Eigenvalues and Separation in Graphs

by

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Abstract

Consider a one to one map $f:V(G)\rightarrow V(H)$ of graphs G and H, where $|G| \leq |H|$. We let $|f| = \min\{dist_H(f(x),f(y)): xy \in E(G)\}$, where $dist_H$ denotes distance in H. Now define sep(G,H), the separation of G into H, to be the maximum of $|f|$ over all such maps f.

Using the Kronecker product of matrices we develop a method for computing, in certain favorable cases, the set of eigenvalues of graphs of the form (GXH)(k). Here GXH refers to the usual graph product and S(k) (for a graph S) is the graph obtained from S by joining two points of V(S) by an edge if and only if they are at distance at most k in S. Let Q(n) and C_nXC_n denote the n-dimensional cube and the nxn "discrete torus" respectively, and $\lambda_{min}(S)$ the smallest eigenvalue of a graph S. We apply our method to analyze $\lambda_{min}(Q(n)(k))$ and $\lambda_{min}((C_nXC_n)(k))$, obtaining

exact values for certain k and asymptotically optimal lower bounds for others. Combining these results with one of the results of Alon and Milman [AM], we obtain bounds for the edge isoperimetric problem in the graphs $Q(n)(k)^c$ and $(C_n X C_n)(k)^c$, where S^c denotes the graph obtained from a graph S by joining two vertices if and only if they are not joined in S. As a corollary we obtain functions b(k,p) (resp. c(k,p)), such that if a graph G on p points and q edges satisies $q > b(k,p)$ (resp. $q > c(k,p)$), then $sep(G,Q(n)) \leq k$ (resp. $sep(G,C_n X C_n) \leq k$).

1. Introduction

Consider a one-to-one map $f:V(G)\to V(H)$ of graphs G and H, where $|G| \leq |H|$. We let $|f| = \min\{dist_H(f(x),f(y)): xy \in E(G)\}$, where $dist_H$ denotes distance in H. Now define sep(G,H), the separation of G into H, to be the maximum of |f| over all such maps f. The idea behind sep(G,H) is to map G into H while keeping pairs of adjacent points in G as far apart as possible in H. This parameter is a natural dual to the much studied bandwidth parameter $B(G,H) = \min_f \max\{ dist_H(f(x),f(y)): xy \in E(G) \}$, obtained by interchanging the min and the max in the definition.

The motivation for a study of sep(G,H) has several aspects. Probably the most intensely studied problem similar to that of determining sep(G,H) is that of $sep(K_p,Q(n))$ in connection with the theory of binary error correcting codes. Here Q(n) is the n-dimensional cube, whose vertex set is $(\mathbb{Z}_2)^n$, with two binary n-tuples adjacent when they differ in exactly one coordinate, and K_p denotes the complete graph on p points. The distance between two n-tuples in Q(n) is then the number of coordinates in which they differ, otherwise known as the Hamming distance. Recall that one of the main subjects of study in coding theory is the construction of "(n,m) codes", where an (n,m) code is a set $C \subset V(Q(n))$ such that $dist(x,y) \geq m$ for $x,y \in C$. Much attention is given to finding the maximum size of an (n,m) code as a function of n and m. The existence of such a C of size p, say, is equivalent to showing that $sep(K_p,Q(n)) \geq m$, and maximizing the size of C is then the same as finding $\max\{p: sep(K_p,Q(n)) \geq m\}$. Because of this connection with coding theory, sep(G,Q(n)) is of particular interest. The study of $sep(K_p,H)$ for any graph H is then also a natural extension of the study of codes in the hypercube.

A second motivation for the study of sep(G,H) involves optimal facility location. Here one views the vertices of G as special facilities to be located in the larger network H. Certain pairs of these facilities may provide services which are in some way redundant, and these pairs are the edges of G. In designing an optimal placement of these facilities (that is, an embedding $f:V(G)\rightarrow V(H)$), one might like to place redundant pairs as far apart as possible. This naturally leads to a study of sep(G,H).

Finally we note that separation is related the notion of r-domination. That is, let d(r,H) (the r-domination number of H) be the minimum number of vertices in any subset C of V(H) such that every vertex of H is within distance at most r of some vertex of C. For any tree T let M(p,T) = max{ r: $sep(K_r,T) \geq p$ }. It was shown in [MM] that $M(p+1,T) + p\cdot d(p,T) \leq |T|$ if $|T| \geq p+1$, and that M(2p+1,T) = d(p,T).

In [MP] we investigated $sep(G_n,P_n)$ where G_n is a graph on n vertices and P_n is the path graph on n vertices. Bounds for $sep(G_n,P_n)$ were obtained in terms of various graph parameters of G_n, and an estimate for $sep(Q(n),P_{2^n})$ was given which was shown to be asymptotically optimal.

We will need the following definitions. For a graph G, let $G^{(k)}$ (resp. G(k)) be the graph with vertex set V(G) where two vertices x,y are joined by an edge if and only if dist(x,y) = k (resp. $dist(x,y)\leq k$). The graph G(k) is often called the <u>k'th power of G</u>, and note that $G^{(k+1)} = G(k+1) - E(G(k))$. We also let G^c denote the graph with vertex set V(G) where two vertices x,y are joined by an edge if and only if x,y are not joined by an edge in G. Notice then that $sep(G,H) \geq k+1$ if and only if G can be embedded as a subgraph of $H(k)^c$, or equivalently, $sep(G,H) \leq k$ if and only if G is not a subgraph of $H(k)^c$. Let S and T be two graphs. The the <u>direct product</u> of S and T, denoted by $S \times T$, is the graph with vertex and edge sets given by: $V(S \times T) = V(S) \times V(T)$, and $E(S \times T) = \{(s,t)(s',t'): s = s' \text{ and } tt'\in E(T), \text{ or } t = t' \text{ and } ss'\in E(S)\}$. We may view $S \times T$ as consisting of |S| copies of T, with a pair of T's joined by a matching (connecting pairs of similar points) precisely when the corresponding pair of points in S are joined by an edge. Naturally $Q(n) = K_2\times K_2\times K_2 ...\times K_2$, where there are n factors in the product. We will also be interested in the graph $C_n\times C_n$ (where C_n is the cycle graph on n vertices), sometimes called the <u>discrete torus</u>.

We denote by A(G) the <u>adjacency matrix</u> of G, that is, the nxn matrix (where n = |V(G)|) whose (i,j) entry is a 1 if vertices i and j are joined by an edge, and is a 0 otherwise. The eigenvalues of A(G) (these are also called the eigenvalues of G) will be denoted $\lambda(1)(G)\geq\lambda(2)(G)\geq\lambda(3)(G)\geq\cdots\geq\lambda(n)(G)$, where G is omitted from the notation when it is fixed by context. The multiset of eigenvalues of G is called the <u>spectrum</u> of G. For a matrix A, the <u>adjacency algebra generated by A</u>, denoted by <u>α(A)</u>, is the set of nxn matrices arising as polynomials in A over the complex numbers.

The purpose of this paper is twofold.

First it is to describe a method of using the Kronecker product of matrices to obtain, in certain favorable cases, the eigenvalues of powers of products of graphs; that is, the eigenvalues of graphs of the form $(G\times H)(k)$.

Second it is to apply this method to obtain the eigenvalues of Q(n)(k) and $(C_n \times C_n)(k)$, and to use the knowledge of these in combination with one of the results in [AM] (to be described in the next section) to obtain bounds for the edge isoperimetric problem in $Q(n)(k)^c$ and $(C_n \times C_n)(k)^c$. As a corollary we obtain functions s(k,p) (resp. t(k,p)) such that if a graph G on p points and q edges satisfies q > s(k,p) (resp. q > t(k,p)), then sep(G,Q(n)) ≤ k (resp. $sep(G, C_n \times C_n) \le k$). Indeed the result of [AM] and its possible connection to separation was the main stimulus for the work reported here.

2. Eigenvalues and Separation in Products

Let H be a regular graph on n points, and let $p \leq n$ be an integer. Consider the function

$$f(H,p) = \max\{|E(G)|: G \text{ is an induced subgraph of H on p points.}\},$$

or equivalently the function

$$\partial(H,p) = \min\{|\partial(G)|: G \text{ is an induced subgraph of H on p points.}\},$$

where $\partial(G)$ is the set of edges of H with one point in G and the other in H-G. The problem of determining f(H,p) (or equivalently $\partial(H,p)$) for each integer p is called the <u>edge isoperimetric problem</u> for H.

The relevance of this problem to separation is that bounds for the function f yield edge bounds for separation. Specifically, suppose we knew $f(H(k)^c,p)$ and consider any graph G satisfying $|G| \leq |H|$. We would then know that if $f(G,p) > f(H(k)^c,p)$ then G cannot be embedded in $H(k)^c$, since a certain p-point induced subgraph of G is denser than the densest p-point induced subgraph of $H(k)^c$. Thus we may conclude that $f(G,p) > f(H(k)^c,p)$ implies $sep(G,H) \leq k$. In particular, if $|E(G)| > f(H(k)^c,|G|)$, then $sep(G,H) \leq k$.

The following result in [BuCS] provides bounds for f(H,p) in terms of the eigenvalues of H. This result can also be derived using results of Alon and Milman [AM, remark 2.4]. Let G be any induced subgraph on p points of a d-regular graph H on n points. Then the average degree $d_1(G)$ of G satisfies

$$d_1(G) \leq \frac{p\lambda_1}{n} + d - \lambda_1 ,$$

where λ_1 is the difference between d and the second largest eigenvalue of H. (For graphs in general, that is, ones which are not necessarily regular, λ_1 is defined as the second smallest eigenvalue of the matrix $Q = \text{diag}(\deg(v))_{v \in H} - A(H)$. For regular H, this definition of λ_1 reduces to the difference given above.). Now let $d = \lambda(1) \geq \lambda(2) \geq ... \geq \lambda(n)$ be the eigenvalues of H, so that $\lambda_1 = d-\lambda(2)$. It follows that

$$f(H,p) \leq \frac{1}{2}\left(p\lambda(2)\left(1 - \frac{p}{n}\right) + d\frac{p^2}{n}\right). \qquad \text{(A)}$$

We see then that an upper bound for f(H,p) would follow from an upper bound for $\lambda(2)$.

Suppose now that H is a d-regular graph on n points, and that also H(k) (and therefore $H(k)^c$) is regular for each integer k, $1 \leq k \leq n$. Then the inequality above applies to $H(k)^c$. Recall now the well known relation between the eigenvalues of a d-regular graph S on n points and the eigenvalues of its complement S^c: if S has the eigenvalues $\lambda(1)=d, \lambda(2),...,\lambda(n)$, then S^c has the eigenvalues n-d-1, $-1-\lambda(2), ...,-1-\lambda(n)$. Hence the second largest eigenvalue of S^c is $-1-\lambda_{min}(S)$, where $\lambda_{min}(S)$ is the smallest eigenvalue of S. Since $\lambda_{min} < 0$ for any graph having an edge, it follows that a lower bound for $\lambda_{min}(S)$ yields an upper bound for $\lambda(2)(S^c)$ and hence an upper bound for $f(H(k)^c)$.

We will use these ideas in a form summarized by the following lemma.

Lemma 0: Let H be a graph on n points, and suppose H(k) is regular for some k. Let d be the regularity degree of $H(k)^c$, and assume $\lambda_{min}(H(k)) \geq b(k)$ for some function b. Let G be a graph on p points and q edges, $p \leq n$.

(a) If $q > \frac{1}{2}\left(p[-1-b(k)]\,(1 - \frac{p}{n}) + d\,\frac{p^2}{n} \right)$, then $sep(G,H) \leq k$.

(b) If $p > n\left(1 - \frac{n - d - 1}{n - d - 1 - b(k)} \right)$, then $sep(K_p,H) \leq k$.

Proof: First we have $\lambda(2)(H(k)^c) = -1-\lambda_{min}(H(k)) \leq -1-b(k)$. Now inequality (A) implies that a necessary condition for G to be embeddable in $H(k)^c$ is that

$$q \leq \frac{1}{2}\left(p[-1-b(k)]\,(1 - \frac{p}{n}) + d\,\frac{p^2}{n} \right). \qquad (*)$$

Part (a) follows since the hypothesis violates the condition. Thus G is not embeddable in $H(k)^c$ so $sep(G,H) \leq k$.

For part (b) we just apply (*) to the complete graph $G = K_p$. Substituting $q = \binom{p}{2}$ and solving the resulting inequality for p, we obtain a necessary condition for embeddability of K_p in $H(k)^c$ which is violated by the hypothesis on p. Hence $sep(K_p,H) \leq k$. ∎

We will be concerned with graphs H which, in addition to having regular k'th powers for all k, have the additional property that $A(H^{(k)})$ is a polynomial in A(H) for each k (that is, $H^{(k)}$ is in the adjacency algebra of H). For such graphs the eigenvalues of H(k) are of course easily obtained as polynomials in the eigenvalues of H. Our object in this section is to describe a method for finding the eigenvalues of powers of direct products of such graphs. Then by lower bounding or determining exactly λ_{min} in these powers, we obtain edge bounds for separation in these graphs using lemma 0.

We now recall some basic definitions from matrix theory. Suppose A and B are mxn and pxq matrices respectively. Then the Kronecker product of A and B, denoted by $A \otimes B$, is the (mp)x(nq) matrix obtained by replacing each entry a_{rs} of A by the pxq matrix $a_{rs}B$. Now suppose A and B are adjacency matrices of graphs S and T, and let $x,x' \in V(S)$ and $y,y' \in V(T)$. The (y,y') entry of the submatrix $a_{xx'}B$ of $A \otimes B$ will be referred to as the ((x,y),(x',y')) entry of $A \otimes B$. Under a suitable ordering of the points in the graph $S \times T$, this entry is the one which corresponds in the adjacency matrix of $S \times T$ to the pair of points {(x,y),(x',y')}.

Lemma 1: Let G and H be two graphs. Then for any k, $1 \le k \le n$, we have

$$A((G\times H)^{(k)}) = \sum_{s+t=k} A(G^{(s)})\otimes A(H^{(t)}).$$

Proof: Set $S = \sum_{s+t=k} A(G^{(s)})\otimes A(H^{(t)})$. For any two points (x,y) and (x',y') in G×H, we will show that if $\mathrm{dist}_{G\times H}((x,y),(x',y')) = k$ then the ((x,y),(x',y')) entry of S is a 1, while if $\mathrm{dist}_{G\times H}((x,y),(x',y')) \ne k$ then the ((x,y),(x',y')) entry of S is a 0.

Suppose that $\mathrm{dist}_{G\times H}((x,y),(x',y')) = k$. Then for some nonnegative integers c and d we must have $\mathrm{dist}_G(x,x') = c$, $\mathrm{dist}_H(y,y') = d$, and $c+d = k$. Thus the (x,x') and (y,y') entries of $A(G^{(c)})$ and $A(H^{(d)})$ respectively are 1's, and hence the ((x,x'),(y,y')) entry of $A(G^{(c)})\otimes A(H^{(d)})$ is a 1. Also for any integer pair (c',d') other than (c,d) we see that either the (x,x') entry of $A(G^{(c')})$ is 0, or the (y,y') entry of $A(G^{(d')})$ is 0. Thus the ((x,y),(x',y')) entry of $A(G^{(c')})\otimes A(H^{(d')})$ must be 0. It follows that the ((x,y),(x',y')) entry of S is a 1.

Suppose now that $\mathrm{dist}_{G\times H}((x,y),(x',y')) \ne k$. Then for every nonnegative integer pair (s,t) summing to k, at least one of the statements "$\mathrm{dist}_G(x,x') = s$" and "$\mathrm{dist}_H(y,y') = t$" is false. Hence either the (x,x') entry of $G^{(s)}$ is a 0 or the (y,y') entry of $H^{(t)}$ is a 0. It follows that the ((x,y),(x',y')) entry of $A(G^{(s)})\otimes A(H^{(t)})$ is a 0. This being true for all such pairs (s,t), it follows that the ((x,y),(x',y')) entry of S is a 0. ∎

The following theorem now gives the eigenvalues of the k'th power (G×H)(k) of the graph product G×H, under the assumption that $A(G^{(i)})$ and $A(H^{(i)})$ are in the adjacency algebra of A(G) and A(H) respectively for all $i \ge 1$. With this assumption let $\underline{p_i(x)}$ and $\underline{q_i(x)}$ be the polynomials such that $p_i(A(G)) = A(G^{(i)})$ and $q_i(A(H)) = A(H^{(i)})$. For eigenvalues λ and μ of G and H respectively, denote by $\lambda^{(s)}$ and $\mu^{(t)}$ the numbers $p_s(\lambda)$ and $q_t(\mu)$.

Theorem 1: Let G and H be graphs for which $A(G^{(i)})$ and $A(H^{(i)})$ are in the adjacency algebra of A(G) and A(H) respectively for all $i \ge 1$. Then the set of eigenvalues of (G×H)(k) is given by

$$\{ \sum_{1\le s+t\le k} \lambda^{(s)}\mu^{(t)} : \lambda,\mu \text{ eigenvalues of G and H respectively listed with multiplicity } \}.$$

Proof: Observe that for any graph Γ we have

$$A(\Gamma(k)) = \sum_{r=1}^{k} A(\Gamma^{(r)})$$

Hence by Lemma 1 and the hypothesis on G and H we have

$$A((G\times H)(k)) = \sum_{1\le s+t\le k} p_s(A(G))\otimes q_t(A(H)).$$

Thus for any eigenvector v of G with eigenvalue λ and eigenvector w of H with eigenvalue μ, we have

$$A((G\times H)(k))(v\otimes w) = \left(\sum_{1\leq s+t\leq k} \lambda^{(s)}\mu^{(t)} \right)(v\otimes w). \qquad (\text{ I })$$

We can now apply some elementary linear algebra to complete the argument. Let $B = \{v_i: 1\leq i \leq n\}$ (resp. $B' = \{w_i: 1\leq i \leq n\}$) be a basis of $\mathbb{R}^n$ consisting of eigenvectors of A(G) (resp. A(H)). Then since $A(G^{(i)})$ (resp. $A(H^{(i)})$) is in the adjacency algebra of A(G) (resp. A(H)) for all $i \geq 1$, we see that B (resp. B') is a basis for $A(G^{(i)})$ (resp. $A(H^{(i)})$) for all $i \geq 1$. But we know in general that if $\{\alpha_i: 1 \leq i \leq n\}$ and $\{\beta_i: 1 \leq i \leq n\}$ are bases of $\mathbb{R}^n$ consisting of eigenvectors for nxn matrices X and Y respectively, then the set $\{\alpha_c\otimes\beta_d: 1 \leq c,d \leq n\}$ is a basis of $\mathbb{R}^{n^2}$ consisting of eigenvectors for $X\otimes Y$. Hence the collection $\{v_c\otimes w_d: 1 \leq c,d \leq n\}$ is an eigenbasis for each term $p_s(A(G))\otimes q_t(A(H))$, and hence an eigenbasis for $A((G\times H)(k))$. The theorem now follows from equation (I) above. ∎

Corollary 1.1: If $\{G_1,G_2,...,G_n\}$ is a set of graphs such that $A(G_i^{(s)})$ is in the adjacency algebra of $A(G_i)$ for all i and all s, then the set of eigenvalues of $(G_1\times G_2\times...\times G_n)(k)$ is given by

$$\{ \sum_{1\leq s_1+s_2+..+s_n\leq k} \lambda_{i_1}^{(s_1)}\lambda_{i_2}^{(s_2)}...\lambda_{i_n}^{(s_n)} : \lambda_{i_t} \text{ an eigenvalue of } G_t \text{ counted with multiplicity, } 1\leq t\leq n\}.$$

3. The Hypercube

Our first application of these results will be toward obtaining the eigenvalues of Q(n)(k), determining $\lambda_{min}(Q(n)(k))$, and then deriving edge bounds for separation in the hypercube. As a notational convenience for binomial coefficients, we let $\binom{n}{m} = 0$ when $m > n$ and we observe that various combinatorial identities still hold under this more general notation.

Lemma 2: The distinct eigenvalues of Q(n)(k) are given by

$$A(f) = \sum_{t=0}^{k} (-1)^t \binom{f-1}{t}\binom{n-f}{k-t} - 1, \quad f=0,1,2,...,n. \qquad (B)$$

Proof: We will apply corollary 1.1 to get a general form for the spectrum of Q(n)(k). This form will then be greatly simplified to get the stated result.

Observe that $Q(n) = K_2\times K_2\times...\times K_2$, where there are n factors in the product, and the eigenvalues of each factor K_2 are +1 and -1. The only powers of K_2 are the 0'th and the 1'st, and the corresponding polynomials are $p_0(x) = 1$ and $p_1(x) = x$. Hence in applying corollary 1.1 to Q(n)(k), we have $s_i = 0$ or 1 for all i. Since $\lambda_{i_t}^{(0)} = p_0(\lambda_{i_t}) = 1$ and $\lambda_{i_t}^{(1)} = p_1(\lambda_{i_t}) = \lambda_{i_t}$, it follows that the expression in the corollary may be decomposed into a sum of k "homogeneous" pieces. The r'th homogeneous piece is the function $H_r(\lambda_{i_1},\lambda_{i_2},\cdots,\lambda_{i_n}) = \sum \lambda_{i_{t_1}}\lambda_{i_{t_2}}\cdots\lambda_{i_{t_r}}$, a sum over all possible r-tuples of elements from the set $\{\lambda_{i_1},\lambda_{i_2},\cdots,\lambda_{i_n}\}$. Now observe that H_r is a symmetric function of $\lambda_{i_1},\lambda_{i_2},\cdots,\lambda_{i_n}$, and is therefore determined simply by the number of the λ's that are -1. Let us then write $H_r(f)$ for the value of H_r when f of the λ's are -1 and the remaining n-f are +1.

The first step in simplifying this form is already done in [CvDS, pp.75-77] and is based on generating functions. From the description of H_r above we see that $H_r(f)$ is the coefficient of $(-1)^r x^{n-r}$ in the polynomial

$$(x+1)^f(x-1)^f = \sum_{i=0}^{f} \binom{f}{i} x^i \sum_{j=0}^{n-f} \left[\binom{n-f}{j}(-1)^{n-f-j}\right]x^j .$$

Thus $H_r(f) = (-1)^r \sum_{i=0}^{n-r} \binom{f}{i} \binom{n-f}{n-r-i}(-1)^{r-f+i}$. Letting $A(f) = \sum_{r=1}^{k} H_r(f)$, we see that the A(f), $0\le f\le n$, are the distinct eigenvalues of Q(n)(k), and they are given by the formula

$$A(f) = \sum_{r=1}^{k}(-1)^r \sum_{i=0}^{n-r} \binom{f}{i} \binom{n-f}{n-r-i}(-1)^{r-f+i} = (-1)^f \sum_{r=1}^{k} \sum_{i=0}^{n-r} \binom{f}{i} \binom{n-f}{n-r-i}(-1)^i .$$

This much was previously known [CvDS]. In order to determine or lower bound λ_{min} effectively (so that lemma 0 can be applied) we proceed to the second step of the simplification. First let us rewrite A(f) as

$$A(f) = (-1)^f \sum_{r=1}^{n} \sum_{i=0}^{n-r} \binom{f}{i} \binom{n-f}{n-r-i}(-1)^i - (-1)^f \sum_{r=k+1}^{n} \sum_{i=0}^{n-r} \binom{f}{i} \binom{n-f}{n-r-i}(-1)^i = (-1)^f[S_1 - S_2] ,$$

where S_1 is the first double sum (with lower limit r=1 on the outer sum) and S_2 is the second double sum. Consider first S_1. If we reassociate terms, bringing together those with a common factor of $\binom{n-f}{n-t}$ for some t, then $S_1 = \sum_{t=1}^{n} \binom{n-f}{n-t} \sum_{i=0}^{t-1}(-1)^i \binom{f}{i}$. Using the identity $\sum_{i=0}^{t-1}(-1)^i \binom{f}{i} = (-1)^{t-1}\binom{f-1}{t-1}$ we get

$$S_1 = \sum_{t=1}^{n}(-1)^{t-1}\binom{f-1}{t-1}\binom{n-f}{n-t}.$$

Similarly the second sum becomes

$$S_2 = \sum_{t=1}^{n-k}(-1)^{t-1}\binom{f-1}{t-1}\binom{n-f}{n-k-t}.$$

Now observe that the only nonzero term in S_1 is the t=f term, so $S_1 = (-1)^{f-1}$. Hence we get

$$A(f) = (-1)^{f+1}\sum_{i=1}^{n-k}(-1)^{i-1}\binom{f-1}{i-1}\binom{n-f}{n-k-i} - 1.$$

We now note that terms with i > f are 0 because of the first factor, while terms with i < f-k are 0 because of the second factor. Also noting that $\binom{n-f}{n-k-i} = \binom{n-f}{k-(f-i)}$ and $\binom{f-1}{i-1} = \binom{f-1}{f-i}$ we let t = f-i be our new index of summation, and it follows that

$$A(f) = \sum_{t=0}^{k} (-1)^t \binom{f-1}{t}\binom{n-f}{k-t} - 1.$$

as desired. ∎

We can now find $\lambda_{min}(Q(n)(k))$ precisely for k odd.

Theorem 2: If k is odd, then $\lambda_{min}(Q(n)(k)) = -1 - \binom{n-1}{k}$.

Proof: We need to minimize A(f) over f = 0,1,...,n. Now when f=n the only nonzero term is the t=k term, and hence with k odd we get $A(n) = -\binom{n-1}{k} - 1$. But for any r we have

$$A(r) \geq -1 - \sum_{t=0}^{k} \left| (-1)^t \binom{f-1}{t}\binom{n-f}{k-t} \right| = -1 - \binom{n-1}{k} = A(n).$$

The theorem is thus proved. ∎

We can now use this result to get a bound for the edge isoperimetric problem on powers of hypercubes, and from that we obtain edge bounds for separation.

Corollary 2.1: Let $N = |Q(n)| = 2^n$, and suppose k is odd. Let G be a graph on p points and q edges.

If $q > \frac{1}{2}\left(p\binom{n-1}{k}(1 - \frac{p}{N}) + \frac{p^2}{N} \sum_{t=k+1}^{n} \binom{n}{t} \right)$, then sep(G,Q(n)) ≤ k.

If $p > N\left(1 - \dfrac{N - \sum_{t=k+1}^{n}\binom{n}{t} - 1}{N - \sum_{t=k+1}^{n}\binom{n}{t} + \binom{n-1}{k}} \right)$, then $sep(K_p,Q(n)) \leq k$.

Proof: This follows from lemma 0 and theorem 2 on noting that $Q(n)(k)^c$ is regular of degree $\sum_{t=k+1}^{n}\binom{n}{t}$. ∎

We now develop the analogue of Theorem 2 when k is even. It will turn out that when n ≤ 2k we find $\lambda_{min}(Q(n)(k))$ exactly, while when n > 2k we obtain lower bounds for $\lambda_{min}(Q(n)(k))$.

Formula (B) (in the statement of Lemma 2) suggests that we view A(f) in a combinatorial way. We imagine a set T of size n - 1 partitioned into subsets L and R of sizes f - 1 and n - f respectively. Let E(k,n-1,f-1) be the number of k-subsets of T having even intersection with L, and O(k,n-1,f-1) the number having odd intersection with L. Then A(f) + 1 = E(k,n-1,f-1) - O(k,n-1,f-1).

For x∈T we let E(±x,k,n-1,f-1) be the size of the subset of E(k,n-1,f-1) consisting of subsets containing or not containing x depending on whether the x is preceded by a + or - respectively. Similarly for a pair x,y∈T we let
E(±x,±y,k,n-1,f-1) be the size of the subset of E(k,n-1,f-1) consisting of subsets containing or not containing x or y depending on whether x or y is preceded by a + or - respectively. Analogous definitions are made for O(±x,k,n-1,f-1) and O(±x,±y,k,n-1,f-1).

Lemma 3: Suppose k is even and $1 \le f-1 \le n-2$.
(a) If $k \ge \frac{n}{2}$, then $E(k,n-1,f-1) \ge \binom{n-2}{k}$ and $O(k,n-1,f-1) \ge \binom{n-2}{k}$.
(b) If $k < \frac{n-1}{2}$, then $E(k,n-1,f-1) \ge \binom{n-2}{k-1}$ and $O(k,n-1,f-1) \ge \binom{n-2}{k-1}$.

Proof: Suppose first that $n \le 2k$. We first reduce to the case f >2. For suppose that f = 2 (the smallest allowed value of f by hypothesis), so that |L| = 1. Then any k-subset counted in E(k,n-1,1) has empty intersection with |L|, so $E(k,n-1,1) = \binom{|R|}{k} = \binom{n-2}{k}$. Also any k-subset counted in O(k,n-1,1) intersects L in one point, so $O(k,n-1,1) = \binom{|R|-1}{k-1} = \binom{n-2}{k-1} \ge \binom{n-2}{k}$ since $n \le 2k$. Hence we are done when f = 2.

We now proceed by induction on n. The condition on f requires that each of L and R has at least one point, so that the smallest value of n to consider is when n-1 = 2 in which case f-1 = 1. But this has already been done (in considering the case f=2), so the base of the induction is done.

Proceeding to the inductive step, we consider first the case n < 2k. Let x∈R, and note that E(k,n-1,f-1) = E(+x,k,n-1,f-1) + E(-x,k,n-1,f-1). But $E(-x,k,n-1,f-1) = E(k,n-2,f-1) \ge \binom{n-3}{k}$, where the inequality follows by induction since $n-1 \le 2k$. Also $E(+x,k,n-1,f-1) = E(k-1,n-2,f-1) \ge \binom{n-3}{k-1}$, where now the inequality follows by induction on noting that $n-1 \le 2(k-1)$ since n < 2k. It follows that $E(k,n-1,f-1) \ge \binom{n-3}{k} + \binom{n-3}{k-1} = \binom{n-2}{k}$ as desired. The corresponding inequality for O(k,n-1,f-1) is done in essentially the same way, and the inductive step is completed.

Next suppose n = 2k. Consider a pair x∈L and y∈R, and note that
E(k,n-1,f-1) = E(+x,-y,k,n-1,f-1) + E(-x,+y,k,n-1,f-1) + E(+x,+y,k,n-1,f-1) + E(-x,-y,k,n-1,f-1).

Observe that there are bijections between the two collections counted in E(+x,-y,k,n-1,f-1) and O(-x,+y,k,n-1,f-1) and between the two collections counted in E(-x,+y,k,n-1,f-1) and O(+x,-y,k,n-1,f-1) under which a given k-subset S corresponds to the k-subset S' obtained by deleting whichever of {x,y} was contained in S and adding whichever of {x,y} was not contained in S. Thus
E(+x,-y,k,n-1,f-1) = O(-x,+y,k,n-1,f-1) and E(-x,+y,k,n-1,f-1) = O(+x,-y,k,n-1,f-1). But the sum of these four quantities is the total number of k-subsets of T which include exactly one of {x,y}, and hence is $2\binom{n-3}{k-1}$. Thus

$$E(+x,-y,k,n-1,f-1) + E(-x,+y,k,n-1,f-1) = \binom{n-3}{k-1}.$$

Now observe that $E(-x,-y,k,n-1,f-1) = O(k,n-3,f-2) \geq \binom{n-4}{k}$, with the inequality following by induction on noting that n-2 < 2k since n = 2k, and f-2 > 1 since f > 2.

We analyze the remaining term E(+x,+y,k,n-1,f-1) by first writing E(+x,+y,k,n-1,f-1) = O(k-2,n-3,f-2). But any k-2 subset S counted in O(k-2,n-3,f-2) defines a complementary k-1 subset S' of T (since n = 2k). Now as S runs over all subsets counted in O(k-2,n-3,f-2), the complementary sets S' formed must all have the same parity of intersection with L (where |L| = f-2). Thus O(k-2,n-3,f-2) = O(k-1,n-3,f-2) or E(k-1,n-3,f-2). Noting that n-2 ≤ 2(k-1) (since n=2k) and f-2 ≥ 1, we can apply the inductive hypothesis to either O(k-1,n-3,f-2) or E(k-1,n-3,f-2) to finally arrive at $E(+x,+y,k,n-1,f-1) \geq \binom{n-4}{k-1}$.

Putting it all together we now have $E(k,n-1,f-1) \geq \binom{n-4}{k-1} + \binom{n-4}{k} + \binom{n-3}{k-1} = \binom{n-2}{k}$, as desired. An analogous argument shows that $O(k,n-1,f-1) \geq \binom{n-2}{k}$, and the lemma is proved if n≤ 2k.

Suppose now that $k < \frac{n-1}{2}$. Now any k-subset S counted in E(k,n-1,f-1) defines a complementary k' subset, where k' = n-1-k. As above we observe that when S runs over all subsets counted in E(k,n-1,f-1), then S' runs over all subsets counted in E(k',n-1,f-1) or over all subsets counted in O(k',n-1,f-1) and the correspondence S↔S' is a bijection in either case. Therefore E(k,n-1,f-1) = E(k',n-1,f-1) and O(k,n-1,f-1) = O(k',n-1,f-1), or E(k,n-1,f-1) = O(k',n-1,f-1) and O(k,n-1,f-1) = E(k',n-1,f-1). Now 2k' ≥ n since $k < \frac{n-1}{2}$ while $\binom{n-2}{k'} = \binom{n-2}{k-1}$. Thus we may apply the first part of the lemma to get $E(k',n-1,f-1) \geq \binom{n-2}{k-1}$ and $O(k',n-1,f-1) \geq \binom{n-2}{k-1}$. It follows that $E(k,n-1,f-1) \geq \binom{n-2}{k-1}$, as desired. The same argument gives $E(k,n-1,f-1) \geq \binom{n-2}{k-1}$, and the lemma is done.

∎

We now apply this lemma to analyze $\lambda_{min}(Q(n)(k))$ in the case when k is even.

<u>Theorem 3</u>: Suppose k is even. Then

(a) If $k \geq \frac{n}{2}$, then $\lambda_{min}(Q(n)(k)) = \binom{n-2}{k} - \binom{n-2}{k-1} - 1$.

(b) If $k = \frac{n-1}{2}$, then $\lambda_{min}(Q(n)(k)) = 2\binom{n-3}{k} + 2\binom{n-3}{k-2} - \binom{n-1}{k} - 1$.

(c) If $k < \frac{n-1}{2}$, then $\lambda_{min}(Q(n)(k)) \geq \binom{n-2}{k-1} - \binom{n-2}{k} - 1$.

<u>Proof</u>: Since $\lambda_{min}(Q(n)(k)) = \min\{A(f): f= 0,1,2...,n\}$, we will minimize or bound from below the expression E(k,n-1,f-1) - O(k,n-1,f-1) = A(f) + 1 over $0 \leq f \leq n$.

Using equation (B) we have $A(0) = n + \binom{n}{2} + \binom{n}{3} + ... + \binom{n}{k}$, $A(1) = A(n) = \binom{n-1}{k} - 1$. These numbers being positive, none of them can be $\lambda_{min}(Q(n)(k))$ or a lower bound for it since the spectrum must contain negative numbers. We can then restrict our attention to the cases $2 \leq f \leq n-1$.

Suppose first that $k \geq \frac{n}{2}$. Set E = E(k,n-1,f-1) and F = O(k,n-1,f-1). Clearly $E + F = \binom{n-1}{k}$, while by lemma 3 we have $-E \leq -\binom{n-2}{k}$. Thus

$$E - F \geq 2\binom{n-2}{k} - \binom{n-1}{k} = \binom{n-2}{k} - \binom{n-1}{k-1},$$

this lower bound for A(f) holding independent of f. But also $A(2)+1 = \binom{n-2}{k} - \binom{n-1}{k-1}$. Hence this lower bound is in fact achieved at f=2, so it must be the actual minimum of A(f)+1 over all f. Part (a) follows.

Now suppose $k = \frac{n-1}{2}$, and recall E = E(+x,k,n-1,f-1) + E(-x,k,n-1,f-1). The cases $n \leq 3$ being trivial, assume $n \geq 4$. Then we know that at least one of the subsets L or R of T has at least two points, and let x be a point in such a subset. The term E(-x,k,n-1,f-1) counts the number of k-subsets in the set T-{x} of size n-2 intersecting L evenly. Now 2k = n-1, |T-{x}| = n-2, and T-{x} partitions into either L-{x}∪R or R-{x}∪L (depending on where x lies). Then the hypothesis of lemma 3, with T-{x} taking the place of T, is satisfied; that is, both components of the partition have at least one point. Therefore $E(-x,k,n-1,f-1) \geq \binom{n-3}{k}$.

Consider the term E(+x,k,n-1,f-1). This counts the (k-1)-subsets of T-{x} intersecting L evenly. But now $k-1 = \frac{n-3}{2} < \frac{n-2}{2} = \frac{|T-\{x\}|}{2}$. Hence by lemma 3 with k-1 taking the place of k and n-1 the place of n, we have $E(+x,k,n-1,f-1) \geq \binom{n-3}{k-2}$.

Combining these bounds we get $E \geq \binom{n-3}{k} + \binom{n-3}{k-2}$. Again noting that $E + F = \binom{n-1}{k}$ we argue as in part (a) to get

$$E - F \geq 2\binom{n-3}{k} + 2\binom{n-3}{k-2} - \binom{n-1}{k},$$

a lower bound for A(f)+1 independent of f. But from equation (B) we have, using a binomial identity, that $A(3)+1 = 2\binom{n-3}{k} + 2\binom{n-3}{k-2} - \binom{n-1}{k}$. Hence the minimum of A(f)+1 is in fact A(3)+1. Part (b) follows.

Finally if $k < \frac{n-1}{2}$, then the lower bound on E from lemma 3 yields the required lower bound for λ_{min} by exactly the same argument as in part (a). This gives part (c), and the theorem is proved.

∎

Corollary 3.1: Let $N = |Q(n)| = 2^n$, and suppose k is even. Let G be a graph on p points and q edges.

(a) Suppose $k \geq \frac{n}{2}$.

If $q > \frac{1}{2}\left(p\left(\binom{n-2}{k-1} - \binom{n-2}{k}\right)\left(1 - \frac{p}{N}\right) + \frac{p^2}{N}\sum_{t=k+1}^{n}\binom{n}{t}\right)$, then $sep(G,Q(n)) \leq k$.

(b) Suppose $k < \frac{n-1}{2}$.

If $q > \frac{1}{2}\left(p\left(\binom{n-2}{k} - \binom{n-2}{k-1}\right)\left(1 - \frac{p}{N}\right) + \frac{p^2}{N}\sum_{t=k+1}^{n}\binom{n}{t}\right)$, then $sep(G,Q(n)) \leq k$.

(c) Suppose $k = \frac{n-1}{2}$.

If $q > \frac{1}{2}\left(p\left(\binom{n-1}{k} - 2\binom{n-3}{k} - 2\binom{n-3}{k-2}\right)\left(1 - \frac{p}{N}\right) + \frac{p^2}{N}\sum_{t=k+1}^{n}\binom{n}{t}\right)$, then $sep(G,Q(n)) \leq k$.

Proof: The various parts follow immediately from lemma 0 after consulting theorem 3 for the lower bound (or exact value) of $\lambda_{min}(Q(n)(k))$ and noting that $Q(n)(k)^c$ is regular of degree $\sum_{t=k+1}^{n}\binom{n}{t}$. ∎

Letting G be the complete graph K_p, we obtain the following.

Corollary 3.2: Let $N = |Q(n)| = 2^n$, and suppose k is even. Then

(a) Suppose $k \geq \frac{n}{2}$.

If $p > N\left(1 - \dfrac{N - \sum_{t=k+1}^{n}\binom{n}{t} - 1}{N - \sum_{t=k+1}^{n}\binom{n}{t} + \binom{n-2}{k-1} - \binom{n-2}{k}}\right)$, then $sep(K_p,Q(n)) \leq k$.

(b) Suppose $k < \frac{n-1}{2}$.

$$\text{If } p > N\left(1 - \frac{N - \sum_{t=k+1}^{n}\binom{n}{t} - 1}{N - \sum_{t=k+1}^{n}\binom{n}{t} + \binom{n-2}{k} - \binom{n-2}{k-1}}\right), \text{ then } sep(K_p,Q(n)) \le k.$$

(c) Suppose $k = \frac{n-1}{2}$.

$$\text{If } p > N\left(1 - \frac{N - \sum_{t=k+1}^{n}\binom{n}{t} - 1}{N - \sum_{t=k+1}^{n}\binom{n}{t} + \binom{n-1}{k} - 2\binom{n-3}{k} - 2\binom{n-3}{k-2}}\right) \text{ then } sep(K_p,Q(n)) \le k.$$

4. The Discrete Torus

In this section we consider the discrete torus $C_n \times C_n$, the main result being a good lower bound on λ_{min} for $(C_n \times C_n)(k)$ (theorem 6). Here is an outline of how this result is obtained. From the connection between $A(C_n)$ and circulant matrices, it follows (as will be seen below) that $A(C_n^{(i)})$ is in the adjacency algebra of $A(C_n)$. Theorem 1 can then be applied, and with the aid of some trigonometric identities we obtain a surprisingly simple formula for the eigenvalues of $(C_n \times C_n)(k)$ when $k < \frac{n}{2}$ (theorem 5). Some calculus then gives a lower bound for the λ_{min} of $(C_n \times C_n)(k)$. The resulting bound for the edge isoperimetric problem in $(C_n \times C_n)(k)$ yields, through lemma 0, edge bounds for separation. The reader should understand that <u>in principle</u> there is no trouble "finding" all eigenvalues for any given graph, and no trouble determining the least number in a finite list of numbers already expressed in exceedingly convenient form, but that the difficulty in estimating λ_{min} generally stems from the fact that theoretical results typically express the list of eigenvalues in exceedingly inconvenient form. The fact that there are well known applications involving λ_{min} justifies the pursuit of estimating its value.

The first two lemmas in this section are aimed at applying Theorem 1 to get a workable expression for the eigenvalues of $(C_n \times C_n)(k)$. The first one, which follows, shows that the hypothesis of the theorem applies.

<u>Lemma 4</u>: For $0 \le k \le n/2$, there is a polynomial $p_k(x)$ such that $A(C_n^{(k)}) = p_k(A(C_n))$. That is, $A(C_n^{(k)})$ is in the adjacency algebra generated by $A(C_n)$ for $0 \le k \le n/2$.

Proof: We proceed by induction on k, starting with $p_0(x) = 1$ and $p_1(x) = x$ as base.

Assuming the lemma is true for all $k \le t \le n/2 - 1$, with $t \ge 2$, we show that $p_{t+1}(x)$ exists. More precisely, letting $A = A(C_n)$ we show for $t \le n/2 - 1$ that

(a) $A(C_n^{(t+1)}) = A\cdot A(C_n^{(t)}) - A(C_n^{(t-1)})$ if $t < n/2 -1$

(b) $A(C_n^{(t+1)}) = \frac{1}{2}\left[A\cdot A(C_n^{(t)}) - A(C_n^{(t-1)})\right]$ if $t = n/2 -1$.

Since $A(C_n^{(t)})$ and $A(C_n^{(t-1)})$ are in $\alpha(A(C_n))$ by the inductive hypothesis, the lemma would follow.

Assume first that $t < n/2 -1$. Observe that the (i,j)'th entry of $A\cdot A(C_n^{(t)})$ is a nonnegative integer u, where u is the number of points y on C_n which are adjacent to i and distance t from j in C_n. Hence under our assumption we have u = 1 if either dist(i,j) = t - 1 or dist(i,j) = t + 1, and u = 0 otherwise. Hence (a) follows.

Next assume that t = n/2 - 1, so that n is even. Then u = 1 if dist(i,j) = t -1, u = 2 if dist(i,j) = t +1, and u = 0 otherwise. Hence (b) follows, completing the proof. ∎

Let $W = A(\vec{C}_n)$ be the n x n adjacency matrix of the directed n-cycle $\vec{C}_n$, its entries $w_{i,j}$ being 1 if $j \equiv i + 1 \pmod n$, and 0 otherwise. Let $\alpha = \exp(2\pi i/n)$. It is easy to verify that the eigenvalues of W are $\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{n-1}$, i.e. the n distinct nth roots of unity. Note that $\mathbb{R}^n$ has a basis $\{v_0,v_1,..,v_{n-1}\}$, where v_i is an eigenvector of W with eigenvalue α^i, and we will use this basis below. W is a circulant matrix of classical importance because its adjacency algebra $\alpha(W)$ is precisely the set of all n x n circulant matrices. In particular, $W^1 + W^{n-1}$ is the adjacency matrix of the undirected n-cycle C_n.

For **X** a complex number or an invertible matrix and a an integer, let

$$X^{<a>} = \begin{cases} X^{a} + X^{-a} & \text{if } 1 \le a < \frac{n}{2} \\ X^{a} & \text{if } a = 0 \text{ or } a = \frac{n}{2} \end{cases}$$

. Then $W^{<0>}, W^{<1>}, W^{<2>}, \ldots, W^{<\lfloor n/2 \rfloor>}$ are matrices of 0's and 1's such that $W^{<d>} = A(C_n^{(d)})$. Since the $W^{<d>}$'s are in $\alpha(W)$, [because $W^{-a} = W^{n-a}$], we are in a position to determine the eigenvalues of $(C_n \times C_n)(k)$ in terms of the eigenvalues of W. Keep in mind that for x an nth root of unity, $x^a + x^{-a} = x^a + x^{n-a}$.

Lemma 5: The eigenvalues of $(C_n \times C_n)(k)$ are, with multiplicities, the numbers

$$A_{\lambda,\mu} = \sum_{\substack{0\le s;t\le n/2 \\ 1\le s+t\le k}} \lambda^{<s>}\mu^{<t>} \qquad (*)$$

where (λ,μ) ranges over all n^2 different ordered pairs of nth roots of unity.

Proof: Because of Lemma 4, the hypothesis of Theorem 1 applies to $(C_n \times C_n)(k)$, and hence the eigenvalues of $(C_n \times C_n)(k)$ are

$\{ \sum_{1 \le s+t \le k} \lambda^{(s)}\mu^{(t)} : (\lambda,\mu)$ ranges over all n^2 ordered pairs of eigenvalues of $C_n\}$. Observe that for $s,t > n/2 \ge \mathrm{diam}(C_n)$ we have $\lambda^{(s)} = \mu^{(t)} = 0$ since $C_n^{(s)}$ is edgeless for $s > n/2$. Therefore in the sum we may restrict the indices s and t by further requiring that $0 \le s,t \le n/2$.

We begin by claiming that $((\alpha^i)^{<1>})^{(s)} = (\alpha^i)^{<s>}$. First observe that $W^{<s>} = A(C_n^{(s)})$ implies that

$$(\alpha^i)^{<s>} = \text{the eigenvalue of } v_i \text{ with respect to } A(C_n^{(s)}).$$

Next observe that since $A(C_n) = W + W^{n-1}$, it follows that

$$(\alpha^i)^{<1>} = \text{the eigenvalue of } v_i \text{ with respect to } A(C_n).$$

Hence by definition of $\lambda^{(s)}$ we have

$$((\alpha^i)^{<1>})^{(s)} = \text{the eigenvalue of } v_i \text{ with respect to } A(C_n^{(s)}),$$

as claimed.

Thus we have

$$\{ \sum_{1 \le s+t \le k} \lambda^{(s)}\mu^{(t)} : (\lambda,\mu) \text{ ranges over all } n^2 \text{ ordered pairs of eigenvalues of } C_n\}$$

$$= \{ \sum_{\substack{0 \le s;t \le n/2 \\ 1 \le s+t \le k}} ((\alpha^i)^{<1>})^{(s)} ((\alpha^j)^{<1>})^{(t)} : 0 \le i,j \le n-1\}$$

$$= \{ \sum_{\substack{0 \le s;t \le n/2 \\ 1 \le s+t \le k}} (\alpha^i)^{<s>} (\alpha^j)^{<t>} : 0 \le i,j \le n-1\}$$

$$= \{ \sum_{\substack{0 \le s;t \le n/2 \\ 1 \le s+t \le k}} \lambda^{<s>}\mu^{<t>} : (\lambda,\mu) \text{ ranges over all } n^2 \text{ different ordered pairs of nth roots of unity}\},$$

completing the proof of the lemma. ∎

While useful as a starting point, the form for the spectrum of $(C_n \times C_n)(k)$ given by Lemma 5 still leaves us far from having $\lambda_{min}((C_n \times C_n)(k))$ explicitly bounded, as would be required for our applications.

Define C(x) = cos x, c(x) = cos($\frac{x}{2}$), S(x) = sin x, and s(x) = sin($\frac{x}{2}$) for all real numbers x. Before tackling the problem of simplifying such a sum of products of sums of nth roots of unity, the following elementary facts will be useful.

Lemma 6: For z any unit complex number and for all real numbers w,x and y,

(1) $\exp(x\,i) - 1 = 2i\, s(x) \exp(\frac{x}{2} i)$

(2) $\mathrm{Re}\left[\exp(w\,i)\,\frac{\exp(x\,i)-1}{\exp(y\,i)-1}\right] = \frac{s(x)\, c(2w+x-y)}{s(y)} = \frac{S(\frac{x}{2})\, C(w + \frac{x-y}{2})}{S(\frac{y}{2})}$

(3) $z + z^{-1} = 2\,\mathrm{Re}\, z$

(4) $\sum_{s=0}^{r} z^s = \frac{z^{r+1} - 1}{z - 1}$ for $z \neq 1$

(5) C(x+y) = C(x)C(y) – S(x)S(y), and c(x+y) = c(x)c(y) – s(x)s(y)

(6) $C(x) + C(y) = 2\, C(\frac{x+y}{2})\, C(\frac{x-y}{2})$, and $c(x) + c(y) = 2\, c(\frac{x+y}{2})\, c(\frac{x-y}{2})$

(7) 2 S(x) S(y) = C(x-y) – C(x+y), and 2 s(x) s(y) = c(x-y) – c(x+y)

(8) 2 S(x) C(y) = S(y+x) – S(y–x), and 2 s(x) c(y) = s(y+x) – s(y–x)

Proof: For (1), $2i\, s(x) \exp(\frac{x}{2} i) = 2i \sin\frac{x}{2}(\cos\frac{x}{2} + i \sin\frac{x}{2}) = -2\sin^2\frac{x}{2} + i\, 2\sin\frac{x}{2}\cos\frac{x}{2}$

$= (\cos x - 1) + i \sin x = \exp(x\,i) - 1$. For (2),

$$\mathrm{Re}\left[\exp(w\,i)\,\frac{\exp(x\,i)-1}{\exp(y\,i)-1}\right] = \mathrm{Re}\left[\exp(w\,i)\,\frac{2i\, s(x)\exp(\frac{x}{2} i)}{2i\, s(y) \exp(\frac{y}{2} i)}\right]$$

$$= \frac{s(x)}{s(y)}\mathrm{Re}\left[\exp(\frac{2w+x-y}{2}\, i)\right] = \frac{s(x)\, c(2w+x-y)}{s(y)}.$$

For (3), it suffices to note that the reciprocal of a unit complex number is its complex conjugate. (4) is the familiar rule for the partial sums of a geometric series. (5) is a familiar trigonometric identity for cos(A+B). Similarly, (6),(7) and (8) are restatements of standard trig identities. ∎

Notice for $k < \frac{n}{2}$ that expression (*) for $A_{\lambda,\mu}$ can be written as the following double sum:

$$A_{\lambda,\mu} = \sum_{\substack{0\le s;t\le n/2\\ 1\le s+t\le k}} \lambda^{<s>}\mu^{<t>}$$

$$= \sum_{r=1}^{k}\left[\left(\sum_{s=0}^{r}(\lambda^{s}+\lambda^{-s})(\mu^{r-s}+\mu^{s-r})\right)-(\lambda^{r}+\lambda^{-r})-(\mu^{r}+\mu^{-r})\right]$$

$$= \sum_{r=1}^{k}\left[-(\lambda^{r}+\lambda^{-r})-(\mu^{r}+\mu^{-r})+\sum_{s=0}^{r}\mu^{r}(\lambda\,\mu^{-1})^{s}+\left[\mu^{r}(\lambda\,\mu^{-1})^{s}\right]^{-1}+\mu^{-r}(\lambda\,\mu)^{s}+\left[\mu^{-r}(\lambda\,\mu)^{s}\right]^{-1}\right]$$

$$= \sum_{r=1}^{k} 2\,\mathrm{Re}\left[-\lambda^{r}-\mu^{r}+\sum_{s=0}^{r}\mu^{r}(\lambda\,\mu^{-1})^{s}+\mu^{-r}(\lambda\,\mu)^{s}\right].$$

The next result indicates a preferable way to express each of the k summands above, where $A(\lambda,\mu,r)$ denotes the rth summand.

Lemma 7: For $\lambda = \exp(x\,i)$ and $\mu = \exp(y\,i)$,

$$A(\lambda,\mu,r) = 2\,\mathrm{Re}\left(-\lambda^{r}-\mu^{r}+\sum_{s=0}^{r}\left[\mu^{r}(\lambda\,\mu^{-1})^{s}+\mu^{-r}(\lambda\,\mu)^{s}\right]\right)$$

$$= \begin{cases} \dfrac{C(y(r+1)) - C(y(r-1)) + C(x(r-1)) - C(x(r+1))}{C(y) - C(x)} & \text{for } 0 \le x < y \le \pi \\[2ex] \dfrac{(r+1)\,S(x(r+1)) - (r-1)\,S(x(r-1))}{S(x)} & \text{for } 0 < x = y < \pi \\[2ex] 4r & \text{for } x = y = 0 \\[2ex] (-1)^{r}\,4r & \text{for } x = y = \pi \end{cases}$$

Proof: For $x = y = 0$, $A(\lambda,\mu,r) = 2\,\mathrm{Re}(-2 + \sum_{s=0}^{r}[1+1]) = 4r$. For $x = y = \pi$, the sum is $2\,\mathrm{Re}(-2(-1)^{r} + \sum_{s=0}^{r}[(-1)^{r}+(-1)^{r}]) = (-1)^{r}\,4r$. For $0 < x = y < \pi$, the sum is

$$2\,\mathrm{Re}\left(-2\lambda^{r} + \sum_{s=0}^{r}[\lambda^{r}+\lambda^{-r}(\lambda^{2})^{s}]\right) = 2\,\mathrm{Re}\left((r-1)\lambda^{r} + \lambda^{-r}\frac{(\lambda^{2})^{r+1}-1}{\lambda^{2}-1}\right) =$$

$$2(r-1)\,C(xr) + 2\frac{S(x(r+1))}{S(x)}\,C\!\left(-xr+\frac{2x(r+1)-2x}{2}\right) = \frac{1}{S(x)}\left[(r-1)\,2S(x)\,C(xr) + 2\,S(x(r+1))\right]$$

$$= \frac{1}{S(x)}\ [(r\text{-}1)\ [S(xr+x) - S(xr\text{-}x)] + 2\ S(x(r+1))] = \frac{1}{S(x)}\ [(r+1)\ S(x(r+1)) - (r\text{-}1)\ S(x(r\text{-}1))].$$

For $0 \le x < y \le \pi$, $A(\lambda,\mu,r)$ equals the sum of $2\ \mathrm{Re}\left(-\lambda^r - \mu^r \right)$ plus the following:

$$2\ \mathrm{Re}\left(\mu^r \frac{(\lambda\ \mu^{-1})^{r+1} - 1}{\lambda\ \mu^{-1} - 1} + \mu^{-r} \frac{(\lambda\ \mu)^{r+1} - 1}{\lambda\ \mu - 1} \right) =$$

$$= \frac{2s\big((x-y)(r+1)\big)}{s(x-y)}\ c\big((x-y)(r+1)-(x-y) + 2yr\big) + \frac{2s\big((x+y)(r+1)\big)}{s(x+y)}\ c\big((x+y)(r+1)-(x+y) - 2yr\big)$$

$$= 2\ \frac{s\big((x-y)(r+1)\big)\ 2\ s(x+y)\ c\big((x+y)r\big) + s\big((x+y)(r+1)\big)\ 2\ s(x-y)\ c\big((x-y)r\big)}{2\ s(x+y)\ s(x-y)}$$

$$= 2\frac{s\big((x-y)(r+1)\big)\big[s\big((x+y)(r+1)\big) - s\big((x+y)(r-1)\big)\big] + s\big((x+y)(r+1)\big)\big[s\big((x-y)(r+1)\big) - s\big((x-y)(r-1)\big)\big]}{C(y) - C(x)}$$

$$= \frac{4\ s\big((x+y)(r+1)\big)\ s\big((x-y)(r+1)\big) - 2\ s\big((x-y)(r+1)\big)\ s\big((x+y)(r-1)\big) - 2\ s\big((x+y)(r+1)\big)\ s\big((x-y)(r-1)\big)}{C(y) - C(x)}$$

$$= \frac{2\ \big[C\big(y(r+1)\big) - C\big(x(r+1)\big)\big] - \big[C\big(yr - x\big) - C\big(xr - y\big)\big] - \big[C\big(yr + x\big) - C\big(xr + y\big)\big]}{C(y) - C(x)}$$

$$= \frac{2\ \big[C\big(y(r+1)\big) - C\big(x(r+1)\big)\big] - \big[C\big(yr - x\big) + C\big(yr + x\big)\big] + \big[C\big(xr - y\big) + C\big(xr + y\big)\big]}{C(y) - C(x)}$$

$$= \frac{2\ \big[C\big(y(r+1)\big) - C\big(x(r+1)\big)\big] - 2\ C(yr)\ C(x) + 2\ C(xr)\ C(y)}{C(y) - C(x)}.$$

Therefore $A(\lambda,\mu,r) =$

$$2\ \mathrm{Re}\left(-\lambda^r - \mu^r \right) + \frac{2\ \big[C\big(y(r+1)\big) - C\big(x(r+1)\big)\big] - 2\ C(yr)\ C(x) + 2\ C(xr)\ C(y)}{C(y) - C(x)}$$

$$= -2\ C(xr) - 2\ C(yr) + \frac{2\ \big[C\big(y(r+1)\big) - C\big(x(r+1)\big)\big] - 2\ C(yr)\ C(x) + 2\ C(xr)\ C(y)}{C(y) - C(x)}$$

$$= \frac{C\big(y(r+1)\big) + \big[C\big(y(r+1)\big) - 2\,C(yr)\,C(y)\big] - \big[C\big(x(r+1)\big) - 2\,C(xr)\,C(x)\big] - C\big(x(r+1)\big)}{C(y) - C(x)}$$

$$= \frac{C\big(y(r+1)\big) + \big[-C(yr)\,C(y) - S(yr)\,S(y)\big] - \big[-C(xr)\,C(x) - S(xr)\,S(x)\big] - C\big(x(r+1)\big)}{C(y) - C(x)}$$

$$= \frac{C\big(y(r+1)\big) - C\big(y(r-1)\big) + C\big(x(r-1)\big) - C\big(x(r+1)\big)}{C(y) - C(x)} \; . \;\blacksquare$$

We are now in a position to find closed form expressions for the eigenvalues $A_{\lambda,\mu}$ of $(C_n \times C_n)(k)$. Each n'th root of unity λ and μ is expressible uniquely in the form $\lambda = \exp(\,x\,i\,)$ and $\mu = \exp(\,y\,i\,)$ for some x,y with $-\pi < x,y \le \pi$. The following theorem finds closed form expressions for $A_{\lambda,\mu}$ for the (roughly) one quarter of the n^2 possible pairs (λ,μ) in which x and y are between 0 and π. After the theorem we shall show how to obtain a similar expression for $A_{\lambda,\mu}$ in the remaining three quarters of the cases, using simple symmetry facts.

Theorem 4: Let $k < \frac{n}{2}$, and let $\lambda = \exp(\,x\,i\,)$ and $\mu = \exp(\,y\,i\,)$ be n'th roots of unity with $0 \le x \le y \le \pi$. Then the corresponding eigenvalue $A_{\lambda,\mu}$ of $(C_n \times C_n)(k)$ is given by

$$A_{\lambda,\mu} = \begin{cases} -1 + \dfrac{C((k+1)y) + C(ky) - C((k+1)x) - C(kx)}{C(y) - C(x)} & \text{for } 0 \le x < y \le \pi \\ -1 + \dfrac{(k+1)\,S((k+1)x) + k\,S(kx)}{S(x)} & \text{for } 0 < x = y < \pi \\ 2k^2 + k & \text{for } x = y = 0 \\ (-1)^k\,(2k+1) - 1 & \text{for } x = y = \pi \end{cases}$$

Proof: Since $A_{\lambda,\mu} = \sum_{r=1}^{k} 2\,\mathrm{Re}\left(-\lambda^r - \mu^r + \sum_{s=0}^{r} \left[\mu^r(\lambda\mu^{-1})^s + \mu^r(\lambda\mu)^s\right]\right)$,
we can use lemma 7 to simplify the k terms in the sum in each of the four cases. We obtain the telescoping sums:

(Case $0 \le x < y \le \pi$): $A_{\lambda,\mu} = \sum_{r=1}^{k} \frac{C(y(r+1)) - C(y(r-1)) + C(x(r-1)) - C(x(r+1))}{C(y) - C(x)}$

$$= \frac{C(y(k+1)) + C(yk) - C(y) - C(0) + C(0) + C(x) - C(xk) - C(x(k+1))}{C(y) - C(x)}$$

$$= -1 + \frac{C((k+1)y) + C(ky) - C(kx) - C((k+1)x)}{C(y) - C(x)},$$

and (case $0 < x = y < \pi$): $A_{\lambda,\mu} = \sum_{r=1}^{k} \frac{(r+1)\, S(x(r+1)) - (r-1)\, S(x(r-1))}{S(x)}$

$$= \frac{(k+1)\, S((k+1)x) + k\, S(kx) - S(x)}{S(x)} = -1 + \frac{(k+1)\, S((k+1)x) + k\, S(kx)}{S(x)},$$

and (case $x = y = 0$): $A_{\lambda,\mu} = \sum_{r=1}^{k} 4r = 2k^2 + 2k$

and (case $x = y = \pi$): $A_{\lambda,\mu} = \sum_{r=1}^{k} (-1)^r\, 4r = (-1)^k (2k + 1) - 1.$ ∎

The previous theorem gives a formula for $A_{\lambda,\mu}$ when $0 \le x \le y \le \pi$, $k < n/2$. However, notice that $A_{\lambda,\mu} = A_{\lambda,\mu^{-1}}$ because $(\mu^{-1})^{<t>} = (\mu)^{<t>}$. Also, notice that $A_{\lambda,\mu} = A_{\mu,\lambda}$ because $\sum_{\substack{0\le s;t\le n/2 \\ 1\le s+t\le k}} \lambda^{<s>}\mu^{<t>}$ is symmetric in λ and μ. These observations together with theorem 4 allow us to easily obtain a closed form expression expression for each eigenvalue of $(C_n \times C_n)(k)$, $k < \frac{n}{2}$. We omit the simple proof for brevity.

Theorem 5: Let $k < \frac{n}{2}$. The complete list of eigenvalues of $(C_n \times C_n)(k)$ is given by

$$A_{\lambda,\mu} = \begin{cases} -1 + \dfrac{\cos((k+1)y) + \cos(ky) - \cos((k+1)x) - \cos(kx)}{\cos(y) - \cos(x)} & \text{for } |x| \neq |y| \\ -1 + \dfrac{(k+1)\sin((k+1)x) + k\sin(kx)}{\sin(x)} & \text{for } 0 \neq |x| = |y| \neq \pi \\ 2k^2 + 2k & \text{for } x = y = 0 \\ (-1)^k(2k+1) - 1 & \text{for } x = y = \pi \end{cases}$$

where λ and μ range over all n'th roots of unity, each expressed in the form $\lambda = \exp(x i)$ and $\mu = \exp(y i)$, with $-\pi < x,y \leq \pi$. ∎

We now pursue the matter of finding and/or lower bounding λ_{min} for the kth power of $C_n \times C_n$ for $k < n/2$. Consider the function $f: (-\pi,\pi] \times (-\pi,\pi] \rightarrow \mathbb{R}$ defined by $f(x,y) = A_{\exp(xi), \exp(yi)}$, the function as written in the previous theorem, where now λ and μ are no longer constrained as being nth roots of unity. Because $f(x,y) = f(-x,y) = f(x,-y)$ for all (x,y) in $(-\pi,\pi) \times (-\pi,\pi)$, the lim inf of $f(x,y)$ over $(-\pi,\pi] \times (-\pi,\pi]$ is the same as that over $[0,\pi] \times [0,\pi]$. By the symmetry of f, the lim inf over $[0,\pi] \times [0,\pi]$ is the same as that over $\{ (x,y) : 0 \leq x \leq y \leq \pi \}$. Letting $F(x) = \cos((k+1)x) + \cos(kx)$ and $G(x) = \cos x$, both functions are differentiable on $[0,\pi]$, and $G'(x) \neq 0$ for all x in $(0,\pi)$. Therefore by Cauchy's Mean Value Theorem [PV], given any particular two numbers $x < y$ in $[0,\pi]$, there exists a corresponding number c in (x,y) such that $\frac{F(y) - F(x)}{G(y) - G(x)} = \frac{F'(c)}{G'(c)}$, i.e. such that $1 + f(x,y) = 1 + f(c,c)$. Therefore we have that $\lambda_{min} \geq$ lim inf of $f(x,y)$ over $[0,\pi] \times [0,\pi]$ = lim inf of $f(x,x)$ over $[0,\pi]$. It is a simple matter to verify that as a function of one variable, $f(x,x)$ is continuous over $[0,\pi]$, proving the following.

Lemma 8: For $k < \frac{n}{2}$, in the k'th power of $C_n \times C_n$ the minimum eigenvalue λ_{min} satisfies $\lambda_{min} \geq \min \{ f(x,x) : x \in [0,\pi] \}$. ∎

Observe that for each k and any given $\varepsilon > 0$ there exists a corresponding N such that for all $n \geq N$, $\lambda_{min} - \min \{ f(x,x) : x \in [0,\pi] \} \leq \varepsilon$. This is simply because the function $f(x,x)$ is continuous on $[0,\pi]$, with $\lambda_{min} = \min \{ f(x,x) : x \in [0,\pi]$, x an integer multiple of $\frac{2\pi}{n}$ }. Therefore in this sense the lemma is asymptotically optimal.

Theorem 6: For $2 \le k < \frac{n}{2}$, in the kth power of $C_n \times C_n$ the minimum eigenvalue λ_{min} satisfies

$$\lambda_{min} \ge \begin{cases} -1 - \dfrac{k+1}{\sin \dfrac{\pi}{k+1}} - k^2 & \text{if } k \text{ is even} \\ -1 - \dfrac{k}{\sin \dfrac{\pi}{k}} - (k+1)^2 & \text{if } k \text{ is odd} \end{cases}.$$

Proof: First consider the function $g_k(x) = \begin{cases} \dfrac{\sin kx}{\sin x} & \text{if } 0 < x < \pi \\ k & \text{if } x = 0 \\ k(-1)^{k+1} & \text{if } x = \pi \end{cases}$ in order to analyze $f(x,x) = -1 + (k+1)\, g_{k+1}(x) + k\, g_k(x)$. Over the interval $0 \le x \le \frac{\pi}{2}$, the minimum of $g_k(x)$ certainly occurs at some x in $[\pi/k, 3\pi/2k] \cap [0,\pi/2]$ (for $k \ge 2$) because $g_k(x) \ge 0$ over $[0,\pi/k]$ and the denominator sin x in $g_k(x)$ is nonnegative and increasing. Therefore $g_k(x) \ge \frac{-1}{\sin(\pi/k)}$ for all x in $[0,\pi/2]$. Next suppose that k is odd. Then over the interval $\pi/2 \le x \le \pi$, the minimum of $g_k(x)$ occurs at some x in
$[\pi - 3\pi/2k, \pi - \pi/k]$ (for $k \ge 3$) because $g_k(x) \ge 0$ over $[\pi - \pi/k, \pi]$ and the denominator sin x in $g_k(x)$ is nonnegative and decreasing. Therefore $g_k(x) \ge \frac{-1}{\sin(\pi - \pi/k)} = \frac{-1}{\sin(\pi/k)}$ for all x in $[\pi/2,\pi]$. Next suppose that k is even. Then over the interval $\pi/2 \le x \le \pi$, the minimum of $g_k(x)$ occurs at some x in $[\pi - \pi/2k, \pi]$ because the denominator sin x in $g_k(x)$ is nonnegative and decreasing. We show that over $(\pi - \pi/2k, \pi)$, $g_k'(x) \ne 0$, so that the minimum over $[\pi - \pi/2k, \pi]$ occurs at one of the endpoints. Now $g_k'(x) = \frac{k \sin x \cos kx - \cos x \sin kx}{\sin^2 x} = 0$ for some x in $(\pi - \pi/2k, \pi)$ would imply that $k \tan x = \tan kx$ for that x. Letting $y = \pi - x$, this would imply that $k \tan y = \tan ky$ for some y in $(0,\pi/2k)$. But $k \tan y$ is increasing at the positive rate $k \sec^2 y$ while $\tan ky$ is increasing at the positive rate $k \sec^2 ky$. Because $\sec^2 \theta$ is increasing on $[0,\pi/2)$, $k \sec^2 y < k \sec^2 ky$, so that $\tan ky$ increases at a faster rate than does $k \tan y$ throughout $(0,\pi/2k)$. Since $\tan ky = k \tan y$ at $y = 0$, we have that $\tan ky > k \tan y$ throughout $(0,\pi/2k)$. Therefore $g_k(x) \ge -k$ for all x in $[\pi/2,\pi]$.

Having shown that $g_k(x) \ge \frac{-1}{\sin(\pi/k)}$ for all x in $[0,\pi]$ for k odd and that $g_k(x) \ge -k$ for all x in $[0,\pi]$ for k even, it follows that $f(x,x) = -1 + (k+1)\, g_{k+1}(x) + k\, g_k(x)$

$$\geq \begin{cases} -1 - \dfrac{k+1}{\sin \frac{\pi}{k+1}} - k^2 & \text{if } k \text{ is even} \\ -1 - \dfrac{k}{\sin \frac{\pi}{k}} - (k+1)^2 & \text{if } k \text{ is odd} \end{cases}$$, proving the result. ∎

Because $x - \frac{x^3}{6} \leq \sin x \leq x$ for all x in $[0,\pi/2]$, observe that the term $\frac{k}{\sin \frac{\pi}{k}}$ appearing in the lower bound just given for λ_{min} is itself bounded, $\frac{k^2}{\pi} \leq \frac{k}{\sin \frac{\pi}{k}} \leq \frac{6k^4}{6\pi k^2 - \pi^3}$. Thus the lower bound is seen to be on the order of $-(1+\frac{1}{\pi})\, k^2$.

Using the lower bound on λ_{min} we obtain the following corollary.

Corollary 6.1: Let G be a graph on p points and q edges, $p \leq n^2$, $k < n/2$.

(a) Suppose k is even.

If $q > \frac{1}{2}\left(p\left(\frac{k+1}{\sin(\pi/(k+1))} + k^2\right)\left(1 - \frac{p}{n^2}\right) + (n^2 - 2k^2 - 2k - 1)\frac{p^2}{n^2}\right)$,

then $sep(G, C_n \times C_n) \leq k$.

(b) Suppose k is odd.

If $q > \frac{1}{2}\left(p\left(\frac{k}{\sin(\pi/k)} + (k+1)^2\right)\left(1 - \frac{p}{n^2}\right) + (n^2 - 2k^2 - 2k - 1)\frac{p^2}{n^2}\right)$,

then $sep(G, C_n \times C_n) \leq k$.

Proof: We note that the degree of regularity of $(C_n \times C_n)(k)$ is $2k^2 + 2k$, so the degree of regularity of $(C_n \times C_n)(k)^c$ is $n^2 - 2k^2 - 2k - 1$. The rest follows from theorem 6 and lemma 0. ∎

Letting G be the complete K_p graph in Corollary 6.1, we immediately obtain the following result. Recalling that finding $sep(K_p, Q_n)$ is a key objective in classical coding theory, this result may be viewed as a coding theory type result for the discrete torus.

Corollary 6.2: Let $p \leq n^2$, $k < n/2$.

(a) Suppose k is even.

If $p > n^2\left(1 - \dfrac{2k^2 + 2k}{3k^2 + 2k + 1 + \frac{k+1}{\sin(\pi/(k+1))}}\right)$, then $sep(K_p, C_n \times C_n) \leq k$.

(b) Suppose k is odd.

If $p > n^2\left(1 - \dfrac{2k^2 + 2k}{3k^2 + 4k + 2 + \frac{k}{\sin(\pi/k)}}\right)$, then $sep(K_p, C_n \times C_n) \leq k$.

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