The Dichotomy Theorem for evolution bi-families ⋆

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Abstract

We prove that the operator G, the closure of the first-order differential operator $-d/dt + D(t)$ on $L_2(\mathbb{R}, X)$, is Fredholm if and only if the not well-posed equation $u'(t) = D(t)u(t)$, $t \in \mathbb{R}$, has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ and the ranges of the dichotomy projections form a Fredholm pair; moreover, the index of this pair is equal to the Fredholm index of G. Here X is a Hilbert space, $D(t) = A+B(t)$, A is the generator of a bi-semigroup, $B(\cdot)$ is a bounded piecewise strongly continuous operator valued function. Also, we prove some perturbations results and consider various examples of not well-posed problems.

Key words: Fredholm operators, Fredholm index, exponential dichotomy, not well-posed equations, input-output method, spectral flow.
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1. Introduction

The Dichotomy Theorem goes back to the classical finite dimensional work in [8,32,33] and [36]. In an appropriate setting, this theorem relates the Fredholm property and Fredholm index of the differential operator $(Lu)(t) = -u'(t)+D(t)u(t)$, $t \in \mathbb{R}$, associated with solutions of an infinite dimensional differential equation on the line,

$$u'(t) = D(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

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and exponential dichotomies on the semi-lines $\mathbb{R}_+$ and $\mathbb{R}_-$ of the equation

$$u'(t) = D(t)u(t). \tag{1.2}$$

For the long history and recent advances on this topic we refer to [17,22,23,37,38]. In particular, there is an important connection of the Dichotomy Theorem and the celebrated Atiyah-Patodi-Singer "Index=Spectral Flow" Theorem, see [3,35].

The main purpose of this paper is to continue both the work of B. Sandstede and A. Scheel in [38], and the work in [22,23], proving the Dichotomy Theorem for the infinite dimensional equation (1.2), for which the initial value problem is well-posed on one half of the space for forward time and for another half for backward time. Specifically, we assume that $D(t) = A + B(t)$, $t \in \mathbb{R}$, the operator A is the generator of a stable bi-semigroup, and $B(\cdot)$ is a bounded piecewise strongly continuous function on $\mathbb{R}$ with values in the set $\mathcal{B}(X)$ of bounded operators on a Hilbert space X.

We recall that A is called the generator of a (uniformly exponentially) stable bi-semigroup provided $A = A_1 \oplus (-A_2)$ in the direct sum decomposition $X = X_1 \oplus X_2$, where A_i is the generator of a uniformly exponentially stable strongly continuous semigroup $\{T_i(t)\}_{t\geq 0}$ on X_i, $i = 1, 2$, see e.g. [8]. Thus, equation (1.2) is not well-posed in the sense that it does not generate an evolution family neither in forward nor in backward time on the entire space. Originated in the pioneering work on elliptic problems on cylindrical domains, see [20,27,28,34,38], this not well-posed setting arises from several sources; here we mention the study of modulated waves that can emerge from traveling waves via Hopf bifurcations (see [34] and [38]), the Morse theory (see [1,2,35]) and the theory of PDE Hamiltonian systems(see [37]).

If equation (1.2) is well-posed then one can interpret solutions of the corresponding inhomogeneous equation (1.1) with $f \in L_2(\mathbb{R}, X)$ in the mild sense. This leads to the definition, via the mild solutions, of an operator, G, which is the closure of $-L$. Also, the operator $-G$ can be characterized as the generator of the evolution semigroup defined by means of the evolution family associated with (1.2), see [10]. The Dichotomy Theorem then is proved either for the operator G in place of L, see [4–6,22,23] or, under some additional regularity (parabolicity) assumptions, directly for L, see [16,17]. Unlike the well-posed setting for which the Dichotomy Theorem is well understood, much less is known when (1.2) is not well-posed. We mention here a fundamental contribution in [34,38], where the Dichotomy Theorem is proved for some important specific choices of A and the operator valued function B. In particular, the C_0-semigroups $\{T_j(t)\}_{t\geq 0}$ are assumed in [34,38] to be analytic.

The first new issue related to the not well-posed setting of the current paper is a proper understanding of the notion of solutions of the inhomogeneous equation (1.1). Indeed, in the setting of the current paper neither of the objects mentioned in the previous paragraph exists: we do not have an evolution family, thus the evolution semigroup, thus G. To bypass these difficulties, in this paper we interpret solutions of the not well-posed inhomogeneous equation using the frequency-domain approach. Applying, formally, the Fourier transform $\mathcal{F}$ in (1.1), and using that $i\mathbb{R} \subseteq \rho(A)$ due to the exponential stability of the bi-semigroup, we obtain the equation

$$(\mathcal{F}u)(\xi) - R(2\pi i\xi, A)\mathcal{F}(B(\cdot)u(\cdot))(\xi) = R(2\pi i\xi, A)(\mathcal{F}f)(\xi), \quad \xi \in \mathbb{R}, \tag{1.3}$$

where $R(\lambda, A) = (\lambda - A)^{-1}$ is the resolvent operator. Introducing the main Green's function $\mathcal{V}$ for equation (1.2) with $B = 0$ by

$$\mathcal{V}(t) = T_1(t)P_1 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \mathcal{V}(t) = -T_2(-t)P_2 \quad \text{for} \quad t \leq 0, \tag{1.4}$$

where P_i are the projections given by the decomposition $X = X_1 \oplus X_2$, $X_i = \operatorname{im} P_i$, $i = 1, 2$, we note that $\mathcal{F}\mathcal{V}(\cdot)x = R(2\pi i \cdot, A)x$ for each $x \in X$. Thus, using convolutions, (1.3) can be alternatively written as

$$u - \mathcal{V} * (B(\cdot)u(\cdot)) = \mathcal{V} * f. \tag{1.5}$$

However, it turns out that the frequency domain formulation (1.3) is more convenient than (1.5), and will be used in what follows. Equation (1.3) gives rise to the operator G (such that $Gu = f$ if and only if (1.3) holds), used in the formulation of the Dichotomy Theorem 1.2 below. As shown in Proposition 2.1, under certain additional regularity assumptions on the operator A, one has $G = -L$. Throughout this paper we will assume the following backward-forward uniqueness property.

Hypothesis 1.1 *If a function u that belongs either to $\ker G$ or to $\ker G^*$ is equal to zero at a point $t \in \mathbb{R}$, then u is identically zero.*

This assumption is widely accepted in the related work, see e.g. [18,23,38]. However, to verify this hypothesis for each particular class of PDEs is a separate and rather challenging problem, cf. [38, Remark 2.5], requiring completely different methods. Therefore, this issue is not addressed in this paper. We also mention that the Fredholm property of G does not imply that Hypothesis 1.1 holds even in the well-posed setting as shown by means of [22, Example 7.2].

To formulate our principal result, we recall that a pair (Y, Z) of subspaces of X is called Fredholm provided that $\alpha(Y, Z) := \dim(Y \cap Z) < \infty$, the sum $Y + Z$ is closed and $\beta(Y, Z) := \operatorname{codim}(Y + Z) < \infty$; the Fredholm index of the pair of subspaces is defined as $\operatorname{ind}(Y, Z) = \alpha(Y, Z) - \beta(Y, Z)$.

Theorem 1.2 [Dichotomy Theorem] *Assume Hypothesis 1.1. Then the operator G is Fredholm on $L_2(\mathbb{R}, X)$ if and only if the following two conditions hold:*

(i) Equation (1.2) has an exponential dichotomy on $\mathbb{R}_+$ with dichotomy projections $\{P_+(t)\}_{t\geq 0}$ and an exponential dichotomy on $\mathbb{R}_-$ with dichotomy projections $\{P_-(t)\}_{t\leq 0}$;

(ii) The pair of subspaces $(\operatorname{im} P_+(0), \ker P_-(0))$ is Fredholm.

Moreover, if G is Fredholm, then $\dim\ker G = \alpha(\operatorname{im} P_+(0), \ker P_-(0))$, $\operatorname{codim}\operatorname{im} G = \beta(\operatorname{im} P_+(0), \ker P_-(0))$ and the Fredholm index of G is computed as

$$\operatorname{ind} G = \operatorname{ind}(\operatorname{im} P_+(0), \ker P_-(0)). \tag{1.6}$$

The main novelty of this result as compared to [5,17,22,23] is the not well-posedness of equations (1.1)–(1.2). Also, in our general not well-posed setting, we prove the Dichotomy Theorem without assuming any asymptotic properties of the operator valued function $D(\cdot)$ or the compactness of the embedding of $\operatorname{dom}(A)$ in X, cf. [3,35], and without assuming that the C_0-semigroups $\{T_j(t)\}_{t\geq 0}$ are analytic, cf. [34,38]. The harder part in the proof of the Dichotomy Theorem is to show the existence of the exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ provided G is Fredholm. Here, we remark the following differences between the well- and not well-posed settings. The first major difficulty is to identify the subspaces where the forward and backward solutions of (1.2) exist provided the operator G is Fredholm. The assumption of backward-forward uniqueness in Hypothesis 1.1 is used in the proof of the semiflow properties of the solutions. Second, our definition of the exponential dichotomy is quite different from the one used in the well-posed setting, see, e.g. [10,13,25]. Indeed, we do not assume that the solutions on unstable fibers have continuations in forward time nor we assume that the respective propagators acting between the unstable fibers are invertible (these assumptions are not natural even when $B = 0$). Thus, the dichotomy defintion is close to the one in [38] although we make adjustments to account for our non-analytic setting. Third, the lack of the evolution families does not allow us to use the discrete time systems as in [19] and [16,22,23]. However, there are some similarities in the proof of the Dichotomy Theorem for the well- and not well-posed cases. Indeed, in both cases, we proceed by "removing" first the kernel and the co-kernel of the Fredholm operator G, cf. [23]. As soon as this is done, cf. [22], we use the so called *input-output method* going back to [13] and [25], cf. also [30,31]. The essence of this approach is to consider a solution u of (1.2) on, say, $[\tau, \infty)$ and a suitable real valued function φ so that the formula $G(\varphi u) = \varphi' u$ holds, see Proposition 3.4. Next, we note that for well-posed equations (1.2) a classical way to treat the dichotomy on $\mathbb{R}_-$ is to reduce it to the dichotomy of the adjoint equation on $\mathbb{R}_+$, cf. [13]. In the present setting we do not need to use the adjoint operators and the change of variables $t = -s$ in (1.2) alone does the job by reducing the study of equation (1.2) for $t \le 0$ to the study of equation $x'(t) = D_\sharp(t)x(t)$, $t \ge 0$, where $D_\sharp(t) = -A - B(-t)$. The key point here is that the operator $-A$ satisfies the same properties as A.

Finally, we address in this paper the natural question if the Fredholm property of the operator G is preserved under a small compact perturbation of the coefficients in equation (1.2), generalizing corresponding results from [23,35,34]. More precisely, we prove in Section 7 that if we add compact operators $K(t)$ in equation (1.2) that satisfy one of the asymptotic conditions $\lim_{|t|\to\infty} \|K(t)\| = 0$ or $\|K(\cdot)\| \in L_2(\mathbb{R})$, then the Fredholm property and the Fredholm index of the operator G are preserved. This result can be used to prove the existence of exponential dichotomy for the perturbation of a dichotomic system and to prove the existence of a bi-family associated with a not well-posed equation, see e.g. Example 8.9.

We emphasize that our proof of the index formula does not require differentiability of $D(\cdot)$. When equation (1.2) is well-posed, an index formula of type (1.6) has been given in [23, Thms.1.1,1.2] and [22, Thm.ED]. This formula has its counterparts in the Morse theory, cf. [1,2,38,39] and the literature therein and, in fact, is related to the Atiyah-Patodi-Singer "Index=Spectral Flow" Theorem, see [3,35]. For a more general than (1.6) index formula see Section 5.

The paper is organized as follows. In Section 2 we give the precise setting of the problem and prove some preliminary results. In Section 3 we prove that the Fredholm property of G implies the existence of the exponential dichotomy for equation (1.2) on $\mathbb{R}_+$. In Section 4 we reduce the study of the dichotomy on $\mathbb{R}_-$ to that on $\mathbb{R}_+$, concluding that the Fredholm property of G implies the dichotomy on $\mathbb{R}_-$. In Section 5 we show that if G is Fredholm then condition (ii) holds, and prove the index formula (1.6). In Section 6 we show sufficiency of conditions $(i), (ii)$ in the Dichotomy Theorem for G to be Fredholm. In Section 7 we give the perturbation results. In Section 8 we discuss examples just to illustrate the main setting of our paper.

2. Preliminaries

Notations: $\mathbb{R}_+ = \{t \in \mathbb{R} : t \ge 0\}$, $\mathbb{R}_- = \{t \in \mathbb{R} : t \le 0\}$, t, s, τ, ξ are real numbers and c is a generic positive constant. We denote by χ_E the characteristic function of E. $\mathcal{S}(\mathbb{R})$ denotes the set of all complex valued Schwartz functions on $\mathbb{R}$, $C_0^\infty(\mathbb{R})$ stands for the set of all smooth complex valued functions with compact support. X is a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. The set of bounded linear operators from a Banach space X to a Banach space Y is denoted by $\mathcal{B}(X,Y)$ and $\mathcal{B}(X) := \mathcal{B}(X,X)$. $\mathcal{K}(X)$ is the set of compact operators on X. For an operator T on a Hilbert space X we use T^*, $\operatorname{dom}(T)$, $\operatorname{gr}(T)$, $\ker T$, $\operatorname{im} T$, $\sigma(T)$, $\rho(T)$, $\sigma_F(T)$ and $T_{|Y}$ to denote the adjoint, domain, graph, kernel, range, spectrum, resolvent set, Fredholm spectrum and the restriction of T on a subspace Y of X. We denote by $R(\lambda,T) = (\lambda - T)^{-1}$ the resolvent operator for $\lambda \in \rho(T)$. $\operatorname{Bor}(\mathbb{R})$ is the σ-algebra of all Borel measurable subsets of $\mathbb{R}$, δ_t is the Dirac measure concentrated at $t \in \mathbb{R}$. $\mathcal{M}(\mathbb{R},X)$ is the set of all X-valued Borel measures with bounded total variation $\|\mu\|$ which is the supremum over all $\sum_{n=0}^{\infty}\|\mu(E_n)\|$, where $(E_n)_{n\geq 0}$ is a sequence of disjoint Borel sets with $\bigcup_{n=0}^{\infty} E_n = \mathbb{R}$. If ν is a complex valued Borel measure, $x \in X$, then we denote by $\nu \otimes x$ the X-valued measure defined by $(\nu \otimes x)(E) = \nu(E)x$ for $E \in \operatorname{Bor}(\mathbb{R})$. $L_p(\mathbb{R},X)$, $p \in [1,\infty]$, $C_b(\mathbb{R},X)$ and $C_0(\mathbb{R},X)$ are the usual spaces of p-Bochner integrable functions $f : \mathbb{R} \to X$ with the norm $\|\cdot\|_p$, the space of all bounded continuous functions, and the space of all continuous functions with $\lim_{t\to\pm\infty} f(t) = 0$. $C_0([\tau,\infty),X)$ and $C_0((-\infty,\tau],X)$ are the spaces of continuous functions with $\lim_{t\to\infty} f(t) = 0$ and $\lim_{t\to-\infty} f(t) = 0$, respectively. $H^s(\mathbb{R},X)$, $s \geq 0$, is the usual Sobolev space of X valued functions. The Fourier transform $\mathcal{F}$ is defined by $(\mathcal{F}\mu)(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi t} d\mu(t)$ for $\mu \in \mathcal{M}(\mathbb{R},X)$. For a function $S : \mathbb{R} \to \mathcal{B}(X)$ satisfying $S(\cdot)x \in L^{\infty}(\mathbb{R},X)$ for all $x \in X$, we define the operator of multiplication by S, $M_S : L_2(\mathbb{R},X) \to L_2(\mathbb{R},X)$, by $(M_S f)(t) = S(t)f(t)$. For a function u defined on a proper subset of $\mathbb{R}$ we keep the same notation u to denote its extension to $\mathbb{R}$ by 0.

Setting: Let A be a linear operator on X such that $A = A_1 \oplus (-A_2)$ in the direct sum decomposition $X = X_1 \oplus X_2$, where A_1 and A_2 are the generators of uniformly exponentially stable strongly continuous semigroups $\{T_i(t)\}_{t\geq 0}$ on X_i, $i = 1,2$. Define $R : \mathbb{R} \to \mathcal{B}(X)$ by $R(\xi) = R(2\pi i\xi, A)$. Let $\mathcal{G} : \mathbb{R}^2 \to \mathcal{B}(X)$ be the Green function for (1.2) with $B = 0$ defined by

$$\mathcal{G}(t,\tau) = T_1(t-\tau)P_1 \text{ for } t \geq \tau \text{ and } \mathcal{G}(t,\tau) = -T_2(\tau - t)P_2 \text{ for } t \leq \tau. \tag{2.1}$$

Note that $\mathcal{G}(t,\tau) = \mathcal{V}(t-\tau)$ for all $t,\tau \in \mathbb{R}$, where $\mathcal{V}$ is defined in (1.4). Also, because the semigroups $\{T_i(t)\}_{t\geq 0}$ are uniformly exponentially stable, we have

$$\mathcal{G}(\cdot,\tau)x \in L_p(\mathbb{R},X) \text{ and } \|\mathcal{G}(\cdot,\tau)x\|_p \leq c\|x\| \tag{2.2}$$

for each $p \in [1,\infty]$, $\tau \in \mathbb{R}$ and $x \in X$. Since $R(\lambda, A_j)x = \int_0^{\infty} e^{-\lambda t} T_j(t)x dt$ for all $x \in X_j$, $j = 1,2$, and $\lambda \in \mathbb{C}$ with $Re\lambda \geq 0$, we calculate:

$$\mathcal{F}(\mathcal{G}(\cdot,\tau)x)(\xi) = e^{-2\pi i\xi\tau} R(\xi)x \tag{2.3}$$

for all $\xi,\tau \in \mathbb{R}$ and $x \in X$, and hence $R(\cdot)x \in L_2(\mathbb{R},X) \cap C_0(\mathbb{R},X)$ by the Riemann-Lebesgue Lemma, and, similarly, $R(\cdot)^*x \in L_2(\mathbb{R},X) \cap C_0(\mathbb{R},X)$ for each $x \in X$. By the Closed Graph Theorem, $\|R(\cdot)x\|_2 \leq c\|x\|$ and $\|R(\cdot)^*x\|_2 \leq c\|x\|$.

Denote by P_i the projections on X_i, $i = 1,2$, given by the decomposition $X = X_1 \oplus X_2$. Let $B : \mathbb{R} \to \mathcal{B}(X)$ be bounded and piecewise strongly continuous, and define $D(t) = A + B(t)$. We define the operators L and G on $L_2(\mathbb{R},X)$ as follows: Let $\operatorname{dom}(L)$ be the set of all $u \in H^1(\mathbb{R},X)$ such that $u(t) \in \operatorname{dom}(A)$ for almost all $t \in \mathbb{R}$ and $Au(\cdot) \in L_2(\mathbb{R},X)$, and define L by $(Lu)(t) = -u'(t) + D(t)u(t)$. Let $\operatorname{dom}(G)$ be the set of all $u \in L_2(\mathbb{R},X)$ such that there exists $f \in L_2(\mathbb{R},X)$ for which the relation $(\mathcal{F} - M_R\mathcal{F}M_B)u = M_R\mathcal{F}f$ holds. Note that this f is unique by the injectivity of M_R, and define G by $Gu = f$. Here and below M_R is the operator of multiplication by $R(\cdot)$.

Proposition 2.1 *If* $\{T_i(t)\}_{t\geq 0}$*,* $i = 1, 2$*, are analytic then* $G = -L$*.*

PROOF. First, assume $u \in \mathrm{dom}(L)$. From the definition of the operator L, we have $(\mathcal{F}u')(\xi) = 2\pi i\xi(\mathcal{F}u)(\xi)$ and, using that A is closed, we obtain $(\mathcal{F}u)(\xi) \in \mathrm{dom}(A)$ and $A(\mathcal{F}u)(\xi) = (\mathcal{F}Au)(\xi)$ for almost all $\xi \in \mathbb{R}$. It follows that $(\mathcal{F}Lu)(\xi) = -2\pi i\xi(\mathcal{F}u)(\xi) + A(\mathcal{F}u)(\xi) + (\mathcal{F}M_B u)(\xi)$ for almost all $\xi \in \mathbb{R}$, which yields $M_R\mathcal{F}Lu = -\mathcal{F}u + M_R\mathcal{F}M_B u$, proving $u \in \mathrm{dom}(G)$ and $Gu = -Lu$. Second, it remains to show $\mathrm{dom}(G) \subseteq \mathrm{dom}(L)$. The analyticity of the stable semigroups $\{T_i(t)\}_{t\geq 0}$ implies $\|R(\xi)\| \leq c(1 + 2\pi|\xi|)^{-1}$ for all $\xi \in \mathbb{R}$. Let $u \in \mathrm{dom}(G)$ and $f = Gu$. From the definition of the operator G we have $\mathcal{F}u = M_R\mathcal{F}(M_B u + f)$, and so, for all $\xi \in \mathbb{R}$

$$\begin{aligned}\|\xi(\mathcal{F}u)(\xi)\| &= |\xi|\|(M_R\mathcal{F}(M_B u + f))(\xi)\| \\ &\leq c(1 + 2\pi|\xi|)^{-1}|\xi|\|(\mathcal{F}(M_B u + f))(\xi)\| \leq c\|(\mathcal{F}(M_B u + f))(\xi)\|.\end{aligned}$$

Hence, the function $\xi \longmapsto \xi(\mathcal{F}u)(\xi)$ belongs to $L_2(\mathbb{R}, X)$, and so $u \in H^1(\mathbb{R}, X)$. Thus, $(\mathcal{F}(u' - M_B u - f))(\xi) = 2\pi i\xi(\mathcal{F}u)(\xi) - (2\pi i\xi - A)(\mathcal{F}u)(\xi) = A(\mathcal{F}u)(\xi)$ for almost all $\xi \in \mathbb{R}$, proving $A\mathcal{F}u \in L_2(\mathbb{R}, X)$. It follows that $u(\xi) \in \mathrm{dom}(A)$ for almost all $\xi \in \mathbb{R}$ and $Au(\cdot) \in L_2(\mathbb{R}, X)$, proving $u \in \mathrm{dom}(L)$.

If the semigroups $\{T_i(t)\}_{t\geq 0}$, $i = 1, 2$, are not analytic, the operator L may not be closed even when one of the projections P_i is zero. Since in the first part of the proof of Proposition 2.1 the analyticity assumption on the C_0-semigroups $\{T_i(t)\}_{t\geq 0}$ is not used, G is always an extension of $-L$. We stress that in what follows the semigroups $\{T_i(t)\}_{t\geq 0}$, $i = 1, 2$, are not assumed to be analytic and recall that M_R is the operator of multiplication by $R(\cdot)$.

Proposition 2.2 *(i)* $\mathrm{dom}(G^*) = \{v \in L_2(\mathbb{R}, X)$: *there exists* $g \in L_2(\mathbb{R}, X)$ *such that* $(\mathcal{F} - M_{R^*}\mathcal{F}M_{B^*})v = M_{R^*}\mathcal{F}g\}$ *and* $G^*v = g$*;*

(ii) $\mathrm{dom}\, G \subseteq C_0(\mathbb{R}, X)$*;*

(iii) $\mathrm{dom}\, G^* \subseteq C_0(\mathbb{R}, X)$.

PROOF. (i) First, we claim that $\mathrm{im}\, M_R$ is a dense subspace of $L_2(\mathbb{R}, X)$ (recall $R(\xi) = R(2\pi i\xi, A)$). Indeed, let $J \subset \mathbb{R}$ be bounded and measurable, $x \in \mathrm{dom}\, A$ and define $f(\xi) = \chi_J(\xi)(\xi x - Ax)$, $\xi \in \mathbb{R}$. Obviously, f is Bochner measurable and $\|f(\xi)\| \leq \chi_J(\xi)\sup_{\xi\in J}|\xi|\|x\| + \|Ax\|$ for all $\xi \in \mathbb{R}$. It follows that $f \in L_2(\mathbb{R}, X)$ and, moreover, $M_R f = \chi_J \otimes x \in \mathrm{im}\, M_R$, proving the claim.

¿From the definition of the operator G we have $G + M_B = \mathcal{F}^*M_R^{-1}\mathcal{F}$, where M_R^{-1} is the algebraic inverse of the injective operator M_R. Since $\mathcal{F}$ is a unitary operator on $L_2(\mathbb{R}, X)$ and M_R is a bounded injective operator with dense range, it follows that $G^* + M_{B^*} = (G + M_B)^* = \mathcal{F}^*(M_R^{-1})^*\mathcal{F} = \mathcal{F}^*M_{R^*}^{-1}\mathcal{F}$, yielding (i).

(ii) First, we will prove that if $u \in \mathrm{dom}\, G$ then u is continuous on $\mathbb{R}$. If $g = M_B u + f \in L_2(\mathbb{R}, X)$, then $\mathcal{F}u = M_R\mathcal{F}g$. By (1.4), $u = \mathcal{V} * g$, which implies

$$\begin{aligned}\|u(t) - u(\tau)\| &= \|\int_{\mathbb{R}}\mathcal{V}(s)(g(t-s) - g(\tau - s))ds\| \\ &\leq \int_{\mathbb{R}} Ne^{-\nu|s|}\|g(t-s) - g(\tau - s)\|ds \\ &\leq c\Big(\int_{\mathbb{R}}\|g(t-s) - g(\tau-s)\|^2 ds\Big)^{1/2} \leq c\|g(t - \tau + \cdot) - g\|_2\end{aligned}$$

for all $t,\tau\in\mathbb{R}$, since the C_0-semigroups $\{T_i(t)\}_{t\geq 0}$, $i=1,2$ are uniformly exponentially stable. The continuity of u follows from the strong continuity of the right translation group on $L_2(\mathbb{R},X)$ and the above estimate. Also,

$$\begin{aligned}\|u(t)\| &= \|(\mathcal{V}*g)(t)\| = \|\int_{\mathbb{R}}\mathcal{V}(t-s)g(s)ds\| \leq N\int_{\mathbb{R}} e^{-\nu|t-s|}\|g(s)\|ds\\ &= N\int_{\mathbb{R}} e^{-\nu/2|t-s|}(e^{-\nu/2|t-s|}\|g(s)\|)ds\\ &\leq N\Big(\int_{\mathbb{R}} e^{-\nu|t-s|}ds\Big)^{1/2}\Big(\int_{\mathbb{R}} e^{-\nu|t-s|}\|g(s)\|^2ds\Big)^{1/2}\\ &= N(2/\nu)^{1/2}\Big(\int_{\mathbb{R}} e^{-\nu|t-s|}p(s)ds\Big)^{1/2},\end{aligned}$$

where $p=\|g(\cdot)\|^2\in L^1(\mathbb{R})$. By a standard convolution argument for real valued functions we obtain $\lim_{t\to\pm\infty}\|u(t)\|=0$, proving $u\in C_0(\mathbb{R},X)$. The proof of (iii) is similar.

Definition 2.3 *(i) We say that a function u, continuous on a compact interval $[a,b]$, is a mild solution of equation* (1.2) *on $[a,b]$ if*

$$(\mathcal{F}u)(\xi)-(M_R\mathcal{F}M_Bu)(\xi)=R(\xi)(e^{-2\pi i\xi a}u(a)-e^{-2\pi i\xi b}u(b)) \text{ for all } \xi\in\mathbb{R}. \tag{2.4}$$

Here $\mathcal{F}$ is the Fourier transform and M_R, M_B are the operators of multiplication by $R(\cdot)$ and $B(\cdot)$, respectively.

(ii) We say that u is a mild solution of equation (1.2) *on $J\subseteq\mathbb{R}$ if $u_{|[a,b]}$ is a mild solution of equation* (1.2) *on $[a,b]$ for any subinterval $[a,b]\subseteq J$;*
(iii) We say that u is an $L_2\cap C_0$-solution of (1.2) *on $J=[a,\infty)$, or $J=(-\infty,b]$, or $J=\mathbb{R}$ if u is a mild solution on J and, in addition, $u\in L_2(J,X)\cap C_0(J,X)$.*

Proposition 2.4 *A continuous function u is a mild solution of equation* (1.2) *on $[a,b]$ if and only if*

$$\begin{aligned}u(t) &= T_1(t-a)P_1u(a)+T_2(b-t)P_2u(b)+\int_a^t T_1(t-s)P_1B(s)u(s)ds\\ &\quad -\int_t^b T_2(s-t)P_2B(s)u(s)ds \quad \text{for all} \quad t\in[a,b].\end{aligned} \tag{2.5}$$

PROOF. Let u be a continuous function on $[a,b]$, and extend it to $\mathbb{R}$ by letting $u(t)=0$ for $t\notin[a,b]$. By (1.4), equation (2.5) is equivalent to

$$u=\mathcal{V}*(\delta_a\otimes u(a)-\delta_b\otimes u(b))+\mathcal{V}*(M_Bu). \tag{2.6}$$

Using the Fourier transform and (2.3), we see that (2.6) is equivalent to (2.4).

Integrating by parts, it is easy to verify that strong (or classical) solutions of equation (1.2) are also mild solutions. For brevity (and recalling that the classical solutions might not exist at all), in what follows we omit the adjective "mild" referring to solutions of equation (1.2). Since, in general, P_i, $i=1,2$, and $B(s)$ do not commute (see examples in [38]), equation (2.5) is much harder to handle than its equivalent frequency domain reformulation given in Definition 2.3. We refer to Section 8 for many concrete examples of not well-posed equations satisfying our setting.

Let $\mathcal{P} = \{P(t)\}_{t\in J} \subseteq \mathcal{B}(X)$ be a family of projections on an interval $J \subseteq \mathbb{R}$. For $t, \tau \in J$, given two families of operators, $U_s(t,\tau) \in \mathcal{B}(\operatorname{im} P(\tau), \operatorname{im} P(t))$, $t \geq \tau$, and $U_u(t,\tau) \in \mathcal{B}(\ker P(\tau), \ker P(t))$, $t \leq \tau$, we say that $\mathcal{U} = (U_s, U_u)$ is a *bi-family adjusted to the projection family* $\mathcal{P}$, if for all $t, s, \tau \in J$ the following holds:

(i) $U_s(t,\tau) = U_s(t,s)U_s(s,\tau)$ provided $t \geq s \geq \tau$;

(ii) $U_u(t,\tau) = U_u(t,s)U_u(s,\tau)$ provided $t \leq s \leq \tau$;

(iii) $U_s(t,t)x = x$ for all $x \in \operatorname{im} P(t)$ and $U_u(t,t)x = x$ for all $x \in \ker P(t)$.

A bi-family $\mathcal{U} = (U_s, U_u)$, adjusted to a projection family $\mathcal{P} = \{P(t)\}_{t\in J}$, is called a *bi-family associated with equation* (1.2) if the following assertions hold:

(i) (Existence) $U_s(\cdot,\tau)x$ is a solution of equation (1.2) on $J \cap [\tau, \infty)$ for each $x \in \operatorname{im} P(\tau)$, and $U_u(\cdot,\tau)x$ is a solution of equation (1.2) on $J \cap (-\infty, \tau]$ for each $x \in \ker P(\tau)$;

(ii) (Uniqueness) If u is a solution of equation (1.2) on $[a,b] \subseteq J$, then

$$u(t) = U_s(t,a)P(a)u(a) + U_u(t,b)(I - P(b))u(b) \text{ for all } t \in [a,b].$$

Definition 2.5 *We say that equation* (1.2) *has an exponential dichotomy on an interval* $J \subseteq \mathbb{R}$*, if there exist a bi-family* $\mathcal{U} = (U_s, U_u)$*, associated with equation* (1.2) *and adjusted to a bounded strongly continuous on* J *projection family* $\mathcal{P} = \{P(t)\}_{t\in J}$*, and positive constants* N, ν *such that for all* $t, \tau \in J$ *the following estimates hold:*

$$\|U_s(t,\tau)\| \leq Ne^{-\nu(t-\tau)} \text{ for } t \geq \tau \text{ and } \|U_u(t,\tau)\| \leq Ne^{\nu(t-\tau)} \text{ for } t \leq \tau. \tag{2.7}$$

This definition of exponential dichotomy goes back to [38], and is more general than the standard definition used for evolution families (see, for example, [10,19] and the literature cited therein). Indeed, if $X_2 = \{0\}$ and $A_2 = 0$, and if the evolution family $\{\Phi(t,\tau)\}_{t\geq\tau}$ associated with equation (1.2), has an exponential dichotomy in the sense of [10, Def.2.6], then equation (1.2) has an exponential dichotomy in the sense of Definition 2.5, setting $U_s(t,\tau) = \Phi(t,\tau)_{|X_s(\tau)}$ and $U_u(t,\tau) = (\Phi(\tau,t)_{|X_u(t)})^{-1}$. However, in Definition 2.5, in contrast to the standard definition of exponential dichotomy, see e.g. [19], we do not assume that the propagator $U_u(t,\tau)$ is invertible on the unstable fibers. Moreover, vectors from the stable fibers can be propagated *only* in forward time, meanwhile vectors from the unstable fibers can be propagated *only* in backward time.

Proposition 2.6 *Let* $\mu \in \mathcal{M}(\mathbb{R}, X)$*,* $u \in L_2(\mathbb{R}, X)$*,* $\varphi \in \mathcal{S}(\mathbb{R})$*, and suppose that* $\mathcal{F}u - M_R\mathcal{F}M_Bu = M_R\mathcal{F}\mu$*. Then, for all* $\xi \in \mathbb{R}$*, we have:*

$$\Big(\mathcal{F}(\varphi u) - M_R\mathcal{F}M_B(\varphi u) - M_R\mathcal{F}(\varphi' u)\Big)(\xi) = R(\xi)\int_{\mathbb{R}} e^{-2\pi i\xi t}\varphi(t)d\mu(t).$$

PROOF. The proof follows by a direct but long computation (we denote $\widehat{u} = \mathcal{F}u$ and use properties of the Fourier transform and convolutions):

$$\begin{aligned}(\widehat{\varphi u} - M_R\widehat{M_B(\varphi u)} - M_R\widehat{\varphi' u})(\xi) &= \widehat{\varphi} * \widehat{u}(\xi) - R(\xi)(\widehat{\varphi} * \widehat{M_B u})(\xi) + \widehat{\varphi}' * \widehat{u}(\xi)) \\ &= \int_{\mathbb{R}} \widehat{\varphi}(\xi-\alpha)\widehat{u}(\alpha)d\alpha - R(\xi)\Big(\int_{\mathbb{R}} 2\pi i(\xi-\alpha)\widehat{\varphi}(\xi-\alpha)\widehat{u}(\alpha)d\alpha + \psi(\xi)\Big) \\ &= \int_{\mathbb{R}} \widehat{\varphi}(\xi-\alpha)(I - 2\pi i(\xi-\alpha)R(\xi))\widehat{u}(\alpha)d\alpha - R(\xi)\psi(\xi),\end{aligned}$$

where we define $\psi(\xi) := \int_{\mathbb{R}} \widehat{\varphi}(\xi-\alpha)\widehat{M_B u}(\alpha)d\alpha$. Thus, the last expression is equal to

$$\int_{\mathbb{R}} \widehat{\varphi}(\xi-\alpha)(R(\alpha)-2\pi i(\xi-\alpha)R(\xi)R(\alpha))(\widehat{M_B u}(\alpha)+\widehat{\mu}(\alpha))d\alpha - R(\xi)\psi(\xi)$$
$$= R(\xi)\int_{\mathbb{R}} \widehat{\varphi}(\xi-\alpha)(\widehat{M_B u}(\alpha)+\widehat{\mu}(\alpha))d\alpha - R(\xi)\psi(\xi) \quad \text{(by the resolvent identity)}$$
$$= R(\xi)\int_{\mathbb{R}} \widehat{\varphi}(\xi-\alpha)\widehat{\mu}(\alpha)d\alpha = R(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i\alpha t}\widehat{\varphi}(\xi-\alpha)d\alpha d\mu(t).$$

Changing variables and using the Fourier transform inversion formula, the last expression is equal to

$$R(\xi)\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i(\xi-\beta)t}\widehat{\varphi}(\beta)d\beta d\mu(t) = R(\xi)\int_{\mathbb{R}} e^{-2\pi i\xi t}\int_{\mathbb{R}} e^{2\pi i\beta t}\widehat{\varphi}(\beta)d\beta d\mu(t)$$
$$= R(\xi)\int_{\mathbb{R}} e^{-2\pi i\xi t}\varphi(t)d\mu(t).$$

Proposition 2.7 *(i) If $\mu \in \mathcal{M}(\mathbb{R}, X)$, $u \in L_2(\mathbb{R}, X)$, $u_{|[a,b]}$ is continuous on $[a, b]$ and $\mathcal{F}u - M_R\mathcal{F}M_B u = M_R\mathcal{F}\mu$, then for all $\xi \in \mathbb{R}$ one has:*

$$\Big(\mathcal{F}(\chi_{(a,b)}u) - M_R\mathcal{F}M_B(\chi_{(a,b)}u)\Big)(\xi)$$
$$= R(\xi)\Big(\int_{(a,b)} e^{-2\pi i\xi t}d\mu(t) + e^{-2\pi i\xi a}u(a) - e^{-2\pi i\xi b}u(b)\Big); \qquad (2.8)$$

(ii) If $\mu \in \mathcal{M}(\mathbb{R}, X)$, $u \in L_2(\mathbb{R}, X)$, $u_{|[a,\infty)}$ is continuous on $[a, \infty)$, $\lim_{t\to\infty} u(t) = 0$, and $\mathcal{F}u - M_R\mathcal{F}M_B u = M_R\mathcal{F}\mu$, then, for almost all $\xi \in \mathbb{R}$,

$$\Big(\mathcal{F}(\chi_{(a,\infty)}u) - M_R\mathcal{F}M_B(\chi_{(a,\infty)}u)\Big)(\xi)$$
$$= R(\xi)\Big(\int_{(a,\infty)} e^{-2\pi i\xi t}d\mu(t) + e^{-2\pi i\xi a}u(a)\Big); \qquad (2.9)$$

(iii) If $\mu \in \mathcal{M}(\mathbb{R}, X)$, $u \in L_2(\mathbb{R}, X)$, $u_{|(-\infty,b]}$ is continuous on $(-\infty, b]$, $\lim_{t\to-\infty} u(t) = 0$ and $\mathcal{F}u - M_R\mathcal{F}M_B u = M_R\mathcal{F}\mu$ then, for almost all $\xi \in \mathbb{R}$,

$$\Big(\mathcal{F}(\chi_{(-\infty,b]}u) - M_R\mathcal{F}M_B(\chi_{(-\infty,b]}u)\Big)(\xi)$$
$$= R(\xi)\Big(\int_{(-\infty,b)} e^{-2\pi i\xi t}d\mu(t) - e^{-2\pi i\xi b}u(b)\Big). \qquad (2.10)$$

(iv) A function $u \in L_2([a, \infty)) \cap C_0([a, \infty))$ is an $L_2 \cap C_0$-solution of equation (1.2) on $[a, \infty)$ if and only if

$$(\mathcal{F}u)(\xi) - (M_R\mathcal{F}M_B u)(\xi) = e^{-2\pi i\xi a}R(\xi)u(a) \quad \textit{for all} \quad \xi \in \mathbb{R}; \qquad (2.11)$$

(v) A function $u \in L_2((-\infty, b]) \cap C_0((-\infty, b])$ is an $L_2 \cap C_0$-solution of equation (1.2) on $(-\infty, b]$ if and only if

$$(\mathcal{F}u)(\xi) - (M_R\mathcal{F}M_B u)(\xi) = -e^{-2\pi i\xi b}R(\xi)u(b) \quad \textit{for all} \quad \xi \in \mathbb{R}. \qquad (2.12)$$

PROOF. (i) Let $(\varphi_n)_{n\geq 1}$ be a sequence of functions in $C_0^\infty(\mathbb{R})$ with the following properties: $0 \leq \varphi_n \leq 1$, $\|\varphi_n'\|_\infty \leq nc$, $\varphi_n(t) = 1$ for any $t \in [a + 1/n, b - 1/n]$ and $\varphi_n(t) = 0$ for any $t \notin (a, b)$. Then for each $n \geq 1$ and each $\xi \in \mathbb{R}$ the following estimate holds:

$$
\begin{aligned}
&\|\mathcal{F}(\varphi_n' u)(\xi) - e^{-2\pi i\xi a}u(a) + e^{-2\pi i\xi b}u(b)\| \\
&\qquad = \| \int_a^{a+1/n} \varphi_n'(t)(e^{-2\pi i\xi t}u(t) - e^{-2\pi i\xi a}u(a))dt \\
&\qquad\qquad + \int_{b-1/n}^{b} \varphi_n'(t)(e^{-2\pi i\xi t}u(t) - e^{-2\pi i\xi b}u(b))dt\| \\
&\qquad \le nc\int_a^{a+1/n} \|e^{-2\pi i\xi t}u(t) - e^{-2\pi i\xi a}u(a)\|dt + \\
&\qquad\qquad nc\int_{b-1/n}^{b} \|e^{-2\pi i\xi t}u(t) - e^{-2\pi i\xi b}u(b)\|dt.
\end{aligned}
$$

This shows $\mathcal{F}(\varphi_n' u)(\xi) \to e^{-2\pi i\xi a}u(a) - e^{-2\pi i\xi b}u(b)$ as $n \to \infty$ for each $\xi \in \mathbb{R}$. Since $u \in L_2(\mathbb{R}, X)$, $\varphi_n \to \chi_{(a,b)}$ pointwise as $n \to \infty$, and $0 \le \varphi_n \le 1$, the Lebesgue's dominated convergence theorem yields $\varphi_n u \to \chi_{(a,b)}u$ in $L_2(\mathbb{R}, X)$, which implies $\mathcal{F}(\varphi_n u) - M_R\mathcal{F}M_B(\varphi_n u) \to \mathcal{F}(\chi_{(a,b)}u) - M_R\mathcal{F}M_B(\chi_{(a,b)}u)$ in $L_2(\mathbb{R}, X)$ as $n \to \infty$. Since $\varphi_n \to \chi_{(a,b)}$ pointwise as $n \to \infty$ and $0 \le \varphi_n \le 1$, from Lebesgue's dominated convergence theorem it also follows that $\int_{\mathbb{R}} e^{-2\pi i\xi t}\varphi_n d\mu(t) \to \int_{(a,b)} e^{-2\pi i\xi t}d\mu(t)$ as $n \to \infty$. Applying Proposition 2.6 with μ, u and $(\varphi_n)_{n\ge 1}$ respectively, we have

$$
\begin{aligned}
&\Big(\mathcal{F}(\chi_{(a,b)}u) - M_R\mathcal{F}M_B(\chi_{(a,b)}u)\Big)(\xi) \\
&\qquad = R(\xi)\Big(\int_{(a,b)} e^{-2\pi i\xi t}d\mu(t) + e^{-2\pi i\xi a}u(a) - e^{-2\pi i\xi b}u(b)\Big)
\end{aligned}
$$

for almost all $\xi \in \mathbb{R}$. Since $\chi_{(a,b)}u \in L_1(\mathbb{R}, X)$, both sides of the above equality are continuous functions of $\xi \in \mathbb{R}$, and so they are equal everywhere, proving (i).

(ii) Since $u \in L_2(\mathbb{R}, X)$ we obtain $\chi_{(a,n)}u \to \chi_{(a,\infty)}u$ in $L_2(\mathbb{R}, X)$ as $n \to \infty$, which implies $(\mathcal{F} - M_R\mathcal{F}M_B)(\chi_{(a,n)}u) \to (\mathcal{F} - M_R\mathcal{F}M_B)\chi_{(a,\infty)}u$ in $L_2(\mathbb{R}, X)$ as $n \to \infty$. From (i), Lebesgue's dominated convergence theorem, and since $\mu \in \mathcal{M}(\mathbb{R}, X)$, we infer $(\mathcal{F} - M_R\mathcal{F}M_B)(\chi_{(a,n)}u)(\xi) \to R(\xi)\Big(\int_{(a,\infty)} e^{-2\pi i\xi t}d\mu(t) + e^{-2\pi i\xi a}u(a)\Big)$ as $n \to \infty$, for each $\xi \in \mathbb{R}$, which proves (ii). The proof of (iii) is similar.

(iv) Let $u \in L_2([a,\infty)) \cap C_0([a,\infty))$. First, assume that u is a solution of equation (1.2) in the sense of Definition 2.3(ii). It follows that $u_{|[a,n]}$ is a solution of equation (1.2) on $[a,n]$ for each $n \in \mathbb{N}$ with $n \ge a$. Passing to the limit as $n \to \infty$, (2.11) follows shortly. Second, assume that (2.11) holds. It follows that $\mathcal{F}u - M_R\mathcal{F}M_B u = M_R\mathcal{F}\mu_a$, where $\mu_a = \delta_a \otimes u(a)$. By (i) we conclude that $u_{|[c,d]}$ is a solution of equation (1.2) on $[c,d]$ for each interval $[c,d] \subseteq [a,\infty)$. The proof of (v) is similar.

Remark 2.8 In the left hand side of formulas (2.8)-(2.10) in Proposition 2.7, the open interval (a,b) can be replaced by an interval closed from one or two sides, say, by $[a,b)$ in (i), the interval (a,∞) can be replaced by $[a,\infty)$ in (ii), and the interval $(-\infty,b)$ can be replaced by $(-\infty,b]$ in (iii). ◇

Proposition 2.9 *(i) Let $S : \mathbb{R} \to \mathcal{B}(X)$ be a strongly continuous function such that $S^*(\cdot)x \in L_2(\mathbb{R}, X)$ for each $x \in X$. If $u, g \in L_2(\mathbb{R}, X)$ satisfy $(\mathcal{F}u)(\xi) = S(\xi)(\mathcal{F}g)(\xi)$, $\xi \in \mathbb{R}$, and u is continuous on $\mathbb{R}$, then*

$$\langle (M_S\mathcal{F}g)(\cdot), x\rangle \in L_1(\mathbb{R}, \mathbb{C}) \text{ and } \langle u(t), x\rangle = \int_{\mathbb{R}} e^{2\pi i\xi t}\langle M_S\mathcal{F}g(\xi), x\rangle d\xi \text{ for all } t \in \mathbb{R}.$$

(ii) If u is a solution of (1.2) on $[a,b]$ and $h \in \ker G^*$, then $\langle u(a), h(a)\rangle = \langle u(b), h(b)\rangle$.

PROOF. (i) Follows from $\mathcal{F}(\langle u(\cdot), x\rangle) = \langle (\mathcal{F}u)(\cdot), x\rangle = \langle \mathcal{F}g, S^*(\cdot)x\rangle \in L_1(\mathbb{R}, \mathbb{C})$.

(ii) From (i) for $S(\xi) = R(\xi)^*$, $u = h$ and $g = M_{B^*}h$ we obtain:

$$\langle x, h(s)\rangle = \int_{\mathbb{R}} e^{-2\pi i\xi s}\langle x, (M_{R^*}\mathcal{F}M_{B^*}h)(\xi)\rangle d\xi \text{ for all } x \in X \text{ and } s \in \mathbb{R}.$$

Using this equation for $x = u(a), u(b)$ and for $s = a, b$, respectively, one has:

$$\begin{aligned}
\langle u(a), h(a)\rangle - \langle u(b), h(b)\rangle &= \int_{\mathbb{R}} \langle e^{-2\pi i\xi a}u(a) - e^{-2\pi i\xi b}u(b), (M_{R^*}\mathcal{F}M_{B^*}h)(\xi)\rangle d\xi \\
&= \int_{\mathbb{R}} \langle R(\xi)(e^{-2\pi i\xi a}u(a) - e^{-2\pi i\xi b}u(b)), (\mathcal{F}M_{B^*}h)(\xi)\rangle d\xi \\
&= \int_{\mathbb{R}} \langle (\mathcal{F}u - M_R\mathcal{F}M_Bu)(\xi), (\mathcal{F}M_{B^*}h)(\xi)\rangle d\xi \quad \text{(using (2.4))} \\
&= \langle \mathcal{F}u - M_R\mathcal{F}M_Bu, \mathcal{F}M_{B^*}h\rangle_{L_2} \\
&= \langle \mathcal{F}u, \mathcal{F}M_{B^*}h\rangle_{L_2} - \langle \mathcal{F}M_Bu, M_{R^*}\mathcal{F}M_{B^*}h\rangle_{L_2} \\
&= \langle u, M_{B^*}h\rangle_{L_2} - \langle \mathcal{F}M_Bu, \mathcal{F}h\rangle_{L_2} \quad \text{(because } h \in \ker G^*) \\
&= \langle M_Bu, h\rangle_{L_2} - \langle M_Bu, h\rangle_{L_2} = 0.
\end{aligned}$$

The following subspaces of X turn out to be an important tool in the study of dichotomy. They are obtained as "traces" of $\ker G$ and $\ker G^*$, cf. Proposition 2.2:

$$X_s = \{u(s) : u \in \ker G\}, \quad X_{s,*} = \{v(s) : v \in \ker G^*\}, \quad s \in \mathbb{R}. \tag{2.13}$$

We note that X_s and $X_{s,*}$ are finite dimensional as soon as G is Fredholm. In this case, we define the operator G_0 as follows: $\mathrm{dom}(G_0) = \{u \in \mathrm{dom}(G) : u(0) \in X_0^\perp\}$, $G_0u = Gu$.

Proposition 2.10 *Assume that G is Fredholm. Then:*

(i) $\|u\|_\infty \le c(\|u\|_2 + \|Gu\|_2)$ for all $u \in \mathrm{dom}(G)$;

(ii) G_0 is a closed injective linear operator;

(iii) $\mathrm{im}\, G_0 = \mathrm{im}\, G$;

(iv) $\|G_0u\|_2 \ge c\max(\|u\|_2, \|u\|_\infty) := \|u\|_{2,\infty}$ for all $u \in \mathrm{dom}(G_0)$.

PROOF. (i) Let $u \in \mathrm{dom}(G)$ and $f = Gu \in L_2(\mathbb{R}, X)$. From the definition of the operator G and from Proposition 2.9(i) for $S(\xi) = R(\xi)$, we have

$$\begin{aligned}
|\langle u(t), x\rangle| &= |\int_{\mathbb{R}} e^{2\pi i\xi t}\langle M_R\mathcal{F}(M_Bu + f)(\xi), x\rangle d\xi| \\
&= |\int_{\mathbb{R}} e^{2\pi i\xi t}\langle \mathcal{F}(M_Bu + f)(\xi), R(\xi)^*x\rangle d\xi| \\
&\le \|R(\cdot)^*x\|_2\|\mathcal{F}(M_Bu + f)\|_2 \\
&\le c\|x\|\, \|M_Bu + f\|_2 \le c\|x\|(\|u\|_2 + \|Gu\|_2),
\end{aligned}$$

for all $t \in \mathbb{R}$ and $x \in X$, proving (i).

(ii) Let $(u_n)_{n\ge 1} \subseteq \mathrm{dom}(G_0)$ and $u, f \in L_2(\mathbb{R}, X)$ are such that $u_n \to u$ in $L_2(\mathbb{R}, X)$ and $G_0u_n \to f$ in $L_2(\mathbb{R}, X)$ as $n \to \infty$. Since G is closed (as a consequence of the fact that M_R, $\mathcal{F}$ and M_B are bounded on $L_2(\mathbb{R}, X)$), we obtain that $u \in \mathrm{dom}(G)$ and $Gu = f$. From (i) we infer that the following estimate holds: $\|u_n(0) - u(0)\| \le \|u_n - u\|_\infty \le$

$c(\|u_n - u\|_2 + \|G_0 u_n - f\|_2)$ for all $n \geq 1$, which shows that $u \in \mathrm{dom}(G_0)$ and $G_0 u = f$. Also, if $u \in \ker G_0$ then $u(0) \in X_0 \cap X_0^\perp = \{0\}$, and so, by Hypothesis 1.1, $u = 0$.

(iii) Let $f = Gu \in \mathrm{im}\, G$ for some $u \in \mathrm{dom}(G)$. Since $u(0) \in X = X_0 \oplus X_0^\perp$, there is $g \in \ker G$ such that $u(0) - g(0) \in X_0^\perp$, which gives $u - g \in \mathrm{dom}(G_0)$ and $G_0(u-g) = G(u-g) = Gu - Gg = Gu = f$, proving $\mathrm{im}\, G \subseteq \mathrm{im}\, G_0$. The proof of (iv) follows from (i), (ii) and (iii) since G is Fredholm.

Remark 2.11 The conclusions of Propositions 2.6, 2.7, 2.9(ii), 2.10, Remark 2.8 remain true if we replace B by B^*, $R(\cdot)$ by $R(\cdot)^*$ and G by G^*. $\diamond$

Let us define bounded linear operators $T(t,\tau) : X_\tau \to X_t$ and $T_*(t,\tau) : X_{\tau,*} \to X_{t,*}$, $t, \tau \in \mathbb{R}$, by the relations

$$T(t,\tau)g(\tau) = g(t),\ g \in \ker G,\ \text{and } T_*(t,\tau)h(\tau) = h(t),\ h \in \ker G^*. \tag{2.14}$$

It follows from Hypothesis 1.1 that these operators are well-defined and invertible.

Proposition 2.12 *Assume that G is Fredholm. Then:*

(i) The functions $t \mapsto T(t,\tau) : \mathbb{R} \to \mathcal{B}(X_\tau, X)$ and $t \mapsto T_*(t,\tau) : \mathbb{R} \to \mathcal{B}(X_{\tau,*}, X)$ are continuous on $\mathbb{R}$ for each $\tau \in \mathbb{R}$;

(ii) The functions $t \mapsto (T(t,\tau)^*)^{-1} : \mathbb{R} \to \mathcal{B}(X_\tau, X)$ and $t \mapsto (T_*(t,\tau)^*)^{-1} : \mathbb{R} \to \mathcal{B}(X_{\tau,*}, X)$ are continuous on $\mathbb{R}$ for each $\tau \in \mathbb{R}$.

PROOF. $T(\cdot,\tau)$ is strongly continuous on $\mathbb{R}$ since $\ker G \subseteq C_0(\mathbb{R}, X)$, cf. Proposition 2.2. Since G is Fredholm, $\dim(\ker G) < \infty$, and so X_τ is finite dimensional. Hence, $T(\cdot,\tau)$ is norm continuous, and similarly for $T_*(\cdot,\tau)$, proving (i). Assertion (ii) follows by the continuity of taking inverses and adjoints.

3. The dichotomy on the positive semiline

Throughout this section we assume that the operator G is Fredholm. In order to prove the existence of the stable fibers $X_s^+(\tau)$ and unstable fibers $X_u^+(\tau)$ on $\mathbb{R}_+$, we proceed as follows. First, we construct the stable fibers $X_s^+(\tau) \subseteq X_{\tau,*}^\perp$ and forward solutions of equation (1.2) corresponding to these fibers. Second, we treat the part $Y_u(\tau)$ of the unstable fibers that is contained in the subspaces $X_{\tau,*}^\perp$, $\tau \in \mathbb{R}$, and the corresponding backward solutions of equation (1.2). A crucial point of this section is the decomposition of $X_{\tau,*}^\perp$ given in Theorem 3.6. To complete the construction of the unstable fibers $X_u^+(\tau)$, we need the existence and uniqueness result given in Proposition 3.7.

We define $X_s^+(\tau)$, $\tau \geq 0$, as the subspace of all $x \in X$ such that there exists an $L_2 \cap C_0$-solution u of equation (1.2) on $[\tau, \infty)$ satisfying $u(\tau) = x$. By Hypothesis 1.1, the solution u with this property is unique; it will be denoted by $U_s^+(\cdot,\tau)x$. As always, we extend the function $t \mapsto U_s^+(t,\tau)x$ from $[\tau,\infty)$ to $\mathbb{R}$ by letting $U_s^+(t,\tau)x = 0$ for $t < \tau$.

Remark 3.1 From Definition 2.3 and Hypothesis 1.1 we conclude the following: (i) $U_s^+(t,\tau) : X_s^+(\tau) \to X$ is a linear operator; (ii) $U_s^+(t,\tau)X_s^+(\tau) \subseteq X_s^+(t)$ for all $t \geq \tau \geq 0$; (iii) $U_s^+(t,s)U_s^+(s,\tau) = U_s^+(t,\tau)$ for all $t \geq s \geq \tau \geq 0$; (iv) $U_s^+(\tau,\tau)x = x$ for all $\tau \geq 0$ and $x \in X_s^+(\tau)$. $\diamond$

Proposition 3.2 *(i) $\|g\|_\infty \leq c\|g(0)\|$ for all $g \in \ker G$;*

(ii) There exist positive N, ν such that $\|U_s^+(t,\tau)\| \leq N e^{-\nu(t-\tau)}$ for all $t \geq \tau \geq 0$;

(iii) The linear subspace $X_s^+(\tau)$ is closed for each $\tau \geq 0$.

PROOF. (i) We recall that X_0 and $\ker G$ are finite dimensional since G is Fredholm. Recalling (2.13), define the linear operator $T_0 : X_0 \to \ker G$ by $T_0 x = g_x$, where $g_x \in \ker G$ is the unique (by Hypothesis 1.1) function with the property $g_x(0) = x$. From Proposition 2.10(i) we obtain $\|g_x\|_\infty \leq c(\|g_x\|_2 + \|Gg_x\|_2) = c\|T_0 x\|_2 \leq c\|x\|$.

(ii) Let $\tau \geq 0$ and $x \in X_s^+(\tau)$. From the definition of $U_s^+(\cdot,\tau)x$, Proposition 2.7(iv) and (2.3) we infer $(\mathcal{F} - M_R\mathcal{F}M_B)U_s^+(\cdot,\tau)x = \mathcal{F}\mathcal{G}(\cdot,\tau)x$ or, equivalently,

$$(\mathcal{F} - M_R\mathcal{F}M_B)(U_s^+(\cdot,\tau)x - \mathcal{G}(\cdot,\tau)x) = M_R\mathcal{F}M_B\mathcal{G}(\cdot,\tau)x.$$

Hence, $U_s^+(\cdot,\tau)x - \mathcal{G}(\cdot,\tau)x \in \operatorname{dom}(G)$ and $G(U_s^+(\cdot,\tau)x - \mathcal{G}(\cdot,\tau)x) = M_B\mathcal{G}(\cdot,\tau)x$. Let $P_0^\perp$ be the orthogonal projection of X onto X_0, and fix $z(\cdot;\tau,x) \in \ker G$ be such that $z(0;\tau,x) = P_0^\perp(\mathcal{G}(0,\tau)x - U_s^+(0,\tau)x)$. Then $U_s^+(\cdot,\tau)x - \mathcal{G}(\cdot,\tau)x + z(\cdot;\tau,x) \in \operatorname{dom} G_0$, and so, using Proposition 2.10(iv), we obtain:

$$\begin{aligned}\|U_s^+(\cdot,\tau)x - \mathcal{G}(\cdot,\tau)x + z(\cdot;\tau,x)\|_\infty &\leq c\|G_0(U_s^+(\cdot,\tau)x - \mathcal{G}(\cdot,\tau)x + z(\cdot;\tau,x))\|_2 \\ = c\|G(U_s^+(\cdot,\tau)x - \mathcal{G}(\cdot,\tau)x)\|_2 &= c\|M_B\mathcal{G}(\cdot,\tau)x\|_2 \leq c\|\mathcal{G}(\cdot,\tau)x\|_2 \leq c\|x\|.\end{aligned}$$

Moreover, from (i), (2.2) and since $U_s^+(0,\tau)x = 0$ for $\tau > 0$ and $U_s^+(0,0)x = x$,

$$\begin{aligned}\|z(\cdot;\tau,x)\|_\infty &\leq c\|z(0;\tau,x)\| = c\|P_0^\perp(\mathcal{G}(0,\tau)x - U_s^+(0,\tau)x)\| \\ &\leq c\|\mathcal{G}(0,\tau)x - U_s^+(0,\tau)x\| \leq c\|x\|.\end{aligned}$$

This shows $\|U_s^+(t,\tau)\| \leq c$ for all $t \geq \tau \geq 0$. The required exponential estimate is then obtained similarly to [13, Sec.III.6.1] as follows. Let $t_1 \geq \tau + 1$. Choose a function $\varphi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq 1$, $\|\varphi'\|_\infty \leq c$, $\varphi(t) = 0$ for $t \notin (\tau, t_1+1)$ and $\varphi(t) = 1$ for $t \in [\tau+1, t_1]$. From Proposition 2.6 for $\mu = \delta_\tau \otimes x$ and $u = \varphi U_s^+(\cdot,\tau)x$ it follows that $\varphi U_s^+(\cdot,\tau)x \in \operatorname{dom}(G_0)$ and $G_0(\varphi U_s^+(\cdot,\tau)x) = \varphi' U_s^+(\cdot,\tau)x$. Proposition 2.10(iv) applied to $u = \varphi U_s^+(\cdot,\tau)x$ yields:

$$\begin{aligned}&\int_{\tau+1}^{t_1} \|U_s^+(t,\tau)x\|^2 dt \leq \|\varphi U_s^+(\cdot,\tau)x\|_2^2 \leq c\|\varphi' U_s^+(\cdot,\tau)x\|_2^2 \\ &= c\int_\tau^{\tau+1} |\varphi'(t)|^2 \|U_s^+(t,\tau)x\|^2 dt + c\int_{t_1}^{t_1+1} |\varphi'(t)|^2 \|U_s^+(t,\tau)x\|^2 dt \leq c\|x\|^2.\end{aligned}$$

Since $U_s^+(\cdot,\tau)x$ is an $L_2 \cap C_0$-solution and $\|U_s^+(t,\tau)\| \leq c$ for all $t \geq \tau \geq 0$, letting $t_1 \to \infty$, we obtain

$$\int_\tau^\infty \|U_s^+(t,\tau)x\|^2 dt \leq c\|x\|^2 \quad \text{for all} \quad x \in X_s^+(\tau) \quad \text{and all} \quad \tau \geq 0.$$

Using standard arguments similarly to [13, Sec.III.6.1], (ii) follows.

(iii) Let $\tau \geq 0$, $(x_n)_{n\geq 1} \subseteq X_s^+(\tau)$, $u_n = U_s^+(\cdot,\tau)x_n$ and $x \in X$ be such that $x_n \to x$. Using the estimate in (ii), we have:

$$\sup_{t\geq\tau} \|u_n(t) - u_m(t)\| \leq \sup_{t\geq\tau} Ne^{-\nu(t-\tau)}\|x_n - x_m\| = N\|x_n - x_m\| \text{ for all } n,m \geq 1.$$

Thus, there exists $u \in C_0([\tau,\infty),X)$ such that $u_n \to u$ as $n \to \infty$ in $C_0([\tau,\infty),X)$. Letting $n \to \infty$ in $\|u_n(t)\| \leq Ne^{-\nu(t-\tau)}\|x_n\|$ yields $\|u(t)\| \leq Ne^{-\nu(t-\tau)}\|x\|$ for all $t \geq \tau$. Applying Lebesgue's dominated convergence theorem to $(u_n)_{n\geq 1}$, one has $(\mathcal{F}u_n)(\xi) \to (\mathcal{F}u)(\xi)$ and $(\mathcal{F}M_B u_n)(\xi) \to (\mathcal{F}M_B u)(\xi)$ as $n \to \infty$, yielding $(\mathcal{F}u - M_R\mathcal{F}M_B u)(\xi) = e^{-2\pi i\xi\tau}R(\xi)x$ for all $\xi \in \mathbb{R}$. By Proposition 2.7(iv), u is an $L_2 \cap C_0$-solution of (1.2) on $[\tau,\infty)$ and thus $x \in X_s^+(\tau)$ by the definition of the latter space.

We denote by $Y_u(\tau)$, $\tau \geq 0$, the subspace of all $x \in X$ such that there exists an $L_2 \cap C_0$-solution v of equation (1.2) on $(-\infty, \tau]$ satisfying $v(\tau) = x$ and $v(0) \in X_0^{\perp}$. By Hypothesis 1.1, this solution v is unique; it will be denoted by $V_u(\cdot, \tau)x$.

Remark 3.3 From Hypothesis 1.1 and Definition 2.3 we conclude: (i) $V_u(t,\tau) : Y_u(\tau) \to X$ is a linear operator; (ii) $V_u(t,\tau)Y_u(\tau) \subseteq Y_u(t)$ for all $\tau \geq t \geq 0$; (iii) $V_u(t,s)V_u(s,\tau) = V_u(t,\tau)$ for all $\tau \geq s \geq t \geq 0$; (iv) $V_u(\tau,\tau)x = x$ for all $\tau \geq 0$ and all $x \in Y_u(\tau)$. ◇

Proposition 3.4 *(i) There exist positive N, ν such that $\|V_u(t,\tau)\| \leq Ne^{\nu(t-\tau)}$ for all $\tau \geq t \geq 0$; (ii) The subspace $Y_u(\tau)$ is closed for each $\tau \geq 0$.*

PROOF. (i) Let $\tau \geq 0$ and $x \in Y_u(\tau)$. From the definition of $V_u(\cdot,\tau)x$, Proposition 2.7(v) and (2.3) we have $(\mathcal{F} - M_R\mathcal{F}M_B)V_u(\cdot,\tau)x = -\mathcal{F}\mathcal{G}(\cdot,\tau)x$, which is equivalent to $\mathcal{F}(V_u(\cdot,\tau)x + \mathcal{G}(\cdot,\tau)x) - M_R\mathcal{F}M_B(V_u(\cdot,\tau)x + \mathcal{G}(\cdot,\tau)x) = -M_R\mathcal{F}M_B\mathcal{G}(\cdot,\tau)x$. This yields $V_u(\cdot,\tau)x + \mathcal{G}(\cdot,\tau)x \in \operatorname{dom}(G)$ and $G(V_u(\cdot,\tau)x + \mathcal{G}(\cdot,\tau)x) = -M_B\mathcal{G}(\cdot,\tau)x$. Recall that $P_0^{\perp}$ is the orthogonal projection onto X_0, cf. the proof of Proposition 3.2(ii). Let $\tilde{z}(\cdot;\tau,x) \in \ker G$ be a function with the property $\tilde{z}(0;\tau,x) = -P_0^{\perp}\mathcal{G}(0,\tau)x$. From the definition of $Y_u(\tau)$ we have $V_u(0,\tau)x \in Y_u(0) \subseteq X_0^{\perp}$, which implies $V_u(0,\tau)x + \mathcal{G}(0,\tau)x + \tilde{z}(0;\tau,x) \in X_0^{\perp}$. This yields $V_u(\cdot,\tau)x + \mathcal{G}(\cdot,\tau)x + \tilde{z}(\cdot;\tau,x) \in \operatorname{dom}(G_0)$ and $G_0(V_u(\cdot,\tau)x + \mathcal{G}(\cdot,\tau)x + \tilde{z}(\cdot;\tau,x)) = -M_B\mathcal{G}(\cdot,\tau)x$. From Proposition 2.10(iv) we derive

$$\begin{aligned}\|V_u(\cdot,\tau)x + \mathcal{G}(\cdot,\tau)x + \tilde{z}(\cdot;\tau,x)\|_\infty &\leq c\|G_0(V_u(\cdot,\tau)x + \mathcal{G}(\cdot,\tau)x + \tilde{z}(\cdot;\tau,x))\|_2 \\ &= c\|M_B\mathcal{G}(\cdot,\tau)x\|_2 \leq c\|\mathcal{G}(\cdot,\tau)x\|_2 \leq c\|x\|.\end{aligned}$$

From (2.2) and Proposition 3.2(i) we have

$$\|\tilde{z}(\cdot;\tau,x)\|_\infty \leq c\|\tilde{z}(0;\tau,x)\| = c\|P_0^{\perp}\mathcal{G}(0,\tau)x\| \leq c\|\mathcal{G}(\cdot,\tau)x\|_\infty \leq c\|x\|.$$

Hence, $\|V_u(t,\tau)\| \leq c$, for all $\tau \geq t \geq 0$.

Let $t_1 < \tau - 1$ and fix $\varphi \in C_0^\infty(\mathbb{R})$ with the properties $0 \leq \varphi \leq 1$, $\|\varphi\|_\infty \leq c$, $\varphi(t) = 1$ for $t \in [t_1, \tau - 1]$ and $\varphi(t) = 0$ for $t \in (-\infty, t_1 - 1] \cup [\tau, \infty)$. The function $\varphi V_u(\cdot,\tau)x$ is continuous on $\mathbb{R}$ and has compact support, and so $\varphi V_u(\cdot,\tau)x \in L_2(\mathbb{R}, X) \cap C_0(\mathbb{R}, X)$ and $\varphi(0)V_u(0,\tau)x \in Y_u(0) \subseteq X_0^{\perp}$. From Proposition 2.6 it follows that $\varphi V_u(\cdot,\tau)x \in \operatorname{dom}(G_0)$ and $G_0(\varphi V_u(\cdot,\tau)x) = \varphi' V_u(\cdot,\tau)x$. Using Proposition 2.10(iv), we infer:

$$\begin{aligned}\int_{t_1}^{\tau-1} \|V_u(t,\tau)x\|^2 dt &\leq \|\varphi V_u(\cdot,\tau)x\|_2^2 \leq c\|\varphi' V_u(\cdot,\tau)x\|_2^2 \\ &\leq c\Big(\int_{t_1-1}^{t_1} \|V_u(t,\tau)x\|^2 dt + \int_{\tau-1}^{\tau} \|V_u(t,\tau)x\|^2 dt\Big) \leq c\|x\|^2.\end{aligned}$$

Since $V_u(\cdot,\tau)x \in L_2((-\infty,\tau], X)$ and $\|V_u(t,\tau)\| \leq c$, for all $0 \leq t \leq \tau$, passing to the limit as $t_1 \to -\infty$, we obtain

$$\int_{-\infty}^{\tau} \|V_u(t,\tau)x\|^2 dt \leq c\|x\|^2 \quad \text{for all} \quad x \in Y_u(\tau) \quad \text{and all} \quad \tau \geq 0.$$

Using an argument similar to [13, Sec.III.6.1], the estimate in (i) follows.

(ii) Let $\tau \geq 0$, $(x_n)_{n\geq 1} \subseteq Y_u(\tau)$, assume $x_n \to x$ as $n \to \infty$, and denote $v_n := V_u(\cdot,\tau)x_n$. Since $\|v_n(t) - v_m(t)\| \leq Ne^{\nu(t-\tau)}\|x_n - x_m\|$, $m, n \geq 1$, $t \leq \tau$ by (i), we have $v_n \to v$ in $C_0((-\infty,\tau], X)$ as $n \to \infty$ for some $v \in C_0((-\infty,\tau], X)$. From (i), we have $\|v_n(t)\| \leq Ne^{\nu(t-\tau)}\|x_n\|$ for all $n \geq 1$ and all $t \leq \tau$, and thus $\|v(t)\| \leq Ne^{\nu(t-\tau)}\|x\|$ for all $t \leq \tau$. Lebesgue's dominated convergence theorem applied to the sequences $(v_n)_{n\geq 1}$

and $(M_B v_n)_{n\geq 1}$ yields $(\mathcal{F}v_n)(\xi) \to (\mathcal{F}v)(\xi)$ and $(\mathcal{F}M_B v_n)(\xi) \to (\mathcal{F}M_B v)(\xi)$ as $n \to \infty$ for each $\xi \in \mathbb{R}$. Using the definition of the subspace $Y_u(\tau)$ and Proposition 2.7(v), we obtain $(\mathcal{F}v)(\xi) - (M_R \mathcal{F} M_B v)(\xi) = -e^{-2\pi i \xi \tau} R(\xi)x$ for all $\xi \in \mathbb{R}$. Also, we have $v(\tau) = x$, and so, from Proposition 2.7(v), it follows that $x \in Y_u(\tau)$, proving (ii).

Proposition 3.5 *(i)* $X_\tau \subseteq X_s^+(\tau) \subseteq X_{\tau,*}^\perp$ *for all* $\tau \geq 0$*; (ii)* $Y_u(\tau) \subseteq X_{\tau,*}^\perp$ *for all* $\tau \geq 0$.

PROOF. Let $x \in X_\tau$ and $g \in \ker G$ with $g(\tau) = x$. Since, cf. Proposition 2.2, $g \in L_2(\mathbb{R}, X) \cap C_0(\mathbb{R}, X)$, we have that $v = g_{|[\tau,\infty)}$ is an $L_2 \cap C_0$-solution of equation (1.2) on $[\tau, \infty)$, and so $x \in X_s^+(\tau)$, by the definition of $X_s^+(\tau)$.

Let $x \in X_s^+(\tau)$ and $h \in \ker G^*$. Then $\mathcal{F}h = M_{R^*}\mathcal{F}M_{B^*}h$, and thus, using Proposition 2.2(i) and Proposition 2.9(i) with $S(\xi) = R(\xi)^*$, we obtain:

$$\begin{aligned}
\langle h(\tau), x\rangle &= \int_{\mathbb{R}} e^{2\pi i \xi\tau} \langle (M_{R^*}\mathcal{F}M_{B^*}h)(\xi), x\rangle d\xi = \int_{\mathbb{R}} \langle (\mathcal{F}M_{B^*}h)(\xi), e^{-2\pi i\xi\tau}R(\xi)x\rangle d\xi \\
&= \langle \mathcal{F}M_{B^*}h, \mathcal{F}U_s^+(\cdot,\tau)x - M_R\mathcal{F}M_B U_s^+(\cdot,\tau)x\rangle_{L_2} \\
&= \langle M_{B^*}h, U_s^+(\cdot,\tau)x\rangle_{L_2} - \langle M_{R^*}\mathcal{F}M_{B^*}h, \mathcal{F}M_B U_s^+(\cdot,\tau)x\rangle_{L_2} \\
&= \langle h, M_B U_s^+(\cdot,\tau)x\rangle_{L_2} - \langle \mathcal{F}h, \mathcal{F}M_B U_s^+(\cdot,\tau)x\rangle_{L_2} = 0,
\end{aligned}$$

proving $x \in X_{\tau,*}^\perp$. The proof of (ii) is similar.

Our next result gives a splitting of the subspace $X_{\tau,*}^\perp$ of finite codimension that is crucial for the construction of unstable fibers.

Theorem 3.6 *If* G *is Fredholm, then* $X_{\tau,*}^\perp = X_s^+(\tau) \oplus Y_u(\tau)$ *for all* $\tau \geq 0$.

PROOF. First, we claim $X_s^+(\tau) \cap Y_u(\tau) = \{0\}$. Take $x \in X_s^+(\tau) \cap Y_u(\tau)$ and define $g : \mathbb{R} \to X$ by $g(t) = U_s^+(t,\tau)x$ for $t > \tau$ and $g(t) = V_u(t,\tau)x$ for $t \leq \tau$. From Proposition 2.7(iv,v) we obtain $\mathcal{F}g - M_R\mathcal{F}M_B g = 0$, which implies $g \in \ker G$. This yields $g(0) \in X_0$. From the definition of $Y_u(\tau)$, we have $g(0) = V_u(0,\tau)x \in X_0^\perp$, which implies $g(0) = 0$, and thus, $g = 0$ by Hypothesis 1.1. Hence, $x = 0$, proving the claim. By Proposition 3.5, to finish the proof the theorem, it suffices to show $X_{\tau,*}^\perp \subseteq X_s^+(\tau) + Y_u(\tau)$. Let $x \in X_{\tau,*}^\perp$ and consider the following equation for u:

$$(\mathcal{F}u - M_R\mathcal{F}M_B u)(\xi) = e^{-2\pi i\xi\tau}R(\xi)x \quad \text{a.e.} \quad \xi \in \mathbb{R}. \tag{3.1}$$

Since $(\mathcal{F}\mathcal{G}(\cdot,\tau)x)(\xi) = e^{-2\pi i\xi\tau}R(\xi)x$ for all $\xi \in \mathbb{R}$, equation (3.1) is equivalent to $\mathcal{F}(u - \mathcal{G}(\cdot,\tau)x) - M_R\mathcal{F}M_B(u - \mathcal{G}(\cdot,\tau)x) = M_R\mathcal{F}M_B\mathcal{G}(\cdot,\tau)x$, which can be written as $G(u - \mathcal{G}(\cdot,\tau)x) = M_B\mathcal{G}(\cdot,\tau)x$. Since G is Fredholm, in order to prove that equation (3.1) has a solution, it is enough to show $M_B\mathcal{G}(\cdot,\tau)x \in (\ker G^*)^\perp$. Indeed, if $h \in \ker G^*$, from Proposition 2.9(i) with $S(\xi) = R(\xi)^*$, it follows that

$$\begin{aligned}
\langle M_B\mathcal{G}(\cdot,\tau)x, h\rangle_{L_2} &= \langle \mathcal{G}(\cdot,\tau)x, M_{B^*}h\rangle_{L_2} = \langle \mathcal{F}\mathcal{G}(\cdot,\tau)x, \mathcal{F}M_{B^*}h\rangle_{L_2} \\
&= \int_{\mathbb{R}} \langle e^{-2\pi i\xi\tau}R(\xi)x, (\mathcal{F}M_{B^*}h)(\xi)\rangle d\xi \\
&= \int_{\mathbb{R}} e^{-2\pi i\xi\tau}\langle x, (M_{R^*}\mathcal{F}M_{B^*}h)(\xi)\rangle d\xi = \langle x, h(\tau)\rangle = 0.
\end{aligned}$$

Furthermore, from Proposition 2.10(iii), it follows that we can choose a solution u of (3.1) of the form $u = \mathcal{G}(\cdot,\tau)x + u(\cdot;\tau,x)$, where $u(\cdot;\tau,x) \in \operatorname{dom} G_0$. Let $x_1 = P_1 x + u(\tau;\tau,x)$ and $x_2 = P_2 x - u(\tau;\tau,x)$. From Proposition 2.2(ii) it follows that $u(\cdot;\tau,x) \in L_2(\mathbb{R},X) \cap C_0(\mathbb{R},X)$ which implies $u_{|[\tau,\infty)} \in L_2([\tau,\infty),X) \cap C_0([\tau,\infty),X)$. We infer that equation (3.1) is equivalent to $\mathcal{F}u - M_R\mathcal{F}M_B u = M_R \mathcal{F}\mu$, where $\mu = \delta_\tau \otimes x$. From Proposition 2.7(iv) it follows that $(\mathcal{F}u_{|[\tau,\infty)})(\xi) - (M_R\mathcal{F}M_B u_{|[\tau,\infty)})(\xi) = e^{-2\pi i\xi\tau}R(\xi)x_1$ for almost all $\xi \in \mathbb{R}$, which implies $u_{|[\tau,\infty)}$ is an $L_2 \cap C_0$-solution of equation (1.2) on $[\tau,\infty)$ with $u(\tau) = x_1$, proving $x_1 \in X_s^+(\tau)$. Define $v : (-\infty,\tau] \to X$ by $v(t) = -u(t)$ for $t < \tau$ and $v(\tau) = x_2$. Notice that $v \in L_2((-\infty,\tau],X) \cap C_0((-\infty,\tau],X)$. Moreover, from Proposition 2.7(v) and Remark 2.8 we have

$$
\begin{aligned}
(\mathcal{F}v)(\xi) - (M_R\mathcal{F}M_B v)(\xi) &= -R(\xi)\Big(\int_{(-\infty,\tau)} e^{-2\pi i\xi t}d\mu(t) - e^{-2\pi i\xi\tau}(-x_2)\Big)\\
&= -e^{-2\pi i\xi\tau}R(\xi)x_2 \quad \text{for all} \quad \xi \in \mathbb{R},
\end{aligned}
$$

proving that v is a solution of equation (1.2) on $(-\infty,\tau]$. By Definition 2.3, $v_1 = v_{|[0,\tau]}$ is a solution of equation (1.2) on $[0,\tau]$, and so, by (2.4), $(\mathcal{F}v_1)(\xi) - (M_R\mathcal{F}M_B v_1)(\xi) = R(\xi)(v(0) - e^{-2\pi i\xi\tau}x_2)$ for all $\xi \in \mathbb{R}$. Next, we will prove $v(0) \in X_{0,*}^\perp$. If $h \in \ker G^*$ then from Proposition 2.9(i) with $S(\xi) = R(\xi)^*$ we obtain:

$$
\begin{aligned}
\langle h(0), v(0)\rangle &= \int_{\mathbb{R}} \langle (M_{R^*}\mathcal{F}M_{B^*}h)(\xi), v(0)\rangle d\xi = \int_{\mathbb{R}} \langle (\mathcal{F}M_{B^*}h)(\xi), R(\xi)v(0)\rangle d\xi\\
&= \int_{\mathbb{R}} \langle (\mathcal{F}M_{B^*}h)(\xi), e^{-2\pi i\xi\tau}R(\xi)x_2\rangle d\xi\\
&\quad + \int_{\mathbb{R}} \langle (\mathcal{F}M_{B^*}h)(\xi), (\mathcal{F}v_1)(\xi) - (M_R\mathcal{F}M_B v_1)(\xi)\rangle d\xi \quad \text{(using (2.4))}\\
&= \int_{\mathbb{R}} e^{2\pi i\xi\tau}\langle (\mathcal{F}M_{B^*}h)(\xi), R(\xi)x_2\rangle d\xi + \langle \mathcal{F}M_{B^*}h, \mathcal{F}v_1\rangle_{L_2}\\
&\quad - \langle \mathcal{F}M_{B^*}h, M_R\mathcal{F}M_B v_1\rangle_{L_2}\\
&= \int_{\mathbb{R}} e^{2\pi i\xi\tau}\langle (M_{R^*}\mathcal{F}M_{B^*}h)(\xi), x_2\rangle d\xi + \langle M_{B^*}h, v_1\rangle_{L_2}\\
&\quad - \langle M_{R^*}\mathcal{F}M_{B^*}h, \mathcal{F}M_B v_1\rangle_{L_2}\\
&= \langle h(\tau), x_2\rangle + \langle h, M_B v_1\rangle_{L_2} - \langle \mathcal{F}h, \mathcal{F}M_B v_1\rangle_{L_2} \quad \text{(because } h \in \ker G^*)\\
&= \langle h(\tau), x_2\rangle = \langle h(\tau), x\rangle - \langle h(\tau), x_1\rangle = 0,
\end{aligned}
$$

since $x \in X_{\tau,*}^\perp$ and $x_1 \in X_s^+(\tau) \subseteq X_{\tau,*}^\perp$. Hence, $v(0) \in X_{0,*}^\perp$.

Case 1. Suppose $\tau = 0$. From Proposition 3.5(i) we have $X_0 \subseteq X_{0,*}^\perp$, and so $X_{0,*}^\perp = X_0 \oplus \Big(X_{0,*}^\perp \cap X_0^\perp\Big)$. Let $z \in \ker G$ be such that $v(0) - z(0) \in X_{0,*}^\perp \cap X_0^\perp$. Notice that $v - \chi_{(-\infty,0]}z$ is a $L_2 \cap C_0$-solution of equation (1.2) on $(-\infty,0]$. Hence, $x_2 - z(0) = v(0) - z(0) \in Y_u(0)$. From Proposition 3.5(i) we have $z(0) \in X_0 \subseteq X_s^+(0)$, which yields $x_1 + z(0) \in X_s^+(0)$. Since $x = x_1 + x_2 = (x_1 + z(0)) + (x_2 - z(0))$, we have $X_{0,*}^\perp = X_s^+(0) \oplus Y_u(0)$.

Case 2. Suppose $\tau > 0$. By Case 1 there exist $y_1 \in X_s^+(0)$ and $y_2 \in Y_u(0)$ such that $v(0) = y_1 + y_2$. Let $v_2 : (-\infty,\tau] \to X$ be defined by $v_2(t) = V_u(t,\tau)y_2$ for $t < 0$ and $v_2(t) = v(t) - U_s^+(t,0)y_1$ for $t \in [0,\tau]$; it is an $L_2 \cap C_0$-solution of equation (1.2) on $(-\infty,\tau]$. Moreover, $v_2(\tau) = v(\tau) - U_s^+(\tau,0)y_1 = x_2 - U_s^+(\tau,0)y_1$ and $v_2(0) = y_2 \in Y_u(0)$.

Hence, $x_2 - U_s^+(\tau,0)y_1 \in Y_u(\tau)$ and $x_1 + U_s^+(\tau,0)y_1 \in X_s^+(\tau)$, and thus $x = x_1 + x_2 = (x_1 + U_s^+(\tau,0)y_1) + (x_2 - U_s^+(\tau,0)y_1)$.

For $x \in X_{0,*}$ and $\tau \geq 0$ we set $y := y(\tau) = (T_*(\tau,0)^*)^{-1}x$, where T_* is defined in (2.14). To construct the unstable fibers $X_u^+(\tau)$, we prove first the following fact.

Proposition 3.7 *For each $x \in X_{0,*}, \tau > 0$ there exits a unique $u \in L_2(\mathbb{R}, X)$ such that:*
(i) $(\mathcal{F}u)(\xi) - (M_R\mathcal{F}M_Bu)(\xi) = R(\xi)(x - e^{-2\pi i\xi\tau}y)$ for all $\xi \in \mathbb{R}$;
(ii) u is continuous on $\mathbb{R} \setminus \{0, \tau\}$;
(iii) there exist one-sided limits $u(0\pm0)$ and $u(\tau\pm0)$ so that: $u(0+0) = u(0)$, $u(\tau-0) = u(\tau)$, $u(0) - u(0-0) = x$, $u(\tau+0) - u(\tau) = -y$, $u(0-0) \in Y_u(0)$ and $u(\tau+0) \in X_s^+(\tau)$.

PROOF. *Existence.* Fix $x \in X_{0,*}$ and $\tau \geq 0$. For all $h \in \ker G^*$ from the definition of $T_*(\tau,0)$ in (2.14) it follows that $\langle x, h(0)\rangle = \langle y, h(\tau)\rangle$; also, Proposition 2.9(i) yields:

$$\begin{aligned}
&\langle M_B(\mathcal{G}(\cdot,0)x - \mathcal{G}(\cdot,\tau)y), h\rangle = \langle \mathcal{G}(\cdot,0)x - \mathcal{G}(\cdot,\tau)y, M_{B^*}h\rangle \\
&\quad = \langle \mathcal{F}(\mathcal{G}(\cdot,0)x - \mathcal{G}(\cdot,\tau)y), \mathcal{F}M_{B^*}h\rangle = \int_{\mathbb{R}} \langle R(\xi)(x - e^{-2\pi i\xi\tau}y), (\mathcal{F}M_{B^*}h)(\xi)\rangle d\xi \\
&\quad = \int_{\mathbb{R}} \langle x - e^{-2\pi i\xi\tau}y, (M_{R^*}\mathcal{F}M_{B^*}h)(\xi)\rangle d\xi = \langle x, h(0)\rangle - \langle y, h(\tau)\rangle = 0.
\end{aligned}$$

Since G is Fredholm, by the orthogonality to $\ker G^*$ we have $M_B(\mathcal{G}(\cdot,0)x - \mathcal{G}(\cdot,\tau)y) \in \operatorname{im} G = \operatorname{im} G_0$. Let $v = \mathcal{G}(\cdot,0)x - \mathcal{G}(\cdot,\tau)y + G_0^{-1}M_B(\mathcal{G}(\cdot,0)x - \mathcal{G}(\cdot,\tau)y)$. Then, $v \in L_2(\mathbb{R}, X)$, v is continuous on $\mathbb{R} \setminus \{0,\tau\}$ and, moreover, the limits $v(0 \pm 0)$ and $v(\tau \pm 0)$ do exist. Also, notice that $v(0+0) = v(0)$. Changing the value of v at τ, we can assume without loss of generality $v(\tau-0) = v(\tau)$. Further, from the definition of v, we have $v(0) - v(0-0) = x$, $v(\tau+0) - v(\tau) = -y$ and $(\mathcal{F}v)(\xi) - (M_R\mathcal{F}M_Bv)(\xi) = R(\xi)(x - e^{-2\pi i\xi\tau}y)$ for all $\xi \in \mathbb{R}$. From Proposition 2.7(v) we obtain $\mathcal{F}(\chi_{(-\infty,0]}v)(\xi) - M_R\mathcal{F}M_B(\chi_{(-\infty,0]}v)(\xi) = -R(\xi)v(0-0)$ for all $\xi \in \mathbb{R}$. Using the same argument as in the proof of Proposition 3.5(i), one can conclude $v(0-0) \in X_{0,*}^{\perp}$. From Theorem 3.6 it follows that there exist $x_1 \in X_s^+(0)$ and $x_2 \in Y_u(0)$ such that $v(0-0) = x_1 + x_2$. Define $g : \mathbb{R} \to X$ by $g(t) = U_s^+(t,0)x_1$ for $t \geq 0$ and $g(t) = v(t) - V_u(t,0)x_2$ for $t < 0$. A simple computation shows $g \in \ker G$, and thus, $x_1 \in X_0$. Then, $u = v - g$ satisfies the same equation as v, there exist one-sided limits, $u(0\pm0)$ and $u(\tau\pm0)$ so that, $u(0+0) = u(0)$, $u(\tau-0) = u(\tau)$, $u(0) - u(0-0) = x$ and $u(\tau+0) - u(\tau) = -y$ and, moreover, $u(0-0) \in Y_u(0)$. Proposition 2.7(iv) yields $u(\tau+0) \in X_s^+(\tau)$, proving the existence of u satisfying (i)-(iii).

Uniqueness. Assume that two functions u_1 and u_2 satisfy conditions (i)-(iii) and let $u_0 = u_1 - u_2$. Then $u_0 \in \ker G \subseteq C_0(\mathbb{R}, X)$ and so $u_0(0) = (x + u_1(0-0)) - (x + u_2(0-0)) = u_1(0-0) - u_2(0-0) \in Y_u(0)$. It follows that $u_0(0) \in X_0 \cap Y_u(0) = \{0\}$, and so, from Hypothesis 1.1, we obtain $u_0 = 0$, proving the uniqueness.

Let $h_1, \dots, h_{d^*} \in \ker G^*$ such that $\langle h_i(0), h_j(0)\rangle = \delta_{ij}$ for all $i, j = 1, \dots, d^*$, where δ_{ij} is the Kronecker Delta and $d^* = \dim\ker G^*$. Let $u_1(\cdot;\tau), \dots, u_{d^*}(\cdot;\tau) \in L_2(\mathbb{R}, X)$ be the functions satisfying properties (i)-(iii) in Proposition 3.7 for $x = h_i(0) \in X_{0,*}$ and $\tau > 0$. Also, we define $u_i(\cdot;0) := \chi_{\{0\}}h_i(0)$, $i = 1, \dots, d^*$. From proposition 2.7(i) it follows that $u_i(\cdot;\tau)_{|[0,\tau]}$ is a solution of equation (1.2) on $[0,\tau]$ for $i = 1, \dots, d^*$. Since

$u_i(0;\tau)-h_i(0)=u_i(0-0;\tau)\in Y_u(0)\subseteq X_{0,*}^{\perp}$, we have $\langle u_i(0;\tau),h_j(0)\rangle=\langle h_i(0),h_j(0)\rangle=\delta_{ij}$ for all $i,j=1,\dots,d^*$. From Proposition 2.9(ii) it follows that

$$\langle u_i(t;\tau),h_j(t)\rangle=\delta_{ij}\quad\text{for all}\quad i,j=1,\dots,d^*,\ t\in[0,\tau]. \tag{3.2}$$

Note that vectors $u_i(t,\tau)$, $i=1,\dots,d^*$, $t\in[0,\tau]$, are linearly independent and denote their span by $H(t,\tau)$. We now define the unstable fibers by the formula

$$X_u^+(\tau)=Y_u(\tau)\oplus H(\tau,\tau),\ \tau\geq 0. \tag{3.3}$$

Let $P_+(t)$ be the projection onto $X_s^+(t)$ parallel to $X_u^+(t)$ and $Q_+(t)=I-P_+(t)$, $t\geq 0$.
Remark 3.8 Equation (3.2), Proposition 3.4(ii) and Theorem 3.6 yield: (i) $X=X_{\tau,*}^{\perp}\oplus H(\tau,\tau)$ for all $\tau\geq 0$; (ii) The projection $P_*(\tau)$ onto $X_{\tau,*}^{\perp}$ parallel to $H(\tau,\tau)$, is given by $P_*(\tau)x=x-\sum_{i=1}^{d^*}\langle x,h_i(\tau)\rangle u_i(\tau;\tau)$; (iii) The subspace $X_u^+(\tau)$ is a closed subspace of X for all $\tau\geq 0$; (iv) $X=X_s^+(\tau)\oplus X_u^+(\tau)$ for all $\tau\geq 0$. ◇

We define the linear operator $W_u(t,\tau):H(\tau,\tau)\to H(t,\tau)$ by $W_u(t,\tau)u_i(\tau;\tau)=u_i(t;\tau)$, $i=1,\dots,d^*$, and recall the definition of $V_u(t,\tau)$ in Remark 3.3.

Proposition 3.9 *(i) $X_u^+(t)=Y_u(t)\oplus H(t,\tau)$ for all $\tau\geq t\geq 0$;*

(ii) If $U_u^+(t,\tau):X_u^+(\tau)\to X_u^+(t)$ is defined by $U_u^+(t,\tau)=V_u(t,\tau)\oplus W_u(t,\tau)$ in the direct sum decomposition in (i), then $U_u^+(t,s)U_u^+(s,\tau)=U_u^+(t,\tau)$ for all $\tau\geq s\geq t\geq 0$;

(iii) $U_u^+(\tau,\tau)x=x$ for all $\tau\geq 0$ and $x\in X_u^+(\tau)$.

PROOF. (i) From (3.2) we have $Y_u(s)\cap H(s,\tau)=\{0\}$ for $\tau\geq s\geq 0$. Moreover, by Remark 3.8(i), it is enough to prove $u_i(s;\tau)-u_i(s;s)\in Y_u(s)$ for all $i=1,\dots,d^*$. Let $\tilde{u}_i(\cdot;s,\tau)=u_i(\cdot;\tau)-u_i(\cdot;s)$ for $i=1,\dots,d^*$. From Propositions 2.7(v) and 3.7, we obtain that $\tilde{u}_i(\cdot;s,\tau)_{|(-\infty,s]}$ is an $L_2\cap C_0$-solution of equation (1.2) on $(-\infty,s]$ with $\tilde{u}_i(0;s,\tau)=u_i(0;\tau)-u_i(0;s)=u_i(0-0;\tau)-u_i(0-0;s)\in Y_u(0)$, proving (i).

(ii) Since $\tilde{u}_i(\cdot;s,\tau)$ is an $L_2\cap C_0$-solution of equation (1.2) on $(-\infty,s]$, from the definition of $V_u(t,s)$ we conclude $V_u(t,s)\tilde{u}_i(s;s,\tau)=\tilde{u}_i(t;s,\tau)$ for all $i=1,\dots,d^*$. Fix $x\in X_u^+(\tau)$. Using the definition of $X_u^+(\tau)$ in (3.3), we find $y\in Y_u(\tau)$ and $a_1,\dots,a_{d^*}\in\mathbb{C}$ such that $x=y+\sum_{i=1}^{d^*}a_iu_i(\tau;\tau)$, and so

$$\begin{aligned}U_u^+(s,\tau)x&=V_u(s,\tau)y+\sum_{i=1}^{d^*}a_iW_u(s,\tau)u_i(\tau;\tau)=V_u(s,\tau)y+\sum_{i=1}^{d^*}a_iu_i(s;\tau)\\&=[V_u(s,\tau)y+\sum_{i=1}^{d^*}a_i\tilde{u}_i(s;s,\tau)]+[\sum_{i=1}^{d^*}a_iu_i(s;s)].\end{aligned}$$

Note that the first expression in $[\cdot]$ belongs to $Y_u(s)$ while the second belongs to $H(s,s)$. From this representation and from Remark 3.3(ii), we obtain:

$$\begin{aligned}U_u^+(t,s)U_u^+(s,\tau)x&=V_u(t,s)\Big(V_u(s,\tau)y+\sum_{i=1}^{d^*}a_i\tilde{u}_i(s;s,\tau)\Big)+\sum_{i=1}^{d^*}a_iW_u(t,s)u_i(s;s)\\&=V_u(t,s)V_u(s,\tau)y+\sum_{i=1}^{d^*}a_iV_u(t,s)\tilde{u}_i(s;s,\tau)+\sum_{i=1}^{d^*}a_iu_i(t;s)\\&=V_u(t,\tau)y+\sum_{i=1}^{d^*}a_i\tilde{u}_i(t;s,\tau)+\sum_{i=1}^{d^*}a_iu_i(t;s)\\&=V_u(t,\tau)y+\sum_{i=1}^{d^*}a_iu_i(t;\tau)=U_u^+(t,\tau)x.\end{aligned}$$

(iii) This follows from the definition of W_u and Remark 3.3(iv).

Next, we will prove the exponential estimate for U_u^+. The proof given below is similar to the proof of [22, Thm. 5.5]. We recall that if $X = Z_1 \oplus Z_2$ is a direct sum decomposition, then we can identify $Z_1^* = Z_2^\perp$.

Proposition 3.10 *There exist positive N, ν so that $\|U_u^+(t,\tau)\| \le Ne^{\nu(t-\tau)}$, $\tau \ge t \ge 0$.*

PROOF. *Claim 1.* $\|T_*(\tau,t)\| \le Ne^{-\nu(\tau-t)}$ for all $\tau \ge t \ge 0$.
From Proposition 3.5(i) we have $g(t) \in X_t \subseteq X_s^+(t)$ and $U_u^+(\tau,t)g(t) = g(\tau)$ for all $g \in \ker G$ and $\tau \ge t \ge 0$ and thus, from Proposition 3.2(i), it follows that $\|g(\tau)\| \le Ne^{-\nu(\tau-t)}\|g(t)\|$ for all $\tau \ge t \ge 0$. By Remark 2.11, we can apply this argument for G^* instead of G, and thus obtain $N, \nu > 0$ such that $\|h(\tau)\| \le Ne^{-\nu(\tau-t)}\|h(t)\|$ for all $\tau \ge t \ge 0$ and $h \in \ker G^*$, proving the claim.

Claim 2. $\|U_u^+(t,\tau)\| \le Ne^{\nu(t-\tau)}$ for all $\tau \ge t \ge 0$.
Let $\tau \ge 0$ and $t \in [0,\tau]$. From Proposition 3.9(i) we have $X_u^+(t) = Y_u(t) \oplus H(t,\tau)$ and $X_u^+(\tau) = Y_u(\tau) \oplus H(\tau,\tau)$, which yields $X_{t,*} \subseteq H(t,\tau)^*$ and $X_{\tau,*} \subseteq H(\tau,\tau)^*$. Since $X = X_s^+(t) \oplus X_u^+(t) = X_{t,*}^\perp \oplus H(t,\tau)$ (from Remark 3.8 and Proposition 3.9(i)) and $X = X_{\tau,*}^\perp \oplus H(\tau,\tau)$ (from Remark 3.8), we have $\dim H(t,\tau)^* = \dim X_{t,*} = \dim H(\tau,\tau)^* = \dim X_{\tau,*} = d^*$, and so $H(t,\tau)^* = X_{t,*}$ and $H(\tau,\tau)^* = X_{\tau,*}$. From the definition of $W_u(t,\tau)$ and Proposition 2.9(ii) we have $\langle W_u(t,\tau)x, h(t)\rangle = \langle x, h(\tau)\rangle$ for all $x \in H(\tau,\tau)$ and all $h \in \ker G^*$, which implies $W_u(t,\tau)^* = T_*(\tau,t)$. By Claim 1, we have $\|W_u(t,\tau)\| = \|T_*(\tau,t)\| \le Ne^{\nu(t-\tau)}$. Since $U_u^+(t,\tau) = V_u(t,\tau) \oplus W_u(t,\tau)$ as an operator from $Y_u(\tau) \oplus H(\tau,\tau)$ to $Y_u(t) \oplus H(t,\tau)$, using Proposition 3.4(i), we infer $\|U_u^+(t,\tau)\| \le Ne^{\nu(t-\tau)}$.

Proposition 3.11 *$\mathcal{U}_+ = (U_s^+, U_u^+)$ is a bi-family associated with equation* (1.2).

PROOF. From Proposition 3.2 and Remark 3.8(iii) it follows that $P_+(t)$ is a bounded projection. From Remark 3.1 and Proposition 3.2, 3.9, 3.10 we have that $\mathcal{U}_+ = (U_s^+, U_u^+)$ is a bi-family. From the definition of $U_s^+(t,\tau)$ for $t \ge \tau \ge 0$ and the definition of $U_u^+(t,\tau)$ for $t \le \tau \le 0$, we have that $U_s^+(\cdot,\tau)x$ is an $L_2 \cap C_0$-solution of equation (1.2) on $[\tau,\infty)$ for all $\tau \ge 0$ and all $x \in X_s^+(\tau)$. Similarly, $U_u(\cdot,\tau)x$ is a solution of equation (1.2) on $[0,\tau]$ for all $\tau \ge 0$ and $x \in X_u(\tau)$.

Let u be a solution of equation (1.2) on $[a,b]$. Let $v : [0,b] \to X$ be defined by $v(t) = U_u^+(t,a)Q_+(a)u(a)$ for $t \in [0,a]$ and $v(t) = u(t) - U_s^+(t,a)P_+(a)u(a)$. From Definition 2.3 it follows that v is a solution of equation (1.2) on $[0,b]$. Let $w : \mathbb{R}_+ \to X$ be defined by $w(t) = v(t) - U_u^+(t,b)Q_+(b)v(b)$ for $t \in [0,b]$ and $w(t) = U_s^+(t,b)P_+(b)v(b)$ for $t > b$. Then w is an $L_2 \cap C_0$-solution of equation (1.2) on $\mathbb{R}_+$. From the definition of $X_s^+(0)$ it follows that $w(0) \in X_s^+(0)$ and $w(t) = U_s^+(t,0)w(0)$. Moreover, one has $w(0) = U_u^+(0,a)Q_+(a)u(a) - U_u^+(0,b)Q_+(b)v(b) \in X_u^+(0)$, which implies $w = 0$. Since $Q_+(b)u(b) = Q_+(b)v(b)$, we obtain $v(t) = U_u^+(t,b)Q_+(b)u(b)$ for all $t \in [a,b]$, and thus, $u(t) = U_s^+(t,a)P_+(a)u(a) + U_u^+(t,b)Q_+(b)u(b)$ for all $t \in [a,b]$.

Theorem 3.12 *If G is Fredholm, then* (1.2) *has an exponential dichotomy on $\mathbb{R}_+$.*

PROOF. Taking into account Proposition 3.11 and the fact that the estimates (2.7) follow from Propositions 3.2(ii) and 3.10, to complete the proof of the theorem it suffices to prove that the projection valued function $P_+(\cdot)$ is strongly continuous and bounded on $\mathbb{R}_+$. Recall the definition of $P_*(\cdot)$ in Remark 3.8.

Step 1. We prove that $P_*(\cdot)$ is strongly continuous on $\mathbb{R}_+$. Let $\tau \geq 0$, $y_i(\tau) = (T_*(\tau,0)^*)^{-1}h_i(0)$, $v_i(\cdot;\tau) = G_0^{-1}(M_B\mathcal{G}(\cdot,0)h_i(0) - M_B\mathcal{G}(\cdot,\tau)y_i(\tau))$ and let $\tilde{v}_i(\cdot;\tau) : \mathbb{R} \to X$ be defined by $\tilde{v}_i(t;\tau) = v_i(t;\tau) + \mathcal{G}(t,0)h_i(0) - \mathcal{G}(t,\tau)y_i(\tau)$ for $t \neq \tau$ and $\tilde{v}_i(\tau;\tau) = v_i(\tau;\tau) + T_1(\tau)P_1h_i(0) + P_2y_i(\tau)$. From the definition of the functions $u_i(\cdot;\tau)$ we have $f_i(\cdot;\tau) = u_i(\cdot;\tau) - \tilde{v}_i(\cdot;\tau) \in \ker G \subseteq C_0(\mathbb{R},X)$. Using the fact that $f_i(\cdot;\tau)$ is continuous on $\mathbb{R}$, we obtain $f_i(0;\tau) = u_i(0-0;\tau) - \tilde{v}_i(0-0;\tau)$. Recall that $P_0^\perp$ is the orthogonal projection onto X_0. Since $f_i(0;\tau) \in X_0$ and $u_i(0-0;\tau) \in Y_u(0) \subseteq X_0^\perp$, it follows that

$$f_i(0;\tau) = -P_0^\perp\tilde{v}_i(0-0;\tau) = -P_0^\perp(v_i(0;\tau) - P_2h_i(0) + T_2(\tau)P_2y_i(\tau)). \tag{3.4}$$

Let $(t_n)_{n\geq 1} \subseteq \mathbb{R}$ and assume $t_n \to \tau$ as $n \to \infty$. We will prove $v_i(\cdot;t_n) \to v_i(\cdot;\tau)$ as $n \to \infty$, uniformly on $\mathbb{R}$. From Proposition 2.10(iv) we have, for all $n \geq 1$:

$$\begin{aligned}\|v_i(\cdot;t_n) - v_i(\cdot;\tau)\|_\infty &\leq c\|G_0(v_i(\cdot;t_n) - v_i(\cdot;\tau))\|_2 \\ &= \|M_B(\mathcal{G}(\cdot,\tau)y_i(\tau) - \mathcal{G}(\cdot,t_n)y_i(t_n))\|_2 \leq c\|\mathcal{G}(\cdot,\tau)y_i(\tau) - \mathcal{G}(\cdot,t_n)y_i(t_n)\|_2\,.\end{aligned}$$

From Proposition 2.12 it follows that y_i is continuous on $\mathbb{R}$, and thus, locally bounded. The definition of the function $\mathcal{G}$ yields $\mathcal{G}(t,t_n)y_i(t_n) \to \mathcal{G}(t,\tau)y_i(\tau)$ as $n \to \infty$ for almost all $t \in \mathbb{R}$ and that there exist constants $C, \alpha > 0$ such that $\|\mathcal{G}(t,t_n)y_i(t_n)\| \leq Ce^{-\alpha|t|}$ for all $n \geq 1$ and all $t \in \mathbb{R}$. From Lebesgue's dominated convergence theorem, it follows that $\|\mathcal{G}(\cdot,t_n)y_i(t_n) - \mathcal{G}(\cdot,\tau)y_i(\tau)\|_2 \to 0$ as $n \to \infty$, which proves $v_i(\cdot;t_n) \to v_i(\cdot;\tau)$ as $n \to \infty$, uniformly on $\mathbb{R}$. Since $v_i(\cdot;\tau) \in \operatorname{dom} G_0 \subseteq C_0(\mathbb{R},X)$ we obtain

$$v_i(t_n;t_n) \to v_i(\tau;\tau) \quad \text{as} \quad n \to \infty. \tag{3.5}$$

From (3.4), the continuity of y_i, the strong continuity of the semigroup $\{T_2(t)\}_{t\geq 0}$ and the uniform convergence of the sequence $(v_i(\cdot;t_n))_{n\geq 1}$ on $\mathbb{R}$, we have $f_i(0;t_n) \to f_i(0;\tau)$ as $n \to \infty$. Moreover, Proposition 2.12 yields

$$f_i(t_n;t_n) = T(t_n,0)f_i(0;t_n) \to T(\tau,0)f_i(0;\tau) = f_i(\tau;\tau) \quad \text{as} \quad n \to \infty. \tag{3.6}$$

From (3.5), (3.6) and since $u_i(t_n;t_n) = \tilde{v}_i(t_n;t_n) + f_i(t_n;t_n) = v_i(t_n;t_n) + f_i(t_n;t_n) + T_1(t_n)P_1h_i(0) + P_2y_i(t_n)$ for all $n \geq 1$, we obtain that $u_i(t_n;t_n) \to u_i(\tau;\tau)$ as $n \to \infty$. From Proposition 3.8(ii) it follows that $P_*(t_n)x \to P_*(\tau)x$ as $n \to \infty$ for each $x \in X$, which proves that $P_*(\cdot)$ is strongly continuous on $\mathbb{R}_+$.

Step 2. We prove that $P_+(\cdot)$ is strongly continuous on $\mathbb{R}_+$. Notice that $P_+(t)$ and $P_*(t) - P_+(t)$ are the projections onto $X_s^+(t)$ and $Y_u(t)$ respectively, associated to the splitting $X_{t,*}^\perp = X_s^+(t) \oplus Y_u(t)$. Define $u : \mathbb{R} \to X$ by $u(t) = U_s^+(t,\tau)P_+(\tau)x$ for $t \geq \tau$ and $u(t) = -V_u(t,\tau)(P_*(\tau)x - P_+(\tau)x)$ for $t < \tau$. From the definition of $X_s^+(\tau)$, $Y_u(\tau)$ and Proposition 2.7 we have $(\mathcal{F} - M_R\mathcal{F}M_B)u = \mathcal{F}\mathcal{G}(\cdot,\tau)P_*(\tau)x$, which implies $G(u - \mathcal{G}(\cdot,\tau)P_*(\tau)x) = M_B\mathcal{G}(\cdot,\tau)P_*(\tau)x$. It follows that $u = \mathcal{G}(\cdot,\tau)P_*(\tau)x + w(\cdot;\tau,x) + \bar{z}(\cdot;\tau,x)$ where $w(\cdot;\tau,x) = G_0^{-1}(M_B\mathcal{G}(\cdot,\tau)P_*(\tau)x)$ and $\bar{z}(\cdot;\tau,x) \in \ker G$ such that $\bar{z}(0;\tau,x) = P_0^\perp(u(0) - \mathcal{G}(0,\tau)P_*(\tau)x)$. Passing to the limit as $t \to \tau + 0$,

$$P_+(\tau)x = P_1P_*(\tau)x + w(\tau;\tau,x) + \bar{z}(\tau;\tau,x). \tag{3.7}$$

Let $(t_n)_{n\geq 1} \subseteq \mathbb{R}$ and assume $t_n \to \tau$ as $n \to \infty$. By Proposition 2.10(iii),

$$\begin{aligned}\|w(\cdot;t_n,x) - w(\cdot;\tau,x)\|_\infty &\leq c\|G_0(w(\cdot;t_n,x) - w(\cdot;\tau,x))\|_2 \\ &\leq c\|M_B(\mathcal{G}(\cdot,t_n)P_*(t_n)x - \mathcal{G}(\cdot,\tau)P_*(\tau)x)\|_2 \\ &\leq c\|\mathcal{G}(\cdot,t_n)P_*(t_n)x - \mathcal{G}(\cdot,\tau)P_*(\tau)x\|_2.\end{aligned}$$

From the definition of $\mathcal{G}$ and since $P_*(\cdot)$ is strongly continuous on $\mathbb{R}_+$, and hence, locally bounded on $\mathbb{R}_+$, we have $\mathcal{G}(\cdot,t_n)P_*(t_n)x \to \mathcal{G}(\cdot,\tau)P_*(\tau)x$ as $n \to \infty$ almost everywhere and there exist constants $C, \alpha > 0$ such that $\|\mathcal{G}(t, t_n)P_*(t_n)x\| \le Ce^{-\alpha|t|}$ for all $n \ge 1$ and $t \in \mathbb{R}$. From Lebesgue's dominated convergence theorem it follows that $w(\cdot; t_n, x) \to w(\cdot; \tau, x)$ as $n \to \infty$, uniformly on $\mathbb{R}$. Since $w(\cdot; \tau, x) \in \operatorname{dom} G$ is continuous, we obtain

$$w(t_n; t_n, x) \to w(\tau; \tau, x) \quad \text{as} \quad n \to \infty. \tag{3.8}$$

Since $w(\cdot; \tau, x), \bar{z}(\cdot; \tau, x) \in \operatorname{dom}(G) \subseteq C_0(\mathbb{R}, X)$, we have $\bar{z}(\tau; \tau, x) = T(\tau, 0)z(0; \tau, x) = T(\tau, 0)P_0^{\perp}(u(0) - \mathcal{G}(0, \tau)P_*(\tau)x) = T(\tau, 0)P_0^{\perp}(u(0-0) - \mathcal{G}(0-0, \tau)P_*(\tau)x)$. Moreover, since $u(0-0) \in Y_u(0) \subseteq X_0^{\perp}$, we infer $\bar{z}(\tau; \tau, x) = -T(\tau, 0)P_0^{\perp}\mathcal{G}(0-0, \tau)P_*(\tau)x = T(\tau, 0)P_0^{\perp}T_2(\tau)P_2P_*(\tau)x$. From Step 1, Proposition 2.12 and since $\{T_2(t)\}_{t \ge 0}$ is a C_0-semigroup we obtain

$$\bar{z}(t_n; t_n, x) \to \bar{z}(\tau; \tau, x) \quad \text{as} \quad n \to \infty. \tag{3.9}$$

From Step. 1, (3.7), (3.8) and (3.9) it follows that $P_+(t_n)x \to P_+(\tau)x$ as $n \to \infty$.

Step 3. We prove that $P_+(\cdot)$ is bounded on $\mathbb{R}_+$. Let $\tau \ge 0$ and $x \in X$. Define $u : \mathbb{R}_+ \to X$ by $u(t) = U_s^+(t, \tau)P_+(\tau)x$ for $t > \tau$ and $u(t) = -U_u^+(t, \tau)Q_+(\tau)x$ for $t \in [0, \tau]$. From the definitions of U_s^+ and U_u^+ and Proposition 2.7(iv),(v) we have $(\mathcal{F}u)(\xi) - (M_R\mathcal{F}M_Bu)(\xi) = R(\xi)(e^{-2\pi i\xi\tau}x - y)$ for all $\xi \in \mathbb{R}$, where $y = U_u^+(0, \tau)Q_+(\tau)x$. From the definition of $\mathcal{G}$ we have

$$(\mathcal{F} - M_R\mathcal{F}M_B)(u - \mathcal{G}(\cdot, \tau)x + \mathcal{G}(\cdot, 0)y) = M_R\mathcal{F}M_B(\mathcal{G}(\cdot, \tau)x - \mathcal{G}(\cdot, 0)y),$$

which is equivalent to $G(u - \mathcal{G}(\cdot, \tau)x + \mathcal{G}(\cdot, 0)y) = M_B(\mathcal{G}(\cdot, \tau)x - \mathcal{G}(\cdot, 0)y)$. Let $h \in \ker G$ be such that $h(0) = P_0^{\perp}(u(0) - \mathcal{G}(0, \tau)x + \mathcal{G}(0, 0)y)$. Then $u - \mathcal{G}(\cdot, \tau)x + \mathcal{G}(\cdot, 0)y - h \in \operatorname{dom} G_0$ and $G_0(u - \mathcal{G}(\cdot, \tau)x + \mathcal{G}(\cdot, 0)y - h) = M_B(\mathcal{G}(\cdot, \tau)x - \mathcal{G}(\cdot, 0)y)$. Hence, $u = \mathcal{G}(\cdot, \tau)x - \mathcal{G}(\cdot, 0)y + h + G_0^{-1}(M_B(\mathcal{G}(\cdot, \tau)x - \mathcal{G}(\cdot, 0)y))$. Since $h(0) = P_0^{\perp}(u(0) - \mathcal{G}(0, \tau)x + \mathcal{G}(0, 0)y)$ from (2.2) and Proposition 3.2(i) we have $\|h\|_\infty \le c\|h(0)\| \le c\|u(0)\| + c\|x\| + c\|y\| = c\|x\| + c\|y\|$. Thus, since $P_+(\tau)x = u(\tau + 0)$, from (2.2) and Propositions 2.10(iv) and 3.10 we obtain the following estimate:

$$\begin{aligned}
\|P_+(\tau)x\| &\le \|\mathcal{G}(\cdot, \tau)x\|_\infty + \|\mathcal{G}(\cdot, 0)y\|_\infty + \|h\|_\infty + \|G_0^{-1}(M_B(\mathcal{G}(\cdot, \tau)x - \mathcal{G}(\cdot, 0)y))\|_\infty \\
&\le c(\|x\| + \|y\| + \|M_B(\mathcal{G}(\cdot, \tau)x - \mathcal{G}(\cdot, 0)y)\|_2) \le c(\|x\| + \|y\| + \|\mathcal{G}(\cdot, \tau)x\|_2 + \|\mathcal{G}(\cdot, 0)y\|_2) \\
&\le c(\|x\| + \|y\|) \le c(\|x\| + Ne^{-\nu\tau}\|Q_+(\tau)x\|) \le c(\|x\| + Ne^{-\nu\tau}(\|x\| + \|P_+(\tau)x\|)).
\end{aligned}$$

Let $a > 0$ be such that $Nce^{-\nu\tau} \le 1/2$ for all $\tau \ge a$. From the above estimate for all $\tau \ge a$, we have $\|P_+(\tau)x\| \le c\|x\| + 1/2\|x\| + 1/2\|P_+(\tau)x\|$, yielding $\|P_+(\tau)\| \le c$. By Step 2, $P_+(\cdot)$ is bounded on $[0, a]$, and therefore on $\mathbb{R}_+$.

4. The dichotomy on the negative semiline

Throughout this section we will assume that G is Fredholm. Let $A_\sharp = -A$ and let $B_\sharp, R_\sharp : \mathbb{R} \to \mathcal{B}(X)$ be defined by $B_\sharp(t) := -B(-t)$ and $R_\sharp(\xi) := R(2\pi i\xi, A_\sharp) = -R(-\xi)$. Since A is the generator of a bi-semigroup and B is bounded and piecewise strongly continuous, we have that $A_\sharp$ is the generator of a bi-semigroup and $B_\sharp$ is bounded and piecewise strongly continuous. Moreover, $R_\sharp(\cdot)x, R_\sharp(\cdot)^*x \in L_2(\mathbb{R}, X) \cap C_0(\mathbb{R}, X)$ for all $x \in X$ and thus, by the Closed Graph Theorem, we have $\|R_\sharp(\cdot)x\|_2 \le c\|x\|$ and $\|R_\sharp(\cdot)^*x\|_2 \le c\|x\|$ for all $x \in X$. Consider the equation

$$x'(t) = D_\sharp(t)x(t), t \in \mathbb{R}, \quad \text{where} \quad D_\sharp(t) = A_\sharp + B_\sharp(t). \tag{4.1}$$

We will define an operator $G_\sharp$ associated to equation (4.1), as follows. Let $\operatorname{dom}(G_\sharp)$ be the set of all $u \in L_2(\mathbb{R}, X)$ such that there exists $f \in L_2(\mathbb{R}, X)$ for which the relation

$\mathcal{F}u - M_{R_\sharp}\mathcal{F}M_{B_\sharp}u = M_{R_\sharp}\mathcal{F}f$, holds and define $G_\sharp u = f$. We define the solutions of equation (4.1) on, say, $[a,b]$ or $(-\infty,a]$ similarly to Definition 2.3, by replacing A by $A_\sharp$, B by $B_\sharp$ and R by $R_\sharp$. Finally, let $\Lambda \in \mathcal{B}(L_2(\mathbb{R},X))$ be the reflection operator defined by $(\Lambda f)(t) = f(-t)$, $t \in \mathbb{R}$.

Proposition 4.1 *Let $b > a \geq 0$, $u \in L_2(\mathbb{R},X)$ and $v = \Lambda u = u(-\cdot)$. Then*

(i) $G_\sharp = -\Lambda G \Lambda$;

(ii) u is a solution of equation (1.2) *on $[-b,-a]$ if and only if v is a solution of equation* (4.1) *on $[a,b]$;*

(iii) u is an $L_2 \cap C_0$-solution of equation (1.2) *on $(-\infty,-a]$ if and only if v is an $L_2 \cap C_0$-solution of equation* (4.1) *on $[a,\infty)$.*

PROOF. (i) Let $u \in \operatorname{dom}(G)$, $f = Gu$ and $g = -\Lambda f$. From the definition of the functions $B_\sharp$ and v we have $\mathcal{F}v(\xi) = \mathcal{F}u(-\xi)$ and $\mathcal{F}M_{B_\sharp}v(\xi) = -\mathcal{F}(B(-\cdot)u(-\cdot))(\xi) = -\mathcal{F}M_B u(-\xi)$ for all $\xi \in \mathbb{R}$. Since $u \in \operatorname{dom}(G)$, it follows that

$$\begin{aligned}(\mathcal{F} - M_{R_\sharp}\mathcal{F}M_{B_\sharp})v(\xi) &= \mathcal{F}u(-\xi) - (-R(-\xi))(-\mathcal{F}M_B u(-\xi))\\ &= (\mathcal{F}u - M_R\mathcal{F}M_B u)(-\xi) = R(-\xi)\mathcal{F}f(-\xi) = -R_\sharp(\xi)\mathcal{F}f(-\xi) = M_{R_\sharp}\mathcal{F}g(\xi)\end{aligned}$$

for all $\xi \in \mathbb{R}$, which proves $v \in \operatorname{dom}(G_\sharp)$ and $G_\sharp v = g = -\Lambda G u$. Hence, $\Lambda \operatorname{dom}(G) \subseteq \operatorname{dom}(G_\sharp)$ and $G_\sharp \Lambda u = -\Lambda G u$ for all $u \in \operatorname{dom}(G)$. Using a similar argument, one can prove $\Lambda \operatorname{dom}(G_\sharp) \subseteq \operatorname{dom}(G)$. Since Λ is invertible and $\Lambda^{-1} = \Lambda$, we have $\operatorname{dom}(G_\sharp) \subseteq \Lambda \operatorname{dom}(G)$, proving $G_\sharp \Lambda = -\Lambda G$; and (ii) and (iii) are similar.

Theorem 4.2 *If G is Fredholm, then* (1.2) *has an exponential dichotomy on $\mathbb{R}_-$.*

PROOF. From Proposition 4.1(i) and since G is Fredholm, we have that $G_\sharp = -\Lambda G \Lambda$ is Fredholm. Applying Theorem 3.12 for equation (4.1) (replacing A by $A_\sharp$, B by $B_\sharp$ and R by $R_\sharp$), we obtain that equation (4.1) has an exponential dichotomy on $\mathbb{R}_+$. From the proof of Theorem 3.12 it follows that we can choose the projection family $\{P_{+,\sharp}(t)\}_{t\geq 0}$ such that $\operatorname{im} P_{+,\sharp}(\tau)$, $\tau \geq 0$, is the set of all $x \in X$ such that there exists an $L_2 \cap C_0$-solution u of equation (4.1) on $[\tau,\infty)$ satisfying $u(\tau) = x$. Let $\mathcal{U}_{+,\sharp} = (U^+_{s,\sharp}, U^+_{u,\sharp})$ be a bi-family adjusted to the bounded, strongly continuous projection family $\{P_{+,\sharp}(t)\}_{t\geq 0}$, satisfying Definition 2.5 for equation (4.1) and $J = \mathbb{R}_+$. Let $P_-(t) = I - P_{+,\sharp}(-t)$ for $t \leq 0$. We infer that $\{P_-(t)\}_{t\leq 0}$ is a bounded, strongly continuous projection family. Let $U^-_s(t,\tau) = U^+_{u,\sharp}(-t,-\tau) \in \mathcal{B}(\operatorname{im} P_-(\tau), \operatorname{im} P_-(t))$ for $\tau \leq t \leq 0$ and let $U^-_u(t,\tau) = U^+_{s,\sharp}(-t,-\tau) \in \mathcal{B}(\ker P_-(\tau), \ker P_-(t))$ for $t \leq \tau \leq 0$. Since $\mathcal{U}_{+,\sharp} = (U^+_{s,\sharp}, U^+_{u,\sharp})$ is a by-family adjusted to the projection family $\{P_{+,\sharp}(t)\}_{t\geq 0}$, it follows that $\mathcal{U}_- = (U^-_s, U^-_u)$ is a bi-family adjusted to the projection family $\{P_-(t)\}_{t\leq 0}$. From Proposition 4.1(ii) it follows that $U^-_s(\cdot,\tau)x = U^+_{u,\sharp}(-\cdot,-\tau)x$ is a solution of equation (1.2) on $[\tau,0]$ for all $\tau \leq 0$ and all $x \in \operatorname{im} P_-(\tau)$. Similarly, from Proposition 4.1(iii) it follows that $U^-_u(\cdot,\tau)x = U^+_{s,\sharp}(-\cdot,-\tau)x$ is an $L_2 \cap C_0$-solution of equation (1.2) on $(-\infty,\tau]$ for all $\tau \leq 0$ and all $x \in \ker P_-(\tau)$. Moreover, if u is a solution of equation (1.2) on $[a,b] \subseteq \mathbb{R}_-$, by Proposition 4.1(ii), $v = u(-\cdot)$ is a solution of equation (4.1) on $[-b,-a]$. It follows that $v(t) = U_{s,\sharp}(t,-b)P_{+,\sharp}(-b)v(-b) + U_{u,\sharp}(t,-a)(I - P_{+,\sharp}(-a))v(-a)$ for all $t \in [-b,-a]$, or equivalently, $u(t) = U^-_s(t,a)P_-(a)u(a) + U^-_u(t,b)(I - P_-(b))u(b)$ for all $t \in [a,b]$, which proves that $\mathcal{U}_-$ is a bi-family associated with equation (1.2). Also, we have the estimates

$$\|U^-_s(t,\tau)\| = \|U^+_{u,\sharp}(-t,-\tau)\| \leq Ne^{\nu(-t+\tau)} = Ne^{-\nu(t-\tau)} \quad \text{for all} \quad \tau \leq t \leq 0,$$

$$\|U_u^-(t,\tau)\| = \|U_{s,\sharp}^+(-t,-\tau)\| \le Ne^{-\nu(-t+\tau)} = Ne^{\nu(t-\tau)} \quad \text{for all} \quad t \le \tau \le 0,$$

proving that equation (1.2) has an exponential dichotomy on $\mathbb{R}_-$.

5. The index formula

Throughout this section we assume that G is Fredholm. Let $\{P_+(t)\}_{t\ge 0}$ and $\{P_-(t)\}_{t\le 0}$ be the dichotomy projections defined in Theorem 3.12 and Theorem 4.2.

Proposition 5.1 *Assume that G is Fredholm. Then the pair $(\operatorname{im} P_+(0), \ker P_-(0))$ is Fredholm, $\dim\ker G = \alpha(\operatorname{im} P_+(0), \ker P_-(0))$, $\operatorname{codim}\operatorname{im} G = \beta(\operatorname{im} P_+(0), \ker P_-(0))$ and $\operatorname{ind} G = \operatorname{ind}(\operatorname{im} P_+(0), \ker P_-(0))$.*

PROOF. First, we claim that $\ker P_-(0) = Y_u(0) \oplus X_0$. As in the definition of the subspace $X_s^+(0) = \operatorname{im} P_+(0)$, cf. Remark 3.1, and since $P_-(t) = I - P_{+,\sharp}(-t)$ for all $t \le 0$, we have that $\ker P_-(0) = \operatorname{im} P_{+,\sharp}$ is the space of all $x \in X$ such that there exists an $L_2 \cap C_0$-solution v of equation (4.1) on $[0,\infty)$ with $v(0) = x$. From Proposition 4.1(iii) it follows that $\ker P_-(0)$ is the set of all $x \in X$ such that there exists an $L_2 \cap C_0$-solution u of equation (1.2) on $(-\infty, 0]$ with $u(0) = x$. From the definition of $Y_u(0)$ and X_0, we infer $Y_u(0) \subseteq \ker P_-(0)$, $X_0 \subseteq \ker P_-(0)$ and $X_0 \cap Y_u(0) = \{0\}$. Thus, to prove the claim it is enough to show that $\ker P_-(0) \subseteq Y_u(0) + X_0$. Let $x \in \ker P_-(0)$. As in Proposition 3.6(i), one can prove $x \in X_{0,*}^\perp$. From Theorem 3.6 we have that there exist $y \in Y_u(0)$ and $z \in X_s^+(0) = \operatorname{im} P_+(0)$ such that $x = y + z$. Let u be the $L_2 \cap C_0$-solution of the equation (1.2) on $(-\infty, 0]$ for which $u(0) = x$ (the uniqueness of this solution follows from Hypothesis 1.1). Then $u - V_u(\cdot, 0)y$ is an $L_2 \cap C_0$-solution of equation (1.2) on $(-\infty, 0]$, and $u(0) - V_u(0,0)y = x - y = z \in \operatorname{im} P_+(0)$. Hence, $z \in X_0$, which proves $x = y + z \in Y_u(0) + X_0$.

By the definition of $\operatorname{im} P_+(0)$, $\ker P_-(0)$ and X_0 we have $\operatorname{im} P_+(0) \cap \ker P_-(0) = X_0$. Moreover, using Proposition 3.5(i) and Theorem 3.6, we have $\operatorname{im} P_+(0) + \ker P_-(0) = X_s^+(0) + Y_u(0) + X_0 = X_s^+(0) + Y_u(0) = X_{0,*}^\perp$. It follows that $\dim(\operatorname{im} P_+(0) \cap \ker P_-(0)) = \dim X_0 = \dim\ker(G) < \infty$ and $\operatorname{codim}(\operatorname{im} P_+(0) + \ker P_-(0)) = \operatorname{codim} X_{0,*}^\perp = \dim X_{0,*} = \dim\ker(G^*)$. Thus, the pair $(\operatorname{im} P_+(0), \ker P_-(0))$ is Fredholm and the required formulas for the defect numbers and the index hold.

The following index formula was proved in [22,23] for the case of well-posed equations: If the operator G is Fredholm then the node operator (acting from $\ker P_-(a)$ to $\ker P_+(b)$ and defined by $N(b,a) = Q_+(b)U(b,a)_{|\ker P_-(a)})$ is Fredholm and, moreover, $\operatorname{ind} G = \operatorname{ind} N(b,a)$. For not well-posed equations we cannot define the operator $N(b,a)$ due to the lack of the evolution family $\{U(t,\tau)\}_{t\ge\tau}$. To bypass this difficulty we replace below the node operator in the latter index formula by a certain subspace $\mathcal{Z}_{a,b} \subseteq X \times X$. This subspace resembles the graph of the node operator, but can be defined when equation (1.2) is not well-posed. One can use this subspace because the Fredholm property of an operator can be described by means of its graph as follows.

Remark 5.2 Consider direct sum decompositions $X = Y_1 \oplus Z_1 = Y_2 \oplus Z_2$ of a Banach space X, and an operator $T \in \mathcal{B}(Z_1, Z_2)$. We view the graph $\operatorname{gr}(T) = \{(z, Tz) : z \in Z_1\}$ as a subspace of $X \times X$. The operator T is Fredholm if and only if the pair of subspaces $(\operatorname{gr}(T), X \times Y_2)$ of $X \times X$ is Fredholm, and moreover, $\dim\ker T = \alpha(\operatorname{gr}(T), X \times Y_2)$,

$\operatorname{codim}\operatorname{im} T = \beta(\operatorname{gr}(T), X \times Y_2)$ and $\operatorname{ind} T = \operatorname{ind}(\operatorname{gr}(T), X \times Y_2)$. Indeed, this follows from the formulas $\operatorname{gr}(T) \cap (X \times Y_2) = \ker T \times \{0\}$ and $\operatorname{gr}(T) + (X \times Y_2) = X \times (\operatorname{im} T \oplus Y_2)$. $\diamond$

Assuming that the operator G is Fredholm, let $\{P_\pm(t)\}_{t\in\mathbb{R}_\pm}$ be the dichotomy projections on $\mathbb{R}_\pm$ obtained in Theorems 3.12 and 4.2. Given $a \leq 0 \leq b$, let us define the subspace $\mathcal{Z}_{a,b}$ as the set of all pairs $(x, Q_+(b)y) \in \ker P_-(a) \times \ker P_+(b)$ such that there exists a solution u of (1.2) on $[a,b]$ satisfying $u(a) = x$ and $u(b) = y$. In general, $\mathcal{Z}_{a,b}$ might not be the graph of a linear operator, as shown in Example 8.10. Recall (2.13).

Proposition 5.3 *If G is Fredholm, then the pair $(\mathcal{Z}_{a,b}, X \times \operatorname{im} P_+(b))$ is a Fredholm pair of subspaces of $X \times X$ and* $\dim\ker G = \dim(\mathcal{Z}_{a,b} \cap (X \times \operatorname{im} P_+(b)))$, $\operatorname{codim}\operatorname{im} G = \operatorname{codim}(\mathcal{Z}_{a,b} + (X \times \operatorname{im} P_+(b)))$ *and* $\operatorname{ind} G = \operatorname{ind}(\mathcal{Z}_{a,b}, X \times \operatorname{im} P_+(b))$.

PROOF. *Claim 1.* $\mathcal{Z}_{a,b} \cap (X \times \operatorname{im} P_+(b)) = X_a \times \{0\}$.
Assume $(x,y) \in \mathcal{Z}_{a,b} \cap (X \times \operatorname{im} P_+(b))$. Then $y \in \operatorname{im} P_+(b) \cap \ker P_+(b) = \{0\}$ and there exists a solution u of equation (1.2) on $[a,b]$ such that $u(a) = x \in \ker P_-(a)$ and $Q_+(b)u(b) = y = 0$. Since, $u(b) \in \operatorname{im} P_+(b)$, by the definition of the projections $P_-(a)$ and $P_+(b)$, there exists $g \in \ker G$ such that $u = g_{|[a,b]}$, which implies $x = u(a) = g(a) \in X_a$, proving $\mathcal{Z}_{a,b} \cap (X \times \operatorname{im} P_+(b)) \subseteq X_a \times \{0\}$. Conversely, $X_a \times \{0\} \subseteq X \times \operatorname{im} P_+(b)$. Since $X_b \subseteq \operatorname{im} P_+(b)$, one has $(g(a), 0) = (g(a), Q_+(b)g(b)) \in \mathcal{Z}_{a,b}$ for all $g \in \ker G$, which implies $X_a \times \{0\} \subseteq \mathcal{Z}_{a,b}$, proving the claim. From Hypothesis 1.1 we know that X_a and $\ker G$ are isomorphic, and thus

$$\dim(\mathcal{Z}_{a,b} \cap (X \times \operatorname{im} P_+(b))) = \dim(X_a \times \{0\}) = \dim X_a = \dim\ker G < \infty, \tag{5.1}$$

proving the first formula for the defect numbers in the proposition.

Claim 2. $\mathcal{Z}_{a,b} + (X \times \operatorname{im} P_+(b)) = X \times X^\perp_{b,*}$.
From Proposition 3.5 it follows that $X \times \operatorname{im} P_+(b) \subseteq X \times X^\perp_{b*}$. If $(x,y) \in \mathcal{Z}_{a,b}$, then there exists a solution u of equation (1.2) on $[a,b]$ such that $u(a) = x \in \ker P_-(a)$ and $y = Q_+(b)u(b)$. Since $x \in \ker P_-(a)$ it follows that there exists an $L_2 \cap C_0$-solution v of equation (1.2) on $(-\infty, b]$ such that $u = v_{|[a,b]}$. Using the same argument as in the proof of Proposition 3.5 one can show $v(b) \in X^\perp_{b,*}$. From Theorem 3.6 we have $X^\perp_{b,*} = \operatorname{im} P_+(b) \oplus Y_u(b)$ and thus $x_1 \in \operatorname{im} P_+(b)$ and $y_1 \in Y_u(b)$ such that $v(b) = x_1 + y_1$. Let $w : \mathbb{R} \to X$ be defined by $w(t) = U^+_s(t,b)x_1$ for $t \geq b$ and $w(t) = v(t) - V_u(t,b)y_1$ for $t < b$. A direct computation shows $(\mathcal{F} - M_R\mathcal{F}M_B)w = 0$, which yields $w \in \ker G$. Since $Y_u(b) \subseteq \operatorname{im} P_+(b)$ and $X_b \subseteq \operatorname{im} P_+(b)$, we infer

$$y = Q_+(b)u(b) = Q_+(b)v(b) = Q_+(b)(y_1 + w(b)) = y_1 \in Y_u(b) \subseteq X^\perp_{b,*},$$

which proves $\mathcal{Z}_{a,b} \subseteq X \times X^\perp_{b*}$. It follows that $\mathcal{Z}_{a,b} + (X \times \operatorname{im} P_+(b)) \subseteq X \times X^\perp_{b,*}$. Conversely, let $(x,y) \in X \times X^\perp_{b,*}$. From Theorem 3.6 we have $y \in X^\perp_{b,*} = \operatorname{im} P_+(b) \oplus Y_u(b)$, and thus, there exists $x_2 \in \operatorname{im} P_+(b)$ and $y_2 \in Y_u(b)$ such that $y = x_2 + y_2$. Since $Y_u(b) \subseteq \ker P_+(b)$, we have $Q_+(b)y_2 = y_2$, which implies $(V_u(a,b)y_2, y_2) = (V_u(a,b)y_2, Q_+(b)V_u(b,b)y_2) \in \mathcal{Z}_{a,b}$. We infer $(x,y) = (V_u(a,b)y_2, y_2) + (x - V_u(a,b)y_2, x_2) \in \mathcal{Z}_{a,b} + (X \times \operatorname{im} P_+(b))$, proving the claim. From Hypothesis 1.1 we know that $X_{b,*}$ and $\ker G^*$ are isomorphic. Thus, $\mathcal{Z}_{a,b} + (X \times \operatorname{im} P_+(b))$ is a closed subspace of $X \times X$ and, finishing the proof:

$$\operatorname{codim}(\mathcal{Z}_{a,b} + (X \times \operatorname{im} P_+(b))) = \operatorname{codim}(X \times X^\perp_{b,*}) = \operatorname{codim} X^\perp_{b,*} = \dim\ker G^* < \infty.$$

6. Sufficiency in the Dichotomy Theorem

Throughout this section we assume that equation (1.2) has exponential dichotomies on $\mathbb{R}_\pm$, cf. Definition 2.5. We denote by $\mathcal{U}_\pm = (U_s^\pm, U_u^\pm)$ the bi-families adjusted to the projection families $\{P_+(t)\}_{t\geq 0}$ and $\{P_-(t)\}_{t\leq 0}$, for $J=\mathbb{R}_+$ and $J=\mathbb{R}_-$, respectively. Also, we assume that the pair $(\operatorname{im} P_+(0), \ker P_-(0))$ of subspaces of X is Fredholm. To show that G is Fredholm we need two technical results.

Proposition 6.1 *(i) The function $U_s^+(\tau,\cdot)P_+(\cdot)x$ is right continuous on $[0,\tau]$ for each $\tau \geq 0$ and $x \in X$, and the function $U_s^-(\tau,\cdot)P_-(\cdot)x$ is right continuous on $(-\infty,\tau]$ for each $\tau \leq 0$ and $x \in X$;*

(ii) The function $U_u^+(\tau,\cdot)P_+(\cdot)x$ is left continuous on $[\tau,\infty)$ for each $\tau \geq 0$ and $x \in X$, and the function $U_u^-(\tau,\cdot)P_-(\cdot)x$ is left continuous on $[\tau,0]$ for each $\tau \leq 0$ and $x \in X$.

PROOF. To show (i) let $\tau \geq 0$, $x \in X$ and $(s_n)_{n\in\mathbb{N}} \subseteq [0,\tau]$ such that $s_n \to s$, $s_n \geq s$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
&\|U_s^+(\tau,s_n)P_+(s_n)x - U_s^+(\tau,s)P_+(s)x\| \\
&\qquad = \|U_s^+(\tau,s_n)P_+(s_n)x - U_s^+(\tau,s_n)U_s^+(s_n,s)P_+(s)x\| \\
&\qquad = \|U_s^+(\tau,s_n)\Big(P_+(s_n)x - U_s^+(s_n,s)P_+(s)x\Big)\| \\
&\qquad \leq N\|P_+(s_n)x - U_s^+(s_n,s)P_+(s)x\|
\end{aligned}$$

for all $n \in \mathbb{N}$. $P_+(\cdot)$ is strongly continuous, and $U_s^+(\cdot,s)P_+(s)x$ is an $L_2 \cap C_0$-solution of equation (1.2) on $[s,\infty)$, we obtain that $U_s^+(\tau,\cdot)P_+(\cdot)x$ is right continuous on $[0,\tau]$ for each $\tau \geq 0$ and $x \in X$. Similarly, $U_s^-(\tau,\cdot)P_-(\cdot)x$ is right continuous on $(-\infty,\tau]$ for each $\tau \leq 0$ and $x \in X$, and the proof of (ii) is analogous.

Our plan is to show the existence of a finite rank operator K such that, for a given $f \in L_2(\mathbb{R},X)$ and $u \in \operatorname{dom}(G)$, we have $Gu = f - Kf$. Let $f \in L_2(\mathbb{R},X)$ and define $Z_f^+ : \mathbb{R}^2 \to X$ by $Z_f^+(t,s) = U_s^+(t,s)P_+(s)f(s)$ if $t \geq s \geq 0$, by $Z_f^+(t,s) = -U_u^+(t,s)Q_+(s)f(s)$ if $0 \leq t < s$ and by $Z_f^+(t,s) = 0$ if $t < 0$ or $s < 0$. Also, define $Z_f^- : \mathbb{R}^2 \to X$ by $Z_f^-(t,s) = U_s^-(t,s)P_-(s)f(s)$ if $s \leq t \leq 0$, by $Z_f^-(t,s) = -U_u(t,s)Q_-(s)f(s)$ if $t < s \leq 0$ and by $Z_f^-(t,s) = 0$ if $t > 0$ or $s > 0$. By a direct verification, it follows that

$$\Big((\mathcal{F} - M_R\mathcal{F}M_B)Z_f^+(\cdot,s)\Big)(\xi) = R(\xi)\Big(e^{-2\pi i\xi s}f(s) + Z_f^+(0,s)\Big),\ \xi \in \mathbb{R},\ s \geq 0. \quad (6.1)$$

$$\Big((\mathcal{F} - M_R\mathcal{F}M_B)Z_f^-(\cdot,s)\Big)(\xi) = R(\xi)\Big(e^{-2\pi i\xi s}f(s) - Z_f^-(0,s)\Big),\ \xi \in \mathbb{R},\ s \leq 0. \quad (6.2)$$

By Proposition 6.1 and estimates (2.7), we have $\|Z_f^\pm(t,s)\| \leq Ne^{-\nu|t-s|}\|f(s)\|$ for all $t,s \in \mathbb{R}$. We define $u_f^\pm : \mathbb{R} \to X$ by

$$u_f^\pm(t) = \int_{\mathbb{R}} Z_f^\pm(t,s)ds. \quad (6.3)$$

From the continuity in the first variable of the evolution operators $U_{s,u}^\pm$, estimates (2.7), and since

$$u_f^+(t) = \int_0^t U_s^+(t,s)P_+(s)f(s)ds - \int_t^\infty U_u^+(t,s)Q_+(s)f(s)ds \text{ for all } t \geq 0,$$
$$u_f^-(t) = \int_{-\infty}^t U_s^-(t,s)P_-(s)f(s)ds - \int_t^0 U_u^-(t,s)Q_-(s)f(s)ds \text{ for all } t \leq 0,$$

we have that u_f^+ is continuous on $(0,\infty)$ and u_f^- is continuous on $(-\infty,0)$. Moreover, since $u_f^+(t) = 0$ for all $t < 0$ and $u_f^-(t) = 0$ for all $t > 0$ and $\|Z_f^\pm(t,s)\| \leq Ne^{-\nu|t-s|}\|f(s)\|$ for all $t,s \in \mathbb{R}$, we obtain, for all $t \in \mathbb{R}$,

$$\|u_f^\pm(t)\| \leq \int_\mathbb{R} Ne^{-\nu|t-s|}\|f(s)\|ds \leq c\Big(\int_\mathbb{R} e^{-\nu|t-s|}\|f(s)\|^2 ds\Big)^{1/2},$$

which proves

$$u_f^\pm \in L_2(\mathbb{R},X), \quad \|u_f^\pm\|_2 \leq c\|f\|_2 \quad \text{and} \quad \lim_{t\to\pm\infty} u_f^\pm(t) = 0. \tag{6.4}$$

Proposition 6.2 *Let* $f \in L_2(\mathbb{R},X)$, $f_+ = \chi_{[0,\infty)}f$, $f_- = \chi_{(-\infty,0]}f$, *and define*

$$x_f^+ = \int_0^\infty U_u^+(0,s)Q_+(s)f(s)ds,\ x_f^- = \int_{-\infty}^0 U_s^-(0,s)P_-(s)f(s)ds.$$

Then:

(i) $((\mathcal{F} - M_R\mathcal{F}M_B)u_f^+)(\xi) = R(\xi)(\mathcal{F}f_+(\xi) - x_f^+)$ *for almost all* $\xi \in \mathbb{R}$*;*

(ii) $((\mathcal{F} - M_R\mathcal{F}M_B)u_f^-)(\xi) = R(\xi)(\mathcal{F}f_-(\xi) - x_f^-)$ *for almost all* $\xi \in \mathbb{R}$*.*

PROOF. (i) *Step 1.* Suppose that $f \in L_1(\mathbb{R},X) \cap L_2(\mathbb{R},X)$. From (6.1) and Proposition 2.7(i) we have

$$\begin{aligned}
&\Big((\mathcal{F} - M_R\mathcal{F}M_B)(\chi_{(-n,n)}Z_f^+(\cdot,s))\Big)(\xi)\\
&\quad = R(\xi)\Big(e^{-2\pi i\xi s}f(s) + Z_f^+(0,s) + e^{2\pi in\xi}Z_f^+(-n,s) - e^{-2\pi in\xi}Z_f^+(n,s)\Big)\\
&\quad = R(\xi)\Big(e^{-2\pi i\xi s}f(s) + Z_f^+(0,s) - e^{-2\pi in\xi}Z_f^+(n,s)\Big)
\end{aligned}$$

for all $\xi \in \mathbb{R}$, all $s \geq 0$ and all $n \in \mathbb{N}$. Repeatedly using the definition of the Fourier transform, it follows that

$$\begin{aligned}
&\Big((\mathcal{F} - M_R\mathcal{F}M_B)(\chi_{(-n,n)}u_f^+)\Big)(\xi)\\
&\quad = \int_{-n}^n \Big(e^{-2\pi i\xi t}(I - R(\xi)B(t))\int_0^\infty Z_f^+(t,s)ds\Big)dt \quad \text{(using (6.3))}\\
&\quad = \int_0^\infty \int_{-n}^n e^{-2\pi i\xi t}(Z_f^+(t,s) - R(\xi)B(t)Z_f^+(t,s))dtds \quad \text{(using Fubini's Theorem)}\\
&\quad = \int_0^\infty \Big((\mathcal{F} - M_R\mathcal{F}M_B)(\chi_{(-n,n)}Z_f^+(\cdot,s))\Big)(\xi)ds\\
&\quad = \int_0^\infty R(\xi)\Big(e^{-2\pi i\xi s}f(s) + Z_f^+(0,s) - e^{-2\pi in\xi}Z_f^+(n,s)\Big)ds \quad \text{(using (6.1) and (2.8))}\\
&\quad = R(\xi)(\mathcal{F}f_+(\xi) - x_f^+ - e^{-2\pi in\xi}u_f^+(n))
\end{aligned}$$

for all $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$. Since $\chi_{(-n,n)}u_f^+ \to u_f^+$ in $L_2(\mathbb{R},X)$ and $u_f^+(n) \to 0$ as $n \to \infty$, we obtain $(\mathcal{F} - M_R\mathcal{F}M_B)(\chi_{(-n,n)}u_f^+) \to (\mathcal{F} - M_R\mathcal{F}M_B)u_f^+$ as $n \to \infty$ in $L_2(\mathbb{R},X)$ and

$\Big((\mathcal{F}-M_R\mathcal{F}M_B)(\chi_{(-n,n)}u_f^+)\Big)(\xi) \to R(\xi)(\mathcal{F}f_+(\xi)-x_f^+)$ as $n\to\infty$ for each $\xi\in\mathbb{R}$, which proves (i) for Step 1.

Step 2. Suppose that $f\in L_2(\mathbb{R},X)$. Let $\{f_n\}\subseteq L_1(\mathbb{R},X)\cap L_2(\mathbb{R},X)$ such that $f_n\to f$ as $n\to\infty$ in $L_2(\mathbb{R},X)$. By (6.4) it follows that $u_{f_n}^+\to u_f^+$ as $n\to\infty$ in $L_2(\mathbb{R},X)$. Similarly, by (2.7), we have that $x_{f_n}^+\to x_f^+$ as $n\to\infty$. From Step 1 and since $(f_n)_+\to f_+$ as $n\to\infty$ in $L_2(\mathbb{R},X)$ we obtain $\Big((\mathcal{F}-M_R\mathcal{F}M_B)(\chi_{(-n,n)}u_f^+)\Big)(\xi)\to R(\xi)(\mathcal{F}f_+(\xi)-x_f^+)$ as $n\to\infty$ for each $\xi\in\mathbb{R}$, which proves (i). The proof of (ii) is similar.

Proposition 6.3 *If equation* (1.2) *has an exponential dichotomy on* $\mathbb{R}_\pm$ *and the pair* $(\operatorname{im}P_+(0),\ker P_-(0))$ *is Fredholm, then the operator* G *is Fredholm.*

PROOF. Let $u\in\ker G$. By Proposition 2.7(i) u is a solution of equation (1.2) on $[0,a]$ for $a>0$. By Definition 2.5 for $J=\mathbb{R}_+$, it follows that $u(t)=U_s^+(t,0)P_+(0)u(0)+U_u^+(t,a)Q_+(a)u(a)$ for all $t\in[0,a]$, where $Q_+(t)=I-P_+(t)$ for $t\geq 0$, which implies $Q_+(t)u(t)=U_u^+(t,a)Q_+(a)u(a)$ for all $a>0$ and all $t\in[0,a]$. Hence, $\|Q_+(0)u(0)\|=\|U_u^+(0,a)Q_+(a)u(a)\|\leq Ne^{-\nu a}\|Q_+(a)\|\|u(a)\|\leq cNe^{-\nu a}\|u(a)\|$ for all $a>0$. Passing to the limit as $a\to\infty$, one has $Q_+(0)u(0)=0$, which proves $u(0)\in\operatorname{im}P_+(0)$. From Proposition 4.1(i) we have $v=u(-\cdot)\in\ker G_\sharp$. Applying the above argument to equation (4.1), one has $P_-(0)u(0)=0$, proving $u(0)\in\ker P_-(0)$. Hence, $\{u(0):u\in\ker G\}\subseteq\operatorname{im}P_+(0)\cap\ker P_-(0)$, which implies that the subspace $\{u(0):u\in\ker G\}$ is finite dimensional. Using Hypothesis 1.1, we conclude that $\ker G$ and $\{u(0):u\in\ker G\}$ are isomorphic, and so $\dim\ker G<\infty$.

Since $(\operatorname{im}P_+(0),\ker P_-(0))$ is a Fredholm pair, there exist closed subspaces Y_+ and Y_- and finite dimensional subspaces Y_0 and Y_* such that $\operatorname{im}P_+(0)\cap\ker P_-(0)=Y_0$, $\operatorname{im}P_+(0)=Y_0\oplus Y_+$, $\ker P_-(0)=Y_0\oplus Y_-$ and $X=Y_0\oplus Y_+\oplus Y_-\oplus Y_*$. Denote by P_{Y_0}, P_{Y_+}, P_{Y_-} and P_{Y_*} projectors associated with the latter splitting. Using notation (6.3) and Proposition 6.2 we let $u_f=u_f^++u_f^-\in L_2(\mathbb{R},X)$ and $x_f=x_f^++x_f^-$. From Proposition 6.2 we have $(\mathcal{F}-M_R\mathcal{F}M_B)u_f=M_R\mathcal{F}f-R(\cdot)x_f$. Define $v_f:\mathbb{R}\to X$ by $v_f(t)=U_s^+(t,0)(P_{Y_0}x_f+P_{Y_+}x_f)$ for $t\geq 0$ and $v_f(t)=-U_u^-(t,0)P_{Y_-}x_f$ for $t<0$. From (2.7) we have $v_f\in L_2(\mathbb{R},X)$ and $(\mathcal{F}-M_R\mathcal{F}M_B)v_f=R(\cdot)(x_f-P_{Y_*}x_f)$. It follows that $(\mathcal{F}-M_R\mathcal{F}M_B)(u_f+v_f)=M_R\mathcal{F}f-R(\cdot)P_{Y_*}x_f$. Recall notation (1.4) and that $\mathcal{F}\mathcal{V}(\cdot)x=R(\cdot)x$ for all $x\in X$. Using this fact, we obtain

$$\begin{aligned}(\mathcal{F}&-M_R\mathcal{F}M_B)(u_f+v_f+\mathcal{V}(\cdot)P_{Y_*}x_f)\\ &=M_R\mathcal{F}f-R(\cdot)P_{Y_*}x_f+(\mathcal{F}-M_R\mathcal{F}M_B)\mathcal{V}(\cdot)P_{Y_*}x_f\\ &=M_R\mathcal{F}f-M_R\mathcal{F}M_B\mathcal{V}(\cdot)P_{Y_*}x_f=M_R\mathcal{F}(f-Kf),\end{aligned}$$

where $Kf=M_B\mathcal{V}(\cdot)P_{Y_*}x_f$. Since Y_* is finite dimensional the operator K is of finite rank. Also, we have $u_f+v_f+\mathcal{V}(\cdot)P_{Y_*}x_f\in\operatorname{dom}(G)$ and $G(u_f+v_f+\mathcal{V}(\cdot)P_{Y_*}x_f)=f-Kf$ for all $f\in L_2(\mathbb{R},X)$. Hence, $\operatorname{im}(I-K)\subseteq\operatorname{im}(G)$. Since K has finite rank, $I-K$ is Fredholm, and thus, $\operatorname{im}G$ is closed and $\operatorname{codim}\operatorname{im}G<\infty$.

7. Perturbations

In this section we will discuss some perturbation results. For this, we will need a reformulation of the Fredholm property of G in terms of some spectral conditions involving

$\mathcal{V}$, see (1.4), and the operator valued function $B(\cdot)$. First, we recall the following fact.

Remark 7.1 If Y is a dense subspace of a Hilbert space X and $E, F: Y \to X$ are two closed operators, and $0 \in \rho(F)$, then E is Fredholm if and only if EF^{-1} is Fredholm. Moreover, $\operatorname{ind} E = \operatorname{ind}(EF^{-1})$. Indeed, this follows from equalities $\ker(EF^{-1}) = F(\ker E)$ and $\operatorname{im}(EF^{-1}) = \operatorname{im} E$. $\diamond$

We introduce the operator $G_{unp} = \mathcal{F}^* M_R^{-1} \mathcal{F}$. Since M_R is a bounded injective operator with dense range, G_{unp} is closed, densely defined and invertible with $G_{unp}^{-1} = \mathcal{F}^* M_R \mathcal{F}$.

Proposition 7.2 *The operator G is Fredholm if and only if $1 \notin \sigma_F(\mathcal{V} * M_B)$. Moreover, $\operatorname{ind} G = \operatorname{ind}(I - \mathcal{V} * M_B)$.*

PROOF. Since $\mathcal{V}(\cdot)x = \mathcal{F}^* R(\cdot)x$ for each $x \in X$, we infer that $\mathcal{F}^* M_R \mathcal{F} M_B = \mathcal{V} * M_B$. By the definition of G, if $u \in \operatorname{dom}(G)$ then

$$\begin{aligned}(I - \mathcal{V} * M_B)u &= (I - \mathcal{F}^* M_R \mathcal{F} M_B)u \\ &= \mathcal{F}^*(\mathcal{F} - M_R \mathcal{F} M_B)u = \mathcal{F}^* M_R \mathcal{F} G u = G_{unp}^{-1} G u.\end{aligned}$$

It follows that $G_{unp}^{-1}G$ is closable and $\overline{G_{unp}^{-1}G} = (G^*(G_{unp}^*)^{-1})^*$, which implies $I - \mathcal{V} * M_B = (G^*(G_{unp}^*)^{-1})^*$. In the course of proof of Proposition 2.2 we proved $G^* = G_{unp}^* - M_{B^*}$, which implies $\operatorname{dom}(G^*) = \operatorname{dom}(G_{unp}^*)$. Using Remark 7.1, it follows that G, equivalently, G^* is Fredholm if and only if $G^*(G_{unp}^*)^{-1}$ is Fredholm if and only if $(G^*(G_{unp}^*)^{-1})^* = I - \mathcal{V} * M_B$ is Fredholm if and only if $1 \notin \sigma_F(\mathcal{V} * M_B)$. Moreover, $\operatorname{ind}(G) = -\operatorname{ind}(G^*) = -\operatorname{ind}(G^*(G_{unp}^*)^{-1}) = \operatorname{ind}(G^*(G_{unp}^*)^{-1})^* = \operatorname{ind}(I - \mathcal{V} * M_B)$.

Next, we will discuss compact perturbations of the following special class. Let $K : \mathbb{R} \to \mathcal{K}(X)$ be a strongly continuous and bounded function. Denote $D_K(t) = A + B(t) + K(t)$, $t \in \mathbb{R}$, and consider the equation

$$u'(t) = D_K(t)u(t), \; t \in \mathbb{R}. \tag{7.1}$$

Also, define the operator G_K as follows: Let $\operatorname{dom}(G_K)$ be the set of all $u \in L_2(\mathbb{R}, X)$ such that the relation $(\mathcal{F} - M_R \mathcal{F} M_{B+K})u = M_R \mathcal{F} f$ holds for some $f \in L_2(\mathbb{R}, X)$ (which is unique by the injectivity of M_R), and define G_K by $G_K u = f$. We remark that $G_K = G - M_K$. The following result is an analog of [23, Prop. 7.6]. Combined with the Dichotomy Theorem, it gives sufficient conditions under which the exponential dichotomy on $\mathbb{R}_\pm$ for equation (1.2) persists for equation (7.1).

Proposition 7.3 *Assume that $\lim_{|t|\to\infty} \|K(t)\| = 0$. Then the operator G is Fredholm if and only if the operator G_K is Fredholm; moreover, $\operatorname{ind}(G) = \operatorname{ind}(G_K)$.*

PROOF. Notice that $I - \mathcal{V} * M_{B+K} = (I - \mathcal{V} * M_B) - \mathcal{V} * M_K$. In view of Proposition 7.2, it suffices to show that $\mathcal{V} * M_K$ is a compact operator. To this aim, for every $n \in \mathbb{N}$ we choose $\varphi_n \in C_0^\infty(\mathbb{R})$ such that $0 \le \varphi_n \le 1$, $\varphi_n(t) = 0$ for $t \notin [-n-1, n+1]$ and $\varphi_n(t) = 1$ for $t \in [-n, n]$, and let $K_n : \mathbb{R} \to \mathcal{K}(X)$ be defined by $K_n(t) = \varphi_n(t)K(t)$. By the assumption $\lim_{|t|\to\infty} \|K(t)\| = 0$, $\gamma_n := \sup_{\tau \in \mathbb{R}} \|K_n(\tau) - K(\tau)\| \to 0$ as $n \to \infty$. This implies that $\mathcal{V} * M_{K_n} \to \mathcal{V} * M_K$ as $n \to \infty$ in the operator norm, because

$$\begin{aligned}\|\Big((\mathcal{V} * M_{K_n} - \mathcal{V} * M_K)f\Big)(t)\| &= \|\int_{\mathbb{R}} \mathcal{V}(t-s)(K_n(s) - K(s))f(s)ds\| \\ &\le \int_{\mathbb{R}} N e^{-\nu|t-s|}(\|K_n(s) - K(s)\|)\|f(s)\|ds \le N\gamma_n \int_{\mathbb{R}} e^{-\nu|t-s|}\|f(s)\|ds\end{aligned}$$

$$\leq c\gamma_n\Big(\int_{\mathbb{R}} e^{-\nu|t-s|}\|f(s)\|^2 ds\Big)^{1/2},$$

for all $n \in \mathbb{N}$, $t \in \mathbb{R}$ and all $f \in L_2(\mathbb{R}, X)$ yields

$$\|(\mathcal{V} * M_{K_n} - \mathcal{V} * M_K)f\|_2 \leq c\gamma_n\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\nu|t-s|}\|f(s)\|^2 dsdt\Big)^{1/2} = c\gamma_n\|f\|_2.$$

Let $L_n : \mathbb{R}^2 \to \mathcal{K}(X)$ be the operator valued function defined by $L_n(t,s) = \mathcal{V}(t-s)K_n(t)$. L_n is strongly continuous on $\mathbb{R}^2 \setminus \{(t,t) : t \in \mathbb{R}\}$, and so, strongly measurable for all $n \in \mathbb{N}$. Moreover $\|L_n(t,s)\| \leq Ne^{-\nu|t-s|}\|K_n(s)\| \leq Nce^{-\nu|t-s|}\varphi_n(s)$ for all $t, s \in \mathbb{R}$ and all $n \in \mathbb{N}$, which proves that $\int_{\mathbb{R}^2}\|L_n(t,s)\|^2 dtds < \infty$ for each $n \in \mathbb{N}$. From [9, Prop.2.1], it follows that $\mathcal{V} * M_{K_n} \in \mathcal{K}(L_2(\mathbb{R}, X))$, since $(\mathcal{V} * M_{K_n}f)(t) = \int_{\mathbb{R}} L_n(t,s)f(s)ds$ for all $n \in \mathbb{N}$, $t \in \mathbb{R}$ and all $f \in L_2(\mathbb{R}, X)$. Since $\mathcal{V} * M_{K_n} \to \mathcal{V} * M_K$ as $n \to \infty$ in the operator norm, we have $\mathcal{V} * M_K \in \mathcal{K}(L_2(\mathbb{R}, X))$, as required.

Next, we will show that the last perturbation result holds if we replace the condition $\lim_{|t|\to\infty}\|K(t)\| = 0$ in Proposition 7.3 by the condition $\|K(\cdot)\| \in L_2(\mathbb{R})$.

Proposition 7.4 *Assume that $\|K(\cdot)\| \in L_2(\mathbb{R})$. Then the operator G is Fredholm if and only if the operator G_K is Fredholm; moreover,* $\mathrm{ind}(G) = \mathrm{ind}(G_K)$.

PROOF. The proof is similar to the proof of Proposition 7.3, but this time we will prove $\mathcal{V} * M_K \in \mathcal{K}(L_2(\mathbb{R}, X))$ directly, using [9, Prop.2.1]. Indeed, $(\mathcal{V} * M_K f)(t) = \int_{\mathbb{R}} L(t,s)f(s)ds$ for all $t \in \mathbb{R}$ and $f \in L_2(\mathbb{R}, X)$, where $L : \mathbb{R}^2 \to \mathcal{K}(X)$ is defined by $L(t,s) = \mathcal{V}(t-s)K(s)$. Since L is strongly measurable, $\|L(t,s)\| \leq Ne^{-\nu|t-s|}\|K(s)\|$ for all $t, s \in \mathbb{R}$, and $\int_{\mathbb{R}}\|K(t)\|^2 dt < \infty$, one has $\int_{\mathbb{R}^2}\|L(t,s)\|^2 dtds < \infty$, and thus [9, Prop.2.1] gives the desired conclusion.

In the special case when $B = 0$, we denote the operator G_K by G_K^0. Notice that $G_K^0 = G_{unp} - M_K$ and recall that G_{unp} is invertible. Propositions 7.3 and 7.4 in this special case yield the following result.

Proposition 7.5 *If the perturbation $K : \mathbb{R} \to X$ satisfies either one of the conditions $\|K(\cdot)\| \in L_2(\mathbb{R})$ or $\lim_{|t|\to\infty}\|K(t)\| = 0$, then G_K^0 is Fredholm with* $\mathrm{ind}(G_K^0) = 0$.

We conclude this section by illustrating how our main theorem applies to show the existence of bi-families associated with an not well-posed equation. The Dichotomy Theorem and Proposition 7.5 imply the following fact.

Proposition 7.6 *Assume that A is a generator of a bi-semigroup and $B(\cdot)$ is a bounded piecewise strongly continuous operator valued function on $\mathbb{R}$.*

(i) If $1 \notin \sigma(\mathcal{V} * M_B)$ then G is invertible and, therefore, there exists an exponentially dichotomic bi-family $\mathcal{U} = (U_s, U_u)$, adjusted to a projection family $\{P(t)\}_{t\in\mathbb{R}}$, associated with equation (1.2) *on $\mathbb{R}$;*

(ii) If $B(t) \in \mathcal{K}(X)$ for each $t \in \mathbb{R}$ and either $\lim_{|t|\to\infty}\|B(t)\| = 0$ or $\|B(\cdot)\| \in L_2(\mathbb{R})$, then G is Fredholm, and, therefore, there exist bi-families $\mathcal{U}_\pm = (U_s^\pm, U_u^\pm)$ adjusted to projection families $\{P_\pm(t)\}_{t\in\mathbb{R}_\pm}$, associated with equation (1.2) *on $\mathbb{R}_\pm$.*

8. Examples and special cases

In this section we present several concrete examples of not well-posed differential equations that fit our setting. We start with a special case of the generator of a stable bi-semigroup, probably, well known.

Proposition 8.1 *Let X_0 be a Hilbert space, $A_0 : \mathrm{dom}(A_0) \subseteq X_0 \to X_0$ be a closed densely defined linear operator, and assume that $\sigma(A_0) = \{\lambda_n : n \in \mathbb{N}\}$ is a discrete set that does not intersect $\mathbb{R}_-$, and, moreover, that there exists an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ in X_0 consisting of eigenvectors of A_0. If $X = \mathrm{dom}(|A_0|^{1/2}) \times X_0$ then the operator $A : \mathrm{dom}(A) \subseteq X \to X$ defined by $\mathrm{dom}(A) = \mathrm{dom}(A_0) \times \mathrm{dom}(|A_0|^{1/2})$ and $A = \left[\begin{smallmatrix} 0 & I \\ A_0 & 0 \end{smallmatrix}\right]$ is the generator of a stable bi-semigroup. Moreover, if $A_0^{1/2}$ is defined via $A_0^{1/2} e_n = \lambda_n^{1/2} e_n$ with $\mathrm{dom}(A_0^{1/2}) = \mathrm{dom}(|A_0|^{1/2})$, then $\tilde{A} = \Psi A \Psi^{-1}$, where*

$$\tilde{A} = \begin{bmatrix} -A_0^{1/2} & 0 \\ 0 & A_0^{1/2} \end{bmatrix}, \ \Psi = 2^{-1/2} \begin{bmatrix} A_0^{1/2} & -I \\ A_0^{1/2} & I \end{bmatrix}, \Psi^{-1} = 2^{-1/2} \begin{bmatrix} A_0^{-1/2} & A_0^{-1/2} \\ -I & I \end{bmatrix},$$

such that $\Psi : \mathrm{dom}(|A_0|^{1/2}) \times X_0 \to X_0 \times X_0$ is bounded and boundedly invertible.

PROOF. The choice of the space X and the domain of A above, corresponds to the fact that the operator Ψ is bounded and boundedly invertible. A simple computation shows that $\tilde{A} = \Psi A \Psi^{-1}$, proving the proposition.

Analytic bi-semigroups.

In this subsection we are concerned with the situation when A is the generator of an *analytic* bi-semigroup. The following two examples are taken from [38, Sec.2,3].

Example 8.2 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth nonlinearity, D_0 be an $(n \times n)$ diagonal matrix with positive entries, $c \in \mathbb{R}$, and consider the following system of reaction diffusion equations:

$$\partial_t u = D_0 \partial_\xi^2 u + c\partial_\xi u + f(u), \quad t, \xi \in \mathbb{R}. \tag{8.1}$$

Let $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$ be a *modulated wave* that satisfies (8.1), i.e., a solution of (8.1) satisfying $q(\xi, t) = q(\xi, t + T)$ for all $\xi, t \in \mathbb{R}$ and some $T > 0$ (see [38] and the literature therein for information on this topic). The linearized equation about q is

$$\partial_t u = D_0 \partial_\xi^2 u + c\partial_\xi u + a(\xi, t)u, \quad t, \xi \in \mathbb{R}, \tag{8.2}$$

and we assume $a(\xi, t) = f_u(q(\xi, t))$ bounded and smooth. If Φ is the monodromy operator, acting on $L_2(\mathbb{R}, \mathbb{C}^n)$, associated with equation (8.2), and $\lambda = e^{\alpha T}$, then the eigenvalue problem $\Phi v = \lambda v$ can be transformed, see [38, Sec.2.2], to a differential equation of the form $V' = D(\xi)V$, $\xi \in \mathbb{R}$, on $X = H^{1/2}(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^n) \times L_2(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^n)$, where $D(\xi)$ can be written as $D(\xi) = A + B(\xi)$ with $\mathrm{dom}(A) = H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^n) \times H^{1/2}(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^n)$, and denoting the $(n \times n)$ unit matrix by $I_{n\times n}$,

$$A = \begin{bmatrix} 0 & I_{n\times n} \\ D_0^{-1}(\partial_t + I_{n\times n}) & 0 \end{bmatrix}, \quad B(\xi) = \begin{bmatrix} 0 & 0 \\ D_0^{-1}(\alpha - I_{n\times n} - a(\xi, \cdot)) & -cD_0^{-1} \end{bmatrix}.$$

Clearly, A satisfies conditions in Proposition 8.1, and so A is the generator of a uniformly exponentially stable analytic bi-semigroup; $B(\cdot)$ is bounded and strongly continuous. $\diamond$

Example 8.3 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth nonlinearity, Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary, Δ_η denote the Laplacian on $L_2(\Omega)$ with $\mathrm{dom}(\Delta_\eta) = H^2(\Omega) \cap H_0^1(\Omega)$, $c \in \mathbb{R}$, and consider the equation

$$\partial_t u = \partial_\xi^2 u + \Delta_\eta u + c\partial_\xi u + f(u), \quad t, \xi \in \mathbb{R}, \eta \in \Omega. \tag{8.3}$$

Let $q : \mathbb{R} \times \Omega \to \mathbb{R}^n$ be a *traveling wave* for (8.3), i.e., $q = q(\xi, \eta)$ is a solution of the following elliptic problem on the cylinder $\mathbb{R} \times \Omega$:

$$\partial_\xi^2 q + \Delta_\eta q + c\partial_\xi q + f(q) = 0, \quad \xi \in \mathbb{R}, \eta \in \Omega. \tag{8.4}$$

The linearization of (8.3) about the traveling wave q is given by

$$\partial_t u = \mathcal{L}u, \text{ where } \mathcal{L}u = \partial_\xi^2 u + \Delta_\eta u + c\partial_\xi u + f_u(q(\xi, \cdot))u, \quad t, \xi \in \mathbb{R}, \eta \in \Omega.$$

The eigenvalue problem $\mathcal{L}v = \lambda v$ can be written as the differential equation of the form $V' = D(\xi)V$, $\xi \in \mathbb{R}$, on $X = H_0^1(\Omega, \mathbb{C}^n) \times L_2(\Omega, \mathbb{C}^n)$, where, cf. [38, Sec.3], $D(\xi) = A + B(\xi)$ with $\mathrm{dom}(A) = H^2(\Omega, \mathbb{C}^n) \times H_0^1(\Omega, \mathbb{C}^n)$,

$$A = \begin{bmatrix} 0 & I_{n\times n} \\ -\Delta_\eta + I_{n\times n} & 0 \end{bmatrix}, \quad B(\xi) = \begin{bmatrix} 0 & 0 \\ \lambda + f_u(q(\xi, \cdot)) - I_{n\times n} & -cI_{n\times n} \end{bmatrix}.$$

The operator A is the generator of a uniformly exponentially stable analytic bi-semigroup by Proposition 8.1; $B(\cdot)$ is bounded and strongly continuous. $\diamond$

Example 8.4 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth nonlinearity, $\mathbb{T}^m = \mathbb{R}^m/2\pi\mathbb{Z}^m$ be the torus, and consider equation (8.3) with $c = 0$ from the previous example where Δ_η is the Laplacian with periodic boundary conditions. Moreover, let us assume this time that (8.3) has a 2π-periodic in t solution $q = q(\xi, \eta, t)$. Linearizing equation (8.3) along q, we obtain the equation

$$\partial_t u = \partial_\xi^2 u + \Delta_\eta u + a(\xi, \eta, t)u \quad t, \xi \in \mathbb{R}, \eta \in \Omega, \tag{8.5}$$

where $a(t, \xi, \eta) = f_u(q(t, \xi, \eta))$. If Φ is the monodromy operator, acting on $L_2(\mathbb{R} \times \mathbb{T}^m, \mathbb{C}^n)$, associated with equation (8.5) (for the definition of the monodromy operator see for example [38, Sec.2]), and $\lambda = e^{2\pi\alpha}$, then the eigenvalue problem $\Phi v = \lambda v$ can be written as the differential equation $V' = D(\xi)V$, $\xi \in \mathbb{R}$, on $X = H^{1/2}([0, 2\pi] \times \mathbb{T}^m, \mathbb{C}^n) \times L_2([0, 2\pi] \times \mathbb{T}^m, \mathbb{C}^n)$, where $D(\xi) = A + B(\xi)$ with $\mathrm{dom}(A) = H^1([0, 2\pi] \times \mathbb{T}^m, \mathbb{C}^n) \times H^{1/2}([0, 2\pi] \times \mathbb{T}^m, \mathbb{C}^n)$, and

$$A = \begin{bmatrix} 0 & I_{n\times n} \\ \partial_t - \Delta_\eta + I_{n\times n} & 0 \end{bmatrix} \quad \text{and} \quad B(\xi) = \begin{bmatrix} 0 & 0 \\ a(\cdot, \xi, \cdot) - \alpha - I_{n\times n} & 0 \end{bmatrix}.$$

Then A satisfies conditions in Proposition 8.1, which proves that A is the generator of a uniformly exponentially stable bi-semigroup. Moreover, the semigroups $\{T_j(t)\}_{t\geq 0}$ are analytic which follows from the fact that $\sigma(A_0) = \{|k|^2 + 1 - il : k \in \mathbb{Z}^m, l \in \mathbb{Z}\}$. The function $B(\cdot)$ is bounded and strongly continuous. $\diamond$

Example 8.5 Let $A = \begin{bmatrix} 1 & -d/dx \\ d/dx & -1 \end{bmatrix}$ be the Dirac operator on $L_2(\mathbb{R}) \times L_2(\mathbb{R})$, see [24, Chap.8,9]. It turns out that A is the generator of a (uniformly exponentially stable) bi-semigroup but it is not the generator of a strongly continuous semigroup. Indeed, passing to the Fourier transform, A is unitary equivalent to the operator $\left[\begin{smallmatrix} 1 & -M_g \\ M_g & -1 \end{smallmatrix}\right]$, where M_g is the operator of multiplication by the function $g(\xi) = 2\pi i\xi$ on $L_2(\mathbb{R})$ with the maximal domain. Diagonalizing the matrix, one can see that A is similar to the operator $\left[\begin{smallmatrix} M_h & 0 \\ 0 & -M_h \end{smallmatrix}\right]$, where $h(\xi) = -\sqrt{1+4\pi^2\xi^2}$. It follows that $\sigma(A) = (-\infty, -1] \cup [1, \infty)$, and so A is not a generator of a C_0-semigroup. Since M_h is the generator of the uniformly exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L_2(\mathbb{R})$ given by $(T(t)f)(\xi) = e^{-t\sqrt{1+4\pi^2\xi^2}}f(\xi)$, $t \geq 0$, $\xi \in \mathbb{R}$, it follows that A is the generator of a stable bi-semigroup. ◇

Example 8.6 Consider the following Swift-Hohenberg equation (important in the study of cellular flows and optical parametric oscillators, see, e. g. [11,12]):

$$\partial_t u = au - (1 + \partial_\xi^2)^2 u - u^3, \tag{8.6}$$

where a is a real parameter. Using notations $v = (1 + \partial_\xi^2)u$ and $W = (u, v)^T$, we can write equation (8.6) as the following gradient equation, cf. [21]:

$$T\partial_t W = D\partial_\xi^2 W + Q\nabla F(W), \quad \text{where} \tag{8.7}$$

$$T = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right],\ D = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right],\ Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right],\ F(u, v) = au^2/2 - u^4/4 - uv + v^2/2.$$

Equation (8.7) exhibits stationary patterns with spatially periodic structure. Let q be a periodic solution of the equation $D\partial_\xi^2 W + Q\nabla F(W) = 0$. Linearizing equation (8.7) about q, we obtain the equation

$$T\partial_t W = D\partial_\xi^2 W + Q\nabla^2 F(q(\xi))W. \tag{8.8}$$

Since the matrix T is not invertible but the matrix D is, it is natural to treat (8.8) as an evolution equation in the ξ-variable. Indeed, with the substitution $Z = \partial_\xi W$ and $V(\xi) = (W(\cdot, \xi), Z(\cdot, \xi))$ we can write (8.8) as $V' = D(\xi)V$, $\xi \in \mathbb{R}$, on $X = H^{1/2}(\mathbb{R}) \times H^{1/4}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \times L_2(\mathbb{R})$, where $D(\xi) = A_0 + B_0(\xi)$ with $\mathrm{dom}(A_0) = H^1(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \times H^{1/2}(\mathbb{R}) \times H^{1/4}(\mathbb{R})$, and

$$A_0 = \left[\begin{smallmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \\ \partial_t + 1 & 0 & 0 & 0 \end{smallmatrix}\right],\ B_0(\xi) = \begin{bmatrix} 0 & I_2 \\ DQ\nabla^2 F(q(\xi)) - DT & 0 \end{bmatrix},\ I_2 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

For for $\alpha > 0$ we introduce the notation

$$Y_\alpha := \{f : \mathbb{R} \to \mathbb{C} : f \text{ is measurable and } \int_{\mathbb{R}} (1 + 4\pi^2|\xi|^2)^\alpha |f(\xi)|^2 d\xi < \infty\}.$$

Let us denote by M_g the operator of multiplication by $g(\xi) = 1 + 2\pi i\xi$ acting from $Y_{1/2}$ to $L_2(\mathbb{R})$ with $\mathrm{dom}(M_g) = Y_1$, and by M_{h_k} the operator of multiplication by $h_k(\xi) = (1 + 4\pi^2\xi^2)^{1/8}\exp(k\pi i/2 + i/4 \arctan(2\pi\xi))$, $k = 0, 1, 2, 3$, acting on $L_2(\mathbb{R})$ with $\mathrm{dom}(M_{h_k}) = Y_{1/4}$. Define the operators A_1 on $Y_{1/2} \times Y_{1/4} \times Y_{1/2} \times L_2(\mathbb{R})$ with $\mathrm{dom}(A_1) = Y_1 \times Y_{1/2} \times Y_{1/2} \times Y_{1/4}$ and A_2 on $L_2(\mathbb{R}) \times L_2(\mathbb{R}) \times L_2(\mathbb{R}) \times L_2(\mathbb{R})$ with $\mathrm{dom}(A_2) = Y_{1/4} \times Y_{1/4} \times Y_{1/4} \times Y_{1/4}$ by

$$A_1 = \left[\begin{smallmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & I & 0 & 0 \\ M_g & 0 & 0 & 0 \end{smallmatrix}\right] \text{ and } A_2 = \left[\begin{smallmatrix} M_{h_0} & 0 & 0 & 0 \\ 0 & M_{h_1} & 0 & 0 \\ 0 & 0 & M_{h_2} & 0 \\ 0 & 0 & 0 & M_{h_3} \end{smallmatrix}\right].$$

Taking the Fourier transform, we see that A_0 is unitary equivalent to A_1. Diagonalizing, we infer that the operators A_1 and A_2 are similar. Notice that $\mathrm{Re}\, h_0(\xi) \geq 1$ and

$\operatorname{Re} h_2(\xi) \le -1$ for all $\xi \in \mathbb{R}$, which proves that $-M_{h_0}$ and M_{h_2} are generators of uniformly exponentially stable C_0-semigroups on $L_2(\mathbb{R})$. Since $\sup_{\xi\ge 1} \operatorname{Re} h_1(\xi) < 0$ and $\inf_{\xi\le -1} \operatorname{Re} h_1(\xi) > 0$, it follows that M_{h_1} can be represented as the sum of the generator of a stable bi-semigroup and a bounded linear operator on $L_2(\mathbb{R})$, and similarly for M_{h_3}. Hence, A_0 is a sum of the generator A of a stable analytic bi-semigroup, and a bounded operator C_0 on X. Thus, $D(\xi) = A + B(\xi)$, where $B(\xi) = C_0 + B_0(\xi)$ gives a bounded strongly continuous function on $\mathbb{R}$.

The spatial dynamics is also used in [14] to study the Swift-Hohenberg equation with ∂_ξ^2 from (8.6) replaced by the two dimensional Laplacian. Unlike the operator A_0, the resulting operator in [14] has a forth order derivative, and so the choice of the space X is different. $\diamond$

Nonanalytic bi-semigroups.

In this subsection we will give examples of equation (1.2) where the C_0-semigroups $\{T_j(t)\}$, $j = 1, 2$, are not analytic.

Example 8.7 Let $b : \mathbb{R} \times [0, 2\pi] \to \mathbb{R}$ be a bounded continuous function, and consider the perturbed wave equation with dumping,

$$\partial_t^2 u = \partial_\eta^2 u + \gamma \partial_\eta u + b(t, \eta) u, \quad t \in \mathbb{R}, \eta \in [0, 2\pi], \tag{8.9}$$

subject to the periodic in η boundary conditions; here $\gamma \ne 0$. Let $X_0 = L_2([0, 2\pi])$ and consider the operator $A_0 = \partial_\eta^2 + \gamma\partial_\eta + 1$ on X_0 with the maximal domain $H^2_{per}([0, 2\pi])$. If X and A are defined as in Proposition 8.1 and $B : \mathbb{R} \to \mathcal{B}(X)$ is defined by $B(t) = \left[\begin{smallmatrix} 0 & 0 \\ b(t,\cdot)-1 & 0 \end{smallmatrix}\right]$, then equation (8.9) can be written in the form $V' = D(t)V$, $t \in \mathbb{R}$, on $X = H^2_{per}([0, 2\pi]) \times L_2([0, 2\pi])$, where $D(t) = A + B(t)$ with $\operatorname{dom}(A) = H^2_{per}([0, 2\pi]) \times H^1_{per}([0, 2\pi])$. Define $e_k : [0, 2\pi] \to \mathbb{C}$ by $e_k(\eta) = e^{ik\eta}$, $k \in \mathbb{Z}$. Then $\{e_k : k \in \mathbb{Z}\}$ is an orthonormal basis in X_0 and $A_0 e_k = (1-k^2+\gamma k i)e_k$ for $k \in \mathbb{Z}$. Notice that Proposition 8.1 applies, and thus it follows that A is the generator of a uniformly stable bi-semigroup. Moreover, $\sigma(A) = \{\pm(1 - k^2 + \gamma k i)^{1/2} : k \in \mathbb{Z}\}$, which implies that the C_0-semigroups $\{T_j(t)\}_{t\ge 0}$, $j = 1, 2$, are not analytic. It is straightforward to see that $B(\cdot)$ is strongly continuous and bounded. $\diamond$

Example 8.8 Consider the KdV equation

$$\partial_t u = 6u\partial_\xi u - \partial_\xi^3 u, \quad t, \xi \in \mathbb{R}, \tag{8.10}$$

and let $u_0(t, \xi) = 2(\cosh(\xi - 4t))^{-2}$ denote its one-soliton solution. The linearization of equation (8.10) about u_0 is given by

$$\partial_t u = 6u_0\partial_\xi u + 6u\partial_\xi u_0 - \partial_\xi^3 u, \quad t, \xi \in \mathbb{R}. \tag{8.11}$$

Let $b : \mathbb{R}^2 \to \mathbb{R}$ be a bounded and smooth function, and consider the perturbed linearized KdV equation

$$\partial_t u = 6u_0\partial_\xi u + 6u\partial_\xi u_0 - \partial_\xi^3 u + b(t, \xi)u, \quad t, \xi \in \mathbb{R}. \tag{8.12}$$

To bypass the difficulties in handling the first term in the RHS of both (8.11) and (8.12), we reduce (8.12) to a first order system of PDEs by treating ξ as the evolution variable. Using the substitution $v = \partial_\xi u$ and $w = \partial_\xi^2 u$, one can see that equation (8.12) is equivalent to

$$\partial_\xi u = v, \quad \partial_\xi v = w, \quad \partial_\xi w = 6u_0 v + 6(\partial_\xi u_0)u + b(t, \xi)u - \partial_t u. \tag{8.13}$$

Denoting $h(\xi) = (u(\cdot,\xi), v(\cdot,\xi), w(\cdot,\xi))^T$, system (8.13) can be written as $h' = D(\xi)h$, $\xi \in \mathbb{R}$, on $X = H^{2/3}(\mathbb{R}) \times H^{1/3}(\mathbb{R}) \times L_2(\mathbb{R})$, where $D(\xi) = A + B(\xi)$ with $\mathrm{dom}(A) = H^1(\mathbb{R}) \times H^{2/3}(\mathbb{R}) \times H^{1/3}(\mathbb{R})$, a natural choice in view of Proposition 8.1, and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\partial_t + 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B(\xi) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6(\partial_\xi u_0)(\cdot,\xi) + b(\cdot,\xi) & 6u_0(\cdot,\xi) & 0 \end{bmatrix}.$$

Since b, u_0 and $\partial_\xi u_0$ are bounded and continuous functions on $\mathbb{R}$, we have that $B(\cdot)$ is bounded and strongly continuous on X. Next, we claim that A can be represented as the sum of the generator of a uniformly exponentially stable bi-semigroup which is not analytic, and a bounded linear operator.

Let M_g be the operator of multiplication by $g(\xi) = 1 - 2\pi i\xi$ from $Y_{2/3}$ to $L_2(\mathbb{R})$ with $\mathrm{dom}(M_g) = Y_1$ and M_{h_k} are the operators of multiplication by $h_k(\xi) = (1 + 4\pi^2\xi^2)^{1/6}\exp(2k\pi i/3 - i/3\arctan(2\pi\xi))$, $k = 0, 1, 2$, on $L_2(\mathbb{R})$ with $\mathrm{dom}(M_{h_k}) = Y_{1/3}$, where the space Y_α, $\alpha > 0$ was defined in Example 8.6. Define the operators A_1 on $Y_{2/3} \times Y_{1/3} \times L_2(\mathbb{R})$ with $\mathrm{dom}(A_1) = Y_1 \times Y_{2/3} \times Y_{1/3}$ and A_2 on $L_2(\mathbb{R}) \times L_2(\mathbb{R}) \times L_2(\mathbb{R})$ with domain $Y_{1/3} \times Y_{1/3} \times Y_{1/3}$ by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ M_g & 0 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} M_{h_0} & 0 & 0 \\ 0 & M_{h_1} & 0 \\ 0 & 0 & M_{h_2} \end{bmatrix}.$$

Taking the Fourier transform, A is unitary equivalent to A_1. Diagonalizing, A_1 is similar to A_2. Since $\mathrm{Re}\, h_0(\xi) \geq 1$ and $\mathrm{Re}\, h_k(\xi) \leq 0$ for all $\xi \in \mathbb{R}$ and $k = 1, 2$, it follows that $M_{h_1 - 1}$, $M_{h_2 - 1}$ and $-M_{h_0}$ are generators of stable C_0-semigroups. Hence, A_2 can be represented as the sum of the generator of a uniformly exponentially stable bi-semigroup which is not analytic, since $\lim_{\xi\to\infty} \arg(h_2(\xi)) = \pi/2$, and a bounded linear operator. ◇

Example 8.9 Consider the functional equation with backward/forward delay,

$$v'(t) = \sum_{j=-m}^{m} A_j(t)v(t+j), \quad t \in \mathbb{R}, \tag{8.14}$$

where the matrix-valued functions $A_j : \mathbb{R} \to \mathbb{C}^{n\times n}$, $-m \leq j \leq m$, are assumed bounded and continuous. Functional equations of this type arise, e.g., as semi-dicretizations of partial differential equations, see, for instance, [18,29]. Denoting $u(t) = (v(t + \cdot), v(t))$, equation (8.14) can be written as the differential equation $u'(t) = D(t)u(t)$, $t \in \mathbb{R}$, on $X = H^1(\mathbb{R}, \mathbb{C}^n) \times \mathbb{C}^n$, where $D(t) : \mathcal{D} \to X$, $\mathcal{D} = \{(v, z) \in H^2(\mathbb{R}, \mathbb{C}^n) \times \mathbb{C}^n : v(0) = z\}$ and $D(t)(v, z) = (v', A_0(t)z + \sum_{1\leq|j|\leq m} A_j(t)v(j))$. Given any matrices $\tilde{A}_j \in \mathbb{C}^{n\times n}$, $-m \leq j \leq m$, we define a constant coefficient operator $A : \mathcal{D} \to X$ by $A(v, z) = (v', \tilde{A}_0 z + \sum_{1\leq|j|\leq m} \tilde{A}_j v(j))$. However, even for $m = n = 1$, $\tilde{A}_{-1} = \tilde{A}_1 = I$ and $\tilde{A}_0 = 0$, the spectrum of A contains eigenvalues whose real part is arbitrarily large and arbitrarily small, and thus, A is not the generator of a C_0-semigroup. Choose $\tilde{A}_j \in \mathbb{C}^n \times \mathbb{C}^n$, $-m \leq j \leq m$, such that A is the generator of a stable non-analytic bi-semigroup, and let $B : \mathbb{R} \to \mathcal{B}(X)$ be defined by $B(t)(v, z) = (0, (A_0(t) - \tilde{A}_0)z + \sum_{1\leq|j|\leq m} (A_j(t) - \tilde{A}_j)v(j))$. Note that $D(t) = A + B(t)$, $t \in \mathbb{R}$, $B(\cdot)$ is strongly continuous and bounded and $B(t)$ has rank 1 for each $t \in \mathbb{R}$. Specifically, assuming Hypothesis 1.1(as in [18]) and that $\tilde{A}_j = \lim_{t\to\pm\infty} A_j(t)$, or $\int_{\mathbb{R}} \|A_j(t) - \tilde{A}_j\|^2 dt < \infty$, one can apply Proposition 7.6(ii) to prove the exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ of equation (1.2). ◇

An example for Section 5.

In the following example (based on [35]), the subspace $\mathcal{Z}_{-1,1}$ from Proposition 5.3 is not the graph of a linear operator from $\ker P_-(-1)$ to $\ker P_+(1)$. The operator A is not the generator of a C_0-semigroup, but $-A$ is the generator of an analytic C_0-semigroup.

Example 8.10 Let $X = L_2([0,1])$ and $D(t) = -d^2/d\xi^2 - c - b(t)$ with $\mathrm{dom}(D(t)) = H^2([0,1]) \cap H_0^1([0,1])$, where b is a smooth function such that $b(t) = b^-$ for all $t \leq -1$ and $b(t) = b^+$ for all $t \geq 1$ and c is a real constant. Notice that we can write $D(t) = A + B(t)$, where $A = -d^2/d\xi^2 + 1$ and $B(t) : L_2([0,1]) \to L_2([0,1])$, $(B(t)f)(\xi) = (-b(t) - c - 1)f(\xi)$. Since the C_0-semigroup generated by $-A$ is analytic and b is a smooth function, the mild solutions of equation (1.2) are also classical solutions. Moreover, from Proposition 2.1, we have $G = -L$, and since it was proved in [35] that L is a Fredholm operator, we obtain that G is a Fredholm operator. We can choose the projection family $\{P_-(t)\}_{t\leq 0}$ such that $x \in \ker P_-(-1)$ if and only if there exists a solution u of equation (1.2) on $(-\infty, -1]$ satisfying $u(-1) = x$. Denoting $u_k(t) = \langle w(t), e_k\rangle_{L_2}$ and $a_k = \langle x, e_k\rangle_{L_2}$, where $e_k(\xi) = e^{2\pi i\xi}$ for $k \in \mathbb{Z}$, the equation $u'(t) = D(t)u(t)$, $t \leq -1$, with the final condition $u(-1) = x$ is equivalent to the system $u_k'(t) = (4k^2\pi^2 - c + b^-)u_k(t)$, $t \leq -1$, with the final condition $u_k(-1) = a_k$, $k \in \mathbb{Z}$. It follows that $\ker P_-(-1) = \{x \in L_2([0,1]) : \langle x, e_k\rangle_{L_2} = 0 \text{ for all } k \in J_0\}$, where $J_0 = \{k \in \mathbb{Z} : 4k^2\pi^2 - c - b^- \leq 0\}$. Similarly, we can choose the projection family $\{P_+(t)\}_{t\geq 0}$ such that $\ker P_+(1) = \ker P_-(-1)$ and that a pair $(x_1, x_2) \in \mathcal{Z}_{-1,1}$ if and only if $(x_1, x_2) \in \ker P_-(-1) \times \ker P_+(1)$ such that $\langle x_2, e_k\rangle_{L_2} = \langle x_1, e_k\rangle_{L_2} e^{2(4k^2\pi^2 - c - b_*)}$ for all $k \in \mathbb{Z}$, where $b_* = \int_{-1}^1 b(s)ds$. It follows that the projection of $\mathcal{Z}_{-1,1}$ on the first component is the set of all $x \in L_2([0,1])$ such that $\langle x, e_k\rangle_{L_2} = 0$ for all $k \in J_0$ and $\sum_{k\in\mathbb{Z}} e^{16k^2\pi^2}|\langle x, e_k\rangle_{L_2}| < \infty$, which proves that $\mathcal{Z}_{-1,1}$ is not the graph of a linear operator from $\ker P_-(-1)$ to $\ker P_+(1)$. $\diamond$

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