On the Separation Number of a Graph

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We consider the following graph labeling problem, introduced by Leung et al. (J. Y-T. Leung, O. Vornberger, and J. D. Witthoff, On some variants of the bandwidth minimization problem. *SIAM J. Comput.* **13** (1984) 650–667). Let G be a graph of order n, and f a bijection from $V(G)$ to the integers 1 through n. Let $|f| = \min\{|f(x) - f(y)|: xy \in E(G)\}$, and define $s(G)$, the separation number of G, to be the maximum of $|f|$ among all such bijections f. We first derive some basic relations between $s(G)$ and other graph parameters. Using a general strategy for analyzing separation number in bipartite graphs, we obtain exact values for certain classes of forests and asymptotically optimal lower bounds for grids and hypercubes.

1. DEFINITIONS AND SOME SIMPLE RESULTS

In recent years there has been much interest in various graph labeling or layout problems. Among these problems, one of particular importance has been the *bandwidth minimization problem*. In this problem we are given a graph $G = (V,E)$, where $|V| = n$, and we are asked to produce a bijection (or layout) $f: V \to [1,n]$ (where $[1,n]$ denotes the set of integers 1 through n inclusive) for which the quantity $\max\{|f(x) - f(y)|: xy \in E(G)\}$ is minimized. This problem is motivated by issues in VLSI design and numerical analysis, and restrictions of it give examples of complete problems for various complexity classes defined by time or space bounds [*S*]. A survey of results on bandwidth may be found in [3].

In this paper we consider the following dual to the bandwidth problem. Given a graph $G = (V,E)$, find a bijection $f: V \to [1,n]$ for which the quantity $\min\{|f(x) - f(y)|: xy \in E(G)\}$ is maximized. This problem and variants of it were studied by Leung, Vornberger, and Witthoff [5]. Their primary concern was to study the complexity of this problem and its variants. In particular they show that the problem "given a graph G and an integer k, is the quantity defined above greater than k?" is

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NP-complete, even for the case $k = 1$ (by a simple reduction from the hamiltonian path problem). Their main results demonstrate the reducibility of various multiprocessor job scheduling problems to variants of the separation problem. Our goal in this report is to establish bounds on the separation number for various classes of graphs.

We use the following notation and conventions. Only finite simple graphs G with vertex set $V(G)$ and edge set $E(G)$ will be considered. As above let $[1,n]$ denote the set of integers from 1 to n inclusive. Let $G \subseteq H$ denote that G is isomorphic to a subgraph of H. We let $\lfloor x \rfloor$, and $\lceil x \rceil$ be the usual floor and ceiling functions, i.e., the greatest (resp. smallest) integer less than (resp. greater than) or equal to x. Let $\chi(G)$, $\Delta(G)$ and $\delta(G)$ denote the chromatic number, maximum degree and minimum degree of G, respectively. A *labeling* of a graph G of order n is a bijection $f:V(G) \to [1,n]$. Let $|f| = \min\{|f(x) - f(y)|:xy \in E(G)\}$. The *separation number* of a graph G is defined as $s(G) = \max\{|f|:f \text{ is a labeling of } G\}$. For definitions of graph theoretic terms not given here, see [1], [2], or [4].

In this first section, we develop some simple relations between the separation number and other natural graph parameters.

Theorem 1. For G of order n with $\delta(G) > 0$, $s(G) \leqslant \lfloor n/2 \rfloor$.

Proof. Let f the a labeling satisfying $|f| = s(G)$. Then $f(x) = \lceil n/2 \rceil$ for some x. Letting y be a neighbor of x, $s(G) \leqslant |f(x) - f(y)| \leqslant n/2$. ∎

Alternatively, the separation number can be conveniently viewed via subgraph containment. Let $H_{n,s}$ (a "host graph") be the graph with $V(H_{n,s}) = [1,n]$, where $ij \in E(H_{n,s})$ if and only if $|i - j| \geqslant s$. For G of order n, it follows that $s(G) = \max\{s:G \subseteq H_{n,s}\}$. Consequently, it will be useful to study these graphs $H_{n,s}$.

Lemma 2. Suppose $s \leqslant n/2$.

(i) For each $i \in [1,s - 1]$, $H_{n,s}$ has exactly two vertices (namely i and $n - i + 1$) of degree $n + 1 - s - i$. The remaining $n + 2 - 2s$ vertices each have degree $n + 1 - 2s$.

(ii) $|E(H_{n,s})| = \binom{n + 1 - s}{2}$.

(iii) $\chi(H_{n,s}) = \lceil n/s \rceil$.

Proof. (i) easily follows from the definition of $H_{n,s}$. Taking one-half of the sum of the degrees of $V(H_{n,s})$, one easily verifies (ii).

For (iii) observe that any subset of $[1,n]$ of cardinality greater than s has its smallest and largest elements differing by at least s. Hence no such set is independent. Therefore $\chi(H_{n,s}) \geqslant n/s$. The proper coloring given by $f(i) = \lceil i/s \rceil$ verifies that $\chi(H_{n,s}) \leqslant \lfloor n/s \rfloor$. ∎

For convenience, in statements of results throughout the rest of the paper, we consider only graphs G with $\delta(G) > 0$. Thus we may assume that $s(G) \leqslant n/2$, so that the facts about $H_{n,s}$ presented in Lemma 2 are relevant.

Theorem 3. For G with n vertices and q edges, with $s = s(G)$, $\chi = \chi(G)$, $\delta = \delta(G)$, $\Delta = \Delta(G)$,

(i) $s \leqslant n + \frac{1}{2}(1 - \sqrt{1 + 8q})$,

(ii) $s < n/(\chi - 1)$,
(iii) $s \leqslant \frac{1}{2}(n - \delta + 1)$,
(iv) $s \leqslant n - \Delta$.

Proof. From (ii) of Lemma 2, $q \leqslant |E(H_{n,s})| = \binom{n+1-s}{2}$, so $0 \leqslant s^2 - (2n + 1)s + n^2 + n - 2q$. Solving this inequality as a quadratic in s, we obtain that either $s \leqslant \frac{1}{2}(2n + 1 - \sqrt{1 + 8q})$ or $s \geqslant \frac{1}{2}(2n + 1 + \sqrt{1 + 8q}) > n$. Clearly the latter is impossible, proving (i).

To prove (ii), observe that $\chi(G) \leqslant \chi(H_{n,s}) = \lceil n/s \rceil$, so $\chi < n/s + 1$.

Finally, (iii) and (iv) follow from $\delta(G) \leqslant \delta(H_{n,s}) = n - 2s + 1$ and from $\Delta(G) \leqslant \Delta(H_{n,s}) = n - s$. ■

2. SEPARATION AND BIPARTITE GRAPHS

Thus far, we have only presented *upper* bounds on the separation number. If G is not constrained to have some nice structure, we will be hard pressed to give useful lower bounds. Indeed (ii) of Theorem 3 tells us that the upper bound $s \leqslant n/2$ can only be attained for G bipartite. But at the other extreme, the complete bipartite graphs have separation number equal to 1! We now develop a strategy for giving constructive lower bounds for the separation number of bipartite graphs. In this section we describe the strategy, and apply it in two examples where the lower bounds produced are easy to evaluate. In the next section we apply the strategy in an example where the evaluation of the lower bound is considerably more involved.

For a bipartite graph B with a specified bipartition M,N with $|M| \leqslant |N|$, we refer to the *minority* $\mathrm{MIN}(B) = |M|$ and the *majority* $\mathrm{MAJ}(B) = |N|$ of B, and refer to M and N as being the minority and majority *sides*, respectively. If B is connected, there is little need to specify a bipartition.

To motivate our strategy, we begin with the example of forests. Later we will generalize the method of proof to obtain our strategy for arbitrary bipartite graphs.

Theorem 4. For F any forest, $s(F) \geqslant \mathrm{MIN}(F)$.

Proof. Given any bipartition X,Y of a forest with $|X| \leqslant |Y|$, there always exists a vertex $y \in Y$ of degree 0 or 1 since the average degree of the majority side vertices is at most $(|X| + |Y| - 1)/|Y|$, which is less than two.

Let a forest F_1 have minority side M_1 and majority side N_1. For each $i \in [1,\mathrm{MAJ}(F_1)]$, recursively define y_i, x_i, F_i, M_i, N_i as follows. Let $y_i \in N_i$ be a vertex of degree 0 or 1 in F_i. If y_i has degree 1 in F_i, choose x_i as its sole neighbor. If M_i is empty, choose $x_i = y_i$. In any other case, choose x_i to be any element of M_i. Let $F_{i+1} = F_i - x_i - y_i, M_{i+1} = M_i - x_i$, $N_{i+1} = N_i - y_i$. Consider the labeling of F_1 given by $f(x_i) = i$ for each $i \in [1,\mathrm{MIN}(F_1)]$ and $f(y_i) = \mathrm{MIN}(F_1) + i$ for each $i \in [1,\mathrm{MAJ}(F_1)]$. By construction, if $x_iy_j \in E(F_1)$ for some $i \in [1,\mathrm{MIN}(F_1)]$, then $i \leqslant j$, so $|f(y_j) - f(x_i)| = \mathrm{MIN}(F_1) + j - i \geqslant \mathrm{MIN}(F_1)$. Therefore $s(F_1) \geqslant \mathrm{MIN}(F_1)$, proving the result. ■

Corollary 4.1. For T a tree with $\mathrm{MAJ}(T) = \Delta$ we have $s(T) = \mathrm{MIN}(T)$.

Proof. $\mathrm{MIN}(T) \leqslant s(T) \leqslant n - \Delta = \mathrm{MIN}(T)$. ■

A bipartite graph is *balanced* if it has a bipartition M,N with $|M| = |N|$.

Theorem 5. If $s(G) = s$, then G contains a set of $n - 2s$ vertices, each of degree at most $n + 1 - 2s$, whose removal yields a balanced bipartite graph of order $2s$.

Proof. This follows from the fact that since $s \leqslant n/2$, the subgraph of $H_{n,s}$ induced by $[1,s] \cup [n + 1 - s,n]$ is a balanced bipartite graph. ∎

We can now categorize those forests for which the upper bound of Theorem 1 is attained.

Theorem 6. For F a forest, $s(F) = \lfloor n/2 \rfloor$ if and only if F is balanced or contains a vertex of degree 1 or 2 whose removal yields a balanced forest.

Proof. Note that balanced graphs are of even order, whereas graphs with a balanced vertex-deleted subgraph are of odd order.

Suppose n is even. If $s(F) = n/2$, then by Theorem 5, F is balanced. Conversely, if F is balanced, then by Theorems 4 and 1, $n/2 \leqslant s(F) \leqslant n/2$.

Suppose $n = 2m + 1$ is odd. If $s(F) = m$, then by Theorem 5, F has a vertex of degree 1 or 2 whose removal yields a balanced forest. Conversely, suppose F has such a vertex v, with balanced bipartition M,N of $F - v$. If F has a bipartition M',N' with $|M'| = m$, then by Theorems 4 and 1, $m \leqslant s(F) \leqslant m$, proving the claim. Therefore we may assume that F has no such bipartition. This implies that v has degree 2, and that v has one of its neighbors x in M and its other neighbor y in N. Then consider $F_1 = F - v - x - y$, with balanced bipartition $M_1 = M - x$, $N_1 = N - y$. Applying the proof of Theorem 4 to F_1, there is a labeling $f:V(F_1) \rightarrow [1,2m - 2]$ for which $|f| \geqslant m - 1$ and $f(M_1) = [1,m - 1]$, $f(N_1) = [m,2m - 2]$. Consider the labeling $g:V(F) \rightarrow [1,n]$ defined as follows. $g(x) = 1$, $g(y) = n$, $g(v) = m + 1$, $g(z) = f(z) + 1$ for all $z \in M_1$, $g(z) = f(z) + 2$ for all $z \in N_1$. One easily verifies that $|g| \geqslant m$, proving the claim. ∎

Notice that Theorem 6 implies that the separation number is $\lfloor n/2 \rfloor$ for complete binary trees.

Recall that the proof of Theorem 4 involves a labeling f which maps the sides X and Y of the bipartition onto disjoint intervals $[1,m]$ and $[m + 1,n]$, respectively. At first glance, this seems to be a good strategy in attempting to maximize $|f|$ for *any* bipartite graph. However, it is not always a *best* strategy, as evidenced by the complete binary tree on seven vertices. Nevertheless, we shall put this strategy to good use.

For bipartite B with bipartition X,Y with $|X| = m$, $|Y| = n - m$, with $Y = \{y_1,y_2, \ldots ,y_{n-m}\}$ (the y_i's in a fixed order), consider all labelings $f:V(B) \rightarrow [1,n]$ for which $f(y_i) = m + i$ for all $i \in [1,n - m]$. Let $s(B,\{y_i\})$ be the maximum value of $|f|$ among all such labelings f. For each $x \in X$, let $j(x) = \min\{i{:}x \text{ is adjacent to } y_i\}$. For each $t \in [1,n - m]$, let $Nb(t) = \{x \in X{:}j(x) = t\}$, let $\text{New}(t) = |Nb(t)|$, and let $B(i) = \sum_{t=1}^{i} \text{New}(t)$, the number of vertices adjacent to any or all elements of $\{y_1,y_2, \ldots ,y_i\}$.

Theorem 7. $s(B,\{y_i\}) = m - \max_i\{B(i) - i\}$.

Proof. Let f be any labeling for which $f(y_i) = m + i$ for all $i \in [1,n - m]$. Among the $B(i)$ many vertices adjacent to $\{y_1,y_2, \ldots ,y_i\}$, there exists $x \in X$ with

$f(x) \geq B(i)$. Now x has a neighbor y_j for some $j \leq i$, so $|f(y_j) - f(x)| \leq m - \max_i\{B(i) - i\}$, hence $|f| \leq m - \max_i\{B(i) - i\}$. Therefore $s(B,\{y_i\}) \leq m - \max_i\{B(i) - i\}$.

Conversely, one can in a natural way construct such a labeling f with $|f| \geq m - \max_i\{B(i) - i\}$. Let $X = \{x_1, x_2, \ldots, x_m\}$. We give a linear order $\langle$ on X as follows. Define $x_h \langle x_i$ if and only if either $j(x_h) < j(x_i)$ or $j(x_h) = j(x_i)$ and $h < i$. One easily verifies that $\langle$ is linear. For x_i the kth least element in the ordering, assign the label $f(x_i) = k$, and as usual let $f(y_i) = m + i$ for each i. For this labeling, if x and y_i are adjacent, it follows that $f(x) \leq B(i)$, so $|f(y_i) - f(x)| \geq (m + i) - B(i) \geq m - \max_i\{B(i) - i\}$. Therefore $|f| = \min\{|f(y_i) - f(x)|: xy_i \in E(B)\} \geq m - \max_i\{B(i) - i\}$, proving the theorem. ∎

We point out a resemblance between Theorem 7 and Hall's Theorem. The *latter* (interpreted appropriately) asserts that the size of a maximum matching in B is:

$|Y| - \max\{|S| - N(S): S \subseteq Y\}$, where $N(S)$ denotes the number of vertices of X adjacent to at least one member of S [2 p. 76 ex. 5.2.6]. The *former* asserts that the maximum value for $|f|$ is:

$m - \max\{N(S_i) - |S_i|: S_i = \{y_1, y_2, \ldots, y_i\} \subseteq Y\}$. While S has free range in Hall's Theorem, notice that we have freedom in choosing the ordering of the y_i's when trying to maximize $|f|$.

We now apply Theorem 7 to find asymptotically good lower bounds for the separation number of m by n grids. For $m \geq n$, let $G_{m,n}$ be the usual Cartesian product graph $P_m \times P_n$ of order mn, where P_k is a path on the points $[1,k]$. Thus (i,j) and (i',j') are adjacent if and only if either $i = i'$ and $|j - j'| = 1$, or $j = j'$ and $|i - i'| = 1$. $G_{m,n}$ is bipartite, so we may take advantage of Theorem 7.

Let $X = \{(i,j) \in V(G_{m,n}): i + j \text{ is odd}\}$ and $Y = \{(i,j) \in V(G_{m,n}): i + j \text{ is even}\}$. Let Y be ordered lexicographically, i.e., $(i,j) < (i',j')$ if and only if either $i < i'$, or $i = i'$ and $j < j'$. Let y_i denote the ith least element of Y.

Theorem 8. $s(G_{m,n},\{y_i\}) = \lfloor (mn - n)/2 \rfloor$. Consequently, $s(G_n) \geq \lfloor (mn - n)/2 \rfloor$.

Proof. New$(i) = 2$ for $1 \leq i \leq n/2$, and New$(i) \leq 1$ for $i > n/2$. Therefore $s(G_{m,n},\{y_i\}) = |X| - \max_i\{B(i) - i\} = \lfloor mn/2 \rfloor - (B(\lfloor n/2 \rfloor) - \lfloor n/2 \rfloor) = \lfloor mn/2 \rfloor - 2\lfloor n/2 \rfloor + \lfloor n/2 \rfloor = \lfloor (mn - n)/2 \rfloor$. ∎

In particular, notice that $s(G_{m,n}) \geq (n/2)$. Also notice that the lower bound $\lfloor (mn - n)/2 \rfloor$ is asymptotically optimal, since the ratio of it compared to the upper bound $\lfloor mn/2 \rfloor$ goes to 1 as $m \to \infty$.

3. SEPARATION OF THE n-CUBE

In this section we describe an asymptotically optimal layout f for the n-cube Q_n. Since Q_n is bipartite we can make use of Theorem 7, again using a certain lexicographic ordering of one side of the bipartition. The choice of labeling f will therefore be easy to describe, and the difficult work is in evaluating $|f|$ so as to obtain a lower bound for $s(Q_n)$.

As usual, we represent the vertices of Q_n by n-digit words over the alphabet $\{0,1\}$, where two vertices are joined by an edge if and only if they disagree in exactly one coordinate. We make occasional use of the symmetries of Q_n by treating it as a binary vector space, with addition modulo 2. We let $wt(v)$, the *weight* of a vertex $v \in Q_n$, be the number of 1's in v. We denote repetition of digits and subwords by exponents, so that $1^2 0^2 (10)^3 = 1100101010$. *Prefix* and *suffix* have their natural meanings, so that the above word has 11 and 11001 among its prefixes, and has 010 as a suffix. Also we let $N(v)$ be the set of points in Q_n adjacent to v.

We make use of the following lexicographic ordering $<$ of the points of Q_n. Define $v < w$ if and only if either $wt(v) < wt(w)$, or $wt(v) = wt(w)$ and the first coordinate in which v and w disagree is a 1 in v and a 0 in w. Letting $v_n(i)$ denote the ith lowest ordered element, we get the ordering $v_n(1) < v_n(2) < \cdots < v_n(2^n)$. This ordering for Q_4 is $0000 < 1000 < 0100 < 0010 < 0001 < 1100 < 1010 < 1001 < 0110 < 0101 < 0011 < 1110 < 1101 < 1011 < 0111 < 1111$.

The ordering we actually use is an ordering e_n of the even weight points of Q_n. This ordering is by definition simply the one induced by the linear order $<$. Thus given two even weight words $x,y \in Q_n$, y is the immediate successor of x under e_n if and only if y is the first even weight word which comes after x under v_n. As n will be fixed in our discussion, we will henceforth refer to $e_n(j)$ (the jth point under e_n) by $e(j)$. For any $1 \leqslant t \leqslant 2^{n-1}$, we let $Nb(t) = N(e(t)) \setminus \bigcup_{i<t} N(e(i))$. We also write $Nb(v)$ to refer to $Nb(t)$, where $v = e(t)$.

The layout f of Q_n may now be described. We let $f(e(i)) = 2^{n-1} + i$. The odd weight words are mapped in the manner specified by Theorem 7. That is, f first maps the n neighbors of $0^n = e(1)$ to $[1,n]$ in any order. Inductively, having defined f on $\bigcup_{i \leqslant t} N(e(i))$, $t \geqslant 1$, we assign $Nb(t+1)$ to the integers $r+1$ through $r + |Nb(t+1)|$, where $r = |\bigcup_{i \leqslant t} N(e(i))|$. We will call such an f a *lexicographic layout* via the even weight words (recall that there are choices of how f is to act on the odd weight words, but these choices do not affect $|f|$).

As an example, we give a layout f of this type for Q_4, written in the order $f^{-1}(1), f^{-1}(2), \ldots, f^{-1}(16)$; namely, $\{1000,0100,0010,0001,1110,1101,1011,0111, 0000,1100,1010,1001, 0110,0101,0011,1111\}$.

Let I_i be the weight 1 vector with its 1 in the ith coordinate. For $v = e(t)$, let $New(t) = New(v) = |Nb(t)|$ for $1 \leqslant t \leqslant 2^{n-1}$. A useful fact is the following.

Lemma 9. Let $e(j)$ have suffix 10^t. Then $Nb(j) = \{e(j) + I_i : n - t < i \leqslant n\}$, and consequently $\text{New}(j) = t$. Note also that $Nb(O^n) = \{I_i : 1 \leqslant i \leqslant n\}$, with $\text{New}(O^n) = n$.

Proof. Let $e(j)$ have weight k, and let $e(j) + I_i$ be adjacent to $e(j)$. We determine for which subscripts i this neighbor is in $Nb(j)$. Let $e(i)$ have its last 1 in the Lth coordinate, so that $e(j) + I_i + I_L$ is the lexicographically least point adjacent to $e(j) + I_i$. Therefore $e(j) + I_i \in Nb(j) \Leftrightarrow e(j) + I_i + I_L = e(j) \Leftrightarrow i = L \Leftrightarrow n - t < i \leqslant n$, proving that $Nb(j) = \{e(j) + I_i : n - t < i \leqslant n\}$. The rest follows easily. ■

$$\text{Let } B(k) = \sum_{t=1}^{k} \text{New}(t), \qquad 1 \leqslant k \leqslant 2^{n-1}.$$

Lemma 10. For any lexicographic layout f of Q_n we have $|f| = 2^{n-1} - \max_k\{B(k) - k\}$.

Proof. By the proof of Theorem 7, $|f| = s(B,\{y_i\})$ for $y_i = e(i)$. ∎

In using a lexicographic layout of Q_n to get a lower bound for $s(Q_n)$, we are therefore reduced to finding the maximum of the function $B(k) - k$ over $1 \leqslant k \leqslant 2^{n-1}$. Let us pick one such f arbitrarily, and fix it for the remainder of the discussion. We now need some definitions.

Let x and y be even weight points with $x = e(r)$ and $y = e(s)$ and $s \geqslant r$. We use open and closed interval notation such as $(x,y]$ and $(e(r),e(s))$ to denote the set of even weight words lexicographically between $e(r)$ and $e(s)$, a closed bracket indicating the inclusion of an endpoint in the interval. It is useful to regard the mapping by f of the points in such an interval $(x,y]$ as a process of successively placing them and their unplaced neighbors at their correct locations (as determined by f) in this sequence; first $e(r+1)$, followed by $Nb(r+1)$, then $e(r+2)$, followed by $Nb(r+2)$, . . . ,then $e(s)$ followed by $Nb(s)$, and we denote this process by $(x \to y]$. We let $value(x) = B(r) - r$. For the interval $(x \to y]$, we let $content(x \to y] = \text{value}(y) - value(x)$. For any interval I, we let $New\ I = \Sigma_{i \in I} \text{New}(i)$. It follows that $B(s) - s = B(r) - r + \text{content}(x \to y]$, so that content$(x \to y]$ measures the contribution to $B(s) - s$ as the points $e(r+1)$, $e(r+2)$, . . . ,$e(s)$ and their neighbors are mapped by f. Since $B(s) - B(r) = \text{New}(e(r) \mapsto e(s)]$, we have that content$(e(r) \to e(s)] = \text{New}(e(r) \to e(s)] + (s - r)$. Later we will use notation $(x \to y \to z]$ for compositions of processes, and it will be useful to know that:

i) content$(x \to z]$ = content$(x \to y]$ + content$(y \to z]$.

ii) value(y) = value(x) + content$(x \to y]$.

iii) value$(y) = n - 1 +$ content$(0^n \to y]$.

iv) For any prefix P of even weight, with x and y of the same weight, content$(Px \to P_y]$ = content$(x \to y]$.

To illustrate these terms we reconsider the example layout of Q_4 above. Here we have value$(x) = 3,4,4$ for $x = 0000,1100,1010$, respectively. Note also that value(1010) = value(0000) + content(0000 → 1010] = 3 + content(0000 → 1010] (illustrating ii) and iii)).

We wish to use the claim in iv) also when the prefix P has odd weight, so we will need the concepts content$(v \to w]$ and value(w) for words $v < w$ of odd weight. We can define the ordering odd$_n$ of the odd weight points of Q_n (in analogy with e_n) as the one induced by the linear order $<$ on all of Q_n. Now interchanging "even" with "odd" and "e_n" with "odd$_n$" in the above description of a lexicographic layout, we obtain a second lexicographic layout g of Q_n, this one via the odd weight words. Thus $g(\text{odd}_n(i)) = 2^{n-1} + i$, thereby mapping the odd weight points to $[2^{n-1}+1, 2^n]$ in the order odd$_n$. Analogously, the even weight points are mapped by g onto $[1,2^{n-1}]$ by neighborhoods. By using the layout g, we can carry over (in the same way) the symbols New(v), value(v), content$(v \to w]$, etc. when v and w have odd weight. Thus, for example, for v an odd weight word with suffix 10^t, it is still true that New$(v) = t$, with the sole exception that New$(10^{n-1}) = n$ (in keeping with New$(0^n) = n$). All other results remain true when interpreted in the context of odd weight words, occasionally requiring that the words "even" and "odd" be interchanged in their statements. Dual proofs based on this extended definition will be omitted for brevity.

Lemma 11. For any $k > 0$, $t > 0$ we have

$$\text{content } (0^{n-k-t}1^k0^t \to 0^{n-k-t+1}\, 1^k0^{t-1}] = \binom{k+t-1}{t-1} - \binom{k+t-1}{t} - 1.$$

Proof. Let $x = e(r) = 0^{n-k-t}1^k0^t$ and $y = e(s) = 0^{n-k-t+1}1^k0^{t-1}$. By Lemma 9 we see that $\text{New}[x \to y) = \binom{k+t-1}{t-1}$, since it counts the number of weight $k+1$ words having $0^{n-k-t}1$ as a prefix. Similarly, $s - r = |[x,y)| = \binom{k+t-1}{t}$, the number of weight k words with $0^{n-k-t}1$ as a prefix. Therefore content $(x \to y] = \text{New}(x \to y] - (s - r) = \text{New}[x \to y) - \text{New}(x) + \text{New}(y) - (s - r) = \binom{k+t-1}{t-1} - t + (t-1) - \binom{k+t-1}{t} = \binom{k+t-1}{t-1} - \binom{k+t-1}{t} - 1.$ ■

Corollary 11.1.
For $k \geqslant 2$, $t \geqslant 1$ we have
content $(0^{n-(k+t)}\, 1^k0^t \to 0^{n-(k+t)+1}\, 1^k0^{t-1}] \geqslant 0$
if and only if $k <$ t. ■

We now define some special words $A(b,n)$ in Q_n by $A(b,n) = 1^{2b-n}(10)^{n-b}$ if $b \geqslant \lceil n/2 \rceil$ and $A(b,n) = 0^{n-2b}(10)^b$ if $b < \lceil n/2 \rceil$. In particular, $A(b,2b) = (10)^b$.

Our ultimate goal is to show that that $B(k) - k$ reaches a maximum at that k for which $e(k) = A(e,n)$, where e is whichever of $\lfloor (n-1)/2 \rfloor$ or $\lfloor (n+1)/2 \rfloor$ is even. It will turn out that $A(b,n)$ has a maximum value among weight b words. We will also see that the sequence $\{\text{value}(A(b,n))\text{: } b \text{ even, } 0 \leqslant b \leqslant n\}$ is unimodal, thereby allowing us to maximize the function $B(k) - k$.

For convenience, we let

$$\alpha_k = \sum_{j=1}^{k-1} \left[\binom{2k-2j}{k-j} - \binom{2k-2j}{k-j+1} \right] = \sum_{j=1}^{k-1} \left[\binom{2j}{j} - \binom{2j}{j+1} \right],$$

the sum of the first $k - 1$ Catalan numbers.

Lemma 12. $\text{content}(1^k0^k \to A(k,2k)] = \alpha_k - (k-1).$

Proof. We observe that the process $(1^k0^k \to A(k,2k)]$ can be decomposed into successive subprocesses as $(1^k0^k \to 101^{k-1}0^{k-1} \to A(k,2k)]$. For the first subprocess, Lemma 11 and observation iv) above give content $(1^k0^k \to 101^{k-1}0^{k-1}] = \binom{2k-2}{k-1} - \binom{2k-2}{k} - 1$. By induction on k, the second subprocess can be assumed to have content

$$\sum_{j=1}^{k-2} \left[\binom{2k-2-2j}{k-1-j} - \binom{2k-2-2j}{k-j} \right] - (k-2)$$
$$= \sum_{j=2}^{k-1} \left[\binom{2k-2j}{k-j} - \binom{2k-2j}{k-j+1} \right] - (k-2).$$

Noting that the content function is additive and hence adding the two quantities, we get the desired result. ■

Let $D(b,n) = \text{content}(A(b,n) \to A(b+2,n)]$. Our next result is the first step in showing that for fixed n the sequence $\{\text{value}(A(b,n)): b \text{ even}, 0 \leq b \leq n\}$ is unimodal.

Theorem 13. Let $b \leq \lfloor (n-1)/2 \rfloor$. Then $D(b,n) \geq 0$ for $b \leq \lfloor (n-3)/2 \rfloor$, and $D(b,n) < 0$ for $b = \lfloor (n-1)/2 \rfloor$.

Proof. We may decompose the process defining $D(b,n)$ into the successive subprocesses $(A(b,n) \to 1^{b+2}0^{n-(b+2)}]$ and $(1^{b+2}0^{n-(b+2)} \to A(b+2,n)]$. Let us denote the first by π_1 and the second by π_2, and compute their contents separately.

It will be convenient to assume first that $b \leq \lfloor (n-5)/2 \rfloor$.

We begin with π_1. If we let π be the process $(1^b0^{n-b} \to 1^{b+2}0^{n-(b+2)}]$ and π_0 the process $(1^b0^{n-b} \to A(b,n)]$, then contentπ_1 = contentπ − contentπ_0 by additivity.

To compute content π_0 we decompose π_0 by $(1^b0^{n-b} \to 0^{n-2b}1^b0^b \to A(b,n)]$. Now applying Lemma 11 successively $n-2b$ times we find that the content of the first part is

$$\left[\binom{b+n-b-1}{n-b-1} - \binom{b+n-b-1}{n-b}\right] + \left[\binom{b+n-b-2}{n-b-2} - \binom{b+n-b-2}{n-b-1}\right] + \cdots + \left[\binom{2b}{b} - \binom{2b}{b+1}\right] - (n-2b)$$
$$= \sum_{j=1}^{n-2b}\binom{n-j}{b} - \sum_{j=1}^{n-2b}\binom{n-j}{b-1} - (n-2b)$$
$$= \sum_{j=2b}^{n-1}\binom{j}{b} - \sum_{j=2b}^{n-1}\binom{j}{b-1} - (n-2b).$$

Using the binomial identity $\Sigma_{m\leq k\leq n}\binom{k}{r} = \binom{n+1}{r+1} - \binom{m}{r+1}$, the content of the first part finally becomes

$$\binom{n}{b+1} - \binom{2b}{b+1} - \binom{n}{b} + \binom{2b}{b} - (n-2b).$$

The content of the second part is $\alpha_b - (b-1)$ by Lemma 12. Hence we have content $\pi_0 = \alpha_b + \binom{n}{b+1} - \binom{2b}{b+1} - \binom{n}{b} + \binom{2b}{b} - (n-b-1)$.

To get content π, we let r and s be defined by $1^b0^{n-b} = e(r)$ and $1^{b+2}0^{n-(b+2)} = e(s)$. Then

$$\begin{aligned}\text{content }\pi &= \text{New}(e(r) \to e(s)] - (s - r)\\ &= \text{New}[e(r) \to e(s)) - \text{New}(r) + \text{New}(s) - |[e(r),e(s))|\\ &= |\{\text{weight } b+1 \text{ words}\}| - (n - b) + (n - b - 2)\\ &\quad - |\{\text{weight } b \text{ words}\}|\\ &= \binom{n}{b+1} - (n-b) + (n-b-2) - \binom{n}{b}\\ &= \binom{n}{b+1} - \binom{n}{b} - 2.\end{aligned}$$

Combining the above two paragraphs we therefore get

$$\begin{aligned}\text{content}\pi_1 &= \text{content}\pi - \text{content}\pi_0\\ &= -\alpha_b + \binom{2b}{b+1} - \binom{2b}{b} + (n - b - 3)\end{aligned}$$

Now consider π_2. Here we observe that π_2 is the same as π_0, with b replaced by $b + 2$. Note that the assumption $b \leqslant \lfloor (n-5)/2 \rfloor$ is used here, since then $b + 2 \leqslant \lfloor (n-1)/2 \rfloor$. This makes valid the application of Lemma 12 (in the derivation of content π_0 with b replaced by $b + 2$). Thus we get

$$\text{content } \pi_2 = \alpha_{b+2} + \binom{n}{b+3} + \binom{2b+4}{b+2} - \binom{n}{b+2} - \binom{2b+4}{b+3} - (n - b - 3).$$

Recalling that $D(b,n)$ is the sum of content π_1 and content π_2 we get after much cancellation

$$\begin{aligned}D(b,n) = {}&\left[\binom{n}{b+3} - \binom{n}{b+2}\right] + \left[\binom{2b+2}{b+1} - \binom{2b+2}{b+2}\right]\\ &+ \left[\binom{2b+4}{b+2} - \binom{2b+4}{b+3}\right]\end{aligned} \tag{A}$$

Clearly the last two differences are nonnegative by the unimodality of the binomial coefficients. Also the first difference is nonnegative by the assumption $b \leqslant \left\lfloor \frac{n-5}{2} \right\rfloor$. Hence the theorem is proved under this assumption.

So assume that $\lfloor b = (n-3)/2 \rfloor$ or $\lfloor (n-1)/2 \rfloor$. We argue separately the cases n even and n odd.

Suppose first that n is even. Let $b = \lfloor (n-3)/2 \rfloor = n/2 - 2$. Since $b + 2 \leqslant \lfloor n/2 \rfloor$, the application of Lemma 12 as above is again valid and we are led to equation (A). Now $n = 2b + 4$, so the first and third differences cancel. Hence we are left with the desired nonnegative second difference, and are done. Now let $b = \left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n}{2} - 1$. Thus $b + 2 = n/2 + 1$, and hence π_2 is now the process $(1^{b+2}0^{n-(b+2)} \to 1^2(10)^b]$. Thus by iv), content π_2 = content$(1^b0^b \to A(b,2b)]$ = $\alpha_b - (b - 1)$. We get $D(b,n)$ = content π_1 + content π_2

$$= -\alpha_b + \binom{2b}{b+1} - \binom{2b}{b} + (n-b-3) + \alpha_b - (b-1)$$
$$= \binom{2b}{b+1} - \binom{2b}{b} < 0, \qquad \text{as desired.}$$

Now suppose n is odd. Letting first $b = \lfloor (n-3)/2 \rfloor$ we have $b + 2 = \binom{n+1}{2}$, and hence π_2 is the process $(1^{b+2}0^{n-(b+2)} \to 1(10)^{b+1}]$ so content π_2 = content $(1^{b+1}0^{b+1} \to A(b+1, 2b+2)] = \alpha_{b+1} - b$. Hence $D(b,n)$ = content π_1 + content π_2 = $-\alpha_b + \binom{2b}{b+1} - \binom{2b}{b} + (n-b-3) + \alpha_{b+1} - b = n - 2b - 3 = 0$, as desired.

Finally let $b = \lfloor (n-1)/2 \rfloor$ with n odd. Then $b + 2 = (n+3)/2$, and now π_2 is the process $(1^{b+2}0^{n-(b+2)} \to 1^3(10)^{b-1}]$. Thus content π_2 = content$(1^{b-1}0^{b-1} \to A(b-1, 2b-2)] = \alpha_{b-1} - (b-2)$. Hence

$$D(b,n) = \text{content } \pi_1 + \text{content } \pi_2$$
$$= -\alpha_b + \binom{2b}{b+1} - \binom{2b}{b} + (n-b-3) + \alpha_{b-1} - (b-2)$$
$$= \binom{2b}{b+1} - \binom{2b}{b} + \binom{2b-2}{b} - \binom{2b-2}{b-1} + n - 2b - 1$$
$$< 0, \text{ as desired.}$$

The theorem is thus proved. ∎

Theorem 14. For $b \geqslant \lfloor (n+1)/2 \rfloor$ we have $D(b,n) < 0$.

Proof. Retaining the notation of Theorem 13, we decompose the process defining $D(b,n)$ into the successive subprocesses π_1 and π_2. Our computations will be similar, only with changes in some parameters.

We start with π_1. We have again content π_1 = content π − content π_0, where π is the process $(1^b0^{n-b} \to 1^{b+2}0^{n-(b+2)}]$ and π_0 is the process $(1^b0^{n-b} \to 1^{2b-n}(10)^{n-b}]$ (the last word being of course $A(b+2, n)$). Note from iv) that content π_0 = content $(1^{n-b}0^{n-b} \to (10)^{n-b}]$, so by Lemma 12, content $\pi_0 = \alpha_{n-b} - (n-b-1)$. Exactly as in the proof of Theorem 13 we have content $\pi = \binom{n}{b+1} - \binom{n}{b} - 2$. Therefore

$$\text{content } \pi_1 = \left[\binom{n}{b+1} - \binom{n}{b} - 2\right] - \left[\alpha_{n-b} - (n-b-1)\right].$$

As for π_2, we note that

$$\text{content } \pi_2 = \text{content}(1^{n-(b+2)}0^{n-(b+2)} \to (10)^{n-(b+2)}]$$
$$= \alpha_{n-(b+2)} - (n-b-3).$$

Now using $D(b,n)$ = content π_1 + content π_2, we obtain after simplification

$$D(b,n) = \left[\binom{n}{b+1} - \binom{n}{b}\right] - \left[\binom{2(n-b)-2}{n-b-1} - \binom{2(n-b)-2}{n-b}\right] - \left[\binom{2(n-b)-4}{n-b-2} - \binom{2(n-b)-4}{n-b-1}\right].$$

By unimodality of the binomial coefficients the first difference is negative for $b \geqslant \lfloor (n+1)/2 \rfloor$ while the second and third differences are each positive and preceded by negative signs. Hence $D(b,n) < 0$, as desired. ∎

Theorems 13 and 14 combine to show the unimodality of the sequence $\{D(b,n)\}$ for fixed n. Our next goal is to prove the maximality property of the words $A(b,n)$.

Lemma 15. For words $x < y$ both of weight k and any word P, we have value $(Py) \geqslant$ value$(Px) \Leftrightarrow$ value$(y) \geqslant$ value(x).

Proof. If k is even,

$$\begin{aligned} &\text{value}(Py) - \text{value}(Px) \\ &= \text{content}(Px \rightarrow Py] - |(Px,Py]| \\ &= \text{content}(x \rightarrow y] - |(x,y]| \qquad \text{(by iv))} \\ &= \text{value}(y) - \text{value}(x). \end{aligned}$$

The dual proof for k odd is omitted. ∎

Lemma 16. For any $b \geqslant 1$ and $n \geqslant b$ we have value$(01A(b-1,n-2)) > (<)$ value$(1A(b-1,n-1))$ for $n > (\leqslant)\ 2b$.

Proof. Consider the compositions of processes $(0^n \rightarrow 1^b0^{n-b} \rightarrow 1A(b-1,n-1)]$ and $(0^n \rightarrow 1^b0^{n-b} \rightarrow 01A(b-1,n-2)]$, and let P and P' be the second steps in them resulting in $1A(b-1,n-1)$ and $01A(b-1,n-2)$, respectively. From the relation between value and content it suffices to show that content $P' > (<)$ content P for $n > (\leqslant)\ 2b$.

Since the prefix 1 is fixed throughout the process P, we have content P = content$(1^{b-1}0^{n-1-(b-1)} \rightarrow A(b-1,n-1)]$. But this is the same as the process π_0 of Theorem 13 with n replaced by $n-1$ and b by $b-1$. Thus

$$\begin{aligned} \text{content } P = \alpha_{b-1} &+ \binom{n-1}{b} + \binom{2(b-1)}{b-1} \\ &- \binom{n-1}{b-1} - \binom{2(b-1)}{b} - (n-b-1). \end{aligned}$$

Now P' is the composition of processes $(1^b0^{n-b} \rightarrow 01^n0^{n-b-1} \rightarrow 01A(b-1,n-2)]$. By Lemma 11, the first part of the composition has content $\binom{r-1}{b} - \binom{n-1}{b-1}$

$-$ 1. The second part, having prefix 01 throughout, is π_0 with n replaced by $n - 2$ and b by $b - 1$. Adding then content π_0 (with the indicated replacements) to the content of the first part we obtain

$$\text{content } P' = \binom{n-1}{b} - \binom{n-1}{b-1} - 1 + \alpha_{b-1} + \binom{n-2}{b} + \binom{2(b-1)}{b-1} - \binom{n-2}{b-1} - \binom{2(b-1)}{b} - (n-b-2).$$

Now after cancellation we obtain

$$\text{content } P' - \text{content } P = \binom{n-2}{b} - \binom{n-2}{b-1} + 1.$$

For $n > 2b$ this is positive, as desired.

Now assume $n \leqslant 2b$. We have a decomposition of P' by $(1^n0^{n-b} \to 01^b0^{n-b-1} \to 01^{2b-n+1}(10)^{n-b-1}]$, of which we will call the first process F and the second S. By Corollary 11.1 and the assumption $n \leqslant 2b$ we know that content $F < 0$. By Lemma 12 we have content $S = \alpha_{n-b-1} - (n - b - 2)$. It follows that content $P' < \alpha_{n-b-1} - (n - b - 2)$. Now P may be written as $(1^{2b-n}1^{n-b}0^{n-b} \to 1^{2b-n}(10)^{n-b}]$. Hence content $P = \alpha_{n-b} - (n - b - 1)$. Now since $\alpha_{n-b} > \alpha_{n-b-1}$, it follows that content $P >$ content P' as desired. ■

We are now ready for our maximality result.

Theorem 17. Among all weight b words of length $n \geqslant b$, a maximum value is achieved by $A(b,n)$.

Proof. We proceed by induction on n, the case $r = b$ being clearly true. Let z be a weight b word of length n at which the stated maximum occurs, and assume to the contrary that this maximum does not occur at $A(b,n)$.

Suppose first that $n > 2b$, so that $A(b,n) = 0^{n-2b}(10)^b$.

Observe first that z must have prefix 0. For assume not and let $z = 1s$, where s is the length $n - 1$ suffix of z. Using induction on weight $b - 1$ words of length $n - 1$ we have value$(s) \leqslant$ value$(A(b - 1,n - 1))$. Hence by Lemmas 15 and 16 we get value $(z) \leqslant$ value$(1A(b - 1,n - 1)) <$ value $(01A(b - 1,n - 2))$. This contradicts the maximality of value(z).

We claim further that z must have prefix 0^{n-2b} 1. Suppose not, and assume first that z is prefixed by 0^j1, say $z = 0^j1z'$, where $j < n - 2b$. The above paragraph implies $j > 0$. We may therefore apply induction to weight b words of length $n - j$ to conclude that value$(A(b,n - j)) \geqslant$ value$(1z')$. Since $0^jA(b,n - j) = A(b,n)$ (by the assumption $n > 2b$), Lemma 15 then implies value$(A(b,n)) \geqslant$ value(z), a contradiction. Next assume that z is prefixed by $0^{n-2b+e}1$, say $z = 0^{n-2b+e}1z''$, where $e \geqslant 1$. Now applying induction to weight b words of length $2b$ we have value$(A(b,2b)) \geqslant$ value$(0^e1z'')$. Since $0^{n-2b}A(b,2b) = A(b,n)$, Lemma 15 again gives value $(A(b,n)) \geqslant$ value(z), a contradiction. The claim is thus proved.

Now write $z = 0^{n-2b}1r$, where r is the length $2b - 1$ suffix of z. We apply induction to weight b words of length $2b - 1$ to see that value$(A(b - 1,2b - 1) \geqslant$ value(r).

Then since $0^{n-2b}1A(b-1,2b-1) = A(b,2b)$, Lemma 15 implies value$(A(b,n)) \geqslant$ value(z), contradicting the fact that the maximum did not occur at $A(b,n)$.

Now assume $n \leqslant 2b$, so that $A(b,n) = 1^{2b-n}(10)^{n-b}$. The argument is similar in outline to the case $n > 2b$, so we only sketch it. We first argue that z is prefixed by 1 by noting that $01A(b-1,n-1) = 0A(b,n-1)$, and using induction and Lemma 16. Induction can then be used to show that z is in fact prefixed by $1^{2b-n+1}0$. Now Lemma 15 and induction give the desired result. ∎

Putting our results together we get the following.

Theorem 18. The even weight word of length n with maximum value is $A(e,n)$, where e is whichever of $\lfloor (n-1)/2 \rfloor$ or $\lfloor (n+1)/2 \rfloor$ is even.

Proof. Theorem 17 tells us that for any b the word $A(b,n)$ has a maximum value among weight b words of length n. Theorems 13 and 14 tell us that the sequence {value$(A(b,n)$:b even) is unimodal for a given n. It follows that the even weight word of maximum value is the $A(e,n)$ such that $D(e-2,n) \geqslant 0$ and $D(e,n) < 0$. Theorems 13 and 14 imply that the e satisfying this is whichever of $\left\lfloor \frac{n-1}{2} \right\rfloor$ or $\left\lfloor \frac{n+1}{2} \right\rfloor$ is even. Our theorem follows. ∎

Finally we use Theorem 18 to reach our goal of providing an asymptotically optimal lower bound for $s(Q_n)$.

Theorem 19. Assume $n \geqslant 3$. Let e be whichever of $\left\lfloor \frac{n-1}{2} \right\rfloor$ or $\left\lfloor \frac{n+1}{2} \right\rfloor$ is even, and let

$$F(e,n) = \binom{n}{e+1} + \binom{2e}{e} - \binom{n}{e} - \binom{2e}{e+1} - (n-e-1) + \sum_{i=1}^{e-1} \left[\binom{2e-2j}{e-j} - \binom{2e-2j}{e-j+1} \right].$$

Then

$$s(Q_n) \geqslant 2^{n-1} - \left[\binom{n-1}{e-1} + (n-e-1) + F(e-1,n-1) \right] \quad \text{if } n \equiv 3 (\text{mod } 4),$$

and

$$s(Q_n) \geqslant 2^{n-1} - \left[\binom{n-1}{e-1} + (n-e-1) + F(e,n) \right] \quad \text{otherwise.}$$

Furthermore, this lower bound is asymptotically optimal.

Proof. By Lemma 10 and Theorem 18, we need only show that value$(A(e,n))$ is the expression in the brackets. We proceed to do this.

Observe that we may reach $A(e,n)$ by the sequence of processes $(0^n \to 0^{n-2}1^2 \to 0^{n-4}1^4 \to \cdots \to 0^{n-(e-2)}\, 1^{e-2} \to 1^e 0^{n-e} \to A(e,n)]$.

We begin by computing value($0^{n-(e-2)}\,1^{e-2}$). Note that for each even k, where $2 \leqslant k \leqslant e - 2$, we have

$$\text{content}(0^{n-(k-2)}\,1^{k-2} \to 0^{n-k}1^{k}] = |\{\text{weight } k + 1 \text{ words}\}| - |\{\text{weight } k \text{ words}\}| = \binom{n}{k+1} - \binom{n}{k}.$$

Also value(0^n) $= n - 1$. Hence value($0^{n-(e-2)}\,1^{e-2}$), which is the sum of all these quantities, is $\Sigma_{k=0}^{e-2}\left[\binom{n}{k+1} - \binom{n}{k}\right] = \binom{n-1}{e-1}$.

To compute value(1^e0^{n-e}), we observe that 1^e0^{n-e} is the immediate successor of $0^{n-(e-2)}1^{e-2}$ in our ordering of even weight points. Hence

$$\text{value}(1^e0^{n-e}) = \text{value}(0^{n-(e-2)})1^{e-2}) + \text{New}(1^e0^{n-e}) = \binom{n-1}{e-1} + n - e - 1.$$

The last step is to compute content($1^e0^{n-e} \to A(e,n)$]. Now recall that for $e \leqslant \lfloor n/2 \rfloor$ this quantity is exactly content π_0 with e playing the role of b, where π_0 appears in the proof of Theorem 13. With e defined as above, we have $e \leqslant \lfloor n/2 \rfloor$ precisely when $n \not\equiv 3(\text{mod } 4)$. Hence observing that with e replacing b we have content $\pi_0 = F(e,n)$, it follows that content($1^e0^{n-e} \to A(e,n)$] $= F(e,n)$ when $n \not\equiv 3(\text{mod } 4)$. Now when $n \equiv 3(\text{mod } 4)$ we have $e = (n + 1)/2$, and in that case $A(e,n) = 1A(e - 1,n - 1)$. Thus content($1^e0^{n-e} \to A(e,n)$] = content ($1^{e-1}0^{n-e} \to A(e - 1,n - 1)$]. The right side, being content π_0 with b replaced by $e - 1$ and n by $n - 1$, is $F(e - 1,n - 1)$. We can therefore conclude that content($1^e0^{n-e} \to A(e,n)$] $= F(e - 1,n - 1)$ or $F(e,n)$ depending on whether $n \equiv 3(\text{mod } 4)$ is true or not respectively.

Finally, since value($A(e,n)$) $=$ value(1^e0^{n-e}) $+$ content ($1^e0^{n-e} \to A(e,n)$], the theorem follows.

To measure how good this lower bound is, we estimate it asymptotically. For simplicity we assume $n \equiv 0(\text{mod } 4)$.

First consider the sum $S = \Sigma_{j=1}^{e-1}\left[\binom{2e-2j}{e-j} - \binom{2e-2j}{e-j+1}\right]$.

Since $\binom{2k}{k} - \binom{2k}{k+1} = \binom{2k}{k}\Big/k$, and $\left[\binom{2k}{k}\Big/k\right] \div \left[\binom{2k-2}{k-1}\Big/k-1\right]$ approaches 4 for large k it follows that $S \sim \binom{2e-2}{e-1} \div (e - 1)$.

Now for $n \equiv 0(\text{mod } 4)$ we have $e = n/2$, and hence the expression in the braces becomes, after cancellation, $\binom{n-1}{e-1} + S$. Now $\binom{n-1}{e-1}$ dominates S, and $\binom{n-1}{e-1} = \frac{1}{2}\binom{n}{n-2} \sim c\,2^{n-1}n^{-1/2}$ (where c is a constant), by the Stirling approximation. Thus we find that $s(Q_n)$ is bounded below by $(1 - cn^{-1/2})2^{n-1}$ for large

n. The same holds in the other possible congruence classes mod 4 for n, only with different constants.

From the trivial upper bound $s(Q_n) \leqslant 2^{n-1}$, we see that the ratio of our lower bound to whatever is the true value must approach 1 as $n \rightarrow \infty$. Hence our lower bound is asymptotically optimal. ■

REFERENCES

[1] M. Behzad, G. Chartrand, and L. Lesniak-Foster, *Graphs and Digraphs,* Prindle, Weber & Schmidt (1979).
[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications,* North Holland, Amsterdam (1976).
[3] P. Z. Chinn, J. Chvatalova, A. K. Dewdney, and N. E. Gibbs, The bandwidth problem for graphs and matrices—a survey. *J. Graph Theory* **6** (1982) 223–254.
[4] F. Harary, *Graph Theory,* Addison-Wesley, Reading, MA (1969).
[5] J. Y-T. Leung, O. Vornberger, and J. D. Witthoff, On some variants of the bandwidth minimization problem. *SIAM J. of Computing* **13** (1984) 650–667.
[6] I. H. Sudborough, Bandwidth constraints on problems complete for polynomial time. *Theoret. Computer Sci.* **26** (1983) 25–52.

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