textit{ and}\\ \text{Prob }(B_Q) &= \frac{\binom{n-4}{2t-4}}{\binom{n}{2t}} \leq \left(\frac{2t}{n}\right)^4.\end{aligned}$$

Let X (resp. Y) denote the number of bad 2-claws (resp. bad $2K_2$'s) contained in T. We have

$$\begin{aligned}E(X) &= \Sigma_L \text{ Prob }(A_L) < \frac{1}{2}mn^2\left(\frac{2t}{n}\right)^3 = \frac{4mt^3}{n},\\ E(Y) &= \Sigma_Q \text{ Prob }(A_Q) < \frac{1}{8}mn^3\left(\frac{2t}{n}\right)^4 = \frac{2mt^4}{n}.\end{aligned}$$

By linearity of expectation $E(X+Y) < \frac{4mt^3+2mt^4}{n} = t$. This implies that there exists a $2t$-subset T' of $V(K_n)$ that contains at most t bad $2K_2$'s or bad 2-claws. By deleting (at most) one vertex of each such bad structure from T' we obtain a subset T'' of T' of order at least t containing no bad 2-claw or bad $2K_2$. Clearly, T'' induces a rainbow complete subgraph of order at least t. ∎

Next, we improve the upper bound on $g(m, K_t)$ to within a constant factor of the lower bound. We need the following result of Duke, Lefmann, and Rödl [10] on uncrowded hypergraphs, although we note that a previous slightly weaker result proven in [1] would be enough for our purpose. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. For a vertex $x \in \mathcal{V}$, let $\deg_{\mathcal{H}}(x)$ denote the degree of x in $\mathcal{H}$, i.e. the number of edges $e \in \mathcal{E}$ containing x, and let $\Delta(H) = \max\{\deg_{\mathcal{H}}(x) : x \in \mathcal{V}\}$. Let $\alpha(\mathcal{H})$ denote the independence number of $\mathcal{H}$, which is defined as the largest size of a subset of $\mathcal{V}$ containing no edge of $\mathcal{H}$.

Theorem 5.4 [10] *For every fixed $k \geq 3$ there is an $a_0 = a_0(k) > 0$ such that the following holds. Let $\mathcal{H} = (V, \mathcal{E})$ be a k-uniform hypergraph on n vertices with maximum degree $\Delta(\mathcal{H}) \leq b^{k-1}$, where $b \gg k$. If $\mathcal{H}$ contains no 2-cycles, then*

$$\alpha(\mathcal{H}) \geq a_0 \cdot \frac{n}{b}(\ln b)^{1/(k-1)}.$$

Theorem 5.5 *Let β be a constant with $0 < \beta < 2$. There exists a positive absolute constant c_2' such that for any positive integer t sufficiently large and $m = m(t) = O\left(t^{2-\beta}\right)$ we have*

$$g(m, K_t) < \frac{c_2' mt^3}{\ln t}.$$

Proof. Let c_2' be a sufficiently large positive constant to be specified later. Let n be any positive integer with $n \geq c_2' mt^3/\ln t$. We show that every m-good edge-coloring of $K = K_n$ contains a rainbow copy of K_t.

Let c be an m-good edge-coloring of K. As in the proof of Theorem 5.3, the number of bad 2-claws in c is at most $\frac{1}{2}m \cdot n^2$ and the number of bad $2K_2$'s in c is at most $\frac{1}{4}m \cdot n^3$.

Now consider a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with vertex set $V = V(K)$, and edge set $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1 = \{\{x,y,z\} : \{x,y,z\}$ is the vertex set of a bad 2-claw in $c\}$, and $\mathcal{E}_2 = \{\{u,v,w,x\} : \{u,v,w,x\}$ is the vertex set of a bad $2K_2$ in $c\}$. So by the above,

$$|\mathcal{E}_1| \leq \frac{1}{2} m \cdot n^2 \quad \text{and} \quad |\mathcal{E}_2| \leq \frac{1}{4} m \cdot n^3. \tag{3}$$

Note that if $Z \subseteq V$ is an independent set of $\mathcal{H}$, then Z induces a rainbow complete subgraph. Our aim thus is to find a large independent set of $\mathcal{H}$. Our strategy is to find a large sub-hypergraph of $\mathcal{H}$ with few edges and no 2-cycles and then use Theorem 5.4 to find a large independent set inside it. For that we first bound the number of 2-cycles formed by edges in $\mathcal{E}_2$.

For $l = 2, 3$, let μ_l denote the number of 2-cycles formed by edges in $\mathcal{E}_2$ in which the two edges intersect at exactly l vertices. We bound μ_2 and μ_3 as follows. Observe that since c is m-good, for any two fixed vertices x, y there are fewer than $2m \cdot n$ bad $2K_2$'s in c having x, y as two of four endpoints. Thus each $\{u,v,w,x\} \in \mathcal{E}_2$ intersects fewer than $\binom{4}{2} \cdot 2 \cdot m \cdot n$ other edges of $\mathcal{E}_2$ at exactly two vertices. Therefore

$$\mu_2 \leq |\mathcal{E}_2| \cdot 2 \cdot \binom{4}{2} \cdot m \cdot n/2 < 2m^2 n^4. \tag{4}$$

To bound μ_3, observe that for any fixed three vertices x, y, z there are at most $3m$ bad $2K_2$'s in c that contain x, y, z (again because c is m-good). Thus, each $\{u,v,w,x\} \in \mathcal{E}_2$ intersects fewer than $\binom{4}{3} \cdot 3m$ other edges of $\mathcal{E}_2$ at exactly three vertices. Therefore

$$\mu_3 \leq |\mathcal{E}_2| \cdot \binom{4}{3} \cdot 3m/2 < (3/2) m^2 n^3. \tag{5}$$

Let $p = \frac{1}{3m^{1/2} n^{3/5}}$. Let Y be a random subset of V with vertices chosen independently, each with probability p. We have $E(|Y|) = p \cdot n$. For sufficiently large pn, we have by standard estimations that

$$\text{Prob}\ (|Y| \leq \frac{7}{8} \cdot p \cdot n) < \frac{1}{10}. \tag{6}$$

Let $\mathcal{H}[Y]$ denote the the sub-hypergraph of $\mathcal{H}$ induced by Y. For $l = 2, 3$, let $\mu_l(Y)$ denote the random variable counting the number of 2-cycles in $\mathcal{H}[Y]$ induced by Y with the two edges intersecting at exactly l vertices. Then

$$E(\mu_2(Y)) = \mu_2 \cdot p^6 \leq 2m^2 n^4 p^6 = 2m^2 n^3 p^5 \cdot (p \cdot n) = \frac{2}{3^5 m^{1/2}} (p \cdot n) \leq \frac{2}{3^5} \cdot p \cdot n \tag{7}$$

$$E(\mu_3(Y)) = \mu_3 \cdot p^5 \leq (3/2) m^2 n^2 p^4 \cdot (p \cdot n) = \frac{1}{54 n^{2/5}} \cdot (p \cdot n) \leq \frac{1}{54} \cdot p \cdot n \tag{8}$$

For $i = 1, 2$, and any $U \subseteq V$ let $\mathcal{E}_i(U) = \mathcal{E}(\mathcal{H}[U]) \cap \mathcal{E}_i$. We have

$$E(|\mathcal{E}_1(Y)|) = |\mathcal{E}_1| \cdot p^3 \leq \frac{1}{2} m \cdot n^2 \cdot p^3 = o(p \cdot n), \tag{9}$$

$$E(|\mathcal{E}_2(Y)|) = |\mathcal{E}_2| \cdot p^4. \tag{10}$$

Now, the probability that $\mu_2(Y) > \frac{1}{8} pn$ is at most $\frac{2}{3^5} / \frac{1}{8}$, since $E(\mu_2(Y)) \leq \frac{2}{3^5} pn$. Similar computations show that the probabilities of the separate events $\mu_3(Y) > \frac{1}{8} pn$, $|\mathcal{E}_1(Y)| > 50mn^2p^3$, and $|\mathcal{E}_2(Y)| > 2|\mathcal{E}_2| p^4$ are at most $\frac{1}{54} / \frac{1}{8}$, $\frac{1}{2}/50$, and $1/2$ respectively. Recall also that Prob $(|Y| \leq \frac{7}{8} \cdot p \cdot n) < \frac{1}{10}$. Since these five probabilities sum to less than 1, we deduce that there exists a subset $Y_0 \subseteq V$ with $|Y_0| \geq \frac{7}{8} \cdot p \cdot n$, $\mu_2(Y_0) \leq \frac{1}{8} \cdot p \cdot n$, $\mu_3(Y_0) \leq \frac{1}{8} \cdot p \cdot n$, $|\mathcal{E}_1(Y_0)| \leq 50mn^2p^3 = o(p \cdot n)$, and $|\mathcal{E}_2(Y_0)| \leq 2|\mathcal{E}_2| \cdot p^4$.

We delete from Y_0 a vertex from each 2-cycle and a vertex from each edge in $\mathcal{E}_1(Y_0)$ to obtain a subset $Y_1 \subseteq Y_0$, with $|Y_1| \geq \frac{1}{2}p \cdot n$, such that the sub-hypergraph $\mathcal{H}[Y_1]$ induced by Y_1 has no 2-cycles, no edges of $\mathcal{E}_1$, and at most $2|\mathcal{E}_2| \cdot p^4$ edges in $\mathcal{E}_2$ (and thus is 4-uniform). Note that $\mathcal{H}[Y_1]$ has average degree at most $4|\mathcal{E}(\mathcal{H}(Y_1))|/|Y_1| \leq 4 \cdot (2|\mathcal{E}_2| \cdot p^4)/(\frac{1}{2}p \cdot n) = \frac{16 \cdot |\mathcal{E}_2| \cdot p^4}{p \cdot n}$.

Finally, delete all vertices in Y_1 with degree larger than

$$\frac{32|\mathcal{E}_2| \cdot p^4}{p \cdot n},$$

noting that there are at most $\frac{1}{2}|Y_1|$ such vertices. We obtain a subset $Z \subseteq Y_1$ with at least $\frac{1}{2}|Y_1| \geq \frac{1}{4} \cdot p \cdot n$ vertices such that the sub-hypergraph $\mathcal{G}$ of $\mathcal{H}$ induced by Z satisfies the assumptions of Theorem 5.4 with

$$\Delta(\mathcal{G}) \leq \frac{32 \cdot |\mathcal{E}_2| \cdot p^4}{p \cdot n} \leq 8 \cdot m \cdot n^2 \cdot p^3 = \frac{8}{27} \frac{n^{1/5}}{m^{1/2}} = b^3,$$

recalling from (3) that $|\mathcal{E}_2| \leq \frac{1}{4}m \cdot n^3$. By our condition $m = O(t^{2-\beta})$, and our choice of $n \geq c_2' mt^3/\ln t$, we have $b^3 = \Omega(\frac{n^{1/5}}{m^{1/2}}) = \Omega(t^\gamma)$ for some positive constant γ. Thus, $b \gg 4$. Also,

$$b = (8 \cdot m \cdot n^2)^{1/3} \cdot p.$$

Applying Theorem 5.4 to $\mathcal{G}$ and choosing a_1, a_2, a_3 to be sufficiently small positive constants, we have

$$\begin{aligned} \alpha(\mathcal{H}) &\geq \alpha(\mathcal{G}) \\ &\geq a_0 \cdot \frac{|Z|}{b} \cdot (\ln b)^{1/3} \\ &\geq a_1 \cdot \frac{p \cdot n}{(m \cdot n^2)^{1/3} \cdot p} \cdot [\ln t]^{1/3} \\ &\geq a_2 (\frac{n}{m})^{1/3} (\ln t)^{1/3} \end{aligned}$$

Recall that $n \geq c_2' mt^3/\ln t$. When c_2' is chosen to be sufficiently large beforehand, the last inequality above yields $\alpha(\mathcal{G}) \geq t$. Thus, c contains a rainbow complete subgraph of order t, completing the proof. ∎

Theorem 5.6 *There exists an absolute constant $c_2'' > 0$ such that for all admissible integers m and t,*

$$g(m, K_t) \leq c_2'' \frac{mt^3}{\ln t}.$$

Proof. Clearly we may assume that t is sufficiently large as specified below. If, say, $m \leq t^{3/2}$, then the result follows from Theorem 5.5 with $\beta = 1/2$. Otherwise, suppose $n = c_2'' \frac{mt^3}{\ln t}$, and let c be an m-good coloring of K_n, where $m > t^{3/2}$. We apply Corollary 4.11 as follows. Set $\varepsilon = 1/2$. Without loss of generality, suppose $m = 2^q \cdot (\frac{t}{2})$ (and assume $m/2^q = t/2$ is large enough as needed in Corollary 4.11 for the given ε). By Corollary 4.11, there is a subset $S \subseteq V(K_n)$ of size $(\frac{c_2'' mt^3}{\ln t})/2^q = \frac{c_2'' t^4}{2 \ln t}$, so that the coloring c restricted to S is $(1+\varepsilon)\frac{t}{2}$-good. In particular, c restricted to S is t-good. By Theorem 5.5 there is a rainbow copy of K_t in S, provided c_2'' is sufficiently large. This completes the proof. ∎

6 Bounds on $g(m, G)$ for general graphs G

In this section, we give bounds on $g(m, G)$ for a general graph G in terms of degrees in G. First, we recall and restate the trivial lower bound on $g(m, G)$ from Lemma 2.3, now in terms of the average degree of G.

Proposition 6.1 *Suppose G is a graph with t vertices, having average degree d, with m any positive integer. Then $g(m, G) > m(dt - 2)/2$.* ■

For dense graphs G, we can considerably improve the lower bound to the following.

Theorem 6.2 *There exists an absolute constant $c_3' > 0$ such that if G is a graph with t vertices, average degree d and at least two edges, then*

$$g(m, G) \geq c_3' \cdot \frac{md^2t}{\ln t + \ln m}.$$

Proof. Let c_3' be a small positive constant to be specified later. Let N be a positive integer with $N = \lceil c_3' \cdot \frac{md^2t}{\ln t + \ln m} \rceil$. Without loss of generality, we assume that m divides N, and let $q = N/m$. We show that there exists an m-good edge-coloring of $K = K_N$ with no rainbow copy of G.

Let $\mathcal{M} = \{M_1, M_2, \cdots, M_N\}$ be any decomposition of $E(K)$ into matchings, where such an $\mathcal{M}$ exists since $\chi'(G) \leq N$. Let $\mathcal{C}$ be the probability space consisting of all m-perfect colorings of $[N]$ using colors from $[q]$, with each m-perfect coloring of $[N]$ being equally likely. For each m-perfect coloring σ in $\mathcal{C}$, let c_σ denote the coloring of $E(K)$ obtained by assigning color $\sigma(j)$ to each edge of matching M_j for each $j = 1, 2, \cdots, N$. Since σ is m-perfect, c_σ is an m-good coloring of $E(K_N)$. Let $X = \{x_1, \cdots, x_t\}$ be a fixed set of t vertices of K. Note that there are at most $t!$ labeled copies of G with vertex set X. Let H denote an arbitrary copy of G with vertex set X, and let A_H denote the event that H is rainbow under c_σ. We estimate Prob (A_H).

If H contains two edges e, e' from the same M_i for some i, then clearly $c_\sigma(e) = c_\sigma(e')$, preventing H from being rainbow. Thus, we have Prob $(A) = 0$ in this case. Next, suppose that no two edges of H are from the same M_i. Let $T = \{j : E(H) \cap M_j \text{ is nonempty}\}$. Then $|T| = e(H)$, and the probability that H is rainbow is equal to the probability that σ assigns distinct colors to the elements of T. For convenience, let $s = |T|$. Note that $s = e(H) = dt/2$. By Lemma 2.5, we have

$$\begin{aligned} \text{Prob } (A_H) &\leq \left[exp(-\frac{m-1}{2N}) \right]^{s(s-1)} \\ &\leq \left[exp(-\frac{m}{4N}) \right]^{s^2/2} \\ &= exp\left[-\frac{md^2t^2}{32N} \right] \end{aligned}$$

Since there are $\binom{N}{t}$ many t-subsets X of $V(G)$, and there are at most $t!$ copies of G with vertex set X, we have

Prob (Some copy of G in K is rainbow)

$$\leq \binom{N}{t} \cdot t! \cdot exp\left[-\frac{md^2t^2}{32N}\right]$$
$$\leq N^t \cdot exp\left[-\frac{md^2t^2}{32N}\right].$$

Recall that $N = \lceil c_3' \cdot \frac{md^2t}{\ln t + \ln m} \rceil$. For a pre-chosen sufficiently small positive constant c_3', we have $N^t \cdot exp\left[-\frac{md^2t^2}{32N}\right] < 1$ (as the logarithm of the left-hand side is negative). Thus we have

$$\text{Prob (Some copy of } G \text{ in } K \text{ is rainbow) } < 1.$$

Hence

$$\text{Prob } (K \text{ contains no rainbow copy of } G) > 0.$$

Thus there exists a coloring σ in $\mathcal{C}$ for which c_σ is an m-good edge-coloring of K containing no rainbow copy of G. ■

Remark 6.3 If G has average degree d on the order of $t = n(G)$, then Theorem 6.2 and Theorem 5.6 imply that $\Omega(mt^3/(\ln t + \ln m)) \leq g(m, G) \leq O(mt^3/\ln t)$.

Finally, we give an upper bound on $g(m, G)$ for general graphs G in terms of the maximum degree of G. The method is based on the proof of Theorem 4.4.

Lemma 6.4 *Suppose $\varepsilon \in (0, 1)$, n is a positive integer, and G is a graph with $t = n(G)$ vertices and maximum degree d.*
If $t < (1-\varepsilon)n - 2d^2\sqrt{mn/\varepsilon} - (md^2/2)t$, then $g(m, G) < n$.

Sketch of proof. We follow the proof of Theorem 4.4 almost exactly, except that we define Type 3 vertices to be vertices w such that there exists $u \in N_j$ for which uw has a color already used on an edge of G_{j-1} and note that there are at most $md^2t/2$ such vertices. ■

Theorem 6.5 *Let G be a graph with t vertices and maximum degree d. For all positive integers m, we have $g(m, G) \leq 2md^2t + 32md^4 + 4t$.*

Proof. Applying Lemma 6.4 with $\varepsilon = 1/2$, we have that if n satisfies

$$t < (1/2)n - 2d^2\sqrt{2mn} - md^2t/2 \tag{11}$$

then $g(m, G) < n$. Rearranging, we get

$$n - 4d^2\sqrt{2m} \cdot \sqrt{n} - (md^2t + 2t) > 0. \tag{12}$$

Letting $y = \sqrt{n}$, $B = 4d^2\sqrt{2m}$, $C = md^2t + 2t$, we can rewrite inequality (12) as

$$y^2 - By - C > 0. \tag{13}$$

Inequality (13) is clearly satisfied if

$$y > \frac{B + \sqrt{B^2 + 4C}}{2}. \tag{14}$$

Inequality (14) is in turn satisfied if

$$n = y^2 > \frac{2B^2 + 2(B^2 + 4C)}{4} = B^2 + 2C.$$

Thus, we have

$$g(m, G) \leq B^2 + 2C = 32md^4 + 2md^2t + 4t.$$

■

Remark 6.6 By Theorem 6.5, we have $g(m, G) = O(mt)$ for graphs G on t vertices with bounded maximum degree.

7 Concluding remarks

- Reacll that $er(p)$ denotes the canonical Ramsey number of p as defined in Theorem A. In [15] Lefmann and Rödl showed that every coloring of $E(K_n)$ in which each color appears at most $n/(\frac{27}{16}p^6)$ times at each vertex yields a rainbow copy of K_p, and using this, concluded that

$$er(p) \leq (\frac{27p^6}{16})^{2(p-2)^2+1}.$$

  Using our improved bounds for $g(m, K_t)$ we can slightly improve their bound by showing, using Theorem 5.6, that

$$er(p) \leq (c_2 \frac{p^3}{\ln p})^{2(p-2)^2+1}.$$

  This improves the constant c' in their upper estimate $er(p) \leq 2^{c'p^2 \ln p}$ by roughly a factor of 2.

- By Theorem 4.3, for every fixed m and d and all $n > n_0(m, d)$, $f(m, G) = n$ for every graph G with n vertices and maximum degree d. It seems plausible to conjecture that this holds with $n_0(m, d) = d^{c_1}m$ for some absolute positive constant c. By Theorem 4.12 this is the case for certain graphs G, but at the moment we are unable to prove it for the general case.

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